THERMODYNAMIC FORMALISM AND MULTIFRACTAL ANALYSIS OF FINER JULIA SETS OF EXPONENTIAL FAMILY

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Abstract. The staring point of this paper is a hyperbolic, i.e. having an attracting periodic orbit, exponential map $f_\lambda(z) = \lambda e^z$, $\lambda, z \in \mathbb{C}$. The thermodynamic formalism of 1^+-tame potentials ϕ, that are locally Hölder continuous and lie within a bounded distance from a function $z \mapsto \kappa \text{Re} z$, $\kappa \in \mathbb{R}$-a fixed number, with respect to the maps F_λ, canonical projections of f_λ to the cylinder, is developed. Notably topological pressure, Perron-Frobenius operators, generalized conformal measures, invariant Gibbs states μ_ϕ and variational principle are analyzed in detail. The real analyticity of Perron-Frobenius operators and topological pressure on appropriate parameters is established. The thermodynamic formalism is then applied to demonstrate real analyticity of the multifractal function $k_{\mu_\phi}(\alpha)$ induced by the Gibbs measure μ_ϕ. If in addition this potential is assumed to be harmonic on a half-cylinder containing the Julia sets $J(F_\lambda)$, then the multifractal function, treated now also as a function of the parameter λ, turns out to be real-analytic in both parameters $\lambda \in \mathbb{C}$ and $\alpha \in (0, +\infty)$.

1. **Introduction**

This article has a multifold character. It can be primarily viewed as a continuation and development of [UZ1] and [UZ2] (comp. also [KU], [CS1] and [CS2]). The same hyperbolic exponential maps

$$f_\lambda(z) = \lambda e^z, \ z \in \mathbb{C},$$

are dealt with and the same, projected onto the infinite cylinder, map F_λ is the main object of interest. The development goes into several directions, to mention only more general class of potentials (1^+-tame functions), formulas for the derivatives of topological pressure, variational principle, systematic exposition of thermodynamic formalism, multifractal analysis of Gibbs states of 1^+-tame functions and multifractal analysis of Gibbs states of a 1^+-harmonic potential, where the dependence on the parameter λ is the most challenging issue. This article shares occasionally some methods and overlaps some topics with [UZ1], [UZ2], [UZ3], [PU] and [Ur]. We include them here for the sake of clarity and completeness, notably since, as was actually indicated above, one of the goals of this article is to provide a compact and possibly complete account of thermodynamic formalism and multifractal analysis of hyperbolic exponential maps. We place below the content of this article, which is followed by a short summary of each section.

Contents:

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In Section 2, Hyperbolic maps - Preliminaries, we define the basic object (the projected map F_λ) of our interest in this paper and we establish its first dyamical properties.

In Section 3, Pressure, Perron-Frobenius Operators and Generalized Conformal Measures, we introduce the second basic object we will be dealing with, the class of tame potentials, especially 1^+-tame potentials, that are locally Hölder continuous and lie within a bounded distance from a function $z \mapsto \kappa \text{Re} z$, $\kappa \in I\!R$ a fixed number. We introduce in this section all the objectcs constituing, appropriate in this context, thermodynamic formalism. We mean hear (pointwise) topological pressure, associated Perron-Frobenius operators, and conformal (Gibbs) measures. We also recall from [UZ1] the definition of the refined Julia set $J_r(F)$. The existence, uniqueness and conservativity of conformal measures is proven. Some introductory properties of Perron-Frobenius operators are established.

The short Section 4, Perron-Frobenius Operators and Their Finer Properties, is devoted to establishspectral (quasi-compactness) properties of Perron-Frobenius operators based on the Ionescu-Tulcea and Marinescu Theorem.

Section 5, Invariant Measures, is also short and entirely follows from the previous one. It contains the proof of the existence and uniqueness of an invariant measures (invariant Gibbs states) μ_ϕ equivalent to the conformal measures m_ϕ. The Radon-Nikodym derivative $d\mu_\phi/dm_\phi$ is a nowhere vanishing locally Hölder continuous function. Some stochastic properties of the dynamical system $F : P \to P$ with respect to the measure μ_ϕ are recorded; notably exponential decay of correlations and the Central Limit Theorem.

