THE FINER GEOMETRY AND DYNAMICS OF THE HYPERBOLIC EXPONENTIAL FAMILY

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Abstract. We consider the maps $f : \mathbb{C} \to \mathbb{C}$ from the exponential family $\{\lambda e^z\}$ that have attracting periodic orbits. We prove that $J_r(f_\lambda)$ (the subset of the Julia set consisting of points that do not escape to infinity under forward iterates of f) has the Hausdorff dimension h_λ less than 2, that the h_λ-dimensional Hausdorff measure of $J_r(f_\lambda)$ is positive and finite on each horizontal strip, and that the h_λ-dimensional packing measure of $J_r(f_\lambda)$ is locally infinite at each point of $J_r(f_\lambda)$. We introduce as our main technical device some map F defined on some strip P of height 2π. This map carries enough information about the dynamics of f to study f itself. In particular the existence and uniqueness of a probability conformal measure m (with an exponent greater than 1) for F and a σ-finite conformal measure for f is proven. We also prove the existence and uniqueness of a Borel probability F-invariant ergodic measure equivalent with the conformal measure m.

1. **Introduction**

Given $\lambda \in \mathbb{C} \setminus \{0\}$ let the entire function $f_\lambda : \mathbb{C} \to \mathbb{C}$ be defined by the formula

$$f_\lambda(z) = \lambda e^z.$$

C. McMullen proved in [McM] that the Hausdorff dimension of the set of points escaping to infinity under forward iterates of f_λ is equal to 2. In this paper we thoroughly investigate the geometric (fractal) and dynamical structure of the complement (in the Julia set $J(f_\lambda)$) of this set which will be denoted in the sequel by $J_r(f_\lambda)$. Although our results apply to all functions f_λ with attracting periodic cycles, [1] we perform our analysis in great detail assuming that $\lambda \in (0, 1/e)$ and treat the general case briefly in Section 6. Since f is periodic with period $2\pi i$, it is natural to identify points which differ by $2k\pi i$ and to consider (instead of f) the map F, our main technical device, defined on some strip P of height 2π. Armed with the map F and the concept of tightness we prove the existence and uniqueness of a probability conformal measure m (with an exponent greater than 1) for F and a σ-finite conformal measure for f. This powerful tool enables us in turn to prove that h_λ, the Hausdorff dimension of the set $J_r(f_\lambda)$, is less than 2, that the h_λ-dimensional Hausdorff measure of $J_r(f_\lambda)$ is positive and finite on each horizontal strip, and that the h_λ-dimensional packing measure of $J_r(f_\lambda)$ is locally infinite at each point of $J_r(f_\lambda)$.

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[1] In a forthcoming paper we treat in the same spirit a large class of non-hyperbolic functions f_λ, including the case when $\lambda \in [1/e, \infty)$.

The fact that $h_\lambda < 2$ shows in particular that the equality of the hyperbolic dimension and the Hausdorff dimension, conjectured in the theory of iteration of rational functions, fails in the context of transcendental entire functions.

Turning towards dynamics, we prove the existence and uniqueness of a Borel probability F-invariant ergodic measure equivalent with the conformal measure m. We do this by applying first the method of Marco Martens to show the existence of a σ-finite F-invariant conservative ergodic measure equivalent with the measure m and checking then that this measure is finite.

Our paper is organized as follows. In Section 2 we prove that for every λ the Hausdorff dimension of the set $J_{bd}(f_\lambda) = \{z \in J(f_\lambda) : \{f^n(z)\}\text{is bounded}\}$ is larger than one. This does not require any assumption about hyperbolicity. We need this fact (which, itself, seems interesting) in Sections 2 and 6 for the proof of the existence of conformal measure and in Section 5 for the existence of a Borel probability F-invariant ergodic measure equivalent with the conformal measure. Notice that Theorem 2.1 was already proved in [Ka] for the case of an attracting fixed point with λ real. In Sections 3-5 we give the detailed proofs of the result described above in the case when f_λ has an attracting fixed point and λ is real. In Section 6 we show how to modify the arguments to make them work in the general case of an attracting periodic orbit. In the Appendix we provide an alternative direct proof of the fact that the Hausdorff dimension of the set $J_r(f_\lambda)$, is less than 2 without using the concept of conformal measures.

2. Bounded Orbits

Let

$$f_\lambda(z) = \lambda e^z, \ \lambda \neq 0.$$

We shall prove the following.

Theorem 2.1. *If $J_{bd}(f_\lambda)$ is the set of all points $z \in J(f)$ such that $\{f_\lambda^n(z)\}_{n\geq 0}$, the forward orbit of z, is bounded, then* $\mathrm{HD}(J_{bd}(f_\lambda)) > 1$.

Proof. Let $\log\lambda$ be the logarithm of λ satisfying $\mathrm{Im}\log\lambda \in (-\pi, \pi]$. Fix $R > 0$ and consider the square

$$S_R = (R, 2R) \times (R, 2R).$$

Let $\Pi = \{z \in \mathbb{C} : 0 \leq \mathrm{Arg}(z) \leq \pi/2\}$ be the first quadrant. For every $k \in \mathbb{Z}$ consider $l_k : \Pi \to \mathbb{C}$, the holomorphic branch of the map inverse to the map $z \mapsto \lambda e^z$ given by the formula

$$l_k(z) = -\log\lambda + \log|z| + i\mathrm{Arg}(z) + 2\pi ik, \ 0 \leq \mathrm{Arg}(z) \leq \pi/2.$$

If $R > e^{|\log\lambda|}$ and $k \geq 1$, then $l_k(S_R) \subset \Pi$ and for every $j \in \mathbb{Z}$

$$\mathrm{Re}(l_j(l_k(z)) = \log|l_k(z)| - \log|\lambda| = \log\Big| -\log\lambda + \log|z| + i\mathrm{Arg}(z) + 2\pi ik\Big| - \log|\lambda|.$$

Define the set I_R to be

$$I_R = \Big\{k \geq 1 : R < \log\Big(-|\log\lambda| + |\log(\sqrt{2}R) + 2\pi ik|\Big) - \log|\lambda| < \log\left(|\log\lambda| + \left|\log(2\sqrt{2}R) + \frac{5\pi}{2}ik\right|\right) - \log|\lambda| < 2R\Big\}$$

and for every $k \in I_R$ put

$$I_{R,k} = \{j \geq 1 : R + 2\pi \leq 2\pi j < 2R - 2\pi\}$$

Notice that for every $k \in I_R$, every $j \in \boldsymbol{Z}$ and every $z \in S_R$, $R < \mathrm{Re}(l_j(l_k(z)) < 2R$ and if $j \in I_{R,k}$, then

$$\mathrm{cl}(l_j \circ l_k(S_R)) \subset S_R.$$

We have produced in this way the finite family of maps

$$G_R = \{l_j \circ l_k : S_R \to S_R\}_{k \in I_R, j \in I_{R,k}}.$$

Each map $g \in G_R$ maps S_R conformally onto some topological disk, whose closure is contained in S_R. Moreover, there exists a neighbourhood $V \supset S_R$ such that each map $g \in G_R$ extends conformally to U and is easy to see that

$$\mathrm{cl}((l_j \circ l_k)(S_R)) \cap \mathrm{cl}((l_{j'} \circ l_{k'})(S_R)) = \emptyset$$

if $(j,k) \neq (j',k')$. Indeed, applying (for various $k \in I_R$) l_k to S_R we obtain a collection of topological disks, each of them being an image of the other by a translation $z \mapsto z + 2m\pi i$ for some $m \in \mathbb{Z}$. Each of these disks is contained in some horizontal strip of height $\frac{\pi}{2}$. So, it is obvious that they are disjoint and there exists a neighbourhood $V \supset S_R$ so that l_k extend conformally to V and $l_k(V) \cap l_{k'}(V) = \emptyset$. The sets $l_j(l_k(V)) \cap l_{j'}(l_{k'}(V))$ are disjoint for $k \neq k'$ since already $l_k(V), l_{k'}(V)$ were disjoint. Also, $l_j(l_k(V)) \cap l_{j'}(l_k(V)) = \emptyset$ for $j \neq j'$ because $l_j, l_{j'}$ are different branches of f_λ^{-1}. We define the compact J_R as follows.

$$J_R = \bigcap_{n \geq 0} \bigcup_{g^n} g^n(S_R)$$

where we take the union over all possible compositions

$$g^n = g_{i_1} \circ \cdots \circ g_{i_n},\ g_{i_1}, \ldots, g_{i_n} \in G_R.$$

The map $f_\lambda|_{J_R} : J_R \to J_R$ is a conformal expanding repellor. In addition, it is easy to see that J_R is a Cantor set. For every $t \in I\!R$ the topological pressure $\mathrm{P}_R(t)$ of the potential $-t\log|f_\lambda'|$ with respect to the reppellor $f_\lambda|_{J_R} : J_R \to J_R$ can be calculated as follows.

