Holomorphic maps for which the unstable manifolds depend on prehistories

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Abstract

For points x belonging to a basic set Λ of an Axiom A holomorphic endomorphism of $\mathbb{P}^2$, one can construct the local stable manifold $W^s_{\varepsilon_0}(x)$ and local unstable manifolds $W^u_{\varepsilon_0}(\hat{x})$.

A priori, $W^u_{\varepsilon_0}(\hat{x})$ should depend on the entire prehistory $\hat{x}$ of x. However, all known examples have all their local unstable manifolds depending only on the base point x.

Therefore a natural problem is to give actual examples where, for different prehistories of points in the basic sets of holomorphic Axiom A maps, we obtain different unstable manifolds. We solve this problem and also show that, by perturbing $(z^2 + c, w^2)$, one can get also maps f_ε which are injective on Λ_ε, their corresponding basic sets, hence the cardinality of the set $(f_\varepsilon|_{\Lambda_\varepsilon})^{-1}(x), x \in \Lambda_\varepsilon$, is not stable under perturbation.

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1 Introduction

Throughout the paper we consider an Axiom A holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ on the 2-dimensional complex projective space.

Denote by $\Omega := \{x \in \mathbb{P}^2, \forall U$ neighbourhood of $x, \exists\, n \geq 1 s.t f^n(U) \cap U \neq \emptyset\}$, the non-wandering set of f.

For the general theory of Axiom A maps, we refer to [2], [7], [8], [9]. If f is an Axiom A map, then Ω is the closure of the periodic points of f, and f is hyperbolic on Ω.

First, let us define the space of prehistories of Ω,

$$\hat{\Omega} := \{(x_n)_{n\leq 0}, f(x_{n-1}) = x_n, x_n \in \Omega, \forall n \leq 0\}.$$

The tangent bundle over $\hat{\Omega}$ is defined by its fibers:

$$T_{\hat{\Omega}}((x_n)_n) = \{(x_n)_n, v, \text{ with } v \in T_x\mathbb{P}^2\}.$$

If f is hyperbolic, then there exists a continuous splitting of the tangent bundle, $T_{\hat{\Omega}}((x_n)_n) = E^s_{x_0} \oplus E^u_{(x_n)_n}$ and constants $C > 0$, $\lambda > 1$, such that:

$$Df_x(E^u_{\hat{x}}) \subseteq E^u_{\hat{f}_x}, Df_x(E^s_x) \subseteq E^s_{f_x},$$

$$||Df^n(v)|| \leq C\lambda^{-n} \cdot ||v||,\ \forall v \in E^s_{x_0},\ \forall n \geq 0$$

and

$$||Df^n(w)|| \geq C\lambda^{n} \cdot ||w||,\ \forall w \in E^u_{\hat{x}},\ \forall n \geq 0$$

Like in the case of diffeomorphisms, the above splitting into stable and unstable directions, implies the existence of local stable and unstable manifolds for every $0 < \varepsilon << 1$:

$$W^s_{\varepsilon_0}(x) := \{y \in \mathbb{P}^2, d(f^n x, f^n y) \leq \varepsilon_0, \forall n \geq 0\},$$

$$W^u_{\varepsilon_0}(\hat{x}) := \{y \in \mathbb{P}^2, y \text{ has a prehistory } \hat{y} = (y_n)_{n\leq 0}, d(x_{-n}, y_{-n}) \leq \varepsilon_0, \forall n \geq 0\}$$

$W^s_{\varepsilon_0}(x)$ and $W^u_{\varepsilon_0}(\hat{x})$ are complex disks.

In case $y \notin W^s_{\varepsilon_0}(x), W^s_{\varepsilon_0} \cap W^s_{\varepsilon_0}(y) = \emptyset$, and the family $(W^s_{\varepsilon_0}(x))_x$ forms a foliation.

$W^u_{\varepsilon_0}(\hat{x})$ depends a priori on the entire prehistory $\hat{x}$, but as will be explained later, in all known examples so far, $W^u_{\varepsilon_0}(\hat{x})$ depends only on the base point x and not on the entire prehistory. Obviously, before one should try to find examples when this is not true, one should obtain examples of families of maps which are not injective on their basic sets.