In Section 6, Variational Principle, the variational principle for topological pressure is proven in the form nearly coinciding with the classical one. The Gibbs measures μ_ϕ are shown to be equilibrium states.

In Section 7, Analyticity of the Pressure Function, the exponential maps is still kept fixed but the potential is let to depent linearly on a real parameter. The corresponding Perron-Frobenius operator is canonically complexified and is demonstrated to depend holomorphically on the (complex) parameter. This central (in this section) technical fact is a source of a number of interesting consequence. Among them real analyticity of topological pressure and other objects like eigenfunctions and contracting "remainders" produced in the process of developing the thermodynamic formalism. A uniform version of exponential decay of correlations finishes the section.

Section 8, Derivatives of the Pressure Function, motivated by the appropriate parts of [PU] establishes formulas for the first and second derivatives of topological pressure. Even in the classical cases of distance expending or subshift of finite type cases, this is not an easy task. In our present context, the calculations, especially of the second derivative, are tedious indeed. One of the sources of technical difficulties is the fact that 1^+-tame functions are unbounded, and therefore, do not belong to the Banch space of bounded locally Hoölder continuous functions.

In the Section 9, Cohomologies, any two 1^+-tame functions are characterized to induce the same invariant Gibbs states if and only if they are cohomologous modulo constant. Working with Rokhlin's natural extention of the map $F : J(F) \to J(F)$ and employing the machinery of conditional measures associated to uncountable measurable partitions, it is proved that a tame function is cohomologous to constant in the class of bounded (locally) Hölder continuous functions on $J(F)$ if and only if its asymptotic variance with respect to one (equivalently all) Gibbs state vanishes.

In the Section 10, Auxiliary properties of the pressure function, the first results directly related to multifractal analysis are proven. The asymptotic behavior of the pressure function is analyzed, the potentials $\phi_{q,t} = -t\log|F'| + q\phi$ (ϕ is a 1^+-tame function) are introduced and the existence of a unique parameter $t = T(q) \in I\!R$ with the property that $\mathrm{P}(\phi_{t,q}) = 0$ is demonstrated.

In Section 11, Hausdorff Dimension of Gibbs States, the local dimension (and consequently the Hausdorff dimension as well) of an invariant Gibbs state is identified as the quotient of entropy and Lyapunov exponent.

In Section 12, Multifractal Analysis of 1^+-Tame Functions, we apply the developed machinery of thermodynamic formalism to undertake the multifractal analysis of the invariant Gibbs measure of a 1^+-tame potential ϕ. The multifractal spectrum of this Gibbs measure μ_ϕ is defined over the set $J_{rr}(F)$ exhibiting some regularity features of consequtive returns to a compact part of $J(F)$. This set is dynamically significant in the sense that each invariant ergodic measure charges it with full mass. The multifractal function turns out then to be real-analytic and to form together with the function $T(q)$ a Legendre transform provided that the functions ϕ and $-\mathrm{HD}(J(F))\log|F'|$ are not cohomologous modulo constant.

Section 13, Quasiconformal Conjugacies, is the first section where the parameter λ is not kept any longer fixed. The local uniform expanding property is established and quasiconformal conjugacies between nearby hyperbolic exponential functions are investigated in detail. Notable, the boundedness of the holomorphic motion generating by these conjugacies is proven in a simpler way than in [UZ2].

Section 14, Finer Analytic Properties of Perron-Frobenius Operators and Topological Pressure, forms a preparatory step for the next section, where the multifractal function is considered as depending also on the parameter λ. Here, two harmonic (on a half-cylinder) tame functions give rise to a 3-parameter ($q, t \in I\!\!R$ and $\lambda \in \mathbb{C}$) family of perron-Frobenius operators. Them main, and a rather technical, result of this section is that this family of operators has a holomorphic extension to a polydisk in $\mathbb{C}^4$. Using then the perturbation theory for linear operators, the real analytic dependence of the topological pressure on the three parameters $q, t \in I\!\!R$ and $\lambda \in \mathbb{C}$ is concluded.