$$\mathrm{P}_R(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{g^n} ||(g^n)'||^t,$$

where, again, we sum up over all possible compositions

$$g^n = g_{i_1} \circ \cdots \circ g_{i_n},\ g_{i_1}, \ldots, g_{i_n} \in G_R.$$

It is well-known (see [PU], comp. [Bo]) that the Hausdorff dimension $t = \mathrm{HD}(J_R)$ of J_R is determined as the unique $t \in I\!R$ for which $\mathrm{P}_R(t) = 0$. Since the function $t \mapsto \mathrm{P}_R(t)$ is strictly decreasing, in order to prove that $\mathrm{HD}(J_R) > 1$, it is enough to show that $\mathrm{P}_R(1) > 0$. Indeed, for $z \in S_R$ and all $k \in I_R$, $j \in I_{R,k}$, we have

$$
\begin{aligned}
|(l_j \circ l_k)'(z)| = & \frac{1}{|l_k(z)| \cdot |z|} \geq \frac{1}{2\sqrt{2}R| - \log\lambda + \log|z| + i\mathrm{Arg}(z) + 2k\pi i|} \geq \\
& \frac{1}{2\sqrt{2}R(|\log\lambda| + |\log|z| + i\mathrm{Arg}(z) + 2k\pi i|)} \geq \\
& \frac{1}{2\sqrt{2}R(|\log\lambda| + |\log|z| + \frac{5}{2}k\pi i|)}
\end{aligned} \tag{2.1}
$$

Let $|(l_j \circ l_k)'| = \inf\{|(l_j \circ l_k)'(z)| : z \in S_R\}$. Fix $t \geq 0$. Then by (2.1)

$$
\begin{aligned}
\mathrm{P}_R(t) &\geq \log \sum_{k \in I_R} \sum_{j \in I_{R,k}} |(l_j \circ l_k)'|^t \\
&\geq \log \sum_{k \in I_R} \left(\frac{1}{2\sqrt{2}R}\right)^t \# I_{R,k} \left| |\log\lambda| + \log(2\sqrt{2}R) + \frac{5}{2}\pi i k\right|^{-t} \\
&\geq t\log\left(\frac{1}{2\sqrt{2}}\right) - t\log R + \log\left(\frac{R}{4\pi}\right) + \log \sum_{k \in I_R} \left(|\log\lambda| + |\log(2\sqrt{2}R) + \frac{5}{2}\pi k|\right)^{-t},
\end{aligned}
$$

where we have used inequality $\# I_{R,k} \geq \frac{R}{4\pi}$, true for all R large enough. It follows from the definition of I_R that $\left(|\log\lambda| + |\log(2\sqrt{2}R) + \frac{5}{2}\pi i k|\right) \leq 4\pi k$, $\min(I_R) \leq \exp\left(\frac{5}{4}R\right)$ and $\max(I_R) \geq \exp\left(\frac{3}{2}R\right)$ for all R sufficiently large. Hence

$$
\begin{aligned}
\mathrm{P}_R(t) &\geq t\log\left(\frac{1}{2\sqrt{2}}\right) - t\log R + \log R - \log(4\pi) + \log \sum_{k=\exp\left(\frac{5}{4}R\right)}^{\exp\left(\frac{3}{2}R\right)} (4\pi k)^{-t} \\
&= t\log\left(\frac{1}{2\sqrt{2}}\right) - \log(4\pi) + \log R - t\log R - t\log(4\pi) + \log \sum_{k=\exp\left(\frac{5}{4}R\right)}^{\exp\left(\frac{3}{2}R\right)} k^{-t}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{P}_R(1) &\geq \log\left(\frac{1}{2\sqrt{2}}\right) - 2\log(4\pi) + \log \sum_{k=\exp\left(\frac{5}{4}R\right)}^{\exp\left(\frac{3}{2}R\right)} k^{-1} \\
&\geq \log\left(\frac{1}{2\sqrt{2}}\right) - 2\log(4\pi) + \log\left(\log\exp\frac{3}{2}R - \log\exp\frac{5}{4}R - C\right) \\
&= \log\left(\frac{1}{2\sqrt{2}}\right) - 2\log 4\pi + \log(\frac{1}{4}R - C)
\end{aligned}
$$

where $C > 0$ is a universal constant. It therefore follows that $\mathrm{P}_R(1) > 0$ for R large enough and, consequently, $\mathrm{HD}(J_R) > 1$. By the definition of the set J_R we have $J_R \subset \{z : f_\lambda^{2n}(z) \in S_R \text{ for all } n \geq 0\}$. Since $|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}(z)}$, we conclude that the forward orbit of each point in J_R is bounded for every $R > 0$. Since J_R is contained in the closure of fixed points (which are necessary contracting) of all compositions of maps forming the system G_R, it is contained in the closure of repelling periodic points of f which in turn is contained in $J(f)$. Hence $J_R \subset J(f)$. Thus $\mathrm{HD}(J_{bd}(f_\lambda)) > 1$. ■

We would like to notice that this result overlaps with those proven in [Ka]. More precisely, it follows from Theorem 2 in [Ka] (even though it is not stated explicitly there) that for $\lambda \in (0, \frac{1}{e})$ we have $\mathrm{HD}(J_{bd}(f_\lambda)) > 1$. Unlike [Ka] however we do not assume that λ is real and belongs to $(0, 1/\mathrm{e})$ nor that there exists an attracting fixed point of f.

The following observation ending this section can be deduced from Theorem 2 in [Ka].

Corollary 2.2. *If $\lambda \in (0, \infty)$, then*

$$\lim_{\lambda \to 0} \mathrm{HD}(J_{bd}(f_\lambda)) = 1.$$

3. Existence of conformal measure

From now on until the last section we assume that $\lambda \in (0, 1/\mathrm{e})$. Then $f = f_\lambda$ has a unique attracting fixed point, $0 \in A_\lambda$, the basin of its immediate attraction and $f_\lambda|_{\mathbb{R}}$ has another (positive, repelling) fixed point which we denote by $q = q_\lambda$. Standard straightforward calculations show that

$$\{z : \mathrm{Re}(z) < q_\lambda\} \subset A_\lambda.$$

Let

$$P = \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) \leq \pi\}$$

and let

$$P_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq q \text{ and } \mathrm{Im}(z) \in (-\pi, \pi]\}.$$

Fix $M > q_\lambda$ and set

$$P_M = \{z \in P : q_\lambda \leq \mathrm{Re}(z) \leq M\}.$$

Let

$$\pi_0 : \mathbb{C} \to P$$

be the projection given by $\pi_0(z) = w$ if and only if $w \in P$ and $\exp(z) = \exp(w)$. We define the map $F = F_\lambda : P \to P$ we intend to work with by the formula

$$F(z) = \pi_0(f(z)) \tag{3.1}$$

In this section we construct a conformal measure for the map $F : P \cap J(f) \to P \cap J(f)$. Recall that a Borel measure m is called t conformal (with $t > 0$) if for any Borel set $A \subset P$ on which F is injective, we have

$$m(F(A)) = \int_A |F'|^t dm.$$

We will frequently use throughout the entire paper the following obvious fact without explicit invoking it.

Theorem 3.1. *For any conformal measure m for $F : J(F) \to J(F)$ and any non-empty open subset U of $J(F)$ (in the relative topology on $J(F)$), $m(U) > 0$.*

Here, instead of the rectangle P_M we consider a slightly modified rectangle. Indeed, notice that there exists $p < q$ so close to q that for every $M > q$, the set

$$\tilde{P}_M = \{z \in P : -\frac{3}{4}\pi < \mathrm{Im} z < \frac{3}{4}\pi, p < \mathrm{Re} z < M\}$$

is disjoint from the forward orbit of 0 under iterates of f. Consider the preimage $F^{-1}(\tilde{P}_M)$. This set is a union of infinitely many topological disks Q_i contained in the strip $-\frac{\pi}{2} < \mathrm{Im} z < \frac{\pi}{2}$ (recall that the points $z \in P$ such that $|\mathrm{Im} z| > \frac{\pi}{2}$ are mapped into the region $\mathrm{Re} z < 0$, thus outside $\tilde{P}_M$). Moreover,

$$\overline{Q_i} \cap \overline{Q_j} = \emptyset.$$

Now, we consider the finite family of disks Q_i^M, whose closures are contained in $\tilde{P}_M$. In this way we obtain the finite iterated function system:

$$\phi_i : \tilde{P}_M \to Q_i^M$$

where ϕ_i is an appropriate holomorphic branch of F^{-1}. Let J_M be the limit set of this system and let m_M be the unique conformal measure. In this case this is simply the normalized Hausdorff measure with the exponent h_M equal to the Hausdorff dimension of J_M.

Remark 3.2. *We have $J_M \subset J_{M+1}$ for all M large enough. In order to see this, take Q_i^M and let Q_i^{M+1}, be the preimage of $\tilde{P}_{M+1}$ under the same holomorphic branch F_*^{-1} of F^{-1}. Then, obviously, $Q_i^{M+1} \supset Q_i^M$. Since $F(Q_i^{M+1} \setminus Q_i^M) \subset \{z \in \tilde{P}_{M+1} : M < \mathrm{Re} z \le M+1\}$ and since the derivative of F_*^{-1} on $\{z \in \tilde{P}_{M+1} : M < \mathrm{Re} z \le M+1\}$ is bounded from above by $C_1 M^{-1}$, we conclude that $\mathrm{diam}(Q_i^{M+1} \setminus Q_i^M) \le C_2 M^{-1}$ for some appropriate constants C_1 and C_2. Since $Q_i^M \subset \{\mathrm{Re} z \le M\}$, this implies that*

$$Q_i^{M+1} \subset \{\mathrm{Re} z \le M+1\}$$

for all M large enough. Hence, each $Q_i^{M+1} \supset Q_i^M$ is (see the definition) used in the construction of J_{M+1}. Thus, the corresponding limit set J_{M+1} contains J_M.