On the other hand we will show that in fact, by perturbing the map $(z^2 + c, w^2)$, one gets in general both maps which are injective and maps which are not injective on their respective basic sets.

In the real case, we would like to mention Przytycki's result ([7]), showing that, if f is any special Anosov endomorphism of a compact manifold M (special meaning that unstable spaces do not depend on prehistories), f not expanding or a diffeomorphism, $\varepsilon > 0$ and $x \in M$ are given, then there exists $g : M \to M$, and G'_x, a set of prehistories of x under g, so that $d(f,g) < \varepsilon$, G'_x is homeomorphic to a Cantor set, and the unstable manifolds $W^u_{\varepsilon_0}(\hat{x}), \hat{x} \in G'_x$ are all mutually distinct. Also in this situation, $G'_x \ni \hat{x} \to E^u_{\hat{x}}$ is a homeomorphism, so the unstable vector spaces depend on the prehistories from G'_x as well.

However, his proof depending on local C^∞ real perturbations does not seem to be extendable to the complex case. We will obtain our examples as perturbations of maps of the form (z^2+c, w^2).

Firstly, in order to understand the problem at hand let us look at some classical examples of holomorphic mappings on $\mathbb{P}^2$.

1) $f[z : w : t] = [P(z : t) : Q(w : t) : t^d]$, with P, Q one-dimensional complex polynomials of degree d, hyperbolic on their Julia sets J_P, respectively J_Q. Then, in the set $t = 1, S_1 = \{$periodic sinks of $P\} \times J_Q \cup J_P \times \{$periodic sinks of $Q\}$. In $t = 0, S_1$ has a basic set given by the Julia set of the map $[P(z : 0) : Q(w : 0) : 0]$. In the example of this form, if $x \in S_1$ is fixed, then all the unstable manifolds $W^u_\varepsilon(\hat{x})$, for $\hat{x} \in \widehat{S}_1$ coincide. So $W^u_\varepsilon(\hat{x})$ does not really depend on $\hat{x}$, it depends only on x.

2) Let ϕ the Serre map, $\phi\colon\ \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, $\phi([z_0 : w_0], [z_1 : w_1]) = [z_0z_1 : w_0w_1 : z_0w_1 + w_0z_1]$, a 2-to-1 covering of $\mathbb{P}^2$. If $f_0\colon\ \mathbb{P}^1 \to \mathbb{P}^1$ is holomorphic and hyperbolic on its Julia set, then there exists f holomorphic on $\mathbb{P}^2$ s.t. $\phi(f_0, f_0) = f \circ \phi$, and f is Axiom A ([2]). The basic sets for S_1 are of the form ϕ ($\{$periodic sink for $f_0\} \times J_{f_0}$) and again it is easy to check the independence on prehistory of the unstable manifolds.

3) There are other examples of Axiom A maps f (even s-hyperbolic) which have basic sets Λ near which the map is injective, like the solenoidal examples ([2], [5]). For these examples the map f behaves like a diffeomorphism, hence the local unstable manifolds depend only on x and they form a foliation.

So, the question arises whether there are examples of Axiom A holomorphic maps on $\mathbb{P}^2$ for which the unstable manifolds $W^u_\varepsilon(\hat{x})$ truly depend on $\hat{x}$, and not only on x. Also is it possible that for every point $x \in \Lambda$, we have different local unstable manifolds $W^u_\varepsilon(\hat{x})$ and $W^u_\varepsilon(\hat{x}')$ whenever the prehistories of $x, \hat{x}$ and $\hat{x}'$ are different?

Obviously, before we attack this problem, there must be found first a non-trivial example of a map g with a basic set Λ_g s.t. $g|_{\Lambda_g}$ is non-injective, since this would guarantee the existence of at least two different prehistories for a given point $x \in \Lambda_g$.

Let $f(z,w) = (z^2, w^2)$. Then f has a basic set $\Lambda = \{0\} \times J_{(w^2)}$. Since f is Axiom A and has no cycles, the Stability Theorem ([7], [8]) will imply that for $\varepsilon > 0$ small enough, $f_\varepsilon(z,w) := (z^2 + \varepsilon w^2, w^2)$ has a basic set Λ_ε close to Λ in the Hausdorff metric and f_ε is hyperbolic on Λ_ε. If $(z,w) \in \Lambda_\varepsilon \Rightarrow |w| = 1$.

