TRANSVERSAL FAMILIES OF SKEW-PRODUCT AXIOM A ENDOMORPHISMS

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ABSTRACT. We study families of Axiom A skew products with the transversality condition and in particular, the Hausdorff dimension of their fibers, by using thermodynamical formalism. The maps we consider can be non-invertible, and the study of their dynamics is influenced greatly by this fact.

We introduce and employ probability measures (constructed from equilibrium measures on the natural extension), which are supported on the fibers of the skew product. A stronger condition, that of Uniform Transversality is then considered in order to obtain a general formula for Hausdorff dimension of fibers for all base points and almost all parameters.

In the end we study a large class of examples of transversal Axiom A families which locally depend linearly on the parameters, and also another class of examples related to complex dynamics.

1. TRANSVERSAL FAMILIES OF SKEW-PRODUCT AXIOM A ENDOMORPHISMS

Recall from [6] that a continuous self-map $f : X \to X$ of a compact metric space (X, ρ) is called open distance expanding, provided that f is open, Lipschitz continuous, and there are three constants $\eta > 0$, $\gamma > 1$ and an integer $k \geq 1$, such that $\rho(f^k(x), f^k(z)) \geq \gamma\rho(x, z)$ whenever $\rho(x, z) \leq \eta$. It is fairly easy to see that changing the metric ρ in a bi-Lipschitz manner, we may assume without loss of generality that $k = 1$. There is an abundance of open distance expanding maps. We want to bring reader's attention now to one particular class of them, called expanding repellers. Let U be a bounded open subset of a Euclidean space $\mathbb{R}^p$ with some $p \geq 1$.

A map $f : U \to \mathbb{R}^p$ is called an expanding repeller if and only if the following conditions are satisfied.

i) $f : U \to \mathbb{R}^p$ is a $C^{1+\gamma}$ endomorphism.
ii) $X = \bigcap_{n=0}^{\infty} f^{-n}(U)$ is a compact f-invariant ($f(X) = X$) subset of U. The map $f : X \to X$ is transitive.
iii) The map $f : X \to X$ is infinitesimally expanding, i.e. there exists $k \geq 1$ such that for all $x \in X$ and for all $v \in \mathbb{R}^p$, we have $||D_x f^k(v)|| \geq 2||v||$.

Clearly, $f : X \to X$ is an open distance (with respect to the Euclidean metric) expanding map.

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Start with $f : X \to X$ an open distance expanding map and suppose it is transitive. Let V be a bounded quasi-convex open subset of $\mathbb{R}^q$, $q \geq 1$. Being D-quasiconvex (with some $D \geq 1$) means that the internal distances are not bigger than Euclidean distances multiplied by D. In what follows quasi-convexity will be used only when the Mean Value Inequality is to be applied. So, in order to simplify notation, we will assume in the sequel that V is convex.

Suppose now that for all $x \in X$ there exists a $C^{1+\gamma}$ conformal endomorphism $\phi_x : V \to V$ conformally extendable to a neighborhood of $\overline{V}$ with the following properties.

(a) $\kappa := \sup\{|(\phi_x)'(y)| : (x, y) \in X \times \overline{V}\} < 1$.

(b) $\underline{\kappa} := \inf\{|(\phi_x)'(y)| : (x, y) \in X \times \overline{V}\} > 0$.

If the conditions (a) and (b) are satisfied, then the map $F : U \times V \to \mathbb{R}^p \times V$ given by the formula

$$F(x, y) = (f(x), \phi_x(y))$$

is called a skew-product Axiom A fiberwise conformal endomorphism provided that it is Lipschitz continuous (with respect to the sum metric on $X \times \mathbb{R}^q$) and the map $(x, y) \mapsto (f(x), \phi'_x(y))$ is also Lipschitz continuous. Denote the common Lipschitz constant by L_F. Set

$$\Lambda = \bigcap_{n=-\infty}^{+\infty} f^{-n}(U \times \overline{V}) = \bigcup_{x\in X} \bigcap_{n=0}^{\infty} \bigcup_{z\in f^{-n}(x)} \phi_z^n(\overline{V}),$$

where $\phi_z^n = \phi_{f^{n-1}(z)} \circ \phi_{f^{n-1}(z)} \circ \ldots \circ \phi_z : \overline{V} \to \overline{V}$ and $F^n(x, y) = (f^n(x), \phi_x^n(y))$; Λ is called the basic set of the endomorphism f. Obviously

$$f(\Lambda) \subset \Lambda \ \text{ and } \ f(Y_x) \subset Y_{f(x)},$$

where

$$Y_x = \bigcap_{n=0}^{\infty} \bigcup_{z\in f^{-n}(x)} \phi_z^n(\overline{V}).$$

Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be the Rokhlin's natural extension (projective limit) of the endomorphism $f : X \to X$. For every $n \geq 0$ let $p_n : \tilde{X} \to X$ be the projection onto nth coordinate of $\tilde{X}$. Put

$$\hat{\Lambda} = \bigcup_{x\in X} p_0^{-1}(x) \times Y_x$$

and define the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$ by the formula

$$\hat{F}(\tilde{x}, y) = (\tilde{f}(\tilde{x}), \phi_{x_1}(y)).$$

Notice that the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$ is a homeomorphism and the mapping

$$((x_n, y_n)_0^{\infty}) \mapsto ((x_n, y_0)_0^{\infty})$$

is a homeomorphism from $\tilde{\Lambda}$, the Rokhlin's natural extension of Λ, to $\hat{\Lambda}$ which establishes a canonical topological conjugacy between the map $\tilde{F} : \tilde{\Lambda} \to \tilde{\Lambda}$ and the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$.

Note that for every $\hat{x} \in \hat{X}$, $\{\phi_{x_n}^n(\overline{V})\}_{n=0}^\infty$ is descending (as $\phi_{x_{n+1}}^{n+1} = \phi_{x_n}^n \circ \phi_{x_{n+1}}$) sequence of compact sets whose diameters, by condition (e) converge to 0. Hence, the intersection

$$\bigcap_{n=0}^\infty \phi_{x_n}^n(\overline{V})$$

is a singleton, and denote its only element by $\pi(\tilde{x})$. So, we have defined a map

$$\pi : \tilde{X} \to \overline{V}.$$

It is easy to see that for every $x \in X$,

$$\pi(p_0^{-1}(x)) = Y_x.$$

Endow $\tilde{X}$ with a metric $\tilde{\rho}$ defined as follows.

$$\tilde{\rho}(\tilde{x}, \tilde{z}) = \sum_{n=0}^\infty \kappa^n \rho(x_n, z_n).$$

We shall prove the following.

Proposition 1.1. *The map $\pi : \tilde{X} \to \overline{V}$ is Lipschitz continuous.*

Proof. We shall first prove the following formula by induction

$$\left\|\phi_{x_n}^n(w) - \phi_{z_n}^n(w)\right\| \le \sum_{j=0}^{n-1} \kappa^j \left|\left|\phi_{x_{j+1}}\left(\phi_{z_n}^{n-j-1}(w)\right) - \phi_{z_{j+1}}\left(\phi_{z_n}^{n-j-1}(w)\right)\right|\right| \tag{1.1}$$

for all $n \ge 1$, all $w \in \overline{V}$ and all $\tilde{x}, \tilde{z} \in \tilde{X}$. Indeed, for $n = 1$ we even have equality. Suppose the formula is true for some $n \ge 1$. Using the Mean Value Inequality we then get

$$\begin{aligned}
&\left\|\phi_{x_{n+1}}^{n+1}(w) - \phi_{z_{n+1}}^{n+1}(w)\right\| = \\
&\quad = \left\|\phi_{x_n}^n\left(\phi_{x_{n+1}}(w)\right) - \phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right) + \phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right) - \phi_{z_n}^n\left(\phi_{z_{n+1}}(w)\right)\right\| \\
&\quad \le \left\|\phi_{x_n}^n\left(\phi_{x_{n+1}}(w)\right) - \phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right)\right\| + \left\|\phi_{x_n}^n\left(\phi_{z_{n+1}}(w)\right) - \phi_{z_n}^n\left(\phi_{z_{n+1}}(w)\right)\right\| \\
&\quad \le \kappa^n ||\phi_{x_{n+1}}(w) - \phi_{z_{n+1}}(w)|| + \sum_{j=0}^{n-1} \kappa^j \left\|\phi_{x_{j+1}}\left(\phi_{z_n}^{n-j-1}\left(\phi_{z_{n+1}}(w)\right)\right) - \phi_{z_{j+1}}\left(\phi_{z_n}^{n-j-1}\left(\phi_{z_{n+1}}(w)\right)\right)\right\| \\
&\quad = \kappa^n \left\|\phi_{x_{n+1}}(w) - \phi_{z_{n+1}}(w)\right\| + \sum_{j=0}^{n-1} \kappa^j \left\|\phi_{x_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right) - \phi_{z_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right)\right\| \\
&\quad = \sum_{j=0}^{n} \kappa^j \left\|\phi_{x_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right) - \phi_{z_{j+1}}\left(\phi_{z_{n+1}}^{n-j}(w)\right)\right\|.
\end{aligned}$$

The inductive proof of formula (1.1) is complete. Continuing the estimates in this formula, we obtain

$$\left\|\phi_{x_n}^n(w) - \phi_{z_n}^n(w)\right\| \le L_F \sum_{j=0}^{n-1} \kappa^j \rho(x_{j+1}, z_{j+1}) \le L_F \sum_{j=0}^{\infty} \kappa^j \rho(x_{j+1}, z_{j+1}).$$

So, letting $n \to \infty$, we get

$$||\pi(\tilde{x}) - \pi(\tilde{z})|| \le L_F \sum_{j=0}^{\infty} \kappa^j \rho(x_{j+1}, z_{j+1}) \le L_F \frac{\rho(\tilde{x}, \tilde{z})}{\kappa}.$$

We are done. $\square$

For every continuous potential $g : \tilde{X} \to \mathbb{R}$ let $\mathrm{P}(g) = \mathrm{P}(\tilde{f}, g)$ be the topological pressure of g with respect to the dynamical system $\tilde{f}\tilde{X} \to \tilde{X}$. For the topological pressure and its basic propersties see for ex. [1] and [6]. Now consider the potential $\zeta = \zeta_F : \tilde{X} \to \mathbb{R}$ given by the formula

$$\zeta(\tilde{x}) = \log|\phi'_{x_0}(\pi(\tilde{x}))|$$

This potential is Hölder continuous because of Proposition 1.1. It is easy to see that the function $t \mapsto \mathrm{P}(\tilde{f}, t\zeta)$ is convex, Lipschitz continuous, strictly decreasing, and

$$\lim_{t\to-\infty} \mathrm{P}(\tilde{f}, t\zeta) = +\infty \text{ and } \lim_{t\to+\infty} \mathrm{P}(\tilde{f}, t\zeta) = -\infty.$$

Thus there exists exactly one $t \in \mathbb{R}$, denoted by h, such that $\mathrm{P}(\tilde{f}, h\zeta) = 0$. Since $\mathrm{P}(\tilde{f}, 0\zeta) = \mathrm{h_{top}}(\tilde{f}) > 0$, we see that $h > 0$. The number h is called Bowen's stable zero of the basic set Λ. Our goal from now on throughout this section is to provide a geometric characterization of this dimension in the framework of smooth families of skew-product Axiom A fiberwise conformal endomorphisms.