Remark 3.3. *Since $J_{bd}(f)\cap P=\bigcup_{N=[q]+1}^{\infty}J_N$, reasoning as in the remark above, it follows from Theorem 2.1 and Remark 3.2 that there exist $h_0>1$ and M_0 such that for every $M>M_0$ $h_M=\mathrm{HD}(J_M)>h_0$.*

Proposition 3.4. *The sequence of measures m_M, $M\in\mathbb{N}$ is tight, i.e. for every $\epsilon>0$ there exists M so large that for every N*

$$m_N(\{z\in P:\mathrm{Re}z>M\})<\epsilon.$$

Proof. Fix $\epsilon>0$, $M>0$ and $N\geq q$. We shall estimate separately the measure m_N of two sets, which cover $\{z\in P:\mathrm{Re}z>M\}$. First, we have

$$m_N(\{x\in J_N:\mathrm{Re}F(x)\geq M\})=\sum_{k\in\boldsymbol{Z}}m_N\Big(\{x\in J_N:f(x)\in[M,N]\times(-\pi,\pi]+2k\pi i\}\Big).$$

If $x\in J_N$ and $f(x)\in[M,N]\times[-\pi,\pi]+2k\pi i$, then

$$|F'(x)|=|f(x)|\geq\frac{1}{2}(M+\pi|k|)\geq\frac{1}{2}(M+|k|)$$

which gives

$$\begin{aligned}m_N(\{x:\mathrm{Re}F(x)\geq M\})&\leq 2\sum_{k=0}^{\infty}m_N(\{x:M\leq\mathrm{Re}x\leq N\})\cdot\frac{2^{h_N}}{(M+k)^{h_N}}\\&\leq 2^{h_N+1}\sum_{k=0}^{\infty}\frac{1}{(M+k)^{h_N}},\end{aligned}\tag{3.2}$$

where, let us recall, h_N is the exponent of the measure m_N. By Remark 3.3 and Remark 3.2 there exists $T>q$ such that $h_N\geq h_T>1$ for all $N\geq T$. If $N\leq M$, then

$$m_N(\{z\in P:\mathrm{Re}z>M\})=0.\tag{3.3}$$

If $M\geq T$ and $N>M$, then it follows from (3.2) that

$$m_N(\{x:\mathrm{Re}F(x)\geq M\})\leq\frac{2^3}{h_N-1}M^{1-h_N}\leq\frac{2^3}{h_T-1}M^{1-h_T}\tag{3.4}$$

Keeping $M\geq T$ and $N>M$ we now estimate the measure of the second set, namely

$$m_N(\{x:M<\mathrm{Re}x<N\text{ and }\mathrm{Re}F(x)<M\}).$$

If $\mathrm{Re}x>M$, then $|f(x)|>\lambda e^M$ and therefore $|\mathrm{Im}f(x)|\geq\sqrt{\lambda^2e^{2M}-M^2}$. Thus,

$$\begin{aligned}m_N\Big(\{x:M<\mathrm{Re}x<N\text{ and }\mathrm{Re}F(x)<M\}\Big)&\leq const\sum_{k\geq(2\pi)^{-1}\sqrt{\lambda^2e^{2M}-M^2}}^{\infty}(2\pi k)^{-h_N}\\&\leq const\cdot\frac{1}{h_N-1}e^{M(1-h_N)}\\&\leq\frac{const}{h_T-1}e^{M(1-h_T)}.\end{aligned}\tag{3.5}$$

Combining this with (3.3) and (3.4) we obtain

$$m_N(\{x : \mathrm{Re}x > M\}) < \epsilon$$

for all N and all M large enough. ■

Since the sequence m_N is tight, it follows from Prochorov's theorem that there exists an increasing to infinity sequence $\{N_i\}_{i=1}^{\infty}$ such that the sequence $\{m_{N_i}\}_{i=1}^{\infty}$ weakly converges to some limit probability measure m. This is the measure we are looking for. Put

$$J(F) = P \cap J(f).$$

We shall prove the following.

Theorem 3.5. *The measure m is h-conformal, where $h = \lim_{i\to\infty} h_{N_i}$ and $m(J(F)) = 1$.*

Proof. Since $J_M \subset J(F)$, $J(F)$ is closed and $m_M(J_M) = 1$ for every $M > p$, it immediately follows from the definition of the measure m that $m(J(F)) = 1$. In view of Remark 3.2, the sequence $\{h_N\}$ is eventually nondecreasing and consequently the limit $\lim_{N\to\infty} h_N$ exists. Notice that each measure m_N is h_N-conformal for $F|_{J_N}$ but not for F itself (the set J_N is not backward invariant). However, if N is large enough, then for every Borel set $A \subset \{z : \mathrm{Re}z < N-1\}$ such that $F|_A$ is one-to-one, we have

$$m_N(F(A)) = \int_A |F'|^{h_N}\, dm_N. \tag{3.6}$$

To verify this, first we claim that

$$F(A) \cap J_N = F(A \cap J_N). \tag{3.7}$$

Indeed, $F(A \cap J_N) \subset F(A) \cap F(J_N) \subset F(A) \cap J_N$. To see the opposite inclusion, let $x \in F(A) \cap J_N$. Take $y \in A$ such that $F(y) = x$. Let Q be the component of $F^{-1}(\tilde{P}_N)$ containing y. We claim that Q is entirely contained in $\tilde{P}_N$, i.e. Q is one of components Q_i^N used in the construction of J_N. Suppose on the contrary, that Q intersects the line $\mathrm{Re}z = N$. Then for some $z \in Q$, $|f(z)| = |F'(z)| = \lambda e^N$. This means that Q is contained in a component of $f^{-1}(P_+ + 2k\pi i)$, where $k \geq (2\pi)^{-1}\sqrt{\lambda^2 e^{2N} - N^2}$. If N is large, this implies that

$$\mathrm{diam}(Q) \leq C\frac{N}{\lambda e^N} < 1.$$

But Q contains a point $y \in A$ and $A \subset P_{N-1}$. This contradiction shows that Q is entirely contained in $\tilde{P}_N$, i.e. Q is one of components Q_i^N used in the construction of J_N. Since $x = F(y) \in J_N$, this implies that $y \in J_N$. The formula (3.7) is proved. Using (3.7) we can write

$$m_N(F(A)) = m_N(F(A) \cap J_N) = m_N(F(A \cap J_N)) = \int_{A\cap J_N} |F'|^{h_N}\, dm_N = \int_A |F'|^{h_N}\, dm_N.$$

Since the sequence $\{m_{N_i}\}$ converges weakly to m, we have

$$m_{N_i}(A) \to m(A)$$

for every Borel set A such that $m(\partial A)=0$. In particular, this holds for every bounded Borel A such that $m(\partial A)=0$ and $m(\partial F(A))=0$. For these sets A, using (3.6), we get

$$m(F(A))=\lim_{i\to\infty} m_{N_i}(F(A))=\lim_{i\to\infty}\int_A |F'|^{h_{N_i}}dm_{N_i}=\int_A |F'|^h dm_{N_i}+\int_A (|F'|^{h_{N_i}}-|F'|^h)dm_{N_i}.$$

The first summand converges to $\int_A |F'|^h dm$. The second summand can be estimated by $\sup_A(|F'|^{h_{N_i}}-|F'|^h)$. This tends to zero, since $|F'|$ is bounded on A and $h_{N_i}\to h$. So,

$$m(F(A))=\int_A |F'|^h dm. \tag{3.8}$$

Now, take an arbitrary Borel set A such that $F_{|A}$ is injective. One can assume that A is bounded. Since $J(F)\subset\{z:\pi/2\le \mathrm{Im}z\le\pi/2\}$, and consequently, in the terminology from [DU1], $\mathrm{Sing}(F:J(F)\to J(F))=\emptyset$, and since $m(J(F))=1$, in order to verify the equality $m(F(A))=\int_A |F'|^h dm$, it is enough to invoke Lemma 2.4 in [DU1] and to apply (3.8). ■

The existence of a conformal measure leads to the following straightforward corollary.

Corollary 3.6. *There exists a σ-finite measure $\tilde m$, which is h-conformal for $f|_{J(f)}$.*

Proof. Define $\tilde m$ on each strip $P_k=P+2k\pi i$ as $m\circ\pi$, where, let us recall, π is the natural projection of P_k onto P. Checking that $\tilde m$ is f-conformal is straightforward. And indeed, assume first that $A\subset P_n$ for some $n\in\boldsymbol{Z}$ and $f|_A$ is injective. Let for every $k\in\boldsymbol{Z}$, $Z_k=f^{-1}(P_k)\cap P$ and let $\tilde A=A-2\pi in$. Then

$$\begin{aligned}\tilde m(f(A))=\tilde m(f(\tilde A))&=\sum_{k\in\boldsymbol{Z}}\tilde m(f(\tilde A\cap Z_k))=\sum_{k\in\boldsymbol{Z}} m(\pi\circ f(\tilde A\cap Z_k))=\sum_{k\in\boldsymbol{Z}} m(F(\tilde A\cap Z_k))\\ &=\sum_{k\in\boldsymbol{Z}}\int_{\tilde A\cap Z_k}|F'|^h dm=\sum_{k\in\boldsymbol{Z}}\int_{\tilde A\cap Z_k}|f'|^h dm=\int_{\tilde A}|f'|^h dm=\int_A |f'|^h d\tilde m\end{aligned}$$

Now, let $A\subset\mathbb{C}$ be an arbitrary Borel set on which f is injective. Let $A_k=A\cap P_k$. Since $A_k\cap A_j=\emptyset$ for $k\neq j$, we get

$$\tilde m(f(A))=\sum_{k\in\boldsymbol{Z}}\tilde m(f(A_k))=\sum_{k\in\boldsymbol{Z}}\int_{A_k}|f'|^h d\tilde m=\int_{A\in\boldsymbol{Z}}|f'|^h d\tilde m.$$

This ends the proof.■

Let

$$I_\infty(F)=\{z\in P:\lim_{n\to\infty}F^n(z)=\infty\},$$

i.e. $I_\infty(F)$ is the set of points escaping to infinity under forward iterates of F. Analogously define

$$I_\infty(f)=\{z\in P:\lim_{n\to\infty}f^n(z)=\infty\},$$

Let

$$J_r(F)=J(F)\setminus I_\infty(f)\ \text{ and }\ J_r(f)=J(f)\setminus I_\infty(F)$$

Notice that $I_\infty(f)\cap P=I_\infty(F)$.