In Section 15, Refined Multifractal Analysis, which is the last section of this paper, a harmonic 1^+-tame potential ϕ is fixed and the Gibbs measures μ_ϕ^λ of the potential ϕ with respect to the dynamical systems $F_\lambda : J(F_\lambda) \to J(F_\lambda)$ are considered. The multifractal function $k_\phi(\lambda, \alpha) = k_{\mu_\phi^\lambda}(\alpha)$ depending on two parameters $\lambda \in \mathbb{C}$ and $\alpha \in (0, +\infty)$ is proven to be real-analytic. This result, actually culminating the whole paper, can be in some sense considered as a refinement of the main result from [UZ2] saying that the Hausdorff dimension of $J_r(F_\lambda)$ is a real-analytic function of the parameter λ.

2. **Hyperbolic maps - Preliminaries**

For every $\lambda \in \mathbb{C} \setminus \{0\}$ let

$$f_\lambda(z) = \lambda e^z.$$

The Fatou set of f_λ consists of those points $z \in \mathbb{C}$ that admit an open neighbourhood U_z such that the family $\{f_\lambda^n|_U\}_{n=1}^\infty$ of iterates of f, restricted to U is normal. The Julia set $J_\lambda(f)$ is defined to be the complement of the Fatou set. We denote by

$$\mathrm{Hyp}$$

the set of all parameters λ such that f_λ has a periodic attracting orbit. Each such parameter λ and the corrsponding map f_λ are called hyperbolic. Fix $\lambda \in \mathrm{Hyp}$. Since the map f_λ is $2\pi i$-periodic, we consider it rather on the cylinder than on $\mathbb{C}$. So, let P be the quotient space (the cylinder),

$$P = \mathbb{C}/\sim,$$

where $z_1 \sim z_2$ if and only if $z_1 - z_2 = 2k\pi i$ for some $k \in \mathbb{Z}$. Let $\pi : \mathbb{C} \to P$ be the natural projection. Since the map $\pi \circ f_\lambda : \mathbb{C} \to P$ is constant on equivalence classes of relation $\sim$, it canonically induces a conformal map

$$F_\lambda : P \to P.$$

The map $F_\lambda : P \to P$ will be the main object of our considerations. We will frequantly skip the parameter λ and write f for f_λ and F for F_λ. The Julia set of F is defined to be

$$J(F) = \pi(J(f)). \tag{2.1}$$

and

$$F(J(F)) = J(F) = F^{-1}(J(F)).$$

The cylinder P is canonically endowed with a Euclidean metric which without confusion will be denoted by the same symbol $|w - z|$ for all $z, w \in P$. Given $M \in I\!R$, we put

$$P_M = \{z \in P : \mathrm{Re}z \le M\} \text{ and } P_+^c = \{z \in P : \mathrm{Re}z \ge \inf\{\mathrm{Re}(J(F)\}\}.$$

In [UZ2] we have checked that f is expanding on its Julia set. Precisely.

Proposition 2.1. *If $\lambda \in \mathrm{Hyp}$, then there exist $c > 0$ and $\xi > 1$ such that for all $z \in J(f_\lambda)$,*

$$|(f_\lambda^n)'(z)| > c\xi^n.$$

We skip its proof at this moment since we will prove a stronger statement in Section 13 as Lemma 13.2. It follows from this proposition that there exists $q \ge 1$ such that

$$|(F^n)'(z)| \ge 2 \tag{2.2}$$

for all $n \ge q$ and all $z \in J(F)$. Since $\inf(\mathrm{Re}(J(F)) > -\infty$, we see that there exists $\rho > 0$ such that

$$|F'(z)| \ge \rho \tag{2.3}$$

for all $z \in J(F)$.

We will need more technical facts.

