ZERO TEMPERATURE LIMITS OF GIBBS-EQUILIBRIUM STATES FOR COUNTABLE ALPHABET SUBSHIFTS OF FINITE TYPE

O. JENKINSON, R. D. MAULDIN, AND M. URBAŃSKI

Abstract. Let Σ_A be a finitely primitive subshift of finite type on a countable alphabet. For appropriate functions $f : \Sigma_A \to I\!R$, the family of Gibbs-equilibrium states $(\mu_{tf})_{t\geq 1}$ for the functions tf is shown to be tight. Any weak*-accumulation point as $t \to \infty$ is shown to be a maximizing measure for f.

1. Introduction

Let Σ_A be a subshift of finite type on a countably infinite alphabet, and suppose that the function $f : \Sigma_A \to I\!R$ has summable variations. Further assumptions on f ensure it has a unique Gibbs-equilibrium state μ_f (see §2 for more details). The purpose of this article is to analyse the behaviour, as $t \to \infty$, of the Gibbs-equilibrium states μ_{tf} of tf. It will be shown that the family $(\mu_{tf})_{t\geq 1}$ is tight, thereby ensuring that it has a weak* accumulation point. Any such accumulation point is shown to be a maximizing measure for the function f (i.e. its f-integral dominates the integral of f with respect to other shift-invariant probability measures). This extends the analogous results of [CG, CLT, J] which were proved in the setting of finite alphabet subshifts of finite type.

The thermodynamic interpretation (cf. [Ru]) of the parameter t is as an inverse temperature of a system, while the measure μ_{tf} describes the equilibrium of the system at temperature $1/t$ (i.e. the one which minimizes the "free energy"). The $t \to \infty$ limit is therefore a zero temperature limit, and an accumulation point of the μ_{tf} can be interpreted as a *ground state*.

If f has a unique maximizing measure then our result asserts that μ_{tf} will converge to that measure. A more intriguing sitution arises when

Date: October 2004.

The first author was partially supported by an EPSRC Advanced Research Fellowship. He acknowledges the warm hospitality of the University of North Texas, where this research was initiated. The second and third authors were supported in part by the NSF Grant no. DMS 0100078.

there are several maximizing measures, in which case we only know that μ_{tf} will accumulate on some non-empty subset of such measures. However, in all known examples it has been observed that the family μ_{tf} *does converge*, so a natural conjecture is that this is always the case; if this conjecture is true then the limit of the μ_{tf} may be regarded as the most "physically relevant" maximizing measure. This problem is open even for finite alphabet subshifts of finite type, though Brémont [Br] has shown that if f depends on only finitely many coordinates then the μ_{tf} do converge (cf. [C, J, PS] for related results).

2. Preliminaries

Let $\Sigma = \mathbb{N}^{\mathbb{N}}$ denote the full shift on the countable alphabet $\mathbb{N} = \{1, 2, \ldots\}$, equipped with the product topology.

Given an adjacency matrix $A : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$, let Σ_A denote the associated subshift of finite type

$$\Sigma_A = \{x \in \Sigma : A(x_n, x_{n+1}) = 1 \text{ for all } n \geq 1\} .$$

We shall suppose that A is *finitely primitive*, i.e. there exists an integer $N \geq 0$, and a finite sub-alphabet $\mathbb{M} \subset \mathbb{N}$, such that for all $x \in \Sigma_A$ and all $i \in \mathbb{N}_A$ there exists $w \in \mathbb{M}^N$ with $iwx \in \Sigma_A$. This implies that the shift map $T : \Sigma_A \to \Sigma_A$, defined by $(Tx)_n = x_{n+1}$, is topologically mixing.

For $n \in \mathbb{N}$, define $\Pi_n : \Sigma_A \to \mathbb{N}^n$ by $\Pi_n(x) = (x_1, \ldots, x_n)$, and $\pi_n : \Sigma_A \to \mathbb{N}$ by $\pi_n(x) = x_n$. If $w \in \mathbb{N}^n$ then the corresponding *cylinder set* in Σ_A is defined by $[w] = \{x \in \Sigma_A : \Pi_n(x) = w\}$. The subshift of finite type Σ_A is compact if and only if $\mathbb{N}_A := \{i \in \mathbb{N} : [i] \neq \emptyset\}$ is finite.