Let m be the h-conformal measure constructed in Theorem 3.5. We shall prove the following.

Proposition 3.7. *There exists $M > 0$ such that for m-a.e. x*

$$\liminf_{n\to\infty} \mathrm{Re} F^n(x) \le M.$$

In particular, $m(I_\infty(F)) = 0$ or equivalently $m(J_r(F)) = 1$.

Proof. Put

$$Y_M = \{z \in P : \mathrm{Re} z > M\}$$

Let $B \subset Y_M$ be an arbitrary Borel set. We shall estimate from above the measure $m(B \cap F^{-1}(B))$. We have

$$m(B \cap F^{-1}(B)) \le m(F^{-1}(B)) = \sum_{k\in\mathbf{Z}} m(x : f(x) \in B + 2k\pi i)$$

If $f(x) \in B + 2k\pi i$, then

$$|F'(x)| = |f'(x)| = |f(x)| > (M^2 + k^2)^{\frac{1}{2}}.$$

Thus

$$m\Big(\{x : F(x) \in B\}\Big) < 2\sum_{k=0}^{\infty} m(B) \cdot \frac{1}{(M^2 + k^2)^{\frac{h}{2}}} < const\, m(B) M^{1-h}.$$

Therefore, in particular, one gets

$$m(B \cap F^{-1}(B)) < \frac{C}{M^{h-1}} m(B) \tag{3.9}$$

for every Borel set $B \subset Y_M$ for some constant C independent of M and B. Since $B \cap F^{-1}(B) \subset Y_M$, one can now use the estimate (3.9) to get inductively

$$m\Big(B \cap F^{-1}(B) \cap \cdots \cap F^{-n}(B)\Big) < (CM^{1-h})^n m(B)$$

This implies that for all M large enough

$$m(\bigcap_{n=0}^{\infty} F^{-n}(Y_M)) = 0$$

and consequently

$$m(\bigcup_{k=0}^{\infty} F^{-k}(\bigcap_{n=0}^{\infty} F^{-n}(Y_M))) = 0$$

The proof is finished. ■

Let us show now that the estimates used in Proposition 3.7 and Proposition 3.4 lead to the following.

Corollary 3.8.

$$m(Y_M) < Ce^{(1-h)M}.$$

for some constant C and all $M \geq 0$ large enough.

Proof. It follows from the proof of Proposition 3.7 that

$$m\Big(\{x \in Y_M : F(x) \in Y_M\}\Big) < m(Y_M)CM^{1-h}$$

and from the proof of Proposition 3.4 (formula (3.5)) with m_N replaced by m that

$$m\Big(\{x \in Y_M : \mathrm{Re}F(x) \leq M\}\Big) < Ce^{(1-h)M}.$$

These two sets cover the whole set Y_M. The first inequality says that (for all M sufficiently large) the first set covers less than, say, one half of the measure of Y_M. Thus,

$$m(Y_M) \leq 2m\Big(\{x \in Y_M : \mathrm{Re}F(x) \leq M\}\Big) < 2Ce^{(1-h)M}$$

and the proof is complete. ■

4. Conformal, Hausdorff and packing measures; Hausdorff dimension

Let again $f = f_\lambda$, $q = q_\lambda$ and $F = F_\lambda$. Recall that

$$J(F) = J(f) \cap ([q,\infty) \times [-\pi,\pi]) = J(f) \cap ([q,\infty) \times [-\pi/2,\pi/2]).$$

Recall also that

$$P_+ = \{z \in \mathbb{C} \colon \mathrm{Re}(z) \geq q \text{ and } \mathrm{Im}(z) \in (-\pi,\pi]\}.$$

Fix some $R > q$. Consider a countable partition $\alpha = \{A_n : n \geq 0\}$ of P_+ defined as follows.

$$A_0 = \{z \in P_+ : \mathrm{Re}z \leq R\},$$

$$A_1 = \{z \in P_+ : R < \mathrm{Re}z \leq R+1\},$$

and

$$A_n = \{z \in P_+ : R+n-1 < \mathrm{Re}z \leq R+n\}$$

for $n \geq 1$. We start this section with two technical lemmas.

Lemma 4.1. *If the constant R is large enough (depending on λ), then for every $k \geq 0$*

$$F(A_k) \supset A_0 \cup A_1 \cup \cdots \cup A_{k+1}$$

Proof. Let $k \geq 1$. Then $f(A_k)$ is an annulus centered at 0 bounded by two circles of radii $\lambda\exp(R+k-1)$ and $\lambda\exp(R+k)$.

Let z_0 be the point in the outer circle such that $\mathrm{Re}z_0 = \lambda\exp(R+k-1)$ and $\mathrm{Im}z_0 > 0$. A straightforward geometrical argument shows that if $R > 0$ is taken so large that for all $k \geq 1$

$$\lambda\exp(R+k-1)(\sqrt{e^2-1}-1) > 4\pi,$$

then $f(A_k)$ contains some rectangle

$$0 < \mathrm{Re}z < \mathrm{Re}z_0, \mathrm{Im}z_0 - 4\pi < \mathrm{Im}z < \mathrm{Im}z_0$$

If moreover $R > 0$ is taken so large that $\mathrm{Re} z_0 = \lambda \exp(R + k - 1) > k + 1 + R$, then this rectangle contains some component of the set $\pi_0^{-1}(A_0 \cup \cdots \cup A_{k+1})$. So, by definition,

$$F(A_k) \supset A_0 \cup \cdots \cup A_{k+1}.$$

It remains to check the case when $k = 0$. But $f(A_0)$ is the annulus of inner radius q and outer radius $\lambda \exp R$. If R is large then this set contains $A_0 \cup A_1 = \{z \in P : q \le \mathrm{Re} z < R + 1\}$. ■

From now on in this section fix the partition α satisfying the statement of the previous lemma. As an immediate consequence of this lemma we get the following.

Corollary 4.2. *For every $k \geq 0$ we have*

$$\lim_{n\to\infty} m(F^n(A_k)) = 1.$$

Lemma 4.3. *For every $x \in J(F)$ and every $r > 0$*

$$\lim_{n\to\infty} m(F^n(B(x, r))) = 1.$$

Proof. For every $k \geq 0$ let $A_k(x)$ be the element of partition α containing $F^k(x)$. Denote by $B_k(x)$ the component of $F^{-k}(A_k(x))$ containing x. Since diameters of A_k are bounded and F is expanding on its Julia set, $\mathrm{diam}(B_k(x)) \to 0$ as $k \to \infty$. So, for some $k \in \mathbb{N}$, $B(x, r) \supset B_k(x)$. Thus, for every $n \geq 0$

$$F^{n+k}(B(x, r)) \supset F^{n+k}(B_k(x)) \supset F^n(A_k)$$

and the lemma follows from Corollary 4.2 . ■

Let us prove the following.

Theorem 4.4. *The h-conformal measure m is a unique t-conformal measure for F with $t > 1$. In addition it is conservative and ergodic.*

Proof Suppose that ν is a t-conformal measure for F with some $t > 1$. The same proof as in the case of the measure m shows that $\nu(I_\infty(F)) = 0$. Let $J_{r,N}(F)$ be the subset of $J_r(F)$ defined as follows: $z \in J_{r,N}(F)$ if the trajectory of z under F has an accumulation point in $\{\mathrm{Re} z < N\}$. Obviously, $\bigcup_N J_{r,N}(F) = J_r(F)$ and by Proposition 3.7 there exists $M > 0$ such that $\nu(J_{r,M}(F)) = m(J_{r,M}(F)) = 1$. Fix $z \in J_{r,N}(F)$. Then there exist $y \in J(F)$ such that $\mathrm{Re} y < N$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y = \lim_{k\to\infty} F^{n_k}(z)$. Considering for k large enough the sets $F_z^{-n_k}(B(y, \pi/4))$ and $F_z^{-n_k}(B(y, \pi/(4K)))$, where $F_z^{-n_k}$ is the holomorphic inverse branch of F^{n_k} defined on $B(y, \pi/2)$ and sending $F^{n_k}(z)$ to z, using conformality of measures m and ν along with Koebe's distortion theorem we easily deduce that

$$B_N(\nu)^{-1}|(F^{n_k})'(z)|^{-t} \le \nu\Big(B(z, c|(F^{n_k})'(z)|^{-1})\Big) \le B_N(\nu)|(F^{n_k})'(z)|^{-t} \tag{4.1}$$