Theorem 1. *For every point $(z,w) \in \Lambda_\varepsilon$, and $\widehat{(z,w)}$, $\widehat{(z,w)}'$, any two different prehistories of (z,w), we have $E^u(\widehat{(z,w)}) \neq E^u(\widehat{(z,w)}')$. In particular $W^u_{\varepsilon_0}(\widehat{(z,w)}) \neq W^u_{\varepsilon_0}(\widehat{(z,w)}')$.*

Proof. In fact we prove more. Namely, we estimate from below the angle between $E^u(\widehat{(z,w)})$ and $E^u(\widehat{(z,w)}')$. Let us consider $(z_0, w_0) \in \Lambda_\varepsilon$ and show that $(z_0, -w_0) \in \Lambda_\varepsilon$ also. Take $U =$

$D(z_0,\eta)\times D(w_0,\eta)$ a small neighborhood of (z_0,w_0); then, from the fact that (z_0,w_0) is a non-wandering point, we get that there exists an increasing sequence of integers n_k s.t. $f_\varepsilon^{n_k}(U)\cap U\neq\emptyset$. If $p_2(z,w)=w\Rightarrow$ for n_k large enough, $p_2\circ f_\varepsilon^{n_k}(U)=S^1$. And $(p_1\circ f_\varepsilon(U))\cap D(z_0,\eta)\neq\emptyset$. So if $V:=D(z_0,\eta)\times D(-w_0,\eta)$, then $f_\varepsilon^{n_k}(V)\cap V\neq\emptyset$, and as η was taken arbitrarily small, we will get that $(z_0,-w_0)\in\Lambda_\varepsilon$. And obviously $f_\varepsilon(z_0,w_0)=f_\varepsilon(z_0,-w_0)$, so $f_\varepsilon|_{\Lambda_\varepsilon}$ is at least 2-to-1. We will now look closer at the definition of hyperbolicity for $f_\varepsilon|_{\Lambda_\varepsilon}$. According to the definition (for example from [2]), there exists a continuous splitting of the tangent bundle

$$T_{\hat{x}}=E^u_{\hat{x}}\oplus E^s_x,$$

and $Df_x(E^u_{\hat{x}})\subseteq E^u_{\hat{f}_x}$, $Df_x(E^s_x)\subseteq E^s_{f_x}$, and $\exists\lambda>1$ with $\|Df^n(v)\|\geq\lambda^n\cdot\|v\|$, $v\in E^u_{\hat{x}}$, $\|Df^n(w)\|\leq\lambda^{-n}\cdot\|w\|$, $w\in E^s_x$. So, a priori the unstable spaces $E^u_{\hat{x}}$ should depend on $\hat{x}$, and in our situation we show they really do.

We have $Df_\varepsilon(z,w)=(\,2z\ 2\varepsilon w\ 0\ 2w\,)$ and assume $E^u_{(\widehat{z,w})}=\{(v_1,v_2)\in\mathbb{C}^2,\ v1=\beta(\widehat{z,w})\cdot v_2\}$, for $(\widehat{z,w})\in\hat{\Lambda}_\varepsilon$; $E^s_{(z,w)}$ is close to the v_1-axis, and $E^u_{(\widehat{z,w})}$ is close to the v_2-axis, so $\beta(\widehat{z,w})<\beta_0<\infty$, β_0 independent of $(\widehat{z,w})\in\hat{\Lambda}_\varepsilon$. Now

$$Df_\varepsilon(z,w)\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}2z & 2\varepsilon w\\ 0 & 2w\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}2zv_1+2\varepsilon wv_2\\ 2wv_2\end{pmatrix}.$$

Denote by $(z_{-i},w_{-i})=(\widehat{z,w})_{-i}$, $(z_{-i},w_{-i})\in\Lambda_\varepsilon$. We know

$$Df_{(z_{-1},w_{-1})}(E^u_{(\widehat{z_{-1},w_{-1}})})=E^u_{(\widehat{z,w})}2z_{-1}v_1+2\varepsilon w_{-1}v_2=\beta(\widehat{z,w})\cdot 2w_{-1}v_2.$$