We shall assume that $f : \Sigma_A \to \mathbb{R}$ has *summable variations*, i.e. that

$$V(f) := \sum_{n=1}^{\infty} \mathrm{var}_n(f) < \infty \,, \tag{1}$$

where

$$\mathrm{var}_n(f) = \sup_{\Pi_n(x) = \Pi_n(y)} |f(x) - f(y)| \,.$$

In particular this implies that f is uniformly continuous (though not necessarily bounded, since $\mathrm{var}_0(f) = \sup_{x,y \in \Sigma_A} |f(x) - f(y)|$ is not included in the above sum).

We also assume that

$$\sum_{i \in \mathbb{N}} \exp(\sup f|_{[i]}) < \infty \,, \tag{2}$$

so in particular f is bounded above, and is unbounded below if and only if $\mathbb{N}_A$ is infinite. The summability condition (2) allows much of the thermodynamic formalism (cf. [Bo, Ru]) for finite alphabet subshifts of finite type to be generalised to the non-compact setting[1]. In particular it is equivalent [MU, Prop. 2.1.9] to the finiteness of the topological pressure

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n y = y} \exp\left(\sup_{x\in[\Pi_n(y)]} \sum_{i=0}^{n-1} f(T^i x) \right) ,$$

and implies the variational characterisation [MU, Thm. 2.1.8]

$$P(f) = \sup\{h(m) + \int f\, dm : m \in \mathcal{M}, \int f\, dm > -\infty\} , \qquad (3)$$

where $\mathcal{M}$ denotes the set of T-invariant Borel probability measures on Σ_A, and $h(m)$ the metric entropy of $m \in \mathcal{M}$.

Moreover, there is (see [MU, Theorems 2.2.4 and 2.3.3]) a unique measure $\mu_f \in \mathcal{M}$ for which there exist constants $C_2 > C_1 > 0$ such that

$$C_1 \le \frac{\mu_f[\Pi_n(x)]}{\exp(\sum_{i=0}^{n-1} f(T^i x) - nP(f))} \le C_2 \qquad (4)$$

for all $x \in \Sigma_A$, $n \ge 1$. In fact [MU, Thm. 2.2.7] we may choose

$$C_2 = \exp\left(4V(f)\right) .$$

The measure μ_f is called the *Gibbs state* for f.

Suppose furthermore that[2]

$$\sum_{i\in\mathbb{N}} \inf(-f|_{[i]}) \exp(\inf f|_{[i]}) < \infty , \qquad (5)$$

so that in particular $\int f\, d\mu_f > -\infty$.

In this case (see [MU, Lem. 2.2.8, Thm. 2.2.9]) μ_f is an *equilibrium state* for f, in the sense that

$$P(f) = h(\mu_f) + \int f\, d\mu_f . \qquad (6)$$

Indeed it is the *unique* equilibrium state for f: if $m \in \mathcal{M} \setminus \{\mu_f\}$ is any other invariant measure with $\int f\, dm > -\infty$, then $h(m) + \int f\, dm <$

---

[1] Our reference to this generalised thermodynamic formalism is [MU] (though see also [Sa1, Sa2]), in which f is assumed to be locally Hölder (i.e. $\mathrm{var}_n(f) \to 0$ exponentially fast). The proofs in [MU] can, however, be easily adapted to the more general case where f has summable variations.

[2] Note that the lefthand side of (5) is well-defined: (2), together with the fact that $\mathrm{var}_1(f) < \infty$, implies that $\inf(-f|_{[i]}) \to \infty$, so that $\inf(-f|_{[i]})$ is positive except for finitely many i.

$P(f)$. Since μ_f is both the unique Gibbs state and the unique equilibrium state for f, we shall henceforth refer to it as the *Gibbs-equilibrium state* for f.

Since f satisfies (1), (2) and (5), so does the function tf for any $t \geq 1$. It follows that each such tf also has a unique Gibbs-equilibrium state μ_{tf}.