and

$$B_N(m)^{-1}|(F^{n_k})'(z)|^{-h} \le m\Big(B(z,c|(F^{n_k})'(z)|^{-1})\Big) \le B_N(m)|(F^{n_k})'(z)|^{-h} \tag{4.2}$$

for all $k \ge 1$ large enough, where $K = 16$ is the constant appearing in the Koebe's distortion theorem and ascribed to the scale $1/2$, $B_N(\nu)$ is some constant depending on ν and N. Let M be fixed as above. Fix now E, an arbitrary bounded Borel set contained in $J_r(F)$ and let $E' = E \cap J_{r,M}(F)$. Since m is regular, for every $x \in E'$ there exists a radius $r(x) > 0$ of the form from (4.1) such that

$$m(\bigcup_{x\in E'} B(x,r(x)) \setminus E) < \epsilon. \tag{4.3}$$

Now by the Besicovič theorem (see [G]) we can choose a countable subcover $\{B(x_i,r(x_i))\}_{i=1}^{\infty}$, $r(x_i) \le \epsilon$, from the cover $\{B(x,r(x))\}_{x\in E'}$ of E, of multiplicity bounded by some constant $C \ge 1$, independent of the cover. Therefore by (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned}
\nu(E') = \nu(E) &\le \sum_{i=1}^{\infty} \nu(B(x_i,r(x_i))) \le B_M(\nu)\sum_{i=1}^{\infty} r(x_i)^t \\
&\le B_M(\nu)B_M(m)\sum_{i=1}^{\infty} r(x_i)^{t-h} m(B(x_i,r(x_i))) \\
&\le B_M(\nu)B_M(m)C\epsilon^{t-h} m(\bigcup_{i=1}^{\infty} B(x_i,r(x_i))) \\
&\le CB_M(\nu)B_M(m)\epsilon^{t-h}(\epsilon + m(E')) = CB_M(\nu)B_M(m)\epsilon^{t-h}(\epsilon + m(E)).
\end{aligned} \tag{4.4}$$

In the case when $t > h$, letting $\epsilon \searrow 0$ we obtain $\nu(E) = 0$ and consequently $\nu(J(F)) = 0$ which is a contradiction. We obtain a similar contradiction assuming that $t < h$ and replacing in (4.4) the roles of m and ν. Thus $t = h$ and letting $\epsilon \searrow 0$ we obtain from (4.4) that $\nu(E) \le CB_M(\nu)B_M(m)m(E)$. Exchanging m and ν, we obtain $m(E) \le CB_M(\nu)B_M(m)\nu(E)$. These two conclusions along with the already mentioned fact that $m(J_r(F)) = \nu(J_r(F)) = 1$, imply that the measures m and ν are equivalent with Radon-Nikodym derivatives bounded away from zero and infinity.

Let us now prove that any h-conformal measure ν is ergodic. Indeed, suppose to the contrary that $F^{-1}(G) = G$ for some Borel set $G \subset J(F)$ with $0 < m(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(F)\setminus G}$

$$\nu_G(B) = \frac{\nu(B\cap G)}{\nu(G)}, \quad \nu_{J(F)\setminus G}(B) = \frac{\nu(B\cap J(F)\setminus G)}{\nu(J(F)\setminus G)}$$

would be h-conformal and mutually singular; a contradiction.

If now ν is again an arbitrary h-conformal measures, then by a simple computation based on the definition of conformal measures we see that the Radon-Nikodyn derivative $\phi = d\nu/dm$ is constant on grand orbits of F. Therefore by ergodicity of m we conclude that ϕ is constant m-almost everywhere. As both m and ν are probability measures, it implies that $\phi = 1$ a.e., hence $\nu = m$.

It remains to show that m is conservative. We shall prove first that every forward invariant ($F(E) \subset E$) subset E of $J(F)$ is either of measure 0 or 1. Indeed, suppose to the contrary that $0 < m(E) < 1$. Since $m(I_\infty(F)) = 0$, it suffices to show that

$$m(E \setminus I_\infty(F)) = 0.$$

Denote by Z the set of all points $z \in E \setminus I_\infty(F)$ such that

$$\lim_{r \to 0} \frac{m(B(z,r) \cap (E \setminus I_\infty(F)))}{m(B(z,r))} = 1. \tag{4.5}$$

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [Fe]), $m(Z) = m(E)$. Since $m(E) > 0$ we find at least one point $z \in Z$. Since $z \in J(F) \setminus I_\infty(F)$, there exists $x \in J(F)$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $x = \lim_{k\to\infty} F^{n_k}(z)$. Let

$$\delta = \min\{\pi/8, q/4\}.$$

Suppose that $m(B(x,\delta) \setminus E) = 0$. By conformality of m, $m(F(Y)) = 0$ for all Borel sets Y such that $m(Y) = 0$. Hence,

$$\begin{aligned} 0 = m\Big(F^n(B(x,\delta) \setminus E)\Big) &\ge m\Big(F^n(B(x,\delta)) \setminus F^n(E)\Big) \\ &\ge m\Big(F^n(B(x,\delta)) \setminus E\Big) \ge m\Big(F^n(B(x,\delta)\Big) - m(E) \end{aligned} \tag{4.6}$$

for all $n \ge 0$. By Lemma 4.3 $\lim_{n\to\infty} m\Big(F^n(B(x,\delta)\Big) = 1$. Then (4.6) implies that $0 \ge 1 - m(E)$ which is a contradiction. Consequently $m(B(x,\delta) \setminus E) > 0$. Hence for every $j \ge 1$ large enough, $m\Big(B(F^{n_j}(z), 2\delta) \setminus E\Big) \ge m\Big(B(x,\delta) \setminus E\Big) > 0$. Therefore, as $F^{-1}(J(F) \setminus E) \subset J(F) \setminus E$, the standard application of Koebe's Distortion Theorem shows that

$$\limsup_{r\to 0} \frac{m(B(z,r) \setminus E)}{m(B(z,r))} > 0$$

which contradicts (4.5). Thus either $m(E) = 0$ or $m(E) = 1$.

Now conservativity is straightforward. One needs to prove that for every Borel set $B \subset J(F)$ with $m(B) > 0$ one has $m(G) = 0$, where

$$G = \{x \in J(F) : \sum_{n \ge 0} \chi_B(F^n(x)) < +\infty\}.$$

Indeed, suppose that $m(G) > 0$ and for all $n \ge 0$ let

$$G_n = \{x \in J(F) : \sum_{k \ge n} \chi_B(F^n(x)) = 0\} = \{x \in J(F) : F^k(x) \notin B \ \text{ for all } \ k \ge n\}.$$

Since $G = \bigcup_{n\ge 0} G_n$, there exists $k \ge 0$ such that $m(G_k) > 0$. Since all the sets G_n are forward invariant we conclude that $m(G_k) = 1$. But on the other hand all the sets $F^{-n}(B)$, $n \ge k$, are of positive measure and are disjoint from G_k. This contradiction finishes the proof. ■

In the proof of the following Theorem as well as in the proofs of Proposition 4.9 and Theorem 4.10 we use various forms of the converse Frostman's type lemmas which can be found for example in [DU2] and in the Chapter 6 of the book [PU].

Theorem 4.5. *If $\lambda \in (0, 1/e)$, then the h-dimensional Hausdorff measure H^h of $J_r(F)$ is finite, the measure H^h of $J_r(f_\lambda)$ is σ-finite and*

$$h_\lambda = \mathrm{HD}(J_{bd}(f_\lambda)) = \mathrm{HD}(J_r(f_\lambda)) < 2,$$

where h_λ is the exponent of the conformal measure $m = m_\lambda$ (see Theorem 3.5 and Theorem 4.4).

Proof. Fix $\lambda \in (0, 1/e)$. Put $f = f_\lambda$ and $h = h_\lambda$. By the definition of the numbers h_N (see the beginning of Section 4) and Theorem 3.5, $h \le \mathrm{HD}(J_{bd}(f))$. It follows from (4.1) applied with the measure m that the h-dimensional Hausdorff measure $\mathrm{H}^h(J_{r,M}(F))$ is finite. Since $m(J_{N,r}(F) \setminus J_{r,M}(F)) = 0$, we deduce in the similar way (using again (4.1)) that $\mathrm{H}^h(J_{r,N}(F) \setminus J_{r,M}(F)) = 0$ for all $N > M$. Since $\bigcup_{N \ge M} J_{r,N}(F) = J_r(F)$, we thus conclude that $\mathrm{H}^h(J_r(F)) = \mathrm{H}^h(J_{r,M}(F)) < \infty$ and consequently, $\mathrm{HD}(J_r(F)) \le h$.

Since $J_r(f) = \bigcup_{n \in \mathbf{Z}} \Big(J_r(F) + 2\pi i n\Big)$, we therefore conclude that $\mathrm{H}^h|_{J_r(f)}$ is σ-finite and $\mathrm{HD}(J_r(f)) \le h$. It therefore remains to demonstrate that $\mathrm{HD}(J_r(F)) < 2$. But otherwise, it would follow from (4.1) and (4.4) with the measure ν replaced by m and m replaced by planar Lebesgue measure, that the planar Lebesgue measure of $J_r(F)$ is positive. This would however contradict McMullen's result from [McM] which finishes the proof. ■

An alternative direct proof, not using the concept of conformal measures, of the fact that $\mathrm{HD}(J_r(f_\lambda)) < 2$ is provided in Corollary 7.3 of the Appendix. Recall that in [DU3] (comp. [PU]) the dynamical dimension, proven in [PU] to be equal to the hyperbolic dimension, was defined as the supremum of Hausdorff dimensions of all probability invariant ergodic measures with positive entropy. It has been conjectured that in the case of rational functions the dynamical dimension and the Hausdorff dimension of the Julia set coincide. Since each Borel probability f_λ-invariant measure is by Poincare's Recurrence Theorem supported on $J_r(f)$, as an immediate consequence of Theorem 4.5 we get the following corollary which disproves this conjecture in the case of transcendental entire functions.