Endow the space $C^{1+\gamma}(\overline{V})$ of all $C^{1+\gamma}$ differentiable endomorphisms from $\overline{V}$ into $\overline{V}$ with the norm $||\cdot||_\gamma$ given by the formula

$$||\phi||_\gamma = ||\phi||_\infty + ||\phi'||_\infty + v_\gamma(\phi'),$$

where

$$v_\gamma(\phi') = \inf\{L > 0 : |\phi'(y) - \phi'(x)| \le L|y - x|^\gamma \text{ for all } x, y \in \overline{V}\}.$$

Obviously $C^{1+\gamma}(\overline{V})$ endowed with this norm becomes a Banach space. Denote the metric induced by the norm $||\cdot||_\gamma$ by ρ_γ. Now fix $d \ge 1$ and an open set $W \subset \mathbb{R}^d$ and consider a family $\Phi = \{\phi_x^\lambda : \overline{V} \to V\}_{(\lambda,x)\in W\times X}$ of maps from $C^{1+\gamma}(\overline{V})$ satisfying the following conditions.

- (af) Conditions (a) and (b) with the same constants $\kappa, \underline{\kappa} \in (0,1)$.
- (bf) The map $(\lambda, x) \mapsto \phi_x^\lambda \in C^{1+\gamma}(\overline{V})$ defined on $W \times X$ is continuous.
- (cf) (Transversality Condition)

$$\forall(x \in X)\,\forall(\lambda_0 \in W)\,\exists(\delta(x,\lambda_0) > 0)\,\exists(C_1 > 0)\,\forall(\tilde{x}, \tilde{y} \in p_0^{-1}(x))\,\forall(r > 0)$$
$$x_1 \ne y_1 \Rightarrow l_d\big(\{\lambda \in B(\lambda_0, \delta(x,\lambda_0)) : ||\pi_\lambda(\tilde{x}) - \pi_\lambda(\tilde{y})|| \le r\}\big) \le C_1 r^q,$$

  where l_d denotes the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and $\pi_\lambda : \tilde{X} \to \overline{V}$ is the canonical projection induced by the skew-product Axiom A fiberwise conformal endomorphism $F_\lambda : U \times \overline{V} : \mathbb{R}^p \times \overline{V}$ given by the formula

$$F_\lambda(x, y) = (f(x), \phi_x^\lambda(y)).$$

Any such family Φ is said to be transversal and the canonically induced family $\overline{\Phi} = \{F_\lambda\}_{\lambda \in W}$ is also called transversal. Transversality will be the crucial issue throughout the paper. We would like to say that a version of the transversality condition, for iterated function systems with overlaps, was used, indirectly, for first time in [5] and then subsequently in [8]. The term transversality was consistently used beginning with the paper [4]. A version of transversality condition for some skew products appeared in [7]; appropriate dimension formulae were obtained there.

For all $\lambda, \lambda' \in W$ put

$$||F_\lambda||_\gamma = \sup\{||\phi_x^\lambda||_\gamma : x \in X\} \text{ and } \overline{\rho}_\gamma(F_\lambda, F_{\lambda'}) = \sup\{\rho_\gamma(\phi_x^\lambda, \phi_x^{\lambda'}) : x \in X\}.$$

Condition (bf) can be now rephrased as follows.

(b'f) The function $\lambda \mapsto F_\lambda$, $\lambda \in W$, is continuous.

In order to prove Bowen's formula for the family $\overline{\Phi}$, we need some auxiliary facts.

Lemma 1.2.

$$\forall(\eta > 0)\exists(\delta > 0)\forall(\lambda_0 \in W)\forall(\lambda \in B(\lambda_0, \delta) \cap W)\forall(\tilde{x} \in \tilde{X})\forall(n \geq 0)$$

$$e^{-\eta n} \leq \frac{||\big(\phi_{x_n}^{\lambda,n}\big)'||}{||\big(\phi_{x_n}^{\lambda_0,n}\big)'||} \leq e^{\eta n}.$$

Proof. Fix $y \in \overline{V}$. Using the Mean Value Inequality and condition (a), we get

$$\begin{aligned}
&||\phi_{x_{n+1}}^{\lambda,n+1}(y) - \phi_{x_{n+1}}^{\lambda_0,n+1}(y)|| \leq \\
&\qquad \leq ||\phi_{x_1}^{\lambda}\big(\phi_{x_{n+1}}^{\lambda,n}(y)\big) - \phi_{x_1}^{\lambda_0}\big(\phi_{x_{n+1}}^{\lambda,n}(y)\big)|| + ||\phi_{x_1}^{\lambda_0}\big(\phi_{x_{n+1}}^{\lambda,n}(y)\big) - \phi_{x_1}^{\lambda_0}\big(\phi_{x_{n+1}}^{\lambda_0,n}(y)\big)|| \\
&\qquad \leq ||\phi_{x_1}^{\lambda} - \phi_{x_1}^{\lambda_0}||_\infty + ||\big(\phi_{x_1}^{\lambda_0}\big)'||_\infty ||\phi_{x_{n+1}}^{\lambda,n}(y) - \phi_{x_{n+1}}^{\lambda_0,n}(y)|| \\
&\qquad \leq ||\phi_{x_1}^{\lambda} - \phi_{x_1}^{\lambda_0}||_\infty + \kappa||\phi_{x_{n+1}}^{\lambda,n}(y) - \phi_{x_{n+1}}^{\lambda_0,n}(y)||.
\end{aligned}$$

Thus, by induction

$$||\phi_{x_n}^{\lambda,n}(y) - \phi_{x_n}^{\lambda_0,n}(y)|| \leq (1-\kappa)^{-1}||\phi_{x_1}^{\lambda} - \phi_{x_1}^{\lambda_0}||_\infty \leq (1-\kappa)^{-1}\overline{\rho}_\gamma(F_\lambda, F_{\lambda_0}).$$

Hence, for every $0 \leq k \leq n$, we get that

$$\begin{aligned}
&||\big(\phi_{x_k}^{\lambda}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big) - \big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda_0,n-k}(y)\big)|| \leq \\
&\qquad \leq ||\big(\phi_{x_k}^{\lambda}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big) - \big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)|| + \\
&\qquad\quad + ||\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big) - \big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda_0,n-k}(y)\big)|| \\
&\qquad \leq ||\big(\phi_{x_k}^{\lambda}\big)' - \big(\phi_{x_k}^{\lambda_0}\big)'||_\infty + v_\gamma\big(\big(\phi_{x_k}^{\lambda_0}\big)'\big)||\phi_{x_n}^{\lambda,n-k}(y) - \phi_{x_n}^{\lambda_0,n-k}(y)||^\gamma \\
&\qquad \leq \overline{\rho}_\gamma(F_\lambda, F_{\lambda_0}) + ||F_{\lambda_0}||_\gamma(1-\kappa)^{-\gamma}\overline{\rho}_\gamma^\gamma(F_\lambda, F_{\lambda_0}) \\
&\qquad \leq \big(1 + (1-\kappa)^{-\gamma}||F_{\lambda_0}||_\gamma\big)\overline{\rho}_\gamma^\gamma(F_\lambda, F_{\lambda_0}),
\end{aligned}$$

where the last inequality was written assuming that $\overline{\rho}_\gamma(F_\lambda, F_{\lambda_0}) \leq 1$. Since $\log|b/a| \leq |b-a|/|b|$, we further get using (af) that

$$\log\frac{|\big(\phi_{x_k}^{\lambda}\big)'\big(\phi_{x_n}^{\lambda,n-k}(y)\big)|}{|\big(\phi_{x_k}^{\lambda_0}\big)'\big(\phi_{x_n}^{\lambda_0,n-k}(y)\big)|} \leq \underline{\kappa}^{-1}\big(1 + (1-\kappa)^{-\gamma}||F_{\lambda_0}||_\gamma\big)\overline{\rho}_\gamma^\gamma(F_\lambda, F_{\lambda_0}).$$

Using the Chain Rule, we therefore get

$$\frac{1}{n}\log\frac{|(\phi_{x_n}^{\lambda,n})'(y)|}{|(\phi_{x_n}^{\lambda_0,n})'(y)|}=\frac{1}{n}\sum_{k=1}^{n}\log\frac{|(\phi_{x_k}^{\lambda})'(\phi_{x_n}^{\lambda,n-k}(y))|}{|(\phi_{x_k}^{\lambda_0})'(\phi_{x_n}^{\lambda_0,n-k}(y))|}\le\underline{\kappa}^{-1}\big(1+(1-\kappa)^{-\gamma}||F_{\lambda_0}||_\gamma\big)\overline{\rho}_\gamma^\gamma(F_\lambda,F_{\lambda_0}).$$

So, the lemma follows by invoking (b'f), the uniform (decreasing W if necessary) continuity of the function $\lambda\mapsto F_\lambda$ and the distortion property of $\phi_{x_n}^\lambda$ on V. □

Our next auxiliary result is this.