But

$$\begin{aligned}v_1=\beta(\widehat{z_{-1},w_{-1}})\cdot v_2&\Rightarrow 2z_{-1}\beta(\widehat{z_{-1},w_{-1}})v_2+2\varepsilon w_{-1}v_2\\ &=\beta(\widehat{z,w})\cdot 2w_{-1}v_2, v_2\neq 0\Rightarrow\beta(\widehat{z,w})=\varepsilon+\frac{z_{-1}}{w_{-1}}\cdot\beta(\widehat{z_{-1},w_{-1}}).\end{aligned}$$

Applying recursively the above identity, gives

$$\begin{aligned}\beta(\widehat{z,w})&=\varepsilon+\frac{z_{-1}}{w_{-1}}\beta(\widehat{z_{-1},w_{-1}})=\varepsilon+\varepsilon\frac{z_{-1}}{w_{-1}}+\frac{z_{-1}}{w_{-1}}\cdot\frac{z_{-2}}{w_{-2}}\beta(\widehat{z_{-2},w_{-2}})=\cdots=\\ &=\varepsilon+\varepsilon\frac{z_{-1}}{w_{-1}}+\cdots+\varepsilon\frac{z_{-1}\cdots z_{-n}}{w_1\cdots w_{-n}}+\frac{z_{-1}\cdots z_{-n-1}}{w_{-1}\cdots w_{-n-1}}\beta(\widehat{z_{-n-1},w_{-n-1}}).\end{aligned}$$

Fix now a point $(z,w)\in\Lambda_\varepsilon$. From our discussion at the beginning of the Example, we know that $(z_{-1},-w_{-1})\in\Lambda_\varepsilon$ too and denote by $(\widehat{z,w})'$ any prehistory of (z,w) s.t. $(\widehat{z,w})'_{-1}=(z_{-1},-w_{-1})$. We will show that $\beta(\widehat{z,w})\neq\beta(\widehat{z,w})'$. So

$$\beta(\widehat{z,w})'=\varepsilon-\varepsilon\frac{z_{-1}}{w_{-1}}+\cdots+\varepsilon\frac{z'_{-1}\cdots z'_{-n}}{w'_{-1}\cdots w'_{-n}}+\frac{z'_{-1}\cdots z_{-n-1}}{w'_{-1}\cdots w'_{-n-1}}\cdot\beta(\widehat{z'_{-n-1},w'_{n-1}}).$$

Hence

$$\begin{aligned}\beta(\widehat{z,w})-\beta(\widehat{z,w})'&=2\varepsilon\frac{z_{-1}}{w_{-1}}+\varepsilon\left(\frac{z_{-1}z_{-2}}{w_{-1}w_{-2}}+\cdots+\frac{z_{-1}\cdots z_{-n}}{w_{-1}\cdots w_{-n}}-\frac{z'_{-1}z'_{-2}}{w'_{-1}w'_{-2}}-\cdots-\frac{z'_{-1}\cdots z'_{-n}}{w'_{-1}\cdots w'_{-n}}\right)\\ &+\left(\frac{z_{-1}\cdots z_{-n-1}}{w_{-1}\cdots w_{-n-1}}\beta(\widehat{z_{-n-1},w_{-n-1}})-\beta(\widehat{z_{-n-1},w_{-n-1}})'\cdot\frac{z'_{-1}\cdots z'_{-n-1}}{w'_{-1}\cdots z'_{-n-1}}\right)\quad(1)\end{aligned}$$

It will be useful to give estimates of $\sup_{(z,w)\in\Lambda_\varepsilon} |z|$ $\inf_{(z,w)\in\Lambda_\varepsilon} |z|$. Assume

$$\begin{aligned}|z_0| = \sup_{(z,w)\in\Lambda_\varepsilon} |z| &\Rightarrow \exists (z,w)\in\Lambda_\varepsilon \text{ s.t. } z^2+\varepsilon w^2 = z_0 \Rightarrow |z_0| \\ &= |z^2+\varepsilon w^2| \le |z|^2+\varepsilon \le |z_0|^2+\varepsilon \Rightarrow |z_0|^2-|z_0|+\varepsilon \ge 0.\end{aligned}$$