A *maximizing measure* for f is a measure $\mu \in \mathcal{M}$ such that $\int f\,d\mu \geq \int f\,dm$ for all $m \in \mathcal{M}$. Our assumptions on f ensure (see [JMU]) that this definition of a maximizing measure is equivalent to requiring that

$$\int f\,d\mu = \sup_{x\in\Sigma_A} \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)\,.$$

They also ensure (see [JMU]) the *existence* of a maximizing measure. The set of maximizing measures for f, which in general is not a singleton, will be denoted $\mathcal{M}_{\max}(f)$. The general properties of a maximizing measure are rather different from those of a Gibbs-equilibrium state. For example the support of a Gibbs-equilibrium state is always the full space Σ_A, whereas a maximizing measure has full support only in the trivial situation where f is cohomologous to a constant (i.e. $f = c+\varphi\circ T-\varphi$ for some $c \in I\!R$ and some bounded continuous φ). This latter fact is because if f is as above then there exists a bounded continuous φ such that the set of maxima of $f+\varphi-\varphi\circ T$ contains the support of a T-invariant measure (see [JMU]).

3. Proofs of results

To prove our main result, Theorem 1, we first require two preparatory lemmas. For the first of these we only use the fact that f is continuous and bounded above.

Lemma 1. *The map*

$$\mathcal{M} \longrightarrow I\!R$$
$$\mu \longmapsto \int f\,d\mu$$

is upper semi-continuous with respect to the weak topology on $\mathcal{M}$.*

Proof. Suppose that $\mu_i \to \mu$ in the weak* topology. That is, $\int g\,d\mu_i \to \int g\,d\mu$ for all bounded continuous functions g. We must prove that

$$\limsup_{i\to\infty} \int f\,d\mu_i \leq \int f\,d\mu\,. \tag{7}$$

Let $f_k \searrow f$ be a sequence of bounded continuous functions converging pointwise to f, for example $f_k = \max(f,-k)$. If $I \in I\!R$ is such that $I > \int f d\mu$ then $\int f_k \, d\mu < I$ for all sufficiently large $k \geq 1$, by the monotone convergence theorem. Choose one such k, and let $\delta > 0$ be arbitrary. Since f_k is a bounded continuous function, and $\mu_i \to \mu$ in the weak* topology, $\int f_k \, d\mu > \int f_k \, d\mu_i - \delta$ for all i sufficiently large. But $\int f_k \, d\mu_i \geq \int f \, d\mu_i$ since $f_k \geq f$, hence

$$I > \int f_k \, d\mu > \int f_k \, d\mu_i \, - \delta \geq \int f d\mu_i \, - \delta$$

for all i sufficiently large. But $\delta > 0$ and $I > \int f \, d\mu$ were arbitrary, so in fact

$$\int f \, d\mu \geq \int f \, d\mu_i \tag{8}$$

for all i sufficiently large, and (7) follows. □

For the second lemma we only use the fact that the μ_{tf} are *Gibbs* states.

Lemma 2. *The family of Gibbs-equilibrium states $(\mu_{tf})_{t\geq 1}$ is tight, i.e. for all $\varepsilon > 0$ there exists a compact set $K \subset \Sigma_A$ such that $\mu_{tf}(K) > 1 - \varepsilon$ for all $t \geq 1$.*

Proof. Given $\varepsilon > 0$, we will find an increasing sequence (n_k) in $\mathbb{N}$ such that the compact set $K = \{x \in \Sigma_A : 1 \leq x_k \leq n_k \ \forall k \in \mathbb{N}\}$ satisfies $\mu_{tf}(K) > 1 - \varepsilon$ for all $t \geq 1$. Now

$$\begin{aligned}
\mu_{tf}(K) &= \mu_{tf}\left(\Sigma_A \setminus \cup_{k=1}^{\infty} \{x \in \Sigma_A : x_k > n_k\}\right) \\
&\geq 1 - \sum_{k=1}^{\infty} \mu_{tf}\left(\{x \in \Sigma_A : x_k > n_k\}\right) \\
&= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_{tf}(\pi_k^{-1}(i)) \\
&= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_{tf}[i] \,,
\end{aligned}$$

so to ensure that $\mu_{tf}(K) > 1 - \varepsilon$ it suffices to choose the integers n_k such that

$$\sum_{i=n_k+1}^{\infty} \mu_{tf}[i] < \frac{\varepsilon}{2^k} \quad \text{for all } k \in \mathbb{N}, t \geq 1 \,. \tag{9}$$