Corollary 4.6. *If $\lambda \in (0, 1/e)$, then the supremum of Hausdorff dimensions of all probability f_λ-invariant ergodic measures is less than the Hausdorff dimension of the Julia set of f_λ.*

Corollary 4.7. *The function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is continuous in the interval $(0, 1/e)$.*

Proof. Fix $\lambda \in (0, 1/e)$ and a sequnce $\lambda_n \in (0, 1/e)$ converging to λ. Let $m = m_\lambda$ and $m_n = m_{\lambda_n}$ be the corresponding conformal measures. Denote their exponents respectively by h and h_n. By Theorem 4.5, $h = \mathrm{HD}(J_r(f_\lambda))$ and $h_n = \mathrm{HD}(J_r(f_{\lambda_n}))$. Consider an arbitrary subsequence $\{n_k\}_{k=1}^{\infty}$ such that the sequence h_{n_k} converges, say to a limit ξ. Since the maps f_{λ_n} and f_λ are topologically conjugate and the conjugating map converges to identity if $n \to \infty$, we conclude that for every compact set $T \subset \mathbb{C}$, $J(f_{\lambda_n}) \cap T$ converges in the Hausdorff metric to $J(f_\lambda) \cap T$. Consequently, any weak* accumulation point of the sequence m_{n_k} is a conformal measure for F_λ. Since, by the proof of Theorem 2.1 and by Theorem 4.5, $\xi > 1$, it therefore follows from Theorem 4.4 that $\xi = h$ and we are done. ■

Remark 4.8. *An alternative proof of this result would use the existence of quasi-conformal conjugacies between the maps f_λ, $\lambda \in (0, 1/e)$ with dilations constants converging to* 1 *when $\lambda \to \lambda_0$.*

Let P^h be the h-dimensional packing measure (see [TT], comp. [PU] for example, for its definition and some basic properties). The last three results of this section provide in a sense a complete description of the geometrical structure of the sets $J_r(F)$ and $J_r(f)$ and also they exhibit the geometrical meaning of the h-conformal measure m.

Proposition 4.9. *We have $\mathrm{P}^h(J_r(f)) = \infty$, in fact $\mathrm{P}^h(G) = \infty$ for every open nonempty subset of $J_r(f)$.*

Proof. Since $m(J_r(F) \cap (P \setminus P_M)) > 0$ for every $M \in \mathbb{R}$, it follows from Birkhoff's ergodic theorem and Theorem 5.2 below (whose proof is obviously independent of the results proven in the remainder of this section) that there exists a set $E \subset J_r(F)$ such that $m(E) = 1$ and

$$\limsup_{n \to \infty} \mathrm{Re} F^n(z) = \infty \tag{4.7}$$

for every $z \in E$. Fix $z \in E$, $n \geq 1$ and consider the ball $B(z, K^{-1}|(F^n)'(z)|^{-1})$, where $K = 16$ is the Koebe constant corresponding to the scale $\frac{1}{2}$. Then

$$B(z, K^{-1}|(F^n)'|(z)|^{-1}) \subset F_z^{-n}(B(F^n(z), 1)),$$

where $F_z^{-n} : B(F^n(z), 1) \to \mathbb{C}$ is the analytic inverse branch of F^n mapping $F^n(z)$ to z. Applying Koebe's Distortion Theorem, conformality of the measure m, and Corollary 3.8, we obtain

$$\begin{aligned} m(B(z, K^{-1}|(F^n)'(z)|^{-1}) &\leq K^h |(F^n)'(z)|^{-h} m(B(F^n(z), 1)) \\ &\leq K^{2h} (K^{-1}|(F^n)'(z)|^{-1})^h m(Y_{\mathrm{Re}F^n(z)-1}) \\ &\leq K^{2h} C \exp((1-h)(\mathrm{Re}F^n(z) - 1))(K^{-1}|(F^n)'(z)|^{-1})^h \end{aligned}$$

Hence, using (4.7), we conclude that

$$\liminf_{r \to 0} \frac{m(B(z, r))}{r^h} = 0.$$

Since $m(G \cap J_r(F)) > 0$ for every non-empty open subset of $J_r(F)$, this implies (see an appropriate Converse Frostman's Type Theorem in [DU2] or [PU]) that $\mathrm{P}^h(G) = \infty$. Since $J_r(f) = \bigcup_{k\in\boldsymbol{Z}}(J_r(F) + 2\pi ik)$, we are therefore done. ■

Theorem 4.10. $0 < \mathrm{H}^h(J_r(F)) < \infty$.

Proof. We know from Theorem 4.5 that $\mathrm{H}^h(J_r(F)) < \infty$ and we are therefore left to show that $\mathrm{H}^h(J_r(F) > 0$. Since $m(J_r(F))) = 1$, it suffices to demonstrate that for every $z \in J_r(F)$ and all $r > 0$ sufficiently small (depending on z)

$$m(B(z,r)) \le Cr^h$$

for some constant $0 \le C < \infty$ independent of z and r. And indeed, put

$$\theta = \min\{\pi, \mathrm{dist}(J(F), \{f^k(0) : k \ge 0\})\}.$$

Fix $z \in J_r(F)$, $0 < r \le \theta(32|f'(z)|)^{-1}$. Since $F : J(F) \to J(F)$ is an expanding map, there exists a largest $n \ge 1$ such that

$$r|(f^n)'(z)| \le \frac{\theta}{32}. \tag{4.8}$$

Thus

$$r|(f^{n+1})'(z)| > \frac{\theta}{32}. \tag{4.9}$$

It follows from the definition of θ that the holomorphic inverse branch $f_z^{-n} : B(f^n(z),\theta) \to \mathbb{C}$ of f^n sending $f^n(z)$ to z, is well-defined. Since $f|_{B(f^n(z),\theta)}$ is 1-to-1 and since, by Koebe's $\frac{1}{4}$-Theorem, $f(B(f^n(z),\theta)) \supset B\Big(f^{n+1}(z), \frac{1}{4}\theta|f'(f^n(z))|\Big)$, we conclude that the holomorphic inverse branch $f_z^{-(n+1)} : B\Big(f^{n+1}(z), \frac{1}{4}\theta|f'(f^n(z))|\Big) \to \mathbb{C}$ of f^{n+1} mapping $f^{n+1}(z)$ to z, is well-defined. Since

$$4r|(f^{n+1})'(z)| = 4r|(f^n)'(z)|\cdot|f'(f^n(z))| = \theta\Big(\frac{32}{\theta}r|(f^n)'(z)|\Big)\cdot\frac{1}{8}|f'(f^n(z))|$$

and since, by (4.8), $\frac{32}{\theta}r|(f^n)'(z)| \le 1$, we conclude that $4r|(F^{n+1})'(z)| \le \frac{1}{8}\theta|f'(f^n(z))|$. Applying Koebe's $\frac{1}{4}$-Theorem again, we see that

$$f_z^{-(n+1)}\Big(B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)\Big) \supset B\Big(z, |(f^{n+1})'(z)|^{-1}r|(f^{n+1})'(z)|\Big) = B(z,r).$$

Since the ball $B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)$ intersects at most $\frac{1}{2\pi}4r|(f^{n+1})'(z)|+1 \le r|(f^{n+1})'(z)|$ horizontal strips of the form $2\pi ik + P$, $k \in \boldsymbol{Z}$, using Koebe's Distortion Theorem, h-conformality of the measure $\tilde{m}$ and, at the end, (4.9), we get

$$\begin{aligned} r^{-h}(m(B(z,r)) &\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)m\Big(\pi_0\Big(B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)\Big)\Big) \\ &\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|) \\ &= K^h(r|(f^{n+1})'(z)|)^{1-h} \le K^h\left(\frac{32}{\theta}\right)^{h-1}, \end{aligned}$$

where $K = 16$ is the Koebe constant corresponding to the scale $1/2$. We are done by applying an appropriate Converse Frostman's Type Theorem in [DU2] or [PU]). ■

As an immediate consequence of this theorem we get the following.

Corollary 4.11. *The h-dimensional Hausdorff measure of the set J_r is positive.*

5. Invariant Measures

In order to prove Theorem 5.2 below we apply a general sufficient condition for the existence of σ-finite absolutely continuous invariant measure proven in [Ma]. In order to formulate this condition suppose that X is a σ-compact metric space, ν is a Borel probability measure on X, positive on open sets, and that a measurable map $T : X \to X$ is given with respect to which the measure ν is quasi-invariant, i.e. $\nu \circ T^{-1} << \nu$. Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure ν. We also assume that $\nu(X \setminus \bigcup_{n\geq 0} A_n) = 0$, and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$\nu(T^{-k}(A_m) \cap A_n) > 0,$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [Mar] reads the following.