Solving this quadratic inequality, we get $|z_0| \le \frac{1-\sqrt{1-4\varepsilon}}{2}$ or $|z_0| \ge \frac{1+\sqrt{1-4\varepsilon}}{2} > 2/3$ for ε small enough. But Λ_ε is close to $\{0\}\times S^1 \Rightarrow$ we can only have $|z_0| \le \frac{1-\sqrt{1-4\varepsilon}}{2} = \frac{2\varepsilon}{(1+\sqrt{1-4\varepsilon})} = \frac{2\varepsilon}{1+\sqrt{1-4\varepsilon}}$. Also

$$\begin{aligned}&|z_0| = |z^2+\varepsilon w^2| \ge \varepsilon - |z|^2 \ge \varepsilon - |z_0|^2 \Rightarrow |z_0|^2+|z_0|-\varepsilon \ge 0 \Rightarrow \\ &|z_0| \ge \frac{-1+\sqrt{1-4\varepsilon}}{2} = \frac{4\varepsilon}{2(1+\sqrt{1+4\varepsilon})} = \frac{2\varepsilon}{1+\sqrt{1+4\varepsilon}}.\end{aligned}$$

Hence

$$0.9\varepsilon < \frac{2\varepsilon}{1+\sqrt{1+4\varepsilon}} \le \sup_{(z,w)\in\Lambda_\varepsilon} |z| \le \frac{2\varepsilon}{1+\sqrt{1-4\varepsilon}} < 1.5\varepsilon \tag{2}$$

for small ε. Let us estimate also $\inf_{(z,w)\in\Lambda_\varepsilon} |z|$; assume $\inf_{(zw)\in\Lambda_\varepsilon} |z|$ is attained at z_0. There is $(z,w)\in\Lambda_\varepsilon$ s.t. $f_\varepsilon(z,w) = (z_0,w_0) \Rightarrow z^2+\varepsilon w^2 = z_0 \Rightarrow |z_0| \ge \varepsilon - |z|^2 \ge \varepsilon - \frac{4\varepsilon^2}{(1+\sqrt{1-4\varepsilon})^2} \ge \varepsilon - (1.5)^2\cdot\varepsilon^2$, by using the estimate for $\sup_{(z,w)\in\Lambda_\varepsilon} |z|$.

But $\varepsilon-(1.5)^2\varepsilon^2 < 1.5\varepsilon < 2(\varepsilon-(1.5)^2\varepsilon^2)$ if $\varepsilon>0$ is small enough. Hence $C_\varepsilon := \inf_{(z,w)\in\Lambda_\varepsilon} |z| \le D_\varepsilon := \sup_{(z,w)\in\Lambda_\varepsilon} |z| < 2\cdot \inf_{(z,w)\in\Lambda_\varepsilon} |z|$. Using (2), we obtain

$$0.4\varepsilon < C_\varepsilon < 1.5\varepsilon. \tag{3}$$

Recalling the identity (1):

$$\begin{aligned}\beta(\widehat{z,w}) - \beta(\widehat{z,w})' = 2\varepsilon\frac{z_{-1}}{w_{-1}} + \varepsilon\left(\frac{z_{-1}z_{-2}}{w_{-1}w_{-2}} + \cdots + \frac{z_{-1}\cdots z_{-n}}{w_{-1}\cdots w_{-n}} - \frac{z'_{-1}z'_{-2}}{w'_{-1}w'_{-1}} - \cdots - \frac{z'_{-1}\cdots z'_{-n}}{w'_{-1}\cdots w'_{-n}}\right) + \\ + \left(\frac{z_{-1}\cdots z_{-n-1}}{w_{-1}\cdots w_{-n-1}}\beta(\widehat{z_{-n-1},w_{-n-1}}) - \frac{z'_{-1}\cdots z'_{-n-1}}{w'_{-1}\cdots w'_{-n-1}}\cdot\beta(\widehat{z_{-n-1},w_{-n-1}})'\right),\end{aligned}$$

and using the inequality $D_\varepsilon < 2\cdot C_\varepsilon$, we get that the absolute value of the second term (the one in the first paranthesis) is $\le 2\varepsilon D_\varepsilon^2\cdot\frac{1-D_\varepsilon^{n-1}}{1-D_\varepsilon}$ and the third term is $\le 2\beta_0 D_\varepsilon^{n-1}$, since β is bounded above by β_0 on $\hat{\Lambda}_\varepsilon$.