Lemma 1.3. *If $\Phi=\{F_\lambda\}_{\lambda\in W}$ is a transversal family of skew-product Axiom A fiberwise conformal endomorphisms, then for every $\beta\in(0,q)$ and for all $x\in X$ there exists a constant $C>0$ such that for all $\tilde z,\tilde w\in p_0^{-1}(x)$ with $z_1\ne w_1$, we have*

$$\int_{B(\lambda_0,\delta(x,\lambda_0))}\frac{d\lambda}{||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||^\beta}\le C.$$

Proof. Applying the transversality condition (cf), we estimate as follows.

$$\begin{aligned}
&\int_{B(\lambda_0,\delta(x,\lambda_0))}\frac{d\lambda}{||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||^\beta}=\\
&\qquad=\int_0^\infty l_d\left(\left\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)):\frac{1}{||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||^\beta}\ge t\right\}\right)dt\\
&\qquad=\beta\int_0^\infty l_d\big(\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)):||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||\le r\}\big)r^{-\beta-1}\\
&\qquad=\beta\int_0^{\delta(x,\lambda_0)} l_d\big(\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)):||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||\le r\}\big)r^{-\beta-1}+\\
&\qquad\quad+\beta\int_{\delta(x,\lambda_0)}^\infty l_d\big(\{\lambda\in B(\lambda_0,\delta(x,\lambda_0)):||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||\le r\}\big)r^{-\beta-1}\\
&\qquad\le C_1\beta\int_0^{\delta(x,\lambda_0)}r^{q-\beta-1}dr+\beta l_d(B(\lambda_0,\delta(x,\lambda_0)))\int_{\delta(x,\lambda_0)}^\infty r^{-\beta-1}dr\\
&\qquad\le C_1\beta(q-\beta)^{-1}(2\delta(x,\lambda_0))^{q-\beta}+\beta l_d(B(\lambda_0,\delta(x,\lambda_0)))\mathrm{diam}(V)^{-\beta}<+\infty.
\end{aligned}$$

□

Lemma 1.4. *Given $\varepsilon,a>0$ put $\eta=\frac{-\varepsilon\log\kappa}{2a+\varepsilon}$ and take $\delta=\delta(\eta)$ coming from Lemma 1.2 ascribed to η. Then for all $\tilde x\in\tilde X$ and all $n\ge0$,*

$$||\lambda-\lambda_0||<\delta\ \Rightarrow\ ||(\phi_{x_n}^{\lambda_0,n})'||_\infty^{a+\frac{\varepsilon}{2}}\le||(\phi_{x_n}^{\lambda,n})'||_\infty^a.$$

Proof. Applying Lemma 1.2, we get

$$\begin{aligned} ||(\phi_{x_n}^{\lambda_0,n})'||_\infty^{a+\frac{\varepsilon}{2}} &\le \exp\big(\eta n\big(a+\frac{\varepsilon}{2}\big)\big)||(\phi_{x_n}^{\lambda,n})'||_\infty^{a+\frac{\varepsilon}{2}} \\ &\le \exp\big(\eta n\big(a+\frac{\varepsilon}{2}\big)\big)\kappa^{\frac{\varepsilon}{2}n}||(\phi_{x_n}^{\lambda,n})'||_\infty^{a} \\ &= \exp\big(-\frac{\varepsilon}{2}\log\kappa n\big)\kappa^{\frac{\varepsilon}{2}n}||(\phi_{x_n}^{\lambda,n})'||_\infty^{a} = ||(\phi_{x_n}^{\lambda,n})'||_\infty^{a}. \end{aligned}$$

□

For every $\lambda \in W$ denote by h_λ the Bowen's stable zero of the basic set Λ_λ. We now shall prove a technical fact, which will easily imply our main result.

Lemma 1.5. *Suppose that $\Phi = \{F_\lambda\}_{\lambda\in W}$ is a transversal family of skew-product Axiom A fiberwise conformal endomorphisms. Then for all $x \in X$ we have*

(a)

$$\forall(\lambda_0 \in W)\,\forall(\varepsilon > 0)\,\exists(\delta > 0)$$

$$\mathrm{HD}(Y_{\lambda,x}) \ge \min\{h_{\lambda_0}, q\} - \varepsilon$$

for l_d-a.e. $\lambda \in B(\lambda_0,\delta)$ and

(b) *If $h_{\lambda_0} > q$, then there exists $\delta > 0$ such that*

$$l_q(Y_{\lambda,x}) > 0$$

for l_d-a.e. $\lambda \in B(\lambda_0,\delta)$.

Proof. Put $h = \min\{h_{\lambda_0}, q\}$. Since the potential $h_{\lambda_0}\zeta_{F_{\lambda_0}}$ is Hölder continuous, there exists a unique equilibrium (Gibbs) state μ for this potential and the dynamical system $\tilde{f} : \tilde{X} \to \tilde{X}$. Since $f : X \to X$ is a distance expanding map, for every $r > 0$ sufficiently small, say $r \in (0,R]$, every $z \in X$ and every $n \ge 0$ there exists a unique continuous inverse branch $f_z^{-n} : B(f^n(z),r) \to X$ of f^n sending $f^n(z)$ to z. We now want to look at the Gibbs measure μ in greater detail. A straightforward adaptation of the proof of Lemma 1.6, p.11 in [1] results in the existence of a Hölder continuous function ζ_+ that is cohomologous to $h_{\lambda_0}\zeta_{F_{\lambda_0}}$ and depends only on the 0th coordinate, in particular ζ_+ can be regarded as a Hölder continuous function defined on X. Then $\mu = \tilde{\mu}_+$, where μ_+ is the Gibbs (equilibrium) state for the potential $\zeta_+ : \tilde{X} \to \mathbb{R}$. Also $\mu_\circ p_n^{-1} = \mu_+$ for all $n \ge 0$, and $\mathrm{P}(\zeta_+) = \mathrm{P}\big(h_{\lambda_0}\zeta_{F_{\lambda_0}}\big) = 0$. Let $\mathcal{L}_+ : C(X) \to C(X)$ be the Perron-Frobenius operator determined by the potential $\zeta_+ : X \to \mathbb{R}$. It is then well-known (see [6], Ch. 4 for ex.) that there exists m_+, a Borel probability measure on X being a fixed point of the dual operator $\mathcal{L}_+^* : C^*(X) \to C^*(X)$. This means that

$$m_+(f(A)) = \int_A e^{-\zeta_+}dm_+$$

whenever A is a Borel subset of X such that $f|_A : A \to f(A)$ is one-to-one. In particular, for every $x \in X$, every $r \in (0,R]$ and every Borel set $A \subset B(f^n(x),r)$

$$m_+\big(f_x^{-n}(A)\big) = \int_A \exp\big(S_n\zeta_+ \circ f_x^{-n}\big)dm_+ \asymp \exp\big(S_n\zeta_+(x)\big)m_+(A), \tag{1.2}$$

where the universal comparability constant is independent of r, x and n. Since (see [6], Ch.4) the Radon-Nikodym derivative $\frac{d\mu_+}{dm_+}$ is a continuous function bounded away from zero and infinity, we get, using (1.2) and cohomology of ζ_+ and $h_{\lambda_0}\zeta_{F_{\lambda_0}}$, for every $r \in (0,R]$, every $z \in X$ and all $n \geq 0$ that

$$
\begin{aligned}
\mu\big(p_n^{-1}\circ f_z^{-n}(B(f^n(z),r))\big) &= \tilde{\mu}_+\big(p_n^{-1}\circ f_z^{-n}(B(f^n(z),r))\big) = \mu_+\big(f_z^{-n}(B(f^n(z),r))\big) \\
&\asymp m_+\big(f_z^{-n}(B(f^n(z),r))\big) \\
&\asymp \exp\big(S_n\zeta_+(x)\big)m_+(B(f^n(z),r)) \\
&\asymp \exp\big(h_{\lambda_0}S_n\zeta_{F_{\lambda_0}}(\tilde{z})\big)\mu_+(B(f^n(z),r)) \\
&= \left|\left(\phi_z^{\lambda_0,n}\right)'(\pi_{\lambda_0}(\tilde{z}))\right|^{h_{\lambda_0}}\tilde{\mu}_+\circ p_0^{-1}(B(f^n(z),r)) \\
&\asymp \left|\left|(\phi_z^{\lambda_0,n})'\right|\right|^{h_{\lambda_0}}\mu\big(p_0^{-1}(B(f^n(z),r))\big),
\end{aligned}
\tag{1.3}
$$

where $\tilde{z}$ was an arbitrary auxiliary point in $p_0^{-1}(z)$ and all the comparability constants appearing in this calculation are independent of r, z and n. Now, fix $x \in X$, $r \in (0,R]$, $n \geq 0$ and $\xi \in f^{-n}(x)$. Put

$$
\mu_{x,n}(\xi) = \overline{\lim_{r\to 0}}\frac{\mu\big(p_n^{-1}\big(f_\xi^{-n}(B(x,r)))\big)}{\mu\big(p_0^{-1}B(x,r))\big)}.
\tag{1.4}
$$

This formula defines a probability measure on the finite set $f^{-n}(x)$. Since for all $n \geq 1$ and all $z \in f^{-(n-1)}(x)$,

$$
\begin{aligned}
\mu_{x,n}\circ f^{-1}(z) &= \sum_{w\in f^{-1}(z)}\mu_{x,n}(w) = \sum_{w\in f^{-1}(z)}\overline{\lim_{r\to 0}}\frac{\mu\big(p_n^{-1}\big(f_w^{-n}(B(x,r)))\big)}{\mu\big(p_0^{-1}(x,r))\big)} \\
&= \overline{\lim_{r\to 0}}\big(\mu\big(p_0^{-1}(x,r))\big)\big)^{-1}\sum_{w\in f^{-1}(z)}\mu\big(p_n^{-1}\big(f_w^{-n}(B(x,r)))\big) \\
&= \overline{\lim_{r\to 0}}\big(\mu\big(p_0^{-1}(x,r))\big)\big)^{-1}\mu\left(\bigcup_{w\in f^{-1}(z)}p_n^{-1}\big(f_w^{-n}(B(x,r)))\big)\right) \\
&= \overline{\lim_{r\to 0}}\frac{\mu\big(p_{n-1}^{-1}\big(f_z^{-(n-1)}(B(x,r)))\big)}{\mu\big(p_0^{-1}(x,r))\big)} \\
&= \mu_{x,n-1}(z),
\end{aligned}
$$

the sequence $\big(\mu_{x,n}\big)_1^\infty$ is consistent with respect to the sequence of maps $\big(f : f^{-n}(x) \to f^{-(n-1)}(x)\big)_1^\infty$ in the sense of Definition 3.6.3 from [3]. It therefore follows from Daniel-Kolmogorov Consistency Theorem (Proposition 3.6.4 in [3]) that there exists a measure μ_x on $p_0^{-1}(x)$ such that $\mu_x\circ p_n^{-1} = \mu_{x,n}$ for all $n \geq 0$. Hence it follows from (1.3) and (1.4) that for all $x \in X$, all $r > 0$, all $n \geq 0$ and all $\xi \in f^{-n}(x)$, we have

$$
\mu_x(p_n^{-1}(\xi)) = \overline{\lim_{r\to 0}}\frac{\mu\big(p_n^{-1}\big(f_\xi^{-n}(B(x,r)))\big)}{\mu\big(p_0^{-1}B(x,r))\big)} \asymp ||(\phi_\xi^{\lambda_0,n})'||^{h_{\lambda_0}}
\tag{1.5}
$$

and the universal comparability constant is independent of r, x, n and ξ.