Theorem 5.1. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure ν. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1} \frac{\nu(A)}{\nu(A_n)} \leq \frac{\nu(T^{-k}(A))}{\nu(T^{-k}(A_n))} \leq K_n \frac{\nu(A)}{\nu(A_n)},$$

then T has a σ-finite T-invariant measure μ absolutely continuous with respect to ν. In addition, μ is equivalent with ν, conservative and ergodic, and unique up to a multiplicative constant. Moreover, for every Borel set $A \subset X$

$$\mu(A) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \nu(T^{-k}(A))}{\sum_{k=0}^{n} \nu(T^{-k}(A_0))}.$$

The main result of this section is the following.

Theorem 5.2. *There exists a probability F-invariant measure μ absolutely continuous with respect to h-conformal measure m. In addition, μ is equivalent with m and ergodic.*

Proof. Let us first prove that there exists a σ-finite ergodic F-invariant measure μ equivalent with m. Let α be the partition constructed at the beginning of Section 4 with the constant $R > 0$ so large as required in Lemma 4.1. In view of Koebe's distortion theorem there exists a constant $K \geq 1$ such that if $F_*^{-n}: P \to P$ is a holomorphic branch of F^{-n}, then for every $k \geq 0$ and all $x, y \in A_k$ we have

$$\frac{|(F_*^{-n})'(y)|}{|(F_*^{-n})'(x)|} \leq K. \tag{5.1}$$

We therefore obtain for all Borel sets $A, B \subset A_k$ with $m(B) > 0$ and all $n \geq 0$ that

$$\frac{m(F_*^{-n}(A))}{m(F_*^{-n}(B))} = \frac{\int_A |(F_*^{-n})'|^h dm}{\int_A |(F_*^{-n})'|^h dm} \leq \frac{\sup_{A_k}\{|(F_*^{-n})'|^h\} m(A)}{\inf_{A_k}\{|(F_*^{-n})'|^h\} m(B)} \leq K^h \frac{m(A)}{m(B)}.$$

Therefore

$$m(F^{-n}(A)) = \sum_* m(F_*^{-n}(A)) \leq \sum_* K^h m(F_*^{-n}(B)) \frac{m(A)}{m(B)} = K^h m(F^{-n}(B)) \frac{m(A)}{m(B)}, \tag{5.2}$$

where the summation is taken over all holomorphic inverse branches of F^n. In view of Lemma 4.3, for every $k \geq 0$ and every $l \geq 0$ there exists $n_{k,l} \geq 0$ such that

$$F^{n_{k,l}}(A_k) \supset A_l \tag{5.3}$$

Applying now (5.2) and (5.3) along with Theorem 4.4 and Theorem 5.1 concludes the proof of the existence of required σ-finite measure μ.

It only remains to show that μ is finite. And indeed, fix $0 < p < q$ with the same requirements as in the definition of $\tilde{P}_M$ in the beginning of Section 4. Each holomorphic branch $F_*^{-j}: P \to P$ of F^{-j} restricted to the set $A_0 \cup A_1 \ldots \cup A_n$ extends in a holomorphically univalent fashion to the set $\{z \in \mathbb{C}: p < \mathrm{Re} z < R + n + 1 \text{ and } -2n\pi \leq \mathrm{Im} z \leq 2n\pi\}$ and it therefore follows from Koebe's distortion theorem there exists a constant $C_1 \geq 1$ such that for every $n \geq 0$, all $x \in A_0$ and all $y \in A_n$, we have

$$\frac{|(F_*^{-j})'(y)|}{|(F_*^{-j})'(x)|} \leq C_1 (Rn)^3.$$

Therefore, using in addition Lemma 3.8, we obtain

$$\frac{m(F_*^{-j}(A_n))}{m(F_*^{-j}(A_0))} \leq C_1 (Rn)^3 \frac{m(A_n)}{m(A_0)} \leq C_1 (Rn)^3 C m(A_0)^{-1} e^{(1-h)Rn}.$$

Hence

$$\frac{m(F^{-j}(A_n))}{m(F^{-j}(A_0))} \leq C_1 (Rn)^3 C m(A_0)^{-1} e^{(1-h)(R+n-1)}$$

and consequently, for every $k \geq 0$,

$$\frac{\sum_{j=0}^k m(F^{-j}(A_n))}{\sum_{j=0}^k m(F^{-j}(A_0))} \leq C_1 (Rn)^3 C m(A_0)^{-1} e^{(1-h)(R+n-1)}.$$

Thus, applying Theorem 5.1 we get

$$\mu(A_n) = \lim_{k\to\infty} \frac{\sum_{j=0}^k m(F^{-j}(A_n))}{\sum_{j=0}^k m(F^{-j}(A_0))} \leq C_1 (Rn)^3 C m(A_0)^{-1} e^{(1-h)(R+n-1)}.$$

Since $R > 0$, we finally get $\mu(J(F)) = \sum_{n\geq 0} \mu(A_n) < \infty$. We are done. ■

6. General hyperbolic case

In this section we outline the argument showing that the phenomenon described above holds also for every map $f_\lambda = \lambda \exp(z)$ such that f_λ has an attracting periodic orbit.

We decided to write the details of the proof for the particular case of the attracting fixed point because the dynamics is very simple in this case. On the other hand, the extension of the arguments for the general hyperbolic case is rather straightforward, but it requires some extra information about the structure of the Julia set (see [BD]). So, in what follows we rely on the description given in [BD], we also use the notation of this paper. We recall it briefly: $z_0, \dots, z_n = z_0$ is an attracting cycle of f. Assume that the singular value 0 is contained in the domain A_1, the immediate basin of attraction of z_1. The topological disk B_{n+1} containing z_1 is chosen so that $0 \in B_{n+1}$, $f^n(B_{n+1}) \subset B_{n+1}$. Then B_n is defined as $B_n = f^{-1}(B_{n+1})$. The set B_n contains some half-plane $\mathrm{Re} z < -M$ and $z_0 \in B_n$.

For $j = 1, \dots, n$ let B_{n-j} be the connected component of $f^{-1}(B_{n-j+1})$ containing z_{n-j}.

Notice that B_1 is contained in the immediate basin of attraction of z_1 and $B_{n+1} \subset B_1$. The set B_0 contains B_n and $f^n(B_0) = B_n$.

For $i < n$, B_{n-i} is a simply connected unbounded set, bounded by a simple curve, a finger in the notation of [BD]. The set B_0 is a complement of a union of infinitely many "fingers" F_i. In order to build an appropriate dynamics, we fix one component ("finger") F_0 of the complement of B_0 (obviously, $F_i = F_0 + 2k\pi i$ - see Fig.3 in [BD]). Let

$$P = F_0 \setminus \pi^{-1}(\bigcup_{i=1}^{n-1} B_i),$$

where π is the natural projection $\pi : \bigcup F_i \to F_0$. Then

$$f(P) \supset \bigcup_k (P + 2k\pi i)$$

and, actually, modifying the set P slightly, we can require that $f(P) \supset \overline{\bigcup_k (P + 2k\pi i)}$. Now, $F : P \cap f^{-1}(\pi^{-1}(P)) \to P$ is defined as $F = \pi \circ f$.

Let

$$J(F) = \{z \in P : F^n \text{ is defined for all } n \geq 0\}.$$

One can easily see that

$$J(f) \cap P = J(F).$$

Now, the whole construction given in previous sections can be repeated. We omit the details and summarize the results in

Theorem 6.1. *Assume that the map $f(z) = \lambda \exp(z)$ has an attracting periodic orbit. Denote by*

$$J_r = \{z \in J(f) : f^n(z) \text{ does not tend to } \infty\}.$$

Then $h = \mathrm{HD}(J_r) < 2$. Moreover, there exists a h-conformal measure m for the map $F : J(F) \to J(F)$ and a σ-finite conformal measure $\tilde{m}$ for $f : J(f) \to J(f)$ satisfying $\tilde{m}(I_\infty(f) = 0$. The h-dimensional Hausdorff measure of $J(F)$ is finite, while the h-dimensional packing measure is infinite. There exists a probability ergodic F-invariant measure μ equivalent to m.

7. Appendix

Our main goal in this appendix is to provide an alternative direct proof of the fact that the Hausdorff dimension of the set $J_r(f_\lambda)$, is less than 2 without using the concept of conformal measures. Let

$$J_{ru}(f_\lambda) = \{z \in J(f_\lambda) : \liminf_{n\to\infty} |f_\lambda^n(z)| < \infty \text{ and } \limsup_{n\to\infty} |f_\lambda^n(z)| = \infty\}.$$

We start with the following.

