GEOMETRY OF HYPERBOLIC JULIA-LAVAURS SETS

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Abstract. Let J_σ be the Julia-Lavaurs set of a hyperbolic Lavaurs map g_σ and let h_σ be its Hausdorff dimension. We show that the upper ball-(box) counting dimension and the Hausdorff dimension of J_σ are equal, that the h_σ-dimensional Hausdorff measure of J_σ vanishes and that the h_σ-dimensional packing measure of J_σ is positive and finite. If g_σ is derived from the parabolic quadratic polynomial $f(z)=z^2+\frac{1}{4}$, then the Hausdorff dimension h_σ is a real-analytic function of σ. As our tool we study analytic dependence of the Perron-Frobenius operator on the symbolic space with infinite alphabet.

1. Introduction

In Section 1 we study analytic dependence of the Perron-Frobenius operator on the symbolic space with infinite alphabet. In Section 2 we collect preliminaries on conformal iterated function systems. The third section is devoted to introduce and explore lattice type conformal iterated function systems. Section 4 introduces Lavaurs maps and sets and provides some tolls needed in Section 5. In Section 5, the last section of our paper, we combine the results from previous sections to complete the proofs of our two main results Theorem 6.1 and Theorem 6.3.

2. Analyticity of the Perron-Frobenius operator on the symbolic space with infinite alphabet

Let I be a countable set, either finite or infinite. Let

$$\mathcal{H}_0=\{g:I^\infty\to \mathbb{C}: g \text{ is bounded and continuous}\}$$

and for every $f\in\mathcal{H}_0$ let

$$||f||_0=\sup\{|f(\omega)|:\omega\in I^\infty\}.$$

Given $\beta>0$ let

$$V_\beta(f):=\sup_{n\ge 1}\{V_{\beta,n}(f)\}<\infty,$$

where

$$V_{\beta,n}(f)=\sup\{|f(\omega)-f(\tau)|\mathrm{e}^{-\beta(n-1)}:\omega,\tau\in E^\infty \text{ and } |\omega\wedge\tau|\ge n\}.$$

Set

$$\mathcal{H}_\beta=\{g\in\mathcal{H}_0:V_\beta(g)<\infty\}.$$

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The elements of this set will be called Hölder continuous functions of order β. The set $\mathcal{H}_\beta$ becomes a Banach space when endowed with the norm

$$||g||_\beta = ||g||_0 + V_\beta(g).$$

For every $i \in I$ we put

$$[i] = \{\omega \in I^\infty : \omega_1 = i\}.$$

We define the class $\mathcal{H}_\beta^s \subset \mathcal{H}_\beta$ as follows

$$\mathcal{H}_\beta^s = \{f \in \mathcal{H}_\beta : \sum_{i\in I} \exp\Big(\sup(\mathrm{Re}(f|_{[i]}))\Big) < \infty\}.$$

and call its elements Hölder summable functions. $L(\mathcal{H}_\beta)$ denotes the space of all bounded (continuous) operators on $\mathcal{H}_\beta$. Finally given $f \in \mathcal{H}_\beta^s$ we define the Perron-Frobenius operator $\mathcal{L}_f : \mathcal{H}_0 \to \mathcal{H}_0$, acting on the space of bounded continuous functions $\mathcal{H}_0$, as follows

$$\mathcal{L}_f(g)(\omega) = \sum_{i\in I} \exp(f(i\omega))g(i\omega).$$

Then $||\mathcal{L}_f||_0 \le \sum_{i\in I} \exp(\sup(\mathrm{Re}(f|_{[i]}))) < \infty$. In fact (see [MU2] and [MU3]) the operator $\mathcal{L}_f$ preseserves the Banach space $\mathcal{H}_\beta$, is bounded on it, and even more, satisfies the so called Ionescu-Tulcea and Marinescu inequality. We start the results and proofs in this section with the following.

Lemma 2.1. *If for every $\omega \in I^\infty$, the function $t \mapsto f_t(\omega) \in \mathbb{C}$ is holomorphic on a domain $G \subset \mathbb{C}$ and the map $t \mapsto \mathcal{L}_{f_t} \in L(\mathcal{H}_\beta)$ is continuous on G, then the map $t \mapsto \mathcal{L}_{f_t} \in L(\mathcal{H}_\beta)$ is holomorphic on G.*

Proof. Let $\gamma \subset G$ be a simple closed rectifiable curve. Fix $g \in \mathcal{H}_\beta$ and $\omega \in I^\infty$. Let $W \subset G$ be a bounded open set such that $\gamma \subset W \subset \overline{W} \subset G$. Since for each $e \in I$ the function $t \mapsto g(e\omega)\exp\big(f_t(e\omega)\big) \in \mathbb{C}$, $t \in G$, is holomorphic and since for every $t \in W$

$$\begin{aligned} ||\sum_{i\in I} g(i\omega)\exp\Big(f_t(i\omega)\Big)||_\infty &\le ||\sum_{i\in I} g(i\omega)\exp\Big(f_t(i\omega)\Big)||_\beta \\ &\le ||g||_\beta \sup\{||\mathcal{L}_{f_z}||_\beta : z \in \overline{W}\} < \infty \end{aligned}$$

by compactness of $\overline{W}$ and continuity of $t \mapsto \mathcal{L}_{f_t}$, we conclude that the function

$$t \mapsto \mathcal{L}_{f_t}(g)(\omega) = \sum_{i\in I} g(i\omega)\exp\Big(f_t(i\omega)\Big) \in \mathbb{C},\ t \in W,$$

is holomorphic. Hence by Cauchy's theorem $\int_\gamma \mathcal{L}_{f_t}g(\omega)dt = 0$. Since the function $t \mapsto \mathcal{L}_{f_t}g \in \mathcal{H}_\beta$ is continuous, the integral $\int_\gamma \mathcal{L}_{f_t}gdt$ exists and for every $\omega \in I^\infty$, we have $\int_\gamma \mathcal{L}_{f_t}gdt(\omega) = \int_\gamma \mathcal{L}_{f_t}g(\omega)dt = 0$. Hence $\int_\gamma \mathcal{L}_{f_t}gdt = 0$. Now, since $t \mapsto \mathcal{L}_{f_t} \in L(\mathcal{H}_\beta)$ is continuous, the integral $\int_\gamma \mathcal{L}_{f_t}dt$ exists and for every $g \in \mathcal{H}_\beta$, $\int_\gamma \mathcal{L}_{f_t}dt(g) = \int_\gamma \mathcal{L}_{f_t}gdt = 0$. Thus $\int_\gamma \mathcal{L}_{f_t}dt = 0$ and in view of Morera's theorem the map $t \mapsto \mathcal{L}_{f_t} \in L(\mathcal{H}_\beta)$ is holomorphic on G. The proof is complete. ■

In order to prove the main result of this section we need several elementary lemmas. In order to formulate them we need to define some class of mappings. Namely, given $i \in I$, we define the mapping

$$i : I^\infty \to I^\infty$$

by setting

$$i(\omega) = i\omega.$$

Lemma 2.2. *If $i \in I$ and and $\rho \in \mathcal{H}_\beta$ then the operator $A_{i,\rho}$ given by the formula*

$$A_{i,\rho}(g)(\omega) = \rho \circ i(\omega) \cdot g \circ i(\omega)$$

acts on the space $\mathcal{H}_\beta$, is continuous and $||A_{i,\rho}||_\beta \le 3||\rho \circ i||_\beta$.