So $|\beta(\widehat{z,w}) - \beta(\widehat{z,w})'| \ge 2\varepsilon C_\varepsilon - 2\varepsilon D_\varepsilon^2\cdot\frac{1-D_\varepsilon^{n-1}}{1-D_\varepsilon} - 2\beta_0 D_\varepsilon^{n-1}$.

When we take $n\to\infty$, and ε small enough, the inequality (3) will imply:

$$|\beta(\widehat{z,w}) - \beta(\widehat{z,w})'| \ge 2\varepsilon C_\varepsilon - 2\varepsilon D_\varepsilon^2\cdot\frac{1}{1-D_\varepsilon} > .5\varepsilon^2.$$

Moreover, we can estimate the difference $\beta(\widehat{z,w}) - \beta(\widehat{z,w})'$ in the same way as before also for other prehistories.

For example, if $(\widehat{z,w})_{-i} = (\widehat{z,w})'_{-i}$, $0 \le i < j$, and $(\widehat{z,w})'_{-j} = (z_{-j}, -w_{-j})$, then, if $C_\varepsilon := \inf_{(z,w)\in\Lambda_\varepsilon} |z|$, $D_\varepsilon := \sup_{(z,w)\in\Lambda_\varepsilon} |z|$, we get

$$0.5\varepsilon^{j+1} \le |\beta(\widehat{z,w}) - \beta(\widehat{z.w})'| \le 2\varepsilon \cdot D_\varepsilon^j + 3\varepsilon D_\varepsilon^{j+1} < 3\varepsilon^{j+1} \cdot 1.5 < 5 \cdot \varepsilon^{j+1} \tag{4}$$

for $\varepsilon > 0$ small enough.

We see therefore that for these maps f_ε, there pass an uncountable collection of local unstable manifolds through every point of their basic sets Λ_ε. □

For $|c|$ small, the map $f_\varepsilon(z,w) = (z^2+c+\varepsilon w^2, w^2)$ has the same characteristics as $(z^2+\varepsilon w^2, w^2)$ on a basic set Λ_ε. A similar calculation as before will show that $\sup_{(z,w)\in\Lambda_\varepsilon} |z| < 3|c|$ and $\inf_{(z,w)\in\Lambda_\varepsilon} |z| \le \frac{3}{4}|c|$, if $0 < \varepsilon < \varepsilon(c)$ and then for $|c|$ small, (1) will give also that $\beta(\widehat{z,w}) \ne \beta(\widehat{z,w})'$.

The results in [4] and [5] applied to the map $f_\varepsilon(z,w) = (z^2+c+\varepsilon w^2, w^2)$ give that $W^u(\hat\Lambda)$ has empty interior and estimate $HD(W^u_\varepsilon(\hat\Lambda))$ and $HD(\widehat{W^u_\varepsilon(\hat\Lambda)})$. For this map we can apply also the conclusions of Theorems 1 and 2 to get $HD(W^s_{\varepsilon'}(x) \cap \Lambda_\varepsilon) = 0$ and so $h_i(f_\varepsilon|\Lambda_\varepsilon) = 0$.

In the course of working on this article, we also came upon the following question:

If $f|_\Lambda$ as d'-to-1 and f_ε is a small perturbation of f, then is $f_\varepsilon|_{\Lambda_\varepsilon} d'$-to-1?

Of course the first case to check would be that of $f(z,w) = (z^2+c, w^2), c$ small, $c \ne 0$ on its basic set $\Lambda = \{p_0(c)\} \times S^1$, with $p_0(c)$ a fixed attracting point of z^2+c. If c is small enough, f is Axiom A (see also [1]); and $f|_\Lambda$ is 2-to-1.