Given $\varepsilon > 0$, let $0 < \delta = \min\{\delta(\eta), \delta(x, \lambda_0)\}$, where $\eta = \frac{-\varepsilon \log \kappa}{2h-\varepsilon}$ comes from Lemma 1.4 with $a = h - \varepsilon$. By the potential-theoretic characterization of Hausdorff dimension (see [2]), it suffices to prove that

$$\begin{aligned} R_x(\lambda) &= \iint_{\overline{V}\times\overline{V}} \frac{d(\mu_x \circ \pi_\lambda^{-1} \times \mu_x \circ \pi_\lambda^{-1})(w,z)}{||w-z||^{h-\varepsilon}} \\ &= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} \frac{d\mu_2(\tilde{w},\tilde{z})}{||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||^{h-\varepsilon}} < +\infty, \end{aligned} \tag{1.6}$$

where $\mu_2 = \mu_x \times \mu_x$ is the product measure on $p_0^{-1}(x) \times p_0^{-1}(x)$. And in turn, in order to prove (1.6), it is enough to show that

$$\int_{B(\lambda_0,\delta)} R_x(\lambda) d\lambda < +\infty.$$

For every $n \geq 1$ and every $\xi \in f^{-n}(x)$, let

$$A_\xi = \{(\tilde{w},\tilde{z}) \in p_0^{-1}(x) \times p_0^{-1}(x) : w_n = z_n = \xi \text{ and } w_{n+1} \neq z_{n+1}\}.$$

By the Mean Value Inequality, we get for all $(\tilde{w},\tilde{z}) \in A_\xi$ that

$$\begin{aligned} ||\pi_\lambda(\tilde{f}^{-n}(\tilde{w})) - \pi_\lambda(\tilde{f}^{-n}(\tilde{z}))|| &= ||(\phi_\xi^{\lambda,n})^{-1}\big(\pi_\lambda(\tilde{w})\big) - (\phi_\xi^{\lambda,n})^{-1}\big(\pi_\lambda(\tilde{z})\big)|| \\ &\leq ||(\phi_\xi^{\lambda,n})'||^{-1} ||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||. \end{aligned} \tag{1.7}$$

By Lemma 1.4, we have

$$||(\phi_\xi^{\lambda,n})'||^{h-\varepsilon} \geq ||(\phi_\xi^{\lambda_0,n})'||^{h-\frac{\varepsilon}{2}} \geq ||(\phi_\xi^{\lambda_0,n})'||^{h_{\lambda_0}-\frac{\varepsilon}{2}}. \tag{1.8}$$

Hence, changing the order of integration, using (1.7), (1.8) and Lemma 1.3 ($(\tilde{f}^{-n}(\tilde{w}))_0 = \xi = (\tilde{f}^{-n}(\tilde{z}))_0$, $(\tilde{f}^{-n}(\tilde{w}))_1 = w_{n+1} \neq z_{n+1} = (\tilde{f}^{-n}(\tilde{z}))_1$), we get

$$\begin{aligned} &\int_{B(\lambda_0,\delta)} R_x(\lambda) d\lambda = \\ &= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} \int_{B(\lambda_0,\delta)} \frac{d\lambda}{||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &= \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} \int_{B(\lambda_0,\delta)} \frac{d\lambda}{||\pi_\lambda(\tilde{w}) - \pi_\lambda(\tilde{z})||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &\leq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} \int_{B(\lambda_0,\delta)} ||(\phi_\xi^{\lambda,n})'||^{\varepsilon-h} \frac{d\lambda}{||\pi_\lambda(\tilde{f}^{-n}(\tilde{w})) - \pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &\leq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}-h_{\lambda_0}} \int_{B(\lambda_0,\delta)} \frac{d\lambda}{||\pi_\lambda(\tilde{f}^{-n}(\tilde{w})) - \pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||^{h-\varepsilon}} d\mu_2(\tilde{w},\tilde{z}) \\ &\leq C \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}-h_{\lambda_0}} d\mu_2(\tilde{w},\tilde{z}). \end{aligned} \tag{1.9}$$

Now, using (1.5), we can continue (1.9) as follows ($A_\xi \subset p_n^{-1}(\xi)$).

$$\begin{aligned}\int_{B(\lambda_0,\delta)} R_x(\lambda)d\lambda &\preceq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}} \mu_x^{-1}(p_n^{-1}(\xi)) d\mu_2 \\ &= \sum_{n=0}^{\infty} \kappa^{\frac{n\varepsilon}{2}} \sum_{\xi\in f^{-n}(x)} \mu_x^{-1}(p_n^{-1}(\xi))\mu_2(A_\xi) \\ &\le \sum_{n=0}^{\infty} \kappa^{\frac{n\varepsilon}{2}} \mu_x(p_0^{-1}(x)) \\ &= \sum_{n=0}^{\infty} \kappa^{\frac{n\varepsilon}{2}} < +\infty,\end{aligned}$$

and we are done with part (a).

(b) Put $\eta = \frac{-\varepsilon \log \kappa}{2h_{\lambda_0}+\varepsilon}$ and determine $\delta = \delta(\eta)$ by Lemma 1.4 with $a = 1$ and ε replaced by $\varepsilon/h_{\lambda_0}$. We use the same setup and notation as in the proof of part (a); in particular μ denotes the same Gibbs state. For every $\lambda \in B(\lambda_0, \delta)$, let

$$\nu_\lambda = \mu_x \circ \pi_\lambda^{-1}.$$

It suffices to show that $\nu_\lambda << l_q$. We shall prove that

$$R = \int_{B(\lambda_0,\delta)} \int \underline{D}(\nu_\lambda, z) d\nu_\lambda(z) d\lambda = \int_{B(\lambda_0,\delta)} \int_{\overline{V}} \underline{D}(\nu_\lambda, z) d\nu_\lambda(z) d\lambda < \infty,$$

where

$$\underline{D}(\nu_\lambda, z) = \liminf_{r\searrow 0} \frac{\nu_\lambda(B(z,r))}{r^q}.$$

Having this, we will have $\underline{D}(\nu_\lambda, z) < +\infty$ for ν_λ-a.e. $z \in \overline{V}$ and Theorem 2.12 in [2] will imply that ν_λ is absolutely continuous with respect to l_q. So, starting the proof that $R < \infty$, we apply Fatou's lemma to get

$$R \le \liminf_{r\searrow 0} \int_{B(\lambda_0,\delta)} \int_{\overline{V}} \frac{\nu_\lambda(B(z,r))}{r^q} d\nu_\lambda(z) d\lambda. \tag{1.10}$$

Now, use the definition of ν_λ to change the variable, write $\nu_\lambda(B(z,r))$ as an integral of the characteristic function, and change the variable once again to obtain

$$\begin{aligned}\int_{\overline{V}} \nu_\lambda(B(z,r)) d\nu_\lambda(z) &= \int_{p_0^{-1}(x)} \mu_x \circ \pi_\lambda^{-1}\big(B(\pi_\lambda(\tilde z), r)\ \big) d\mu_x \circ \pi_\lambda^{-1}(\tilde z) \\ &= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} \mathbb{1}_{\pi_\lambda^{-1}\left(B(\pi_\lambda(\tilde z),r)\right)}(\tilde w) d\mu_x \circ \pi_\lambda^{-1}(\tilde w) d\mu_x \circ \pi_\lambda^{-1}(\tilde z) \\ &= \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} \mathbb{1}_{\{\tilde w\in \tilde X : ||\pi_\lambda(\tilde w)-\pi_\lambda(\tilde z)||<r\}} d\mu_2(\tilde w, \tilde z).\end{aligned}$$

Inserting this to (1.10) and changing the order of integration, gives

$$\begin{aligned} R &\le \liminf_{r\searrow 0} r^{-q} \iint_{p_0^{-1}(x)\times p_0^{-1}(x)} l_d\big(\{\lambda\in B(\lambda_0,\delta) : ||\pi_\lambda(\tilde{w})-\pi_\lambda(\tilde{z})||<r\}\big) d\mu_2(\tilde{w},\tilde{z}) \\ &= \liminf_{r\searrow 0} r^{-q} \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} l_d\big(\{\lambda\in B(\lambda_0,\delta) : ||\pi_\lambda(\tilde{w})-\pi_\lambda(\tilde{z})||<r\}\big) d\mu_2(\tilde{w},\tilde{z}). \end{aligned}$$

By (1.7), Lemma 1.4 with $a=1$ and ε replaced by $\varepsilon/h_{\lambda_0}$, and (cf), we get for all $(\tilde{w},\tilde{z})\in A_\xi$ that

$$\begin{aligned} & l_d\big(\{\lambda\in B(\lambda_0,\delta) : ||\pi_\lambda(\tilde{w})-\pi_\lambda(\tilde{z})||<r\}\big) \le \\ &\quad \le l_d\big(\{\lambda\in B(\lambda_0,\delta) : ||\pi_\lambda(\tilde{f}^{-n}(\tilde{w}))-\pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||<r||(\phi_\xi^{\lambda,n})'||^{-1}\}\big) \\ &\quad \le l_d\big(\{\lambda\in B(\lambda_0,\delta) : ||\pi_\lambda(\tilde{f}^{-n}(\tilde{w}))-\pi_\lambda(\tilde{f}^{-n}(\tilde{z}))||<r||(\phi_\xi^{\lambda_0,n})'||^{-\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}\}\big) \\ &\quad \le C_1 r^q ||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)}. \end{aligned}$$

Thus

$$\begin{aligned} R &\preceq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \iint_{A_\xi} ||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)} d\mu_2 \\ &\le \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} ||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)} \mu_2(A_\xi) \\ &\le \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} ||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)} \mu_x^2(p_n^{-1}(\xi)). \end{aligned}$$

But it follows from (1.5) that

$$\begin{aligned} ||(\phi_\xi^{\lambda_0,n})'||^{-q\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)} &\le ||(\phi_\xi^{\lambda_0,n})'||^{-(h_{\lambda_0}-\varepsilon)\left(1+\frac{\varepsilon}{2h_{\lambda_0}}\right)} \\ &= ||(\phi_\xi^{\lambda_0,n})'||^{-h_{\lambda_0}} ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}+\frac{\varepsilon^2}{2h_{\lambda_0}}} \\ &\le \mu_x^{-1}(p_n^{-1}(\xi)) ||(\phi_\xi^{\lambda_0,n})'||^{\frac{\varepsilon}{2}} \\ &\le \kappa^{\frac{\varepsilon n}{2}} \mu_x^{-1}(p_n^{-1}(\xi)). \end{aligned}$$

Hence,

$$R \preceq \sum_{n=0}^{\infty} \sum_{\xi\in f^{-n}(x)} \kappa^{\frac{\varepsilon n}{2}} \mu_x(p_n^{-1}(\xi)) = \sum_{n=0}^{\infty} \kappa^{\frac{\varepsilon n}{2}} \mu_x(p_0^{-1}(x)) = \sum_{n=0}^{\infty} \kappa^{\frac{\varepsilon n}{2}} < +\infty.$$

We are done. $\square$

We are now in position to provide a short simple proof of the following main result of this section.