Proof. Fix $g \in \mathcal{H}_\beta$, $\omega \in I^\infty$. Then

$$|A_{i,\rho}(g)(\omega)| = |\rho(i\omega)||g(i\omega)| \le ||\rho \circ i||_\infty ||g||_\infty \le ||\rho \circ i||_\beta ||g||_\beta. \tag{2.1}$$

Fix now in addition $\tau \in I^\infty \setminus \{\omega\}$ such that $|\omega \wedge \tau| \ge 1$. Then

$$\begin{aligned} A_{i,\rho}(g)(\omega) - A_{i,\rho}(g)(\tau)| &= |\rho(i\omega)g(i\omega) - \rho(i\tau)g(i\tau)| \\ &= |\rho(i\omega)(g(i\omega) - g(i\tau)) + g(i\tau)(\rho(i\omega) - \rho(i\tau))| \\ &\le ||\rho \circ i||_\infty |g(i\omega) - g(i\tau)| + ||g||_\infty |\rho(i\omega) - \rho(i\tau)| \\ &\le ||\rho \circ i||_\beta ||g||_\beta \mathrm{e}^{-\beta|\omega\wedge\tau|} + ||g||_\beta ||\rho \circ i||_\beta \mathrm{e}^{-\beta|\omega\wedge\tau|}. \end{aligned}$$

Hence $V_\beta(A_{i,\rho}(g)) \le 2||\rho \circ i||_\beta ||g||_\beta$ and combining this with (2.1), we conclude that $||A_{i,\rho}(g)|| \le 3||\rho \circ i||_\beta ||g||_\beta$. consequently $A_{i,\rho}$ acts on the space $\mathcal{H}_\beta$, is continuous and $||A_{i,\rho}||_\beta \le 3||\rho \circ i||_\beta$. The proof is complete. ■

Similarly (only easier) one proves the following.

Lemma 2.3. *If $\rho, g \in \mathcal{H}_\beta$, then $\rho g \in \mathcal{H}_\beta$ and $||\rho g||_\beta \le 3||\rho||_\beta ||g||_\beta$.*

Lemma 2.4. *If $f \in \mathcal{H}_\beta$, then $e^f \in \mathcal{H}^b$ and if $\rho : Y \to \mathcal{H}_\beta$ is a continuous mapping defined on a compact set Y, then $e^\rho : Y \to \mathcal{H}_\beta$ is also continuous.*

Proof. By Lemma 2.6.3, $||f^n||_\beta \le 3^n ||f||_\beta^n$. Hence, the series $\mathrm{e}^f = \sum_{n=0}^\infty \frac{f^n}{n!}$ converges in $\mathcal{H}_\beta$ and the first part of our lemma is proven. The second part follows now from the remark that for every $y \in Y$, $||\rho(y)||_\beta \le \sup\{||\rho(x)||_\beta : x \in Y\} < \infty$ and the series $\sum_{n=0}^\infty \frac{\rho(y)^n}{n!}$ converges uniformly on Y. The proof is complete. ■

Lemma 2.5. *For every $R > 0$ there exists $M = M_R \ge 1$ such that if $|z - \xi| \le R$, then $|e^\xi - e^z| \le M e^{\mathrm{Re} z}|z - \xi|$.*

Proof. Looking at the Taylor's series expansion of the exponential function about 0, we see that there exists a constant $M \ge 1$ such $|\mathrm{e}^w - 1| \le M|w|$, if $|w| \le R$. Hence $|\mathrm{e}^\xi - \mathrm{e}^z| = |\mathrm{e}^z||\mathrm{e}^{|z-\xi|} = 1| \le \mathrm{e}^{\mathrm{Re} z} M|z - \xi|$. The proof is complete. ■

Lemma 2.6. *If $f \in \mathcal{H}_\beta$, then for every $i \in I$*

$$||e^{f\circ i}||_\beta \le 2M_{||f||_\beta} \exp\Big(\sup \mathrm{Re}\Big(f|_{[i]}\Big)\Big)||f||_\beta.$$

Proof. Fix $\omega \in I^\infty$ such that $A_{i\omega_1} = 1$. Then $|\mathrm{e}^{f(i\omega)}| = \mathrm{e}^{\mathrm{Re} f(i\omega)} \le \exp\Big(\sup \mathrm{Re}\Big(f|_{[i]}\Big)\Big)$, whence

$$||\mathrm{e}^{f\circ i}||_\beta \le \exp\Big(\sup \mathrm{Re}\Big(f|_{[i]}\Big)\Big). \tag{2.2}$$

Fix now in addition $\tau \in I^\infty \setminus \{\omega\}$ with $|\tau \wedge \omega| \ge 1$. Using the Lemma 2.5, we get

$$|\mathrm{e}^{f(i\omega)} - \mathrm{e}^{f(i\tau)}| \le M_{||f||_\beta} \mathrm{e}^{ref(\tau)}|f(i\omega) - f(i\tau)| \le M_{||f||_\beta} \exp\Big(\sup \mathrm{Re}\Big(f|_{[i]}\Big)\Big)||f||_\beta \mathrm{e}^{-\beta|\tau\wedge\omega|}.$$

Thus, $V_\beta(\mathrm{e}^{f\circ i}) \le \exp\Big(\sup \mathrm{Re}\Big(f|_{[i]}\Big)\Big)||f||_\beta$. Combining this and (2.2) completes the proof. ■

Lemma 2.7. *If $\rho : Y \to \mathcal{H}_\beta$ is a continuous mapping defined on a metric space Y, then for every $i \in I$, the function $y \mapsto A_{i,\rho(y)} \in L(\mathcal{H}_b)$, $y \in Y$, is continuous.*

Proof. Fix $y_0 \in Y$ and take $\delta > 0$ so small that for every $y \in B(y_0, \delta)$, $||\rho(y) - \rho(y_0)||_\beta \le \epsilon/3$. Then for $y \in B(y_0, \delta)$, we have in view of Lemma 2.2 the following.

$$||A_{i,\rho(y)} - A_{i,\rho(y_0)}||_\beta = ||A_{i,\rho(y)-\rho(y_0)}||_\beta \le 3||\rho(y) - \rho(y_0)||_\beta \le 3.$$

The proof is complete. ■

Our main theorem in this section is the following.

Theorem 2.8. *If G is an open connected subset of $\mathbb{C}$, the function $t \mapsto f_t \in \mathcal{H}^s_\beta$, $t \in G$, is continuous and the function $t \mapsto f_t(\omega) \in \mathbb{C}$, $t \in G$, is holomorphic for every $\omega \in I^\infty$, then the function $t \mapsto \mathcal{L}_{f_t} \in L(\mathcal{H}_\beta)$, $t \in G$, is holomorphic.*

Proof. In view of Lemma 2.2 it suffices to demonstrate that the function $t \mapsto \mathcal{L}_{f_t} \in L(\mathcal{H}_\beta)$, $t \in G$, is continuous. So, fix $t_0 \in G$ and $\delta > 0$ so small that $B(t_0, 2\delta) \subset G$ and $||f_t - f_{t_0}||_\infty \le ||f_t - f_{t_0}||_\beta \le 1$ for all $t \in B(t_0, 2\delta)$. By Lemmas 2.7 and 2.4, for all $i \in I$, the function $t \mapsto A_{i,\mathrm{e}^{f_t}} \in L(\mathcal{H}_\beta)$, $t \in \overline{B(t_0, \delta)}$, is continuous. Since

$$\mathcal{L}_{f_t} = \sum_{i\in I} A_{i,\mathrm{e}^{f_t}},$$

it therefore suffices to demonstrate that the series $\sum_{i\in I} A_{i,\mathrm{e}^{f_t}}$ converges uniformly on $\overline{B(t_0, \delta)}$. And indeed, in view of Lemma 2.2 and Lemma 2.6, for every $i \in I$ and every $t \in \overline{B(t_0, \delta)}$ we have

$$||A_{i,\mathrm{e}^{f_t}}||_\beta \le 3||\exp(f_t \circ i)||_\beta \le 6M \exp\Big(\sup \mathrm{Re}\Big(f_t|_{[i]}\Big)\Big)M_1,$$

where $M_1 = \sup\{||f_t||_\beta : t \in \overline{B(t_0, \delta)}\}$ is finite due to continuity of the function $t \mapsto f_t \in \mathcal{H}^s_\beta$ on the compact set $\overline{B(t_0, \delta)}$, and $M = M_{M_1}$ in the sense of Lemma 2.5. Now, in view of our choice of δ, we can continue the above stimates as follows.

$$\begin{aligned}
||A_{i,\mathrm{e}^{f_t}}||_\beta &\le 6MM_1 \exp\Big(\sup \mathrm{Re}\Big(f_{t_0}|_{[i]}\Big) + ||f_t - f_{t_0}||_\infty\Big) \\
&\le 6MM_1 \exp\Big(\sup \mathrm{Re}\Big(f_{t_0}|_{[i]}\Big) + 1\Big) \le 6MM_1 \exp\Big(\sup \mathrm{Re}\Big(f_{t_0}|_{[i]}\Big)\Big).
\end{aligned}$$