Theorem 2. *If $c \ne 0$, small, the map $f_\varepsilon(z,w) = (z^2+a\varepsilon z+b\varepsilon w+c+d\varepsilon zw+e\varepsilon w^2, w^2)$ is injective on its basic set Λ_ε close to $p_0(c) \times S^1$, as long as $|c| < c(a,b,d,e)$, $b \ne 0$, and $\varepsilon < \varepsilon(a,b,c,d,e)$.*

Proof. Assume that $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ for two points $(z,w),(z',w') \in \Lambda_\varepsilon$

$$\Rightarrow z^2 + \varepsilon(az+bw+dzw+ew^2) + c = z'^2 + \varepsilon(az'+bw'+dz'w'+ew^2) + c$$
$$\Rightarrow (z^2 - z'^2) + \varepsilon(a(z-z') + b(w-w') + d(zw-z'w') + e(w^2-w'^2)) = 0.$$

Assume $w \ne w' \Rightarrow w' = -w$ since $w^2 = w'^2$

$$\Rightarrow z^2 - z'^2 + \varepsilon(a(z-z') + 2bw + dw(z+z')) = 0.$$
$$\Rightarrow (z-z')(z+z'+\varepsilon a) = -2bw \cdot \varepsilon - d\varepsilon w(z+z').$$

Let $p_0(c)$ be a fixed attracting point for $z \to z^2+c$, c small. And consider $\alpha := \sup_{(z,w)\in\Lambda_\varepsilon} |z-p_0(c)|$; let (z_0,w_0) a point in Λ_ε where the supremum in Λ_ε is attained so $\alpha = |z_0 - p_0(c)|$, and $p_0^2(c)+c = p_0(c)$. Now, we can find a point $(z,w) \in \Lambda_\varepsilon$ s.t.

$$\begin{aligned} f_\varepsilon(z,w) = (z_0,w_0) &\Rightarrow z_0 = z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 \\ &\Rightarrow z_0 - p_0(c) = z^2 - p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \\ &= (z-p_0(c))^2 + 2zp_0(c) - 2p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \Rightarrow \\ &\alpha \le \alpha^2 + 2|p_0(c)|\alpha + K\varepsilon, \end{aligned}$$

with $K = K(a,b,d,e)$ a suitable majoration constant.

Now $\alpha^2 + \alpha(2|p_0(c)| - 1) + K\varepsilon \geq 0$, and since $\alpha \ll 1$ (since Λ_ε is very close to $\{p_0(c)\} \times S^1$), we obtain:

$$0 \leq \alpha \leq \frac{2K\varepsilon}{1 - 2|p_0(c)| + \sqrt{(1-2|p_0(c)|)^2 - 4K\varepsilon}} \leq K' \cdot \varepsilon, \qquad K' = K'(a,b,c,d,e).$$

From (1) one gets $2|b|\varepsilon \leq 2K'\varepsilon \cdot (\varepsilon a + 3|p_0(c)|) + 3d\varepsilon|p_0(c)|$ for $\varepsilon < \varepsilon(c)$, since z, z' are ε-close to $p_0(c)$.

So $0 < |b| \leq K'(\varepsilon a + 3|p_0(c)|) + 3d|p_0(c)|$, which is a contradiction if $b \neq 0$, and c is small in comparison to b, $c < c(a,b,d,e)$ (c small will imply that $|p_0(c)|$ is also small, and we can always reduce ε accordingly).

Hence we proved $w' = w$. Then, from $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ it follows that $z^2 - z'^2 + \varepsilon(a(z-z') + dw(z-z')) = 0$. If $z \neq z'$ we would then get $z+z'+\varepsilon(a+dw) = 0 \Rightarrow |z+z'| = \varepsilon|a+dw| \leq \varepsilon\cdot(|a|+|d|)$. But z, z' are both close to $p_0(c) \neq 0$, so if we choose $\varepsilon < \varepsilon(a,b,c,d)$ small enough, the above inequality gives a contradiction. In conclusion $z = z', w = w' \Rightarrow f_\varepsilon|_{\Lambda_\varepsilon}: \ \Lambda_\varepsilon \to \Lambda_\varepsilon$ is an injective map. $\square$

As remarked at the end of Section 3 of [6], if $f_\varepsilon|_{\Lambda_\varepsilon}$ is injective, then $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$, where h is the topological entropy and h_i is the preimage entropy.

In particular this example implies that the basic set Λ_ε is not even a Jordan curve. Indeed, if Λ_ε were a Jordan curve, the Nitecki-Przytycki Theorem in Section 1 of [6] would imply that $h_i(f_\varepsilon|_{\Lambda_\varepsilon})$ is zero, but as we saw, $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$.

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