Theorem 1.6. *Suppose that* $\Phi = \{F_\lambda\}_{\lambda \in W}$ *is a transversal family of skew-product Axiom A fiberwise conformal endomorphisms. Then the function* $\lambda \mapsto h_\lambda$ *is continuous on* W *and for all* $x \in X$ *there exists a Borel set* $W_x \subset W$ *such that* $l_d(W \setminus W_x) = 0$ *and*

(a)
$$\mathrm{HD}(Y_{\lambda,x}) = \min\{h_\lambda, q\} \text{ for all } \lambda \in W_x.$$

(b)
$$l_d\big(\{\lambda \in W : h_\lambda > q \text{ and } l_d(Y_{\lambda,x}) > 0\}\big) = l_d\big(\{\lambda \in W : h_\lambda > q\}\big).$$

Proof. Continuity of the function $\lambda \mapsto h_\lambda$ is an immediate consequence of the thermodynamic formalism for Smale's spaces ($f : \tilde{X} \to \tilde{X}$) and condition (bf). Inequality $\mathrm{HD}(Y_{\lambda,x}) \le \min\{h_\lambda, q\}$ is known for all skew-product Axiom A fiberwise conformal endomorphisms. Proving (a) suppose for the contrary that for some $x \in X$, $l_d(Z) > 0$, where $Z = \{\lambda \in W : \mathrm{HD}(Y_{\lambda,x}) < \min\{h_\lambda, q\}\}$. Then there is $\varepsilon > 0$ such that $l_d(Z_\varepsilon) > 0$, where $Z_\varepsilon = \{\lambda \in W : \mathrm{HD}(Y_{\lambda,x}) < \min\{h_\lambda, q\} - 2\varepsilon\}$. Let λ_0 be a Lebesgue density point of Z_ε. So, there exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0]$,

$$l_d(Z_\varepsilon \cap B(\lambda_0, \delta)) > 0. \tag{1.11}$$

By the continuity of the function $\lambda \mapsto \min\{h_\lambda, q\}$ there exists $\delta_1 \in (0, \delta_0)$ such that $\min\{h_\lambda, q\} < \min\{h_{\lambda_0}, q\} + \varepsilon$ for all $\lambda \in B(\lambda_0, \delta_1)$. Combining this with (1.11), we conclude that
$$l_d\big(\{\lambda \in B(\lambda_0, \delta) : \mathrm{HD}(Y_{\lambda,x}) < \min\{h_{\lambda_0}, q\} - \varepsilon\}\big) > 0$$
for all $\delta \le \delta_1$. This directly contradicts item (a) of Lemma 1.5, and the proof of item (a) of our present theorem is complete. To finish the proof, that is to demonstrate item (b), note that it directly follows from item (b) of Lemma 1.5. We are done. □

An interesting question arises of when we can find a universal set W' of full measure in W such that item (a) holds for all $x \in X$ and all $\lambda \in W'$. We provide below two sufficient conditions.

Corollary 1.7. *Suppose that* $\Phi = \{F_\lambda\}_{\lambda \in W}$ *is a transversal family of skew-product Axiom A fiberwise conformal endomorphisms and the function* $x \mapsto \mathrm{HD}(Y_{\lambda,x})$, $x \in X$, *is upper semi-continuous, for all* $\lambda \in W$. *Then the function* $\lambda \mapsto h_\lambda$ *is continuous on* W *and there exists a measurable set* $W' \subset W$ *such that* $l_d(W \setminus W') = 0$ *and*
$$\mathrm{HD}(Y_{\lambda,x}) = \min\{h_\lambda, q\}$$
for all $\lambda \in W'$ *and all* $x \in X$.

Proof. Suppose on the contrary that there exists a measurable set W_+ such that $l_d(W_+) > 0$ and for every $\lambda \in W_+$ there exists $x_\lambda \in X$ such that $\mathrm{HD}(Y_{\lambda,x}) < \min\{h_\lambda, q\}$. Fix $\mathcal{B}$, a countable base of topology on X. Since the function $x \mapsto \mathrm{HD}(Y_{\lambda,x})$, $x \in X$, is upper semi-continuous, for every $\lambda \in W_+$ there exists a set $B_\lambda \in \mathcal{B}$ such that $\mathrm{HD}(Y_{\lambda,x}) < \min\{h_\lambda, q\}$ for all $x \in B_\lambda$. For every $B \in \mathcal{B}$, let $W_+(B) = \{\lambda \in W_+ : B = B_\lambda\}$. Since the family $\mathcal{B}$ is countable and $l_d(W_+) > 0$, either there exists $B \in \mathcal{B}$ such that $l_d(W_+(B)) > 0$ or $W_+(B)$ is not measurable. Thus, in any case, there exists $B \in \mathcal{B}$ and a measurable set $U \subset W_+(B)$ such that $l_d(U) > 0$. Fix $z \in B$. Then $\mathrm{HD}(Y_{\lambda,z}) < \min\{h_\lambda, q\}$ for all $\lambda \in U$ contrary to Theorem 1.6(a). We are done. □

Another way to guarantee the existence of a universal set W' as in the corollary above, is to strenghten the transversality condition (cf) as follows.

(c'f) (Uniform Transversality Condition) There exists $C_2 > 0$ such that for all $x \in X$, $\forall \tilde{x}, \tilde{y} \in p_0^{-1}(x), x_1 \neq y_1$, and $\forall r > 0$, we have

$$l_d\big(\lambda \in W : ||\pi_\lambda(\tilde{x}) - \pi_\lambda(\tilde{y})|| \le r\big) \le C_2 r^q.$$

All that has to be done then, is to replace $R_x(\lambda)$ in formula (1.6) by $\sup_{x \in X} R_x(\lambda)$. We thus get the following.

Theorem 1.8. *Suppose that $\Phi = \{F_\lambda\}_{\lambda \in W}$ is a uniformly transversal family of skew-product Axiom A fiberwise conformal endomorphisms. Then the function $\lambda \mapsto h_\lambda$ is continuous on W and there exists a measurable set $W' \subset W$ such that $l_d(W \setminus W') = 0$ and*

$$\mathrm{HD}(Y_{\lambda,x}) = \min\{h_\lambda, q\}$$

for all $\lambda \in W'$ and all $x \in X$.

2. EXAMPLES

We shall now describe a vast class of transversal families of skew product Axiom A fiberwise conformal endomorphisms. We begin with the following elementary auxiliary facts.

Lemma 2.1. *For all $\eta > 0$, $\theta > 0$ and $l > 0$ there exists a constant $C(\eta, \theta, l) \ge 1$ with the following property. If $g : \Delta \to \mathbb{R}$ is a C^1-differentiable function such that*

(a) *Δ is a closed segment of $\mathbb{R}$ with $|\Delta| \le l$,*
(b) *$|g'(x)| \le \theta$ for all $x \in \Delta$,*
(c) *if $x \in \Delta$ and $|g(x)| \le \eta$, then $|g'(x)| \ge \eta$,*

then for every $r > 0$,

$$l_1\big(\{x \in \Delta : |g(x)| \le r\}\big) \le C(\eta, \theta, l) r.$$

Proof. We may assume without loss of generality that $r < \min\{\eta, l\}/2$. It follows from condition (c) that the set $g^{-1}(0)$ is finite. Let $a < b$ be a closest pair of points in this set. Assume without loss of generality that $g'(a) \ge \eta$. Since $g(a) = g(b) = 0$, using the continuity of the function g', we deduce from (c) that there exists a point $w \in (a, b)$ such that $g(w) = \eta$. Fix a minimal w with this property. it then follows from the Mean Value Theorem that $\eta = g(w) - g(a) \le \theta|w - a| \le \theta|b - a|$. Hence $|b - a| \ge \eta/\theta$, and therefore

$$\#g^{-1}(0) \le \theta l/\eta. \tag{2.1}$$

Suppose now that $z \in \Delta$ and $|g(z)| \le r$. Assume without loss of generality that $0 \le g(z) \le r$. Let $a \le \xi \le z$ be the largest number such that $g(\xi) = 0$ if such a number exists, or else, let $\xi = a$. In either case $0 \le g(t) \le r < \eta$ and $g'(t) \ge \eta$ for all $t \in [\xi, z]$. By the Mean Value Theorem there exists $u \in [\xi, z]$ such that $r \ge g(z) - g(\xi) = g'(u)(z - \xi) \ge \eta(z - \xi)$. Thus $z \in (\xi - \frac{r}{\eta}, \xi + \frac{r}{\eta})$ and therefore $g^{-1}([-r, r]) \subset B(\partial\Delta \cup g^{-1}(0), r/\eta)$. So we conclude that $l_1(g^{-1}([-r, r])) \le 2\eta^{-1}(2 + \theta l \eta^{-1})r$.

□

As a straightforward consequence of this lemma, we get the following.

Lemma 2.2. *Let* $U \subset \mathbb{R}^d$ *be a compact convex set with* $\mathrm{diam}(U) \leq l$. *Suppose that* $g : U \to \mathbb{R}$ *is a* C^1*-differentiable function with the following properties.*

(a) *There exists* $1 \leq i \leq d$ *such that* $\left|\frac{\partial g}{\partial x_i}(x)\right| \leq \theta$ *for all* $x \in U$.

(b) *If* $x \in U$ *and* $|g(x)| \leq \eta$, *then* $\left|\frac{\partial g}{\partial x_i}(x)\right| \geq \eta$.

Then for every $r > 0$,

$$l_d\big(\{x \in U : |g(x)| \leq r\}\big) \leq (2l)^{d-1} C(\eta, \theta, l) r.$$

Proof. Assume without loss of generality that $i = d$. For every $x \in \mathbb{R}^{d-1}$ let $\Delta_x = \{t \in \mathbb{R} : (x,t) \in U\}$. Since U is a convex compact set with $\mathrm{diam}(U) \leq l$ it follows that $\mathrm{diam}(\hat{U}) \leq l$, where $\hat{U} = \{x \in \mathbb{R}^{d-1} : \Delta_x \neq \emptyset\}$. Applying Fubini's Theorem and Lemma 2.1, we then get that

$$\begin{aligned} l_d\big(\{x \in U : |g(x)| \leq r\}\big) &= \int_U 1\!\!1_{g^{-1}([-r,r])}(z) dl_d(z) = \int_{\hat{U}} \int_{\Delta_x} 1\!\!1_{g^{-1}([-r,r])}(x,t) dt dl_{d-1}(x) \\ &= \int_{\hat{U}} l_1\big(\{t \in \Delta_x : |g(x,t)| \leq r\}\big) dl_{d-1}(x) \leq C(\eta,\theta,l) l_{d-1}(\hat{U}) r \\ &\leq (2\mathrm{diam}(\hat{U}))^{d-1} C(\eta,\theta,l) r \leq (2l)^{d-1} C(\eta,\theta,l) r. \end{aligned}$$

We are done. □

Passing to the actual examples, let $f : X \to X$ be a topologically exact open distance expanding map for which there exist closed mutually disjoint sets $X_1, X_2, \dots, X_d$ such that $X = \cup_{i=1}^d X_i$, $f(X_i) = X$ for all $i = 1, 2, \dots, d$ and $f|_{X_i}$ is injective for all $i = 1, 2, \dots, d$. The model that we have in mind here is that of an expanding map $f : I_1 \cup \ldots \cup I_d \to [0,1]$ where $I_1, \dots, I_d$ are closed mutually disjoint subintervals of $[0,1]$, $f(I_j) = [0,1], \forall j$, and $f|_{I_j}$ is injective. Then we will take as the compact space X, the set $I_* = \{x \in I_1 \cup \ldots \cup I_d, f^m(x) \in I_1 \cup \ldots \cup I_d, \forall m \geq 0\}$. So, in this case, $X_i = I_* \cap I_i, i = 1, \dots, d$.