Since by summability of the function f_{t_0}, the series $\sum_{i\in I}\exp\Big(\sup\mathrm{Re}\Big(f_{t_0}|_{[i]}\Big)\Big)$ converges, the proof is complete. ■

3. **Preliminaries on conformal iterated function systems**

In [MU1] we have provided the framework to study infinite conformal iterated function systems. We shall recall first this notion and some of its basic properties. Let I be a countable index set with at least two elements and let $S=\{\phi_i : X\to X : i\in I\}$ be a collection of injective contractions from a compact metric space X into X for which there exists $0<s<1$ such that $\rho(\phi_i(x),\phi_i(y))\le s\rho(x,y)$ for every $i\in I$ and for every pair of points $x,y\in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let I^n denote the space of words of length n, I^∞ the space of infinite sequences of symbols in I, $I^*=\bigcup_{n\ge1}I^n$ and for $\omega\in I^n$, $n\ge1$, let $\phi_\omega=\phi_{\omega_1}\circ\phi_{\omega_2}\circ\cdots\circ\phi_{\omega_n}$. If $\omega\in I^*\cup I^\infty$ and $n\ge1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2\ldots\omega_n$. Since given $\omega\in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n\ge1$, converge to zero and since they form a decreasing family, the set

$$\bigcap_{n=0}^{\infty}\phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty\to X$. The main object of our interest will be the limit set

$$J=\pi(I^\infty)=\bigcup_{\omega\in I^\infty}\bigcap_{n=1}^{\infty}\phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J=\bigcup_{i\in I}\phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.

An iterated function system S is said to be conformal if X is a compact connected subset of a Euclidean space$\mathbb{R}^d$ for some $d\ge1$ and the following conditions are satisfied.

- **(a):** Open Set Condition (OSC) $\phi_i(\mathrm{Int}_{\mathbb{R}^d}(X))\cap\phi_j(\mathrm{Int}_{\mathbb{R}^d}(X))=\emptyset$ for all $i,j\in I$, $i\ne j$.
- **(b):** There exists an open connected set V such that $X\subset V\subset\mathbb{R}^d$ and all maps ϕ_i, $i\in I$, extend to C^1 conformal diffeomorphisms of V into V. (Note that for $d=1$ this just means that all the maps ϕ_i, $i\in I$, are C^1 diffeomorphisms, for $d\ge2$ the word conformal mean either holomorphic or antiholomoerphic and for $d\ge3$ the maps ϕ_i, $i\in I$ are Möbius transformations. The proof of the last statement can be found in [BP] for example, where it is called Liouville's theorem)
- **(c):** There exist $\gamma,l>0$ such that for every $x\in\partial X\subset\mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x,\gamma,l)\subset\mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.
- **(d):** Bounded Distortion Property(BDP). There exists $K\ge1$ such that

$$|\phi'_\omega(y)|\le K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

In fact throughout the whole paper we will need one more condition which (comp. [MU1]) can be considered as a strengthening of (BDP).

(e): There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big||\phi'_i(y)| - |\phi'_i(x)|\Big| \leq L||\phi'_i|||y-x|^\alpha.$$

for every $i \in I$ and every pair of points $x, y \in V$.

Remark 3.1. *Note that for $d = 2$, decreasing V if necessary, conditions (e) and (d) are satisfied due to Koebe's distortion theorem. In case $d \geq 3$ these were proved in [Ur].*

Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets C of V

$$\mathrm{diam}(\phi_\omega(C)) \leq ||\phi'_\omega||\mathrm{diam}(C)$$

and, for an appropriate V,

$$\mathrm{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||,$$

where $D \geq 1$ is a constant depending only on V. Moreover,

$$\mathrm{diam}(\phi_\omega(X)) \geq D^{-1}||\phi'_\omega||$$

and

$$\phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r),$$

for every $x \in X$, every $0 < r \leq \mathrm{dist}(X, \partial V)$, and every word $\omega \in I^*$.

Frequently, refering to (BDP) we will mean either (BDP) itself or one of the above properties. Notice that for simplicity and clarity of our exposition we assumed the open set U appearing in the open set condition to be $\mathrm{Int}(X)$.

As was demonstrated in [MU1], conformal iterated function systems naturally break into two main classes, irregular and regular. This dichotomy can be determined from either the existence of a zero of a natural pressure function or, equivalently, the existence of a conformal measure. The topological pressure function, P is defined as follows. For every integer $n \geq 1$ define

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi'_\omega||^t.$$

and

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(t).$$

For a conformal system S, we sometimes set $\psi_S = \psi_1 = \psi$. The finiteness parameter, $\theta = \theta_S$, of the system S is defined by

$$\theta_S = \inf\{t : \psi(t) < \infty\}.$$

In [MU1], it was shown that the topological pressure function $\mathrm{P}(t)$ is non-increasing on $[0,\infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$ and $\mathrm{P}(d) \leq 0$. Of course, $\mathrm{P}(0) = \infty$ if and only if I is infinite. In [MU1] (see Theorem 3.15) we have proved the following characterization of the Hausdorff dimension of the limit set J, which will be denoted by $\mathrm{HD}(J) = h_S$.

Theorem 3.2. $\mathrm{HD}(J) = \sup\{\mathrm{HD}(J_F) : F \subset I \textit{ is finite}\} = \inf\{t : \mathrm{P}(t) \le 0\}$. *If* $\mathrm{P}(t) = 0$, *then* $t = \mathrm{HD}(J)$.

In [MU4] the following formula for the upper ball (box)-counting dimension of the limie set of a conformal IFS was provided.

Theorem 3.3. *For every $x \in X$ the following holds.*

$$\overline{\mathrm{BD}}(J) = \max\{\mathrm{HD}(J), \overline{\mathrm{BD}}(\{\phi_i(x) : i \in I\})\}.$$

We call the system S regular if there is t such that $\mathrm{P}(t) = 0$. It follows from [MU1] that t is unique. Also, the system is regular if and only if there is a t-conformal measure. Recall that a Borel probability measure m is said to be t-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \ne j$. We call the system S hereditarily regular if each cofinite subsystem of S is regular. According to [MU1] hereditarily regular equivalently means that $\mathrm{P}(\theta) = \psi(\theta) = \infty$.

4. Lattice type iterated function systems

We call a conformal IFS $S = \{\phi_i\}_{i \in I}$ in the complex plane $\mathbb{C}$ weakly lattice type if $I = \mathbb{N} \times \mathbb{Z}$ and there exist an integer $q \ge 1$, a finite set $F \subset I$ and $b > 1$ such that the following conditions are satisfied.

(A):

$$b^{-1}|m + ni| \le \mathrm{dist}(\phi_{m,n}(X)^{-q}, 0) \le b|m + ni|.$$

for all $(m, n) \in \mathbb{N} \times \mathbb{Z} \setminus F$.

(B): $|\phi_i(z)| \asymp |\phi_i(z)|^{1+q}$ for all $i \in I \setminus F$.

(C): $\mathrm{diam}\Big(\phi_i(X)^{-q}\Big) \le b$ for all $i \in I \setminus F$.

(D): For all $(m, n), (k, l) \in \mathbb{N} \times \mathbb{Z} \setminus F$,

$$\mathrm{dist}\Big(\phi_{m,n}(X)^{-q}, \phi_{k,l}(X)^{-q}\Big) \le b|(m + ni) - (k + li)|.$$

Increasing b if necessary, it follows from (A) and (C) that

(A'): $\mathrm{Dist}(\phi_{m,n}j(X)^{-q}, 0) \le b|m + ni|$.

A weakly lattice type IFS is called lattice type if in addition there exists $T : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, a holomorphic univalent branch of $z^{-1/q}$ such that

$$T\Big((\phi_i(X))^{-q}\Big) = \phi_i(X)$$

for all $i \in I \setminus F$.

Examples of lattice type iterated function systems have implicitely appeared in the last section of [MU1] as well as in [DSZ] and [Zi]. The examples from [DSZ] and [Zi] form the main object of this paper and are treated in detail in the next section. We shall prove the following.