Lemma 7.1. *If $\lambda \in (0, \infty)$, then*

$$\limsup_{\lambda\to 0} \mathrm{HD}(J_{ru}(f_\lambda)) \le 1.$$

Proof. Fix $\lambda \in (0, 1/e)$. Given an integer $k \ge 2$, consider the set

$$J_k(M) = \{z \in P_M \cap J(f_\lambda) : \mathrm{Re}(f^k(z)) \le M \text{ and } \mathrm{Re}(f^j(z)) > M \text{ for all } j = 1, \dots, k-1\}$$

and define the map $F_k : J_k(M) \to P_M$ by the formula

$$F_k(z) = \pi_0(f^k(z)) = F^k(z).$$

If $z \in J_k(M)$, then $\mathrm{Re}(f^{k-1}(z)) > M$ and therefore $|f^k(z)| > \lambda \mathrm{e}^M$. Since $\mathrm{Re}(f^k(z)) \le M$, this implies that $|\mathrm{Im}(f^k(z))| > \sqrt{\lambda^2 \mathrm{e}^{2M} - M} \ge \lambda \mathrm{e}^M/2$ for all M large enough. Since in addition $\mathrm{Re}(f^j(z)) > M$ for every $z \in J_k(M)$ and all $j = 0, 1, \dots, k-1$, we therefore conclude that for every $w \in F_k(J_k(M))$ and every $t > 1$ we have

$$\begin{aligned}\sum_{z\in F_k^{-1}(w)} |F_k'(z)|^{-t} &\le \sum_{|n|\ge\frac{\lambda \mathrm{e}^M}{4\pi}} \left(\frac{1}{2\pi|n|}\right)^t \left(\sum_{n=-\infty}^{+\infty} \frac{1}{(M^2 + (2\pi n)^2)^{t/2}}\right)^{k-1} \\ &\le \frac{(4\pi)^{t-1}}{t-1} \lambda^{1-t} (M^{1-t}\Sigma_t)^{k-1}\end{aligned} \tag{7.1}$$

where $\Sigma_t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{(1+u^2)^{t/2}} du$. Since all the sets $J_k(M)$, $k \ge 2$, are mutually disjoint, putting $J_\infty(M) = \bigcup_{k\ge 2} J_k(M)$, we can define the map $F_\infty : J_\infty(M) \to P_M$ by the requirement that

$F_\infty|_{J_k(M)} = F_k$. It therefore follows from (7.1) that for every $w \in J(f) \cap P_M$, we get

$$
\begin{aligned}
\sum_{z \in F_\infty^{-1}(w)} |F_\infty'(z)|^{-t} &\le \frac{(4\pi)^{t-1}}{t-1}(\lambda \mathrm{e}^M)^{1-t} \sum_{j=1}^{\infty} (\Sigma_t M^{1-t})^j \\
&= \frac{(4\pi)^{t-1}}{t-1}(\lambda \mathrm{e}^M)^{1-t} \Sigma_t M^{1-t} \frac{1}{1-\Sigma_t M^{1-t}} \\
&\le 2\Sigma_t \frac{(4\pi)^{t-1}}{t-1}(\lambda \mathrm{e}^M)^{1-t} M^{1-t}
\end{aligned}
\tag{7.2}
$$

for all M large enough. Fix now $k \ge 1$ and define

$$
E_k(M) = \Big\{ z \in P_M \cap J(f) : \mathrm{Re}(f^j(z)) \le M \text{ for all } j = 0, 1, \ldots, k-1
$$

$$
\text{and } F^{k-1}(z) \in J_\infty(M) \Big\}.
$$

Put $E_\infty(M) = \bigcup_{k \ge 1} E_k(M)$. Since the sets $E_k(M)$, $k \ge 1$, are mutually disjoint, we can define the map $G : E_\infty(M) \to P_M$ by setting

$$
G(z) = F_\infty(F^{k-1}(z))
$$

if $z \in E_k(M)$.

Note that $E_1(M) = J_\infty(M)$ and $G|_{E_1(M)} = F$. Since $\mathrm{Re}(f^j(z)) \ge q_\lambda$ for all $z \in J(f_\lambda)$ and all $j \ge 0$, we get for all $w \in P_M \cap J(f_\lambda)$ that

$$
\begin{aligned}
\sum_{z \in G^{-1}(w)} |G'(z)|^{-t} &\le \left(2\Sigma_t \frac{(4\pi)^{t-1}}{t-1}(\lambda \mathrm{e}^M)^{1-t} M^{1-t} \right) \sum_{k=1}^{\infty} \left(\sum_{n=-\infty}^{+\infty} \frac{1}{(q_\lambda^2 + (2\pi n)^2)^{t/2}} \right)^{k-1} \\
&\le 2\Sigma_t \frac{(4\pi)^{t-1}}{t-1}(\lambda \mathrm{e}^M)^{1-t} M^{1-t} \sum_{k=0}^{\infty} (q_\lambda^{1-t}\Sigma_t)^k.
\end{aligned}
$$

Fix now $\lambda > 0$ so small that q_λ is so large that $q_\lambda^{1-t}\Sigma_t < 1/2$. Then for all $w \in P_M \cap J(f_\lambda)$, we get

$$
\sum_{z \in G^{-1}(w)} |G'(z)|^{-t} \le C_t (M\mathrm{e}^M)^{1-t}. \tag{7.3}
$$

for some constant C_t depending on t and independent of M. Now, there exists $0 < p_\lambda < q_\lambda$ such that $\{z \in \mathbb{C} : \mathrm{Re}(z) > p_\lambda\} \cap \overline{\{f_\lambda^n(0) : n \ge 0\}} = \emptyset$. Cover the set $Q_M = \{z \in \mathbb{C} : p_\lambda \le \mathrm{Re}(z) \le M+1\}$ by the family $\Re_M$ of non-overlapping rectangles intersecting $G(E_\infty(M))$ of the form $\Delta \times [-\frac{3}{2}\pi, \frac{3}{2}\pi]$ with the lengths of Δ equal to 1. For every element $R \in \Re_M$ fix one element $w_R \in R \cap G(E_\infty(M))$. Then the family $\{G_z^{-1}(R) : R \in \Re_M, z \in G^{-1}(w_R)\}$ covers $E_\infty(M)$, where $G_z^{-1} : Q_M :\to \mathbb{C}$ is the holomorphic branch of G sending w to z. It follows from Koebe's distortion theorem and (7.3) that if $R \in \Re_M$ and $v \in R$, then

$\sum_{z\in G^{-1}(w_R)}|(G_z^{-1})'(v)|^t \le C_t'(Me^M)^{1-t}$ for some constant C_t' independent of M. Consequently

$$\begin{aligned}\sum_{R\in\Re_M}\sum_{z\in G^{-1}(w_R)}\operatorname{diam}^t(G_z^{-1}(R))) &\le \sum_{R\in\Re_M}\sum_{z\in G^{-1}(w_R)}|(G_z^{-1})'(v_z)|^t\operatorname{diam}^t(R)\\ &\le (3\pi+1)^tC_t'\sum_{R\in\Re_M}(Me^M)^{1-t}\\ &\le (3\pi+1)^tC_t'(M+1)(Me^M)^{1-t},\end{aligned}$$

where $v_z\in R$ is chosen so that $|(G_z^{-1})'(v_z)|=\sup_{v\in R}\{|(G_z^{-1})'(v)|\}$. Since for every $N\ge 1$,

$$J_{ru}(f_\lambda)\cap\{z\in \mathbb{C}\colon -\pi\le \operatorname{Im}(z)\le\pi\}\subset\bigcup_{M\ge N}E_\infty(M),$$

since

$$\sum_{M\ge N}\sum_{R\in\Re_M}\sum_{z\in G^{-1}(w_R)}\operatorname{diam}^t(G_z^{-1}(R)))\le(3\pi+1)^tC_t'\sum_{M=N}^\infty(M+1)(Me^M)^{1-t}$$

and since $\lim_{N\to\infty}\left((3\pi+1)^tC_t'\sum_{M=N}^\infty(M+1)(Me^M)^{1-t}\right)=0$, we conclude that

$$\operatorname{HD}(J_{ru}(f_\lambda)\cap\{z\in\mathbb{C}\colon -\pi\le\operatorname{Im}(z)\le\pi\})\le t.$$

Since

$$J_{ru}(f_\lambda)=\bigcup_{n\in\mathbb{Z}}\Big(J_{ru}(f_\lambda)\cap\{z\in\mathbb{C}\colon -\pi\le\operatorname{Im}(z)\le\pi\}+2\pi in\Big),$$

we conclude that $\operatorname{HD}(J_{ru}(f_\lambda))\le t$. The proof is finished. ■

Let

$$J_r(f_\lambda)=\{z\in J(f_\lambda):\liminf_{n\to\infty}|f_\lambda^n(z)|<\infty\}.$$

Since $J_r(f_\lambda)=J_{bd}(f_\lambda)\cup J_{ru}(f_\lambda)$, combining Lemma 7.1 and Corollary 2.2, we obtain the following.

Theorem 7.2. *If $\lambda\in(0,\infty)$, then*

$$\lim_{\lambda\to 0}\operatorname{HD}(J_r(f_\lambda))=1.$$

Corollary 7.3. *If $|\lambda|<1/e$ and $\lambda\ne 0$, then* $\operatorname{HD}(J_r(f_\lambda))<2$.

Proof. The following theorem has been proven in [As] as Corollary 1.3 (comp. [GL], Theorem 5, p.13.

Theorem 7.4. *If $f:\Omega\to\Omega'$ is a K-quasiconformal homeomorphism and $E\subset\Omega$ is a compact set, then*

$$\operatorname{HD}(f(E))\le\frac{2K\operatorname{HD}(E))}{2+(K-1)\operatorname{HD}(E)}.$$

Although Astala's result is stated for compact sets E only, it is in fact true for all subsets E of Ω. And indeed, assuming first that $\overline{E} \subset G$ and the closure $\overline{E}$ is compact, we see that Theorem II.8.1 from [LV] applies and Astala's proof goes step by step through. Now, it suffices to notice that the Hausdorff dimension is σ-stable and each subset of Ω is a countable union of sets whose closures are compact subsets of Ω. In particular quasiconformal maps send sets whose Hausdorff dimension is less than 2 into sets with Hausdorff dimension less than 2. Since all the maps f_λ with $|\lambda| < 1/e$ and $\lambda \neq 0$ are mutually quasiconformally conjugate, combining Theorem 7.2 and Theorem 7.4, we obtain our corollary. ■

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