Returning to the general case of the dynamical system $f : X \to X$ as above, consider $\lambda = (\lambda_1, \dots, \lambda_d) \in B_d(0,\eta) \subset \mathbb{R}^d$, for some small enough $\eta > 0$, and fix Lipschitz continuous functions $\phi_1, \dots, \phi_d : X \times [0,1] \times B_d(0,\eta) \to (0,1)$. So $\phi_1, \dots, \phi_d$ are functions of $(x,y,\lambda) \in X^* := X \times [0,1] \times B_d(0,\eta)$. Let us assume also that $\phi_1(x,\cdot,\cdot), \dots, \phi_d(x,\cdot,\cdot)$ are C^2 differentiable functions of (y,λ), with derivatives in (y,λ) depending Lipschitz continuously on (x,y,λ), and that there exist constants $\alpha, \alpha' > 0$ with $0 < \alpha' < |\frac{\partial}{\partial y}\phi_i| < \frac{1}{4}$ on X^*, for all $i = 1, \dots, d$ and $|\frac{\partial}{\partial \lambda_j}\phi_i| < \alpha$ on X^*, for all $i, j = 1, \dots, d$. If $\phi_i \leq \beta$ on X^*, for $i = 1, \dots, d$, then we assume also that $\eta + \beta < 1$. We define now the parametrized maps $F_\lambda : X \times [0,1] \to X \times (0,1)$ by the formula

$$F_\lambda(x,y) = (f(x), \lambda_i + \phi_i(x,y,\lambda)),$$

if $x \in X_i, i = 1, \ldots, d$. Due to the conditions that we imposed on the functions $\phi_1, \ldots, \phi_d$, one can see that F_λ is well defined and it is a skew-product Axiom A fiberwise conformal endomorphism. In this case, $\phi_x^\lambda(y) = \lambda_i + \phi_i(x, y, \lambda)$, for $x \in X_i, i = 1, \ldots, d$. We see that $0 < \alpha' < |(\phi_x^\lambda)'| < \frac{1}{4}, x \in X, \lambda \in B_d(0, \eta)$, so condition (af) from the definition of a transversal family is satisfied automatically. For this family, the set of parameters is $W = B_d(0, \eta) \subset \mathbb{R}^d$.

Theorem 2.3. *The family $\{F_\lambda\}_{\lambda \in B_d(0,\eta)}$ is uniformly transversal, and therefore, the assertions of Theorem 1.8 hold.*

Proof. For every $w \in X$ let $i(w) \in \{1, \ldots, d\}$ be uniquely determined by the property that $w \in X_{i(w)}$. Fix $1 \le k \le d$ and a prehistory $\tilde{w} \in \tilde{X}$. We have that

$$\pi_\lambda(\tilde{w}) = \lim_n \left(\phi_{w_1}^\lambda \circ \ldots \circ \phi_{w_n}^\lambda\right)(\zeta) = \phi_{w_1}^\lambda \circ \ldots \circ \phi_{w_n}^\lambda(\pi_\lambda(\tilde{w}_n)),$$

where $\tilde{w}_n = (w_n, w_{n+1}, \ldots)$. Notice also that the limit above is uniform in ζ. So,

$$\frac{\partial}{\partial \lambda_j}(\phi_{w_1}^\lambda \circ \phi_{w_2}^\lambda(\zeta)) = \frac{\partial}{\partial \lambda_j}(\lambda_{i(w_1)} + \phi_{i(w_1)}(w_1, \lambda_{i(w_2)} + \phi_{i(w_2)}(w_2, \zeta, \lambda), \lambda))$$

For the derivative of $\phi_{w_1}^\lambda \circ \ldots \circ \phi_{w_n}^\lambda$ with respect to λ_j we obtain a similar formula, and then using that $|\frac{\partial}{\partial y}\phi_i| < \frac{1}{4}, i = 1, \ldots, d$, one proves that the map $\lambda \to \pi_\lambda(\tilde{w})$ is differentiable for every $\tilde{w} \in \tilde{X}$, and the derivative is continuous with respect to $\tilde{w}$. Let us assume first that $i(w_n) \neq k, \forall n \ge 1$. We have then $\pi_\lambda(\tilde{w}) = \lambda_{i(w_1)} + \phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda)$. Therefore

$$\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}) = \frac{\partial}{\partial y}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda) \cdot \frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1) + \frac{\partial}{\partial \lambda_k}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda).$$

Hence $|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \le \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + |\frac{\partial}{\partial \lambda_k}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda)|$. Thus by induction we get

$$\begin{aligned}
|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| &\le \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + |\frac{\partial}{\partial \lambda_k}\phi_{i(w_1)}(w_1, \pi_\lambda(\tilde{w}_1), \lambda)| \\
&\le \frac{1}{4}(\frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_2)| + |\frac{\partial}{\partial \lambda_k}\phi_{i(w_2)}(w_2, \pi_\lambda(\tilde{w}_2, \lambda)|) + \alpha \\
&\le \alpha + \frac{1}{4}\alpha + \frac{1}{4^2}\alpha + \ldots \\
&= \alpha \cdot \frac{4}{3}
\end{aligned}$$

Let us consider now the case when there exists $n \ge 1$ with $i(w_n) = k$, and assume that n is chosen as the smallest integer with this property (for $k \ge 1$ fixed). If $i(w_1) = k$, then

$$|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \le 1 + \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + \alpha \le 1 + \frac{1}{4}(1 + \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_2)| + \alpha) + \alpha \le \ldots \le (1 + \alpha) \cdot \frac{4}{3},$$

as one can see by induction, and using the fact that the derivative of the function $\lambda \to \pi_\lambda(\tilde{w})$ is bounded in $\tilde{w} \in \tilde{X}$. In the case when $i(w_1) \neq k$, but there exists $n \ge 2$ with $i(w_n) = k$, we obtain similarly that

$$|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \le \frac{1}{4}|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w}_1)| + \alpha \le \alpha + \frac{1}{4}(1 + \alpha) \cdot \frac{4}{3} = \alpha + \frac{1 + \alpha}{3}.$$

In conclusion, in all cases we get

$$|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})| \leq \frac{4}{3}(1+\alpha)$$

for all $k = 1, \ldots, d$ and all $\tilde{w} \in \tilde{X}$. Consider now $\tilde{x}, \tilde{z} \in p_0^{-1}(x)$, with $x_1 \neq z_1$, and define, for $\lambda \in B_d(0, \eta)$,

$$g(\lambda) := \pi_\lambda(\tilde{z}) - \pi_\lambda(\tilde{x}) = \lambda_{i(z_1)} + \phi_{i(z_1)}(z_1, \pi_\lambda(\tilde{z}_1), \lambda) - \lambda_{i(x_1)} - \phi_{i(x_1)}(x_1, \pi_\lambda(\tilde{x}_1), \lambda)$$

Let us put $k := i(z_1)$ and $j := i(x_1)$. Then using the estimate obtained above, we infer that $|\frac{\partial}{\partial \lambda_k} g(\lambda)| \leq (1+\alpha) \cdot \frac{8}{3}$. On the other hand, from the formula

$$g(\lambda) = \lambda_k + \phi_k(z_1, \pi_\lambda(\tilde{z}_1), \lambda) - \lambda_j - \phi_j(x_1, \pi_\lambda(\tilde{x}_1), \lambda),$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial \lambda_k} g(\lambda) = 1 &+ \frac{\partial}{\partial y}\phi_k(z_1, \pi_\lambda(\tilde{z}_1), \lambda) \cdot \frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{z}_1) + \frac{\partial}{\partial \lambda_k}\phi_k(z_1, \pi_\lambda(\tilde{z}_1), \lambda) - \\ &- \frac{\partial}{\partial y}\phi_j(x_1, \pi_\lambda(\tilde{x}_1), \lambda) \cdot \frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{x}_1) - \frac{\partial}{\partial \lambda_k}\phi_j(x_1, \pi_\lambda(\tilde{x}_1), \lambda).\end{aligned}$$

Hence using the above estimate on the supremum of $|\frac{\partial}{\partial \lambda_k}\pi_\lambda(\tilde{w})|$, we have

$$|\frac{\partial}{\partial \lambda_k} g(\lambda)| \geq 1 - \frac{1}{4} \cdot \frac{4}{3} \cdot (1+\alpha) - \alpha - \frac{1}{4} \cdot \frac{4}{3} \cdot (1+\alpha) - \alpha = 1 - \frac{2}{3}(1+4\alpha)$$

We want $1 > \frac{2}{3}(1+4\alpha)$, so it is enough to take $\alpha < \frac{1}{8}$. Thus we have verified the hypothesis of Lemma 2.2, and the parametrized family $\{F_\lambda\}_{\lambda \in B_d(0,\eta)}$ is uniformly transversal. □

Therefore, we can apply the conclusion of Theorem 1.8 in order to obtain an estimate for the Hausdorff dimension of the fibers $Y_{\lambda,x}$ of F_λ; recall that for this family, $W = B_d(0, \eta)$.

Corollary 2.4. *If $f : I_1 \cup \ldots \cup I_d \to [0,1]$ and $X = I_*$ satisfy the assumptions of Theorem 2.3, and if there exist constants a, b with $0 < a < b < \frac{1}{4}$ such that $a \leq |\frac{\partial}{\partial y}\phi_i(x, y, \lambda)| \leq b$ for all $(x, y, \lambda) \in X \times [0,1] \times B_d(0, \eta)$ and $i = 1, \ldots, d$, then there exists a measurable set $W' \subset W$, with $l_d(W \setminus W') = 0$, such that for all $x \in X, \lambda \in W'$ we have:*

$$\min\left\{1, \frac{\log d}{|\log a|}\right\} \leq \mathrm{HD}(Y_{\lambda,x}) \leq \min\left\{1, \frac{\log d}{|\log b|}\right\}$$

In particular, one obtains:

(a) $\mathrm{HD}(Y_{\lambda,x}) > 0, x \in X, \lambda \in W'$.
(b) *if* $|a| \geq \frac{1}{d}$, *then* $\mathrm{HD}(Y_{\lambda,x}) = 1$, *for all* $x \in X, \lambda \in W'$.