Theorem 4.1. *If $S = \{\phi_i\}_{i \in I}$ is a weakly lattice type system, then $\theta_S = \frac{2q}{q+1}$, S is hereditarily regular and*

(a): $\overline{\mathrm{BD}}(J) = \mathrm{HD}(J)$.
(b): $\mathrm{H}^h(J) = 0$.

If S is a lattice type system, then in addition

(d): $0 < \mathrm{P}^h(J) < \infty$.

Proof. Let us first determine the θ number of the system S. Using (B), (A) and (A') we get for every $t > 0$ and every $z \in X$

$$\eta(t) := \sum_{(m,n)\in \mathbb{N}\times\mathbb{Z}\setminus F} ||\phi'_{m,n}||^t \asymp \sum_{(m,n)\in \mathbb{N}\times\mathbb{Z}\setminus F} |\phi_i(z)|^{t(1+q)} \asymp \sum_{(m,n)\in \mathbb{N}\times\mathbb{Z}\setminus F} |m+ni|^{-t\frac{q+1}{q}}$$

and this series converges if and only if $t\frac{q+1}{q} > 2$ or equivalently $t > \frac{2q}{q+1}$. Hence $\theta_S = \frac{2q}{q+1}$ and since $\eta\left(\frac{2q}{q+1}\right) = \infty$, the system S is hereditarily regular.

In order to prove (a) fix $x \in X$, $r > 0$ and consider $n \geq 0$ such that $2^n r \leq \mathrm{Dist}(0,X)^q$. Rescaling the system appropriately we may assume without loosing generality that $\mathrm{Dist}(0,X)^q = 1$. Let

$$I_{q,n}(r) = \{i \in I : 2^{n-1}r \leq |\phi_i(x)^q| < 2^n r\}.$$

Notice that if $i,j \in I_{q,n}(r)$ and $|\phi_j(x)^{-q} - \phi_i(x)^{-q}| \leq r(2^n r)^{-2}$, then $|\phi_j(x)^q - \phi_i(x)^q| \leq |\phi_j(x)^q| \cdot |\phi_i(x)^q| \leq r(2^n r)^{-2} < r$ which means that $\phi_j(x)^q \in B(\phi_i(x)^q, r)$. Hence denoting by $N_{q,n}(r)$ the minimal number of balls needed to cover the set $Y_{q,n}(r) = \{\phi_j(x)^q : j \in I_{q,n}(r)\}$, we get

$$N_{q,n}(r) \leq C_1 \frac{(2^n r)^{-2}}{(r(2^n r)^{-2})^2} = C_1 2^{2n} \tag{4.1}$$

for some universal constant $C_1 > 0$. Also, using (A) and (A') we get

$$N_{q,n}(r) \leq \#\{\phi_i(x)^{-q} : i \in I_{q,n}(r)\} \leq \#\{(k,l) : b^{-1}(2^n r)^{-1} \leq |k+li|\beta(2^{n-1}r)^{-1}\} \leq C_2(2^n r)^{-2}$$

for some universal constant $C_2 > 0$. We will use this estimate and (4.1) in the following form.

$$N_{q,n}(r) \leq C_3 \begin{cases} 2^{2n} & \text{if } \ 2^{2n}r \leq 1 \\ 2^{-2n}r^{-2} & \text{if } \ 2^{2n}r \geq 1 \ \text{ and } \ 2^n r \leq 1, \end{cases} \tag{4.2}$$

where $C_3 = \max\{C_1, C_2\}$. Given $0 < r_1 \leq r_2 G$ let $A(r_1, r_2) = \{z \in \mathbb{C} : r_1 \leq |z| < r_2\}$. Given a set $A \subset \mathbb{C}$ by $A^{1/q}$ we denote the full inverse-image of A under the map $z \mapsto z^q$. Fix $j \geq 1$. Since $A\Big((2^{j-1}r)^q, (2^j r)^q\Big)^{1/q} = A(2^{j-1}r, 2^j r)$, since $B^{1/q}$ is contained in a union of at most q

balls of radii $2^{j(q-1)}r^q \cdot \frac{1}{q}\Big((2^{j-1}r)^q\Big)^{\frac{1}{q}-1} = \frac{1}{q}2^{(q-1)}r$ if B is a ball of radius $2^{j(q-1)}r^q$ contained in $A\Big((2^{j-1}r)^q, (2^jr)^q\Big)$, and since $\frac{(2^jr)^q}{2^{j(q-1)}r^q} = 2^j$, we obtain

$$N_j\left(\frac{2^{q-1}}{qr}\right) \le qN_{q,j}(2^{j(q-1)}r^q),$$

where $N_k(s)$ is the minimal number of balls with radii s needed to cover the set $\{\phi_i(x) : 2^{k-1}s \le |\phi_i(x)| < 2^k s\}$. Therefore, applying (4.2), denoting by $N(s)$ the minimal number of balls with radii s needed to cover the set $\{\phi_i(x) : i \in I\}$ and $l_1 = \min\{j : 2^{2j}2^{j(q-1)}r^q \ge 1\}$, $l_2 = \min\{j : 2^j 2^{j(q-1)}r^q \ge 1\}$, we obtain

$$\begin{aligned}
N\left(\frac{2^{q-1}}{q}r\right) &\le 1 + \sum_{j=1}^{l_2} N_j\left(\frac{2^{q-1}}{q}r\right) \le 1 + q\sum_{j=1}^{l_2} N_{q,j}(2^{j(q-1)}r^q) \\
&\le 1 + q\sum_{j=1}^{l_1} 2^{2j} + \sum_{j=l_1-1}^{l_2} 2^{-2j}r^{-2} \le \mathrm{const}(2^{2l_1} + r^{-2}2^{-2l_1}) \\
&\le \mathrm{const}(r^{-\frac{2q}{q+1}} + r^{-2}r^2) \le \mathrm{const}\left(\frac{2^{q-1}}{q}r\right)^{-\frac{2q}{q+1}}.
\end{aligned}$$

Hence $\overline{\mathrm{BD}}(\{\phi_i(x) : i \in I\}) \le \frac{2q}{q+1}$. Applying therefore Theorem 3.3 along with the fact that $\mathrm{HD}(J) \ge \theta$ and the proven above equality $\theta = \frac{2q}{q+1}$, we conclude that $\overline{\mathrm{BD}}(J) = \mathrm{HD}(J)$.

We shall now prove part (b) saying that $\mathrm{H}^h(J) = 0$. So, fix $r > 0$ and consider the set

$$I(r) = \{(m,n) \in I\!\!N \times \boldsymbol{Z} \setminus F : b^{-1}|m + ni| > r^{-q}\}.$$

It follows from (A) that if $(m,n) \in I(r)$, then $\mathrm{Dist}(\phi_{m,n}j(X), 0) \le b^{1/q}|m+ni|^{-1/q} < r$ which means that $\phi_{m,n}j(X) \subset B(0,r)$. Hence, using (B) and (A), and denoting by m the h-conformal measure which exists since the system S has been proved to be hereditarily regular, we get

$$\begin{aligned}
m(B(0,r)) &\ge \sum_{(m,n)\in I(r)} m(\phi_{m,n}j(X)) \ge \sum_{(m,n)\in I(r)} K^{-h}||\phi'_{m,n}||^h \\
&\succeq \sum_{(m,n)\in I(r)} \mathrm{dist}(\phi_{m,n}j(X), 0)^{h(q+1)} \asymp \sum_{(m,n)\in I(r)} |m+ni|^{-h\frac{q+1}{q}} \\
&\asymp \iint_{x^2+y^2>(br^{-q})^2} (x^2+y^2)^{-h\frac{q+1}{2q}} \\
&= \int_0^{2\pi}\int_{br^{-q}}^{\infty} u^{-h\frac{q+1}{q}+1}\,du\,d\theta \asymp (r^{-q})^{2-h\frac{q+1}{q}} = r^{h(q+1)-2q},
\end{aligned} \tag{4.3}$$

where the relation $\alpha(r) \succeq \beta(r)$ means that $\alpha(r) \ge C\beta(r)$ for some constant $C > 0$ independent of r.