We now show that such a choice is possible. First, the Gibbs property (4), with $n = 1$ and f replaced by tf, gives

$$\mu_{tf}[i] \leq e^{4tV(f)} \exp\left(\sup\{tf|_{[i]}\} - P(tf)\right) \,. \tag{10}$$

Now let $m \in \mathcal{M}$ be any measure for which $I := \int f\, dm$ is finite (e.g. we may take m to be supported on a periodic orbit). From (3) we have

$$P(tf) - tI = P\left(t(f-I)\right) \geq \int t(f-I)\, dm + h(m) \geq 0\,,$$

so together with (10) we deduce that

$$\begin{aligned} \mu_{tf}[i]_A &\leq e^{4tV(f)} \exp\left(\sup\{t(f-I)|_{[i]}\}\right) e^{-P(t(f-I))} \\ &\leq e^{4tV(f)} \exp\left(\sup\{t(f-I)|_{[i]}\}\right) \qquad (11) \\ &= \exp\left(t\left(4V(f) - I + \sup\{f|_{[i]}\}\right)\right)\,. \end{aligned}$$

The summability condition (2) implies that $\sup f|_{[i]} \to -\infty$ as $i \to \infty$, with the convention that $f|_{[i]} = -\infty$ if $[i] = \emptyset$. In particular there exists $J \in I\!N$ such that if $i \geq J$ then

$$4V(f) - I + \sup f|_{[i]} < 0\,.$$

So if $t \geq 1$ and $i \geq J$ then $t(4V(f) - I + \sup f|_{[i]}) < 4V(f) - I + \sup f|_{[i]} < 0$, and from (11) we obtain

$$\mu_{tf}[i] \leq e^{4V(f)-I} e^{\sup f|_{[i]}}\,. \qquad (12)$$

The summability condition (2) means there exists $n_k \geq J$ such that

$$\sum_{i=n_k+1}^{\infty} e^{\sup f|_{[i]}} < \frac{\varepsilon}{2^k} e^{I-4V(f)}\,,$$

and combined with (12) we deduce (9), as required. □

Theorem 1. *The family of Gibbs measures $(\mu_{tf})_{t\geq 1}$ has a weak* accumulation point as $t \to \infty$. Any such accumulation point μ is a maximizing measure for f, and $\int f\, d\mu = \lim_{t\to\infty} \int f\, d\mu_{tf}$.*

Proof. By Lemma 2 the family $(\mu_{tf})_{t\geq 1}$ is tight, so by Prohorov's theorem [Bi, p. 37] there exists at least one weak* accumulation point.

Now suppose μ is any such accumulation point. If $p(t) = P(tf)$ for $t \geq 1$ then $p'(t) = \int f\, d\mu_{tf}$ (cf. [MU, Prop. 2.6.13]). But (3) implies that p is convex, so that $t \mapsto p'(t) = \int f\, d\mu_{tf}$ is non-decreasing, and bounded above by $\sup f$. It follows that the limit $\lim_{t\to\infty} p'(t) = \lim_{t\to\infty} \int f\, d\mu_{tf}$ *exists* and is *finite*. Moreover, Lemma 1 gives

$$\lim_{t\to\infty} \int f\, d\mu_{tf} \leq \int f\, d\mu\,. \qquad (13)$$

In particular $\int f\, d\mu > -\infty$. From (3) and (6) it follows that

$$\int tf\, d\mu_{tf} + h(\mu_{tf}) \geq \int tf\, d\mu + h(\mu)\,,$$

so

$$\int f\,d\mu_{tf} + \frac{h(\mu_{tf})}{t} \geq \int f\,d\mu + \frac{h(\mu)}{t}\,. \tag{14}$$

Now $h(\mu_{tf}) = P(tf) - t\int f\,d\mu_{tf} = p(t) - tp'(t)$, so

$$\frac{d}{dt}h(\mu_{tf}) = -tp''(t) < 0$$

for $t \geq 1$. Therefore $h(\mu_{tf})$ is a decreasing function of $t \geq 1$, and in particular is bounded, so letting $t \to \infty$ in (14) gives

$$\lim_{t\to\infty} \int f\,d\mu_{tf} \geq \int f\,d\mu\,.$$

Combining this with (13) we see that $\lim_{t\to\infty}\int f\,d\mu_{tf} \geq \int f\,d\mu$, as required.