Proof. We notice that, since $\zeta_\lambda(\tilde{x}) = \log|(\phi_x^\lambda)'(\pi_\lambda(\tilde{x}))|$, we get $\log a \leq \zeta_\lambda(\tilde{x}) \leq \log b$, hence

$$h_{top}(\tilde{f}|_{\tilde{X}}) + t\log a \leq P(\tilde{f}, t\zeta_\lambda) \leq h_{top}(\tilde{f}|_{\tilde{X}}) + t\log b$$

Now, let us recall that $h_{top}(\tilde{f}|_{\tilde{X}}) = h_{top}(f|_X)$. Also due to the fact that $f|_X$ is topologically conjugated to $\sigma_d : \Sigma_d^+ \to \Sigma_d^+$, the one-sided shift acting on the full symbol space Σ_d^+ generated by d symbols, we have that $h_{top}(f|_X) = \log d$. Therefore, using Theorem 2.3, we obtain the announced estimates of $\mathrm{HD}(Y_{\lambda,x})$, for all $x \in X$, and $\lambda \in W'$. □

We will study in the sequel two other types of examples related to complex dynamics, which satisfy the uniform transversality condition, and hence Theorem 1.8 can be applied to them. The first such example is the family

$$F_\lambda(z,w) = (f(z), h(z) + \frac{1}{2}w + \lambda z)$$

Here we assume that $(z,w) \in U \times V \subset \mathbb{C} \times \mathbb{C}$, the set $U = \Delta(0,2)$ is the disk of center 0 and radius 2 in $\mathbb{C}$, the set $V \subset \mathbb{C}$ is open, bounded and convex; assume also that the function $f(z)$ is close enough to a map of the form $z \to z^2 + c$, with $|c|$ small, and that $X = J(f)$, is the Julia set of f (hence f can be considered expanding on X). We will take also h to be a complex valued Lipschitz continuous map defined in a neighbourhood of X; then since $|h|$ is bounded on X, we can take the bounded sets V and $W \subset \mathbb{C}$ in such a way that the map $F_\lambda : U \times \overline{V} \to \mathbb{C} \times V$ is well defined for all $\lambda \in W$; for example one can take $W = \Delta(0,1), V = \Delta(0,M)$, where $M > 2(\sup\limits_X |h| + 2)$.

Theorem 2.5. *The parametrized family $\{F_\lambda\}_{\lambda \in W}$, defined above, satisfies the uniform transversality condition.*

Proof. Recall that by our definition, $\pi_\lambda(\tilde{z}) = \lim_{n\to\infty} \phi^\lambda_{z_1} \circ \phi^\lambda_{z_2} \circ \ldots \circ \phi^\lambda_{z_n}(\zeta)$, where in general $\phi^\lambda_z(w) := h(z) + \frac{1}{2}w + \lambda z$. Hence

$$\phi^\lambda_{z_1} \circ \phi^\lambda_{z_2}(\zeta) = h(z_1) + \frac{1}{2}(h(z_2) + \frac{1}{2}\zeta + \lambda z_2) + \lambda z_1 = h(z_1) + \frac{1}{2}h(z_2) + \lambda z_1 + \frac{1}{2}\lambda z_2 + \frac{1}{4}\zeta.$$

It can be shown by induction that

$$\pi_\lambda(\tilde{z}) = [h(z_1) + \frac{1}{2}h(z_2) + \frac{1}{4}h(z_3) + \ldots] + \lambda(z_1 + \frac{1}{2}z_2 + \frac{1}{4}z_3 + \ldots).$$

Put

$$A(\tilde{z}) := h(z_1) + \frac{1}{2}h(z_2) + \frac{1}{4}h(z_3) + \ldots, \quad \text{and} \quad B(\tilde{z}) = z_1 + \frac{1}{2}z_2 + \frac{1}{4}z_3 + \ldots.$$

We shall consider now two prehistories $\tilde{z}, \tilde{z}' \in p_0^{-1}(z)$, with $z_1 \neq z'_1$. Let $g(\lambda) := \pi_\lambda(\tilde{z}) - \pi_\lambda(\tilde{z}') = A(\tilde{z}) + \lambda B(\tilde{z}) - A(\tilde{z}') - \lambda B(\tilde{z}')$. Let us notice now that since f is close to the map $z \to z^2 + c$, we have $J(f)$ close to the circle S^1, if c is small enough, and also it follows that z'_1 is close to $-z_1$; consequently $z'_2 \approx iz_2$ or $z'_2 \approx -iz_2$. This means that $|z'_2 - z_2| \approx \sqrt{2}$. Hence $|z'_2 - z_2 + \frac{1}{2}(z'_3 - z_3) + \ldots| \leq \sqrt{2.2} + \frac{1}{2}(2.1 + \frac{1}{2}2.2 + \ldots) \leq \sqrt{2.2} + 2.2$, where we assumed that f to be so close to $z^2 + c$, and $|c|$ to be so small that $|z'_2 - z_2| < \sqrt{2.2}$ and $X \subset \Delta(0, 1.1)$. Thus

$$|B(\tilde{z}) - B(\tilde{z}')| \geq 1.9 - \frac{1}{2}(\sqrt{2.2} + 2.2) > 0.2,$$

if $\tilde{z}, \tilde{z}' \in \tilde{X}, z = z', z_1 \neq z'_1$. Therefore if $|g(\lambda)| = |A(\tilde{z}) - A(\tilde{z}') + \lambda(B(\tilde{z}) - B(\tilde{z}'))| < r$, then

$$\left|\lambda + \frac{A(\tilde{z}) - A(\tilde{z}')}{B(\tilde{z}) - B(\tilde{z}')}\right| < \frac{r}{|B(\tilde{z}) - B(\tilde{z}')|} < \frac{r}{0.2}$$

whenever $z = z'$ and $z_1 \neq z_1'$. This implies that $\lambda \in B(\frac{A(\tilde{z})-A(\tilde{z}')}{B(\tilde{z})-B(\tilde{z}')}, \frac{r}{0.2})$. Hence

$$l_2(\{\lambda : |g(\lambda)| < r\}) \leq 25\pi r^2$$

for all $r > 0$. Thus we proved that the Uniform Transversality Condition is satisfied for this family. □

Another example, with a more complicated dynamics is presented below. Let us consider $f(z) = z^2 + c$, for $|c|$ small enough; thus f has a Julia set denoted by X, close to the unit circle; then we have that f is expanding on X. Assume also that h is a complex valued Lipschitz continuous function defined on a neighbourhood of X, that $0.4 < |h(z)| < 0.6$, for $z \in X$, and that $|h(z) + h(z')| > \frac{3}{2}$ for $z^2 = -z'^2 - 2c, z \in X$, and $|c|$ small. We take then λ to be a complex parameter with $|\lambda| < \frac{1}{6}$, and consider the parametrized family

$$F_\lambda(z, w) = (f(z), h(z) + \frac{1}{5}w^2 + \lambda z^2)$$

Theorem 2.6. *In the above setting, for any λ from $W := \{\lambda \in \mathbb{C}, |\lambda| < \frac{1}{6}\}$ and $z \in X$, the map $F_\lambda(z, \cdot)$ defined above, invariates the domain $V := \{w \in \mathbb{C}, \frac{1}{30} < |w| < 1\}$, and $\{F_\lambda\}_{\lambda \in W}$ satisfies the Uniform Transversality condition.*

Proof. Without loss of generality we will assume that $c = 0$. Due to the way we defined h and X, we have that $|h(z) + \frac{1}{5}w^2 + \lambda z^2| \leq 0.6 + \frac{1}{5} + \frac{1}{6} < 1$ for $(z, w, \lambda) \in X \times V \times W$. Also, $|h(z) + \frac{1}{5}w^2 + \lambda z^2| \geq 0.4 - \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. Therefore F_λ preserves the domain V. Let us check now the other conditions required for Uniform Transversality. Firstly, $|\frac{\partial}{\partial w}\phi_z^\lambda| = |\frac{2w}{5}| < \frac{2}{5}$, and $|\frac{\partial}{\partial w}\phi_z^\lambda| > \frac{1}{75} > 0$, for all $z \in X, w \in V$, where $\phi_z^\lambda(w) := h(z) + \frac{w^2}{5} + \lambda z^2$. We shall prove by induction that for all $n \geq 1$ there exist functions A_n, B_n and C_n such that for all $\tilde{z} = (z, z_1, z_2, \ldots) \in \tilde{X}$, we have

$$\phi_{z_1}^\lambda \circ \ldots \circ \phi_{z_n}^\lambda(w) = A_n(z_n) + \lambda B_n(z_n, \lambda) + wC_n(z_n, w, \lambda).$$

For $n = 1$, we get $\phi_{z_1}^\lambda = h(z_1) + \lambda z_1^2 + \frac{w^2}{5}$, so $A_1(z) = h(z), B_1(z, \lambda) = z^2, C_1(z, w, \lambda) = \frac{w}{5}$. We want now to calculate the formula for $\phi_{z_1}^\lambda \circ \ldots \phi_{z_{n+1}}^\lambda$ and to get recurrence formulas for A_n, B_n, C_n. From above,

$$\begin{aligned}
&\phi_{z_1}^\lambda \circ \ldots \circ \phi_{z_{n+1}}^\lambda(w) = \\
&\quad = \phi_{z_1}^\lambda(A_n(z_{n+1}) + \lambda B_n(z_{n+1}, \lambda) + wC_n(z_{n+1}, w, \lambda)) \\
&\quad = h(z_1) + \lambda z_1^2 + \frac{1}{5}[A_n(z_{n+1}) + \lambda B_n(z_{n+1}, \lambda) + wC_n(z_{n+1}, w, \lambda)]^2 \\
&\quad = h(z_1) + \lambda z_1^2 + \frac{1}{5}[A_n(z_{n+1})^2 + \lambda^2 B_n(z_{n+1}, \lambda)^2 + w^2 C_n(z_{n+1}, w, \lambda)^2 + \\
&\qquad + 2\lambda A_n(z_{n+1})B_n(z_{n+1}, \lambda) + 2\lambda w B_n(z_{n+1}, \lambda)C_n(z_{n+1}, w, \lambda) + 2A_n(z_{n+1})wC_n(z_{n+1}, w, \lambda)] \\
&\quad = h(z_1) + \frac{1}{5}A_n(z_{n+1})^2 + \lambda[z_1^2 + \frac{2}{5}A_n(z_{n+1})B_n(z_{n+1}, \lambda) + \frac{\lambda}{5}B_n(z_{n+1}, \lambda)^2] + \\
&\qquad + wC_n(z_{n+1}, w, \lambda) \cdot [\frac{2\lambda}{5}B_n(z_{n+1}, \lambda) + \frac{2}{5}A_n(z_{n+1}) + \frac{wC_n(z_{n+1}, w, \lambda)}{5}].
\end{aligned}$$