Hence
$$\lim_{r\to 0}\frac{m(B(0,r))}{r^h}\ge \lim_{r\to 0} r^{h(q+1)-2q}=\infty$$
since $h<2$. The proof of part (b) is finished.

The fact that $\mathrm{P}^h(J)>0$ follows immediately from Lemma 4.3 in [MU1]. Assuming that the system S is of lattice type we shall now prove that $\mathrm{P}^h(J)<\infty$. Notice that if $0<r<|x|$, then
$$(B(x,r))^{-1}=B\left(\frac{|x|^2}{|x|^2-r^2}\cdot\frac{1}{x},\frac{r}{|x|^2-r^2}\right).$$
If in addition
$$r\le\frac{1}{2q}|x|. \tag{4.4}$$
then
$$\frac{\frac{2r}{|x|^2-r^2}}{\left|\frac{|x|^2}{|x|^2-r^2}\cdot\frac{1}{x}\right|}=\frac{2r}{|x|}<\frac{1}{q}$$
and therefore the map $z\mapsto z^q$ is univalent on the ball
$$B\left(\frac{|x|^2}{|x|^2-r^2}\cdot\frac{1}{x},\frac{r}{|x|^2-r^2}\right).$$
Thus the map $z\mapsto z^{-q}$ is univalent on the ball $B(x,r)$ and applying Koebe's distortion theorem we get

$$B(x,r/2))^{-q}\supset B(x^{-q},(2K)^{-1}q|x|^{-(q+1)}r), \tag{4.5}$$
where K is the Koebe constant corresponding to the ration of radii equal to $1/2$. Now note that due to (B) and (A) there exists a constant $C_4>0$ such that for all $i\in I\setminus F$, and all $x\in\phi_i(X)$,

$$\mathrm{diam}(\phi_i(X))\ge C_4|x|^{q+1}. \tag{4.6}$$
Fix $i\in I\setminus F$, $x\in\phi_i(X)$, and $1\ge r\ge C_5\mathrm{diam}(\phi_i(X))$ for some large positive constant C_5 which will be determined later in the course of the proof.

Assume first that (4.4) is satisfied. Then
$$=q(2K)^{-1}r|x|^{-(q+1)}\ge q(2K)^{-1}C_5\mathrm{diam}(\phi_i(X))C_4\mathrm{diam}(\phi_i(X))^{-1}=q(2K)^{-1}C_5C_4.$$
Therefore, if $C_5>0$ is sufficiently large, then it follows from (D), (C) and (4.5) that

$$\#Q(r)\le\beta(r|x|^{-(q+1)})^2 \tag{4.7}$$
for some universal constant $\beta>0$, where
$$Q(r)=\{j\in I\setminus F:\phi_j(X)^{-q}\subset B(x,r/2)^{-q}\}.$$

Applying (4.4) we deduce that for every $j \in Q(r)$

$$\mathrm{dist}(\phi_j(X), 0) \geq |x| - r \geq \left(1 - \frac{1}{2q}\right)|x| \geq \frac{1}{2}|x|. \tag{4.8}$$

Since our system is of lattice type and since $T(B(x, r/2)^{-q}) = B(x, r/2)$, we deduce that $Q(r) = \{j \in I \setminus F : \phi_j(X) \subset B(x, r/2)\}$. Using therefore (B), (4.8), (4.7), (4.6) and since $r \geq C_5 \mathrm{diam}(\phi_i(X))$, we get

$$\begin{aligned} m(B(x,r)) &\succeq \sum_{j \in Q(r)} m(\phi_j(X)) \succeq \sum_{j \in Q(r)} K^h ||\phi_j'||^h \succeq \sum_{j \in Q(r)} \mathrm{dist}(\phi_j(X), 0)^{h(q+1)} \\ &\succeq \sum_{j \in Q(r)} |x|^{h(q+1)} \succeq |x|^{h(q+1)} (r|x|^{-(q+1)})^{-2} \succeq r^2 |x|^{(q+1)(h-2)} \\ &\succeq r^2 \mathrm{diam}(\phi_i(X))^{(h-2)} \succeq r^2 r^{(h-2)} = r^h. \end{aligned} \tag{4.9}$$

Suppose now that $x \in \phi_i(X)$ and $1 \geq r \geq 4qC_5\mathrm{diam}(\phi_i(X))$. If $r \leq \frac{1}{2q}|x|$, then (4.9) remains true and we get

$$m(B(x,r)) \succeq r^h. \tag{4.10}$$

If $\frac{1}{2q}|x| \leq r \leq 2|x|$, then $1 \geq \frac{r}{4q} \geq C_5\mathrm{diam}(\phi_i(X))$ and $\frac{r}{4q} \leq \frac{1}{2q}|x|$. So, (4.9) is true with r replaced by $\frac{r}{4q}$ and we get

$$m(B(x,r)) \geq m\left(B\left(x, \frac{r}{4q}\right)\right) \succeq \left(\frac{r}{4q}\right)^h = r^h. \tag{4.11}$$

Finally, if $r \geq 2|x|$, then $B(x,r) \supset B(0, r/2)$ and using (4.3), we get $m(B(x,r)) \succeq r^{h+q(h-2)} \geq r^h$. Combining this, (4.10), (4.11) and applying Lemma 4.10 from [MU1] complete the proof. ■

5. **Preliminaries on Julia-Lavaurs sets**

We start this section by recalling some facts about Leau-Fatou flowers. We consider a holomorphic function of the form $f(z) = z + az^{q+1} + O(z^{q+2})$, $a \neq 0$, i.e. a germ of an holomorphic function tangent to the identity at 0. Performing a preliminary linear conjugation one may as well assume that $a = 1$, so that we can write

$$f(z) = z(1 + z^q) + O(z^{q+2}).$$

The dynamics of f near 0 is described by the Leau-Fatou flower, whose construction we briefly recall. We have q repelling half-lines given by

$$z^q > 0 \Leftrightarrow \mathrm{Arg}(z) = \frac{2k\pi}{q},\ k = 0, ...q-1$$

and q attracting ones given by

$$z^q < 0 \Leftrightarrow \operatorname{Arg}(z) = \frac{\pi}{q} + \frac{2k\pi}{q},\ k = 0, ...q-1.$$

There are q sectors $S_j^{\pm}$ enumerated in the trigonometric order, S_0^- being the sector between $\mathbb{R}_+$ and the next repelling half-line and S_0^+ being the sector between two consecutive attracting half-lines whose bissector is $\mathbb{R}_+$. Each such sector has a bisector which is a repelling or attracting half-line; in the first case the sector is called repelling and labeled with + and in the second it is called attracting and labeled with -. The key to understand the local dynamics is the change of variable

$$Z = \tau(z) = -\frac{1}{qz^q}.$$

It transforms each attracting sector bijectively into $\mathbb{C} \setminus \mathbb{R}_-$ and S_j^+ into $\mathbb{C} \setminus \mathbb{R}_+$. Moreover, in this variable (if z and $f(z)$ belong to the same sector) the mapping f takes on the form

$$F(Z) = Z + 1 + O(Z^{-1/q})$$

which is close to the translation by 1 denoted by T_1 for $|Z|$ appropriately large

Proposition 5.1. *There is a "parabola-shaped" curve included in $\mathbb{C} \setminus \mathbb{R}_-$ whose exterior is forward invariant under F. More precisely one can take as such a domain the exterior of the curve $y = \pm\sqrt{R^2 - x^2},\ x \in [-R/\sqrt{2}, R],\ y = \pm(R/\sqrt{2} + C|x + R/\sqrt{2}|^{1-1/q}$ for R, C large enough.*

For the proof one may consult the Exposé 9, paragraphe 3 in [DH]. This region corresponds in the z-variable to a "petal" $P_j^- \subset S_j^-$. and we denote by $\mathcal{P}_j^-$ the petal in the Z-variable. One can of course build similar petals for f^{-1}; this gives backwards invariant petals in repelling sectors.

We now introduce Fatou coordinates: we have so far a coordinate $Z = \tau(z)$ (we call it approximate Fatou coordinate) conjugating f into F which is "almost" a translation; we now construct a change of variable transforming f exactly in T_1.