We now show that μ is f-maximizing. If not then there exists $\nu \in \mathcal{M}$ with $\int f\,d\nu - \int f\,d\mu = \varepsilon > 0$. Now f is bounded above, so $\int f\,d\nu < \infty$. Moreover $P(f) < \infty$, so (3) and (6) imply that $h(\nu) < \infty$. We can then define the affine map $l_\nu : I\!R \to I\!R$ by $l_\nu(t) = h(\nu) + t\int f\,d\nu$. Now $t \mapsto p'(t) = \int f\,d\mu_{tf}$ is a function which increases to its limit $\int f\,d\mu$, so in particular

$$\int f d\mu \geq \int f\,d\mu_{tf} = p'(t) \quad \text{for all } t \geq 1\,,$$

and hence

$$l'_\nu(t) = \int f\,d\nu = \int f\,d\mu + \varepsilon \geq p'(t) + \varepsilon$$

for all $t \geq 1$. Therefore $l_\nu(t) > p(t)$ for all sufficiently large t. That is, $h(\nu) + \int tf\,d\nu > P(tf)$ for all sufficiently large t, contradicting (3). Therefore μ is f-maximizing. □

Note that in the case of a finite alphabet subshift of finite type Σ_A, the identity $\int f\,d\mu = \lim_{t\to\infty}\int f\,d\mu_{tf}$ in Theorem 1 follows immediately from the fact that μ is a weak* accumulation point of μ_{tf}, since the continuous function f is automatically bounded on the compact space Σ_A.

In the finite alphabet case some extra information is known about μ: it is of maximal entropy within the class of f-maximizing measures (see [CG, CLT, J]). In the infinite alphabet case this is an open problem:

Question 1. *If μ is a weak* accumulation point of $(\mu_{tf})_{t\geq 1}$, is it the case that*

$$h(\mu) = \sup_{m\in\mathcal{M}_{max}(f)} h(m)\ ? \tag{15}$$

An approach to proving (15) is to first show that

$$h(\mu) = \lim_{t\to\infty} h(\mu_{tf}) = \inf_{t\geq 1} h(\mu_{tf})\,. \tag{16}$$

The second equality in (16) is certainly true in the inifinite alphabet case, since $t \mapsto h(\mu_{tf})$ is decreasing (as noted in the proof of Theorem 1), and bounded below. Moreover

$$\lim_{t\to\infty} h(\mu_{tf}) \geq h(\mu)\,, \tag{17}$$

since μ_{tf} is the equilibrium state for tf, while μ is f-maximizing and hence tf-maximizing for $t \geq 0$, so

$$h(\mu_{tf}) - h(\mu) \geq \int tf\,d\mu - \int tf\,d\mu_{tf} \geq 0$$

for all $t \geq 1$.

We do not know, however, if *equality* holds in (17):

Question 2. *If μ is a weak* accumulation point of $(\mu_{tf})_{t\geq 1}$, is it the case that $h(\mu) = \lim_{t\to\infty} h(\mu_{tf})$?*

As noted above, in the finite alphabet case the answer is affirmative; this is proved by combining (17) with the well known fact [Wa, Thm. 8.2] that the entropy map $\nu \mapsto h(\nu)$ is upper semi-continuous on $\mathcal{M}$.

By contrast, for infinite alphabet subshifts of finite type the entropy map is in general *not* upper semi-continuous. To see this, let Σ be the full shift on $\mathbb{N}$, and define the probability vector P_n by

$$P_n = (1 - n^{-1}\,,\, \underbrace{(nk_n)^{-1}, \ldots, (nk_n)^{-1}}_{k_n \text{ terms}}, 0, 0, \ldots)\,,$$

where $k_n = \lceil e^{n^2} \rceil$. Let μ_n be the Bernoulli measure corresponding to P_n (so the support of μ_n is the full shift on the symbols $\{1, \ldots, n+1\}$). Then

$$h(\mu_n) \;=\; -(1 - n^{-1}) \log(1 - n^{-1}) + n^{-1} \log(nk_n) \;>\; n\,.$$

In particular $h(\mu_n) \to \infty$ as $n \to \infty$, whereas the weak* limit of (μ_n) is the Dirac measure concentrated on the fixed point $(1, 1, \ldots)$. This measure has zero entropy, so the entropy map is not upper semi-continuous.