Thus we obtain the following recurrence formulas, with $\tilde{z} = (z, z_1, \ldots, z_n, \ldots) \in \tilde{X}$:

$$A_{n+1}(z_{n+1}) = h(z_1) + \frac{1}{5}A_n(z_{n+1})^2,$$

$$B_{n+1}(z_{n+1}, \lambda) = z_1^2 + \frac{2}{5}A_n(z_{n+1})B_n(z_{n+1}, \lambda) + \frac{\lambda}{5}B_n(z_{n+1}, \lambda)^2,$$

$$C_{n+1}(z_{n+1}, w, \lambda) = C_n(z_{n+1}, w, \lambda) \cdot (\frac{2\lambda}{5}B_n(z_{n+1}, \lambda) + \frac{2}{5}A_n(z_{n+1}) + \frac{wC_n(z_{n+1}, w, \lambda)}{5})$$

Now we want to prove that $\sup|A_n| < 0.7, \sup|B_n| < 1.5, \sup|C_{n+1}| < \frac{1}{2}\sup|C_n|$, for $z \in X, w \in V, \lambda \in W$. The first two inequlities are satisfied at the level $n = 1$, due to our assumptions on F_λ. If $|A_n| < 0.7$, then $|A_{n+1}| < 0.6 + \frac{1}{5}(0.7)^2 < 0.7$. So we proved the first inequality for all $n \geq 1$. Now, assume that $|B_n| < 1.5$; then $|B_{n+1}| < 1 + \frac{2}{5} \cdot 0.7 \cdot 1.5 + \frac{1}{30}(1.5)^2 = 1 + \frac{21}{50} + \frac{3}{40} < 1.5$. Thus we proved also the inequality $\sup|B_n| < 1.5, \forall n \geq 1$. Last, it is clear that $|C_1| < \frac{1}{5}$ on V. Assume that $|C_n| < \frac{1}{5}$; then $|C_{n+1}| < \frac{1}{5}(\frac{2}{30} \cdot \frac{3}{2} + \frac{2}{5} \cdot 0.7 + \frac{1}{25}) < \frac{1}{10}$, and from the recurrence formula for C_{n+1}, we obtain also $\sup|C_{n+1}| < \frac{1}{2}\sup|C_n|, \forall n \geq 1$. This last inequality tells us that $\sup|C_n| \to 0$ when $n \to \infty$. Consequently, in general, for $\tilde{z} \in \tilde{X}$, we have

$$\pi_\lambda(\tilde{z}) = A(\tilde{z}) + \lambda B(\tilde{z}, \lambda)$$

Let us consider now $\tilde{z}, \tilde{z}' \in p_0^{-1}(z)$ and $g(\lambda) := \pi_\lambda(\tilde{z}) - \pi_\lambda(\tilde{z}') = A(\tilde{z}) - A(\tilde{z}') + \lambda(B(\tilde{z}, \lambda) - B(\tilde{z}', \lambda))$. We have from the recurrence formulas above that $B(\tilde{z}, \lambda) = z_1^2 + \frac{2}{5}A(\tilde{z}_1)B(\tilde{z}_1, \lambda) + \frac{\lambda}{5}B(\tilde{z}_1, \lambda)^2$; also we have $A(\tilde{z}_1) = h(z_2) + \frac{1}{5}A(\tilde{z}_2)^2$. Hence we can deduce

$$\begin{aligned}
\pi_\lambda(\tilde{z}) &= A(\tilde{z}) + \lambda(z_1^2 + \frac{2}{5}A(\tilde{z}_1)B(\tilde{z}_1, \lambda) + \frac{\lambda}{5}B(\tilde{z}_1, \lambda)^2) \\
&= A(\tilde{z}) + \lambda[z_1^2 + \frac{2}{5}(h(z_2) + \frac{1}{5}A(\tilde{z}_2)^2) \cdot (z_2^2 + \frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) + \frac{\lambda}{5}B(\tilde{z}_1, \lambda)^2]
\end{aligned}$$

Similarly we show that

$$\pi_\lambda(\tilde{z}') = A(\tilde{z}') + \lambda[z_1'^2 + \frac{2}{5}(h(z_2') + \frac{1}{5}A(\tilde{z}_2')^2) \cdot (z_2'^2 + \frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2) + \frac{\lambda}{5}B(\tilde{z}_1', \lambda)^2]$$

Therefore, recalling that $z_1^2 = z_1'^2$, we obtain that:

$$\begin{aligned}
g(\lambda) &= A(\tilde{z}) - A(\tilde{z}') + \lambda\{\frac{2}{5}[(h(z_2)z_2^2 - h(z_2')z_2'^2) + h(z_2)(\frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \\
&\quad + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - h(z_2')(\frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2) + \frac{1}{5}A(\tilde{z}_2)^2(z_2^2 + \frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \\
&\quad + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - \frac{1}{5}A(\tilde{z}_2')^2(z_2'^2 + \frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2)] + \frac{\lambda}{5}(B(\tilde{z}_1, \lambda)^2 - B(\tilde{z}_1', \lambda)^2)\} \\
&= A(\tilde{z}) - A(\tilde{z}') + \lambda\{\frac{2}{5}[(h(z_2)z_2^2 - h(z_2')z_2'^2) + D(\tilde{z}, \tilde{z}', \lambda) + E(\tilde{z}, \tilde{z}', \lambda)] + \frac{\lambda}{5}G(\tilde{z}, \tilde{z}')\},
\end{aligned}$$

where

$$D(\tilde{z}, \tilde{z}', \lambda) := h(z_2)(\frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - h(z_2')(\frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2),$$

$$E(\tilde{z}, \tilde{z}', \lambda) := \frac{1}{5}A(\tilde{z}_2)^2(z_2^2 + \frac{2}{5}A(\tilde{z}_2)B(\tilde{z}_2, \lambda) + \frac{\lambda}{5}B(\tilde{z}_2, \lambda)^2) - \frac{1}{5}A(\tilde{z}_2')^2(z_2'^2 + \frac{2}{5}A(\tilde{z}_2')B(\tilde{z}_2', \lambda) + \frac{\lambda}{5}B(\tilde{z}_2', \lambda)^2),$$

and

$$G(\tilde z, \tilde z', \lambda) = B(\tilde z_1, \lambda)^2 - B(\tilde z_1', \lambda)^2$$

But we can estimate $|D(\tilde z, \tilde z', \lambda)|$ as follows:

$$|D(\tilde z, \tilde z', \lambda)| \le 2 \cdot 0.6 \cdot (\frac{2}{5} \cdot 0.7 \cdot 1.5 + \frac{1}{30} \cdot \frac{9}{4}) < 0.6.$$

Also we obtain:

$$\begin{aligned}|E(\tilde z, \tilde z', \lambda)| &= 2 \sup |\frac{1}{5} A(\tilde z_2)^2 (z_2^2 + \frac{2}{5} A(\tilde z_2) B(\tilde z_2, \lambda) + \frac{\lambda}{5} B(\tilde z_2, \lambda)^2)| \\ &\le 2 \cdot \frac{1}{5} \cdot (0.7)^2 (1 + \frac{2}{5} \cdot 0.7 \cdot 1.5 + \frac{1}{5} \cdot \frac{1}{6} \cdot (1.5)^2) < \frac{2}{5}.\end{aligned}$$

Notice that

$$|G(\tilde z, \tilde z', \lambda)| \le 3,$$

for all $\tilde z, \tilde z' \in \tilde X, \lambda \in W$, from the estimate for $|B_n|$. Combining all the above we obtain $|\frac{2}{5}[(h(z_2)z_2^2 - h(z_2')z_2'^2) + D(\tilde z, \tilde z', \lambda) + E(\tilde z, \tilde z', \lambda)] + \frac{\lambda}{5} G(\tilde z, \tilde z', \lambda)| \ge \frac{2}{5}(|h(z_2)z_2^2 - h(z_2')z_2'^2| - 0.6 - \frac{2}{5}) - \frac{1}{5} \cdot \frac{1}{6} \cdot 3 = \frac{2}{5}(|h(z_2)z_2^2 - h(z_2')z_2'^2| - 1) - 0.1$. Thus, since $z_2^2 = z_1 - c$, we will obtain

$$|B(\tilde z, \lambda) - B(\tilde z', \lambda)| \ge \frac{2}{5}(|h(z_2)z_2^2 - h(z_2')z_2'^2| - 1) - 0.1 > \gamma > 0,$$

where $\gamma > 0$ is small enough, when $|c|$ is small enough and $|h(z_2) + h(z_2')| > \frac{6}{4}$ for $z_2^2 = -z_2'^2 - 2c, z_2 \in X$. This means that now we can prove Uniform Transversality for F_λ, as in the previous Theorem. □

Another example of a complex parametrized family with Uniform Transversality is

$$F_\lambda(z, w) = (z^2, z^2 + \lambda_1 z + \lambda_2 z w^2),$$

with $W = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, |\lambda_1| < \frac{1}{50}, \frac{1}{10} < |\lambda_2| < \frac{1}{8}\}, V := \{w \in \mathbb{C}, \frac{1}{2} < |w| < 1.5\}$. Then it can be shown that $F_\lambda(z, \cdot) : \overline{V} \to V$ is well defined for $z \in S^1, \lambda \in W$, and $\exists\ \kappa, \underline{\kappa} \in (0, 1)$ such that $\underline{\kappa} \le |(\phi_z^\lambda)'| \le \kappa$ on V. For this example it can be proved similarly that $\{F_\lambda\}_{\lambda \in W}$ is a parametrized family with Uniform Transversality.

Therefore, for all the examples we have given in this section, the conclusions of Theorem 1.8 apply, and we can write, for almost all parameters λ, the Hausdorff dimension of **all** fibers (thus the stable dimension in our case), by means of the thermodynamic formalism on $\tilde X$.

References

[1] R. Bowen, Equilibrium states and the ergodic theory for Anosov diffeomorphisms. Lect. Notes in Math. 470, Springer, 1975.

[2] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability, Cambridge University Press, 1995.

[3] K. R. Parthasarathy, Introduction to Probability and Measure, Hindustan Book Agency, 2005.

[4] Y. Peres, B. Solomyak, Absolute continuity of Bernoulli convolutions, a simple proof, Math. Res. Lett. 3 (1996), 231-239.

[5] M. Pollicott, K. Simon, The Hausdorff dimension of λ-expansions with deleted digits. Trans. Amer. Math. Soc. 347 (1995), 967-983.

[6] F. Przytycki, M. Urbański, Fractals in the Plane - the Ergodic Theory Methods, available on the web:http://www.math.unt.edu/~urbanski, to appear in Cambridge Univ. Press.

[7] K. Simon, Hausdorff dimension for noninvertible maps, Ergodic Theory Dynam. Sys. 13 (1993), 199-212.

[8] B. Solomyak, On the random series $\sum \lambda n$ (an Erdos problem). Ann. of Math. 142 (1995), 611-625.

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