To do this we consider in a sector (attracting or repelling) S a subdomain U as follows. S and U have the same bissector and the angle made by U is $3/4$ of the S-angle. The small real ϵ is chosen so that $U \subset P$ (the petal) and the constant $3/4$ is chosen so that $U_j^+ \cap U_j^- \neq \emptyset$.

Theorem 5.2. *There exists $\varphi_j^{\pm} : U_j^{\pm} \longrightarrow \mathbb{C}$ holomorphic and injective such that, whenever z and $f(z)$ belong to $U_j^{\pm}$,*

$$\varphi_j^{\pm}(f(z)) = T_1(\varphi_j^{\pm}(z))$$

and these mappings are unique up to additive constants. Similarly

$$\Phi_j^{\pm} \circ F = T_1 \circ \Phi_j^{\pm},$$

where $\Phi_j^{\pm} = \varphi_j^{\pm} \circ \tau^{-1} :\ \mathcal{U}_j^{\pm} \longrightarrow \mathbb{C}$.

For the proof, see [Zi]. From now on we focus on the mappings

$$h(z) = e^{2\pi i \frac{p}{q}} z + z^2. \tag{5.1}$$

The new feature here is that h is entire, inducing a global dynamical system. In particular each attracting petal of the flower is contained in a component of the Fatou set that we will call a **Fatou petal**.

Moreover the mapping h induces a permutation between the Fatou petals and classical results of Fatou and Julia imply that each orbit of Fatou petals under h must contain a critical point (of h). Since there is only one we must have $\nu = 1$ and $f = h^q$ has q petals. Moreover f is a branched covering of degree two of each petal onto itself.

Proposition 5.3. *Each attracting Fatou coordinate φ_j^- has an extension to the corresponding Fatou petal as a holomorphic mapping satisfying $\varphi_j \circ f = T_1 \circ \varphi_j^-$.*

Proof. Let P_j be the Fatou petal containing U_j^-. If $z \in P_j$, then there exists $n \geq 1$ such that $f^n(z) \in U_j^-$ and we simply define

$$\varphi_j^-(z) = T_{-n} \circ \varphi_j^- o f^n(z)$$

which does not depend on n. ∎

Remark 5.4. *The extended φ_j^- is holomorphic but not bijective: any precritical point for f is a critical point of φ_j^-.*

The situation for repelling Fatou petals is different. We set $\psi_j = \varphi_j^{+-1}$ which is defined in some left half-plane $\{\mathrm{Re} Z \leq -C\}$.

Proposition 5.5. *The function ψ_j extends holomorphically to the entire plane $\mathbb{C}$.*

Proof. For $Z \in \mathbb{C}$ there exists $n \in \mathbb{N}$ such that $\mathrm{Re}(Z - n) \leq -C$ and we may define

$$\psi_j(Z) = f^n \circ \psi_j \circ T_{-n}(Z)$$

which is again independent of n. ∎

The critical points of ψ_j are the images under φ_j^+ of the points in the postcritical set of f. We recall that f induces in each (Fatou) petal a self-covering of degree 2. Moreover in each of these there exists a non-tangential access to the parabolic point. It follows that if we conjugate this mapping with the Riemann map of the petal onto the unit disk sending the critical point to 0 and the parabolic fixed point to 1 (this makes sense since the parabolic point is accessible) we obtain a Blaschke product of the form

$$b(z) = e^{i\theta} \frac{z + a}{1 + \overline{a}z} \frac{z + b}{1 + \overline{b}z}$$

and moreover the local dynamics of f at the parabolic point shows that the point 1 must be a parabolic point for b.

Theorem 5.6. $b(z) = \frac{3z^2+1}{z^2+3}$.

The proof is given in [Zi] As a corollary all the actions of all f's (meaning whatever p/q is) are conjugated, meaning in particular that the dynamics of f in each petal is conjugated to the dynamics of $z \longmapsto z^2 + 1/4$ in its filled-in Julia set.

We now pass to deal with Lavaurs maps we continue the study of the quadratic maps h defined above.

Definition 5.7. *If $\sigma \in \mathbb{C}$, then the corresponding Lavaurs maps are defined in the petal $\mathcal{P}_j$ as*

$$g_\sigma^+ = \Psi_{j+1} \circ T_\sigma \circ \varphi_j^-, \; g_\sigma^- = \Psi_j \circ T_\sigma \circ \varphi_j^-;$$

The "raison d'etre" of this definition is the following theorem, for the proof of which we refer to [Zi].

Theorem 5.8. *If $|\mathrm{Arg}\alpha| \leq \frac{\pi}{4}$ (resp $|\mathrm{Arg}\alpha - \pi| \leq \frac{\pi}{4}$) and $\alpha \longrightarrow 0$ in such a way that there exists $N_\alpha \in \mathbb{N}$ (resp $N_\alpha \in -\mathbb{N}$) with*

$$N_\alpha - \frac{1}{\alpha q} \longrightarrow \sigma \in \mathbb{C}$$

then

$$h_\alpha^{N_\alpha q} \longrightarrow g_\sigma^+ \; (resp \; \longrightarrow g_\sigma^-)$$

uniformly on compact subsets of $\mathcal{P}_j$, where $h_\alpha(z) = e^{2i\pi(p+\alpha)/q}z + z^2$.

In order to understand this situation more geometrically, let us consider quadratic polynomials $P_c(z) = z^2 + c$, where c is a parabolic parameter in the main cardioid of the Mandelbrot set, ie $c = e^{2i\pi\theta}/2 - e^{4i\pi\theta}/4$ with $\theta = p/q$ so that P_c is affinely conjugated to $z \longmapsto e^{2i\pi\theta}z + z^2$. If we take c' near c then $z \longmapsto z^2 + c'$ is affinely conjugated to $z \longmapsto (1 + \sqrt{1-4c'})z + z^2$ where the square root is the one that gives $1 + \sqrt{1-4c} = e^{2i\pi\theta}$. Assume now that

$$c' = c + \epsilon i\pi e^{2i\pi\theta}(1 - e^{2i\pi\theta}).$$

An easy computation shows that

$$c' = e^{2i\pi\theta'} = e^{2i\pi\frac{p+\alpha}{q}}, \; \alpha = \epsilon q + o(\epsilon).$$

Let us denote by $\mathbf{t}$ the unit tangent at the cardioid at c. Then $c' = c + 2\epsilon\pi\sin(2\pi\theta)$. Putting together all the results, we get [Zi] the

Theorem 5.9. *If $P_\epsilon(z) = z^2 + c + \epsilon\mathbf{t}$ and if $\epsilon \longrightarrow 0$ in such a way that there exists $N_\epsilon \in \mathbb{N}$ such that*

$$-\frac{2\pi\sin(2\pi\frac{p}{q})}{q^2\epsilon} + N_\epsilon \longrightarrow \sigma \in \mathbb{C},$$

then $P_\epsilon^{qN_\epsilon}$ converges on every compact subset of every petal of P_c to the Lavaurs map g_σ.

The introduction of the Lavaurs maps allows us to define a new dynamics, the dynamics of (h, g_σ): we say that a point z escapes by (h, g_σ) if there exists $k \geq 0$ such that $g_\sigma^l(z)$ is well defined for $0 \leq l \leq k$ but $g_\sigma^k(z) \notin K(h)$. The filled-in Julia-Lavaurs set $K(h, g_\sigma)$ is then defined as the set of points which do not escape by (h, g_σ). It is a non-empty compact set whose boundary is by definition the Julia-Lavaurs set $J(h, g_\sigma)$. Douady has shown in [Do] that $K(h, g_\sigma) = J(h, g_\sigma)$ if the critical point ω escapes by (h, g_σ). We focus on the set Σ of phases σ such that ω escapes at once, i.e. such that $g_\sigma(\omega) \notin K(h, g_\sigma)$. We call such a phase and the corresponding Lavaurs map hyperbolic. This set is the union of two strips (depending whether we use $g_\sigma^\pm$) each of them containing the real axis (this corresponds to $\epsilon \in I\!R$ in the last theorem). Douady has shown in [Do] that if $\sigma \in \Sigma$ then with the hypothesis of the last theorem, $J(P_\epsilon)$ converges in the Hausdorff topology towards $J(h, g_\sigma)$. If $\sigma \in \Sigma$ then $J(h, g_\sigma)$ consists of the union of $J(h)$ and "butterflies" attached at each preparabolic point; they all reproduce the butterflies attached at the parabolic point, $2q$ of them, 2 per Fatou petal.