Proposition 2.2. *There exists a sequence $\{w_n\}_{n=1}^\infty \subset J(F)$ such that $\lim_{n\to\infty} w_n = +\infty$ and $F(w_n) = w_n$ for all $n \ge 1$.*

Proof. Since the function $g(z) = \lambda e^z - z$ has an essential singularity at infinity, it follows from Paicard's theorem that for every $n \ge 1$, the set $g(\{z \in \mathbb{C} : |z| \ge n\})$ omits at most two points in $\mathbb{C}$. In particular, there exists $z_n \in \mathbb{C}$ with $|z| \ge n$ and $q_n \in \{0, 1, 2\}$ such that $g(z_n) = 2\pi i q_n$. Suppose now that $z_n - z_m \in 2\pi i \boldsymbol{Z}$ for some $m, n \ge 1$. then

$$z_n = \lambda e^{z_n} - g(z_n) = \lambda e^{z_n} - 2\pi i q_n = \lambda e^{z_m} - 2\pi i q_m + 2\pi i(q_m - q_n) = z_m + 2\pi i(q_m - q_n).$$

Since $|q_m - q_n| \le 2$ and since $\lim_{k\to\infty}|z_k| = +\infty$, passing to a subsequence, we may therefore assume that $z_n - z_m \notin 2\pi i\boldsymbol{Z}$ for all $m, n \ge 1$. This equivalently means that all the points $\pi(z_n) \in P$, $n \ge 1$, are mutually distinct. Now

$$F(\pi(z_n)) = \pi(f_\lambda(z_n)) = \pi(z_n + g(z_n)) = \pi(z_n + 2\pi i q_n) = \pi(z_n), \tag{2.4}$$

So, all the points $\pi(z_n)$, $n \ge 1$, are fixed points of F. In particular, the sequence $\{\pi(z_n)\}_{n=1}^\infty$ cannot have any accumulation point in P since othrwise $F = \mathrm{Id}$. Therefore, passing yet to another sequence, we may assume that either $\lim_{n\to\infty}\pi(z_n) = -\infty$ or $\lim_{n\to\infty}\pi(z_n) = +\infty$. Let us rule out the former possibility. Indeed, if $\lim_{n\to\infty}\pi(z_n) = -\infty$, then $\lim_{n\to\infty}\mathrm{Re}(z_n) = -\infty$, and therefore $|z_n| = |\lambda e^{z_n} - 2\pi i q_n| \le |\lambda||e^{z_n}| + 2\pi|q_n| \le |\lambda|e^{\mathrm{Re}(z_n)} + 4\pi$ is uniformly bounded. This contradiction shows that

$$\lim_{n\to\infty}\pi(z_n) = +\infty. \tag{2.5}$$

Since

$$|F'(\pi(z_n))| = |f'(z_n)| = |f(z_n)| = |g(z_n) + z_n| = |z_n + 2\pi i q_n| \ge |z_n| - 4\pi \ge n - 4\pi,$$

we see that $|F'(\pi(z_n))| \ge 2$ for all $n \ge 18$. Hence $\pi(z_n)$ is a repelling fixed point of F for every $n \ge 18$ and it therefore belongs to $J(F)$. This along with (2.5) completes the proof. ■

Lemma 2.3. *For every compact set $L \subset J(F)$ and every $r > 0$ there exists $q = q(L, r) \ge 0$ such that for every $z \in J(F)$ there exists $p \in \{0, 1, 2, \dots, q\}$ such that $F^p(B(z, r)) \supset L$.*

Proof. First observe that if $A \subset \mathbb{C}$ is an open rectangle with the following properties

(a) the edges of A are parallel to the coordinate axis,
(b) $A \cap J(F) \ne \emptyset$,
(c) the (vertical) height of A is $\ge 4\pi$.
(d) the (horizontal) width of S is large enough, say $\ge \Delta$ for some $\Delta > 0$, so that $\pi(f(A)) \supset L$,then $F(\pi(A)) = \pi(f(A)) \supset L$.