Of course this absence of upper semi-continuity does not rule out an affirmative answer to Question 2. In this case Question 1 could also be answered affirmatively, by the following argument. If $h(\mu) = \sup_{m\in\mathcal{M}_{\max}(f)} h(m)$ were *not* true then we could find $m \in \mathcal{M}_{\max}(f)$

with $h(m) - h(\mu) = \varepsilon > 0$. The affirmative answer to Question 2 then gives

$$h(m) - \lim_{t\to\infty} h(\mu_{tf}) = \varepsilon\,,$$

so that

$$h(m) - h(\mu_{tf}) \geq \frac{\varepsilon}{2} \tag{18}$$

for sufficiently large $t \geq 1$.

But m is f-maximizing, so $\int f\,dm \geq \int f\,d\mu_{tf}$ for all $t \geq 1$, and therefore

$$\int tf\,dm \geq \int tf\,d\mu_{tf} \tag{19}$$

for all $t \geq 1$. Combining (18) and (19) gives

$$h(m) + \int tf\,dm > h(\mu_{tf}) + \int tf\,d\mu_{tf}$$

for $t \geq 1$ sufficiently large. But this is a contradiction, because μ_{tf} is an equilibrium state for the function tf.

References

[Bi] P. Billingsley, *Convergence of probability measures*, John Wiley & Sons, New York, 1968.

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Mathematics vol. 470, Berlin-Heidelberg-New York, Springer-Verlag, 1975.

[Br] J. Brémont, *On the behaviour of Gibbs measures at temperature zero*, *Nonlinearity* **16** (2003), 419–426.

[C] Z. N. Coelho, Entropy and ergodicity of skew-products over subshifts of finite type and central limit asymptotics, *Ph.D. Thesis*, Warwick University, (1990).

[CLT] G. Contreras, A. O. Lopes, & Ph. Thieullen, Lyapunov minimizing measures for expanding maps of the circle, *Ergod. Th. Dyn. Sys.*, **21** (2001), 1379–1409.

[CG] J.-P. Conze & Y. Guivarc'h, Croissance des sommes ergodiques, *manuscript*, circa 1993.

[J] O. Jenkinson, Geometric barycentres of invariant measures for circle maps, *Ergod. Th. & Dyn. Sys.*, **21** (2001), 511–532.

[JMU] O. Jenkinson, R. D. Mauldin & M. Urbański, Ergodic optimization for non-compact dynamical systems, *preprint.*

[MU] R. D. Mauldin & M. Urbański, *Graph directed Markov systems: geometry and dynamics of limit sets*, Cambridge University Press, 2003.

[PS] M. Pollicott & R. Sharp, Rates of recurrence for $\mathbb{Z}^q$ and $\mathbb{R}^q$ extensions of subshifts of finite type, *Jour. London Math. Soc.*, **49** (1994), 401–416.

[Ru] D. Ruelle, *Thermodynamic Formalism*, Encyclopaedia of Mathematics and its Applications, vol. 5, Addison-Wesley, 1978.

[Sa1] O. Sarig, Thermodynamic formalism for countable Markov shifts, *Ergod. Th. Dyn. Sys.*, **19** (1999), 1565-1593.

[Sa2] O. Sarig, Characterization of existence of Gibbs measures for countable Markov shifts, *Proc. Amer. Math. Soc.*, **131** (2003), 1751–1758.
[Wa] P. Walters, *An introduction to ergodic theory*, Springer, 1981.

Oliver Jenkinson; School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK
omj@maths.qmul.ac.uk, http://www.maths.qmul.ac.uk/∼omj

R. Daniel Mauldin; Department of Mathematics, University of North Texas, P.O. Box 311430, Denton, TX 76203-1430, USA
mauldin@unt.edu, http://www.math.unt.edu/∼mauldin

Mariusz Urbański; Department of Mathematics, University of North Texas, P.O. Box 311430, Denton, TX 76203-1430, USA
urbanski@unt.edu, http://www.math.unt.edu/∼urbanski