As in [DSZ], one can define a Markov partition describing the dynamics of (h, g_σ). First of all since the dynamics of $f = h^q$ in each petal is the same as the dynamics of $z^2 + 1/4$ in the "cauliflower" (its filled-in Julia set), we have in each petal a decomposition in pieces $A_{0,n}$ analogous to [DSZ]; moreover the different "layers" of the butterflies attached to the parabolic point are sent by some iterates of g_σ to the boundary of some petal; this allows us to transfer the decomposition in pieces $(A_{0,n})$ inside these butterflies. The proof that this decomposition gives a Markov partition for the dynamics of (h, g_σ) follows the same lines as [DSZ]. The family of its (holomorphic) branches forms a conformal iterated function system refered to as the DSZ iterated function system.

The following theorem has been proved in [DSZ] in the case when $q = 1$ and in the full quadratic case (5.1) in [Zi].

Theorem 5.10. *The DSZ iterated function system associated with each hyperbolic Lavaurs map is a lattice type conformal iterated function systems.*

Let us now restrict ourselves to the Lavaurs maps generated by the parabolic map $f(z) = z^2 + \frac{1}{4}$. Let

$$W = \{\sigma \in I\!\!C : g_\sigma(0) \notin K_{1/4}\}.$$

Let

$$\rho : I^\infty \to I^\infty$$

be the shift map, i.e.

$$\rho(\{\omega_n\}_{n=1}^\infty) = \{\omega_n\}_{n=2}^\infty.$$

For all $\sigma, \sigma_0 \in W$ and all $\omega \in I^\infty$ let

$$\psi_\omega(\sigma) = \frac{(\phi_{\omega_1}^\sigma)'(\pi_\sigma(\rho(\omega)))}{(\phi_{\omega_1}^{\sigma_0})'(\pi_{\sigma_0}(\rho(\omega)))}.$$

The following result follows from Lemma 7.1 in [UZ].

Proposition 5.11. *For every $\sigma_0 \in W$ there exists a radius $r > 0$ such that $B(\sigma_0, 2r) \subset W$ and for every $\omega \in I^\infty$*

$$M_1 = \sup_{\omega \in I^\infty} \sup_{\sigma \in B(\sigma_0, 2r)} \{|\psi_\omega'(\sigma)|\} < \infty,$$

In particular each function $\psi_\omega : B(\sigma_0, 2r) \to \mathbb{C}$ *is a Bloch function and its Bloch's norm* $||\psi_\omega||_{\mathcal{B}} \le M_1$.

6. Measures and dimension for Julia-Lavaurs sets

As an immediate consequence of Theorem 5.10 and Therem 4.1 we get the following first main theorem of this section.

Theorem 6.1. *If* g_σ *is a hyperbolic Lavaurs map corresponding to the polynomial of the form* $h(z) = e^{2\pi i \frac{p}{q}} z + z^2$, *then*

$$\overline{\mathrm{BD}}(J_\sigma) = \mathrm{HD}(J_\sigma),\ \mathrm{H}^h(J_\sigma) = 0,\ 0 < \mathrm{P}^h(J) < \infty.$$

Let us now restrict ourselves to the Lavaurs maps generated by the parabolic map $f(z) = z^2 + \frac{1}{4}$. We will need the following

Lemma 6.2. *For every* $\omega \in I^\infty$ *the function* $\sigma \mapsto \pi_\sigma(\omega)$, $\sigma \in W$, *is holomorphic.*

Proof. Fix $x \in J(f)$. Since each function $(\sigma, z) \mapsto \phi_i^\sigma(z)$, $i \in I$, is holomorphic in both variables σ and z, the function

$$\sigma \mapsto \zeta_n^\omega(\sigma) := \phi_{\omega_1}^\sigma \circ \phi_{\omega_2}^\sigma \circ \ldots \circ \phi_{\omega_n}^\sigma(x)$$

is holomorphic for every integer $n \ge 1$. Since the functions ζ_n^ω are uniformly bounded, one can choose from them a subsequence uniformly convergent on compact subsets of W. Since for every σ the sequence $\zeta_n^\omega(\sigma)$ converges to $\pi_\sigma(\omega)$, we conclude that the function $\sigma \mapsto \pi_\sigma(\omega)$ is holomorphic. ■

The second main theorem of this section is the following.

Theorem 6.3. *The function* $\sigma \mapsto \mathrm{HD}(J_\sigma)$, $\sigma \in W$, *is real-analytic.*

Proof. Consider the function

$$\sigma \mapsto \zeta(\sigma) : I^\infty \to \mathbb{C},\ \sigma \in W,$$

given by the formula

$$\zeta(\sigma)(\omega) = \log |(\phi_{\omega_1}^\sigma)'(\pi_\sigma(\rho(\omega)))|$$

Fix now $\omega \in I^\infty$, $\sigma_0 \in W$ and similarly as in the previous section, consider the function

$$\psi_\omega(\sigma) = \frac{(\phi_{\omega_1}^\sigma)'(\pi_\sigma(\rho(\omega)))}{(\phi_{\omega_1}^{\sigma_0})'(\pi_{\sigma_0}(\rho(\omega)))}.$$

In view of Proposition 5.11 there exists a radius $r > 0$ such that $B(\sigma_0, 2r) \subset W$,

$$M_1 = \sup_{\omega \in I^\infty} \sup_{\sigma \in B(\sigma_0, 2r)} \{|\psi_\omega'(\sigma)|\} < \infty,$$

the function $\psi_\omega : B(\sigma_0, 2r) \to \mathbb{C}$ is a Bloch function and its Bloch's norm $||\psi_\omega||_{\mathcal{B}} \le M_1$. Combining formula (4) and Proposition 4.1 on p.73 in [Po] we therefore conclude that there exists a universal constant $M_2 > 0$ such that

$$|\log \psi_\omega(\sigma))| \le M_2$$

for all $\sigma \in B(\sigma_0, r)$, where the branch $\log \psi_\omega(\sigma))$ is determined by the condition $\log \psi_\omega(\sigma_0)) = 0$. In view of Lemma 6.2 the function $\log \psi_\omega(\sigma))$ is holomorphic and let

$$\log \psi_\omega(\sigma)) = \sum_{n=0}^{\infty} a_n(\omega)(\sigma - \sigma_0)^n$$

be its Taylor series expansion on $B(\sigma_0, 2r)$. By Cauchy's inequalities

$$|a_n(\omega)| \le \frac{M_2}{r^n} \tag{6.1}$$

for all $n \ge 0$. For every $z = x + iy \in B(\sigma_0, r) \subset W \subset \mathbb{C}$, we have

$$\begin{aligned}
\mathrm{Re}\log \psi_\omega &= \mathrm{Re}\left(\sum_{n=0}^{\infty} a_n(\omega)((x - \mathrm{Re}\sigma_0) + (y - \mathrm{Im}\sigma_0)i)^n\right) \\
&= \sum_{p,q=0}^{\infty} \mathrm{Re}\left(a_{p+q}\binom{p+q}{q} i^q\right)(x - \mathrm{Re}\sigma_0)^p (y - \mathrm{Im}\sigma_0)i)^q \\
&= \sum c_{p,q}(x - \mathrm{Re}\sigma_0)^p (y - \mathrm{Im}\sigma_0)i)^q,
\end{aligned}$$

where, due to (6.1), $|c_{p,q} \le |a_{p+q}2^{p+q} \le M_2 r^{-(p+q)} 2^{p+q}$. Hence $\mathrm{Re}\log\psi_\omega$ extends by the same power series expansion $\sum c_{p,q}(x - \mathrm{Re}\sigma_0)^p(y - \mathrm{Im}\sigma_0)i)^q$ to a complex-valued analytic function, denoted by the same symbol $\mathrm{Re}\log\psi_\omega$, on the polydisk $\mathbb{D}_{\mathbb{C}^2}(0, r/4)$, and in addition

$$|\mathrm{Re}\log\psi_\omega| \le 4M_2 \quad \text{on} \quad \mathbb{D}_{\mathbb{C}^2}(0, r/4).$$

So,

$$\tilde{\zeta}(\sigma)(\omega) = \mathrm{Re}\log\psi_\omega(\sigma) + \log|(\phi_{\omega_1}^{\sigma_0})'(\pi_{\sigma_0}(\rho(\omega)))|$$

extends $\zeta(\sigma)(\omega)$ on the polydisk $\mathbb{D}_{\mathbb{C}^2}(0, r/4)$ and for every $t \in \mathbb{C}$ we have

$$\begin{aligned}
|\exp\Big(t\tilde{\zeta}(\sigma)(\omega)\Big)| &= \exp\Big(\mathrm{Re}(t\mathrm{Re}\log\psi_\omega(\sigma) + t\log|(\phi_{\omega_1}^{\sigma_0})'(\pi_{\sigma_0}(\rho(\omega)))|\Big) \\
&= \exp\Big(\mathrm{Re}(t\mathrm{Re}\log\psi_\omega(\sigma))\Big) \cdot |(\phi_{\omega_1}^{\sigma_0})'(\pi_{\sigma_0}(\rho(\omega)))|^{\mathrm{Re}t} \\
&\le \exp\Big(|t|\mathrm{Re}\log\psi_\omega(\sigma)\Big)|(\phi_{\omega_1}^{\sigma_0})'(\pi_{\sigma_0}(\rho(\omega)))|^{\mathrm{Re}t} \\
&\le \exp(4M_2|t|)|(\phi_{\omega_1}^{\sigma_0})'(\pi_{\sigma_0}(\rho(\omega)))|^{\mathrm{Re}t}.
\end{aligned}$$

Therefore

$$\sum_{i \in I} ||\exp\Big(t\tilde{\zeta}(\sigma)\Big)|| < \infty \tag{6.2}$$

for every $\sigma \in D_{\mathbb{C}^2}(0, r/4)$. Since all the maps ϕ_i^σ, $\sigma \in B_{\mathbb{C}}(\sigma_0, r)$ are uniform contractions with some uniform contraction factor $0 < s < 1$, we get

$$|\pi_\sigma(\omega) - \pi_\sigma(\tau)| \le \operatorname{diam}(X) s^{|\omega\wedge\tau|},$$

and using then Koebe's distortion theorem for the modulus and the argument we conclude that if $|\omega \wedge \tau| \ge 1$, then

$$\begin{aligned}
|\log\psi_\omega(\sigma)) - |\log\psi_\tau(\sigma))| &= |\log\phi_{\omega_1}^\sigma)'(\pi_\sigma(\rho(\omega))) - \log\phi_{\tau_1}^\sigma)'(\pi_\sigma(\rho(\tau)))| \\
&\le K_1|\pi_\sigma(\rho(\omega)) - |\pi_\sigma(\rho(\tau))| \le K_1 s^{|\omega\wedge\tau|-1} \\
&= \frac{K_1}{s} s^{|\omega\wedge\tau|}
\end{aligned}$$

for all $\sigma \in B_{\mathbb{C}}(\sigma_0, r)$ and some universal constant K_1. Hence, using Cauchy's inequalities again, we conclude that $|a_n(\omega) - a_n(\tau)| \le r^{-n}\frac{K_1}{s}s^{|\omega\wedge\tau|}$ for every $n \ge 0$. Thus

$$|c_{p,q}(\omega) - c_{p,q}(\tau)| \le 2^{p+q} r^{-(p+q)} \frac{K_1}{s} s^{|\omega\wedge\tau|}$$

and therefore

$$|\mathrm{Re}\log\psi_\omega(\sigma) - \mathrm{Re}\log\psi_\tau(\sigma)| \le \frac{4K_1}{s} s^{|\omega\wedge\tau|}$$

for all $\sigma \in D_{\mathbb{C}^2}(\sigma_0, r/4)$. Consequently

$$|\tilde\zeta(\sigma)(\omega) - \tilde\zeta(\sigma)(\tau)| = |\mathrm{Re}\log\psi_\omega(\sigma) - \mathrm{Re}\log\psi_\tau(\sigma)| \le \frac{4K_1}{s} s^{|\omega\wedge\tau|}$$

for all $\sigma \in D_{\mathbb{C}^2}(\sigma_0, r/4)$. Combining this and (6.2) we conclude that $t\tilde\zeta(\sigma) \in \mathcal{H}_s^{-\log s}$ for all $(\sigma, t) \in D_{\mathbb{C}^2}(\sigma_0, r/4) \times \{t \in \mathbb{C} : \mathrm{Re}t > \frac{2q}{q+1}\}$. Thus the operator $\mathcal{L}_{t\tilde\zeta(\sigma)}$ is well-defined and acts on $\mathcal{H}^s_{-\log s}$ for these (σ, t). Since for every $\omega \in I^\infty$, the function $(\sigma, t) \mapsto t\tilde\zeta(\sigma)$, $(\sigma, t) \in D_{\mathbb{C}^2}(\sigma_0, r/4) \times \{t \in \mathbb{C} : \mathrm{Re}t > \frac{2q}{q+1}\}$ is continuous and since the function $(\sigma, t) \mapsto t\tilde\zeta(\sigma)(\omega)$, $(\sigma, t) \in D_{\mathbb{C}^2}(\sigma_0, r/4) \times \{t \in \mathbb{C} : \mathrm{Re}t > \frac{2q}{q+1}\}$ is analytic for every $\omega \in I^\infty$, in view of Theorem 2.8, the function $(\sigma, t) \mapsto \mathcal{L}_{\tilde\zeta(\sigma,t)}$, $(\sigma, t) \in D_{\mathbb{C}^2}(\sigma_0, r/4) \times \{t \in \mathbb{C} : \mathrm{Re}t > \frac{2q}{q+1}\}$ is also analytic. Since, in view of Theorem 2.3.3 and Theorem 2.4.6 from [MU2] for every $(\sigma, t) \in W \times (\frac{2q}{q+1}, \infty)$ has a simple isolated eigenvalue $\lambda(\sigma, t)$, applying the perturbation theory for linear operators (see [Ka]) , we conclude that there exists an open set $B(\sigma_0, r/4) \times (\frac{2q}{q+1}, \infty) \subset \tilde U_{\sigma_0} \subset D_{\mathbb{C}^2}(\sigma_0, r/4) \times \mathbb{C}$ and an analytic function $\lambda : \tilde U \to \mathbb{C}$ giving simple isolated eigenvalues for operators $\mathcal{L}_{\tilde\zeta(\sigma,t)}$. Since by Theorem 2.3.3 from [MU2], for every $(\sigma, t) \in W \times (\frac{2q}{q+1}, \infty)$, $\mathrm{P}(\sigma, t) := \mathrm{P}(\zeta(\sigma, t)) = \mathrm{e}^{\lambda(\sigma,t)}$, we deduce that the function $(\sigma, t) \mapsto \mathrm{P}(\sigma, t)$ defined on the set $B(\sigma_0, 4/4) \times (\frac{2q}{q+1}, \infty)$, is real-analytic. By Theorem 3.2 and regularity of our system S_σ (see Theorem 5.10 and Theorem 4.1) for every $\sigma \in W$ there exists exactly one $h_\sigma \in (\frac{2q}{q+1}, \infty)$ such that $\mathrm{P}(\sigma, h_\sigma) = 0$ and $h_\sigma = \mathrm{HD}(J_\sigma)$. Since by Proposition 2.6.13 from [MU2] $\frac{\partial \mathrm{P}}{\partial t}(\sigma, t) = \int \zeta(\sigma)\, d\tilde\mu_{\sigma,t} < 0$, where $\tilde\mu_{\sigma,t}$ is the Gibbs state (see [MU2]) for the potential $t\zeta(\sigma)$, it therefore follows from the implicit function theorem that $\sigma \mapsto \mathrm{h}_\sigma$, $\sigma \in B(\sigma_0, 4/4)$

is real-analytic. Since σ_0 was an arbitrary point of W, we finally conclude that the function $\sigma \mapsto \mathrm{h}_\sigma$, $\sigma \in$, is real-analytic. ■

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