Since $|f'(z)| = |f(z)| = |\lambda|e^{\mathrm{Re}(z)}$ and since the map f restricted to any ball of radius 2 is 1-to-1, using Koebe's distortion theorem, we conclude that for all $z \in \mathbb{C}$ with $\mathrm{Re} z$ large enough, say $\mathrm{Re} z \ge M$, we have that $f\Big(B(z, \min\{r, \delta\})\Big)$ contains a rectangle A satisfying all the properties (a)-(d). Consequently

$$\begin{aligned} F^2\Big(B(\pi(z), \min\{r, \delta\})\Big) &= F^2\Big(\pi(B(\pi(z), \min\{r, \delta\})\Big) = \pi \circ f^2\Big(B(z, \min\{r, \delta\})\Big) \\ &\supset \pi(f(A)) = F(\pi(A)) \supset L. \end{aligned}$$

It follows from Theorem 5.2 in [UZ1] (and the remark that the measure μ produced there has full support in $J(F)$) that the set of recurrent points is dense in $J(F)$, and it therefore follows from the closing lemma (Proposition 2.5) that the set of periodic points of F is dense in $J(F)$. Thus, for every $z \in J(F) \cap P_M$ and every $R > 0$ there exists a periodic point $w \in B(z, R/2)$, say of period $s \ge 1$. Since the point w is repelling, there exists $\eta \in (0, \mathbb{R}/2)$ such that $F^s(w, \eta)) \supset B(w, \eta)$. Since $w \in J(F)$, we have that $\bigcup_{k=0}^\infty F^{sk}(B(w, \eta)) \supset J(F)$. But all the sets $F^{sk}(B(w, \eta))$ are open in the cylinder P and, as L is compact, there exists

$k \geq 0$ such that $F^{sk}(B(w,\eta)) = \bigcup_{j=0}^{k} F^{sj}(B(w,\eta)) \supset L$. In particular, $F^{sk}(B(z,R)) \supset L$ and denote by $p(z,R) \geq 0$ the least integer $j \geq 0$ such that $F^j(B(z,R)) \supset L$. We claim that

$$q(R) := \sup\{p(z,R) : J(F) \cap P_M\} < \infty.$$

Indeed, suppose on the contrary that there exists a sequence $\{z_n\}_{n=1}^\infty \subset J(F) \cap P_M$ such that $p(z_n,R) \geq n$ for all $n \geq 1$. Since $J(F) \cap P_M$ is a compact set, passing to a subsequence, we may assume that the sequence $\{z_n\}_{n=1}^\infty$ converges to some point $z_\infty \in J(F) \cap P_M$. Since for $n \geq 1$ large enough $B(z_\infty, R/2) \subset B(z_n,R)$, we see that $p(z_\infty, R/2) \geq p(z_n,R) \geq n$, and consequently $p(z_\infty, R/2) = \infty$. This contradiction shows that $q(R) < \infty$. Setting therefore $q = \max\{2, q(R)\}$ we complete the proof. ■

Following the long tradition in the theory of hyperbolic systems, given $\alpha > 0$ a sequence $\{x_n\}_{n=0}^\infty \subset J(F)$ is called and α-pseudoorbit of F if $|F(x_n) - x_{n+1}| \leq \alpha$ for all $n \geq 0$. A sequence $\{x_n\}_{n=0}^\infty \subset J(F)$ is said to be α-shadowed by a point $x \in J(F)$ if $|x_n - F^n(x)| \leq \alpha$ for all $n \geq 0$. We then also say that the point x α-shadows the sequence $\{x_n\}_{n=0}^\infty$. Since $F : J(F) \to J(F)$ is a continuous expanding map (see (2.2) and using (2.3), the following two facts can be proved in exactly the same way (one only needs to take an additional care of the fact that an iterate of F and not F itself is distance expanding) as their counterparts in [PU].

Proposition 2.4. *(Shadowing Lemma) For every $\beta \in (0,1]$ there exists $\alpha > 0$ such that if $\{x_n\}_{n=0}^\infty \subset J(F)$ is an α-pseudoorbit, then there is a unique point $x \in J(F)$ that β-shadows the pseudoorbit $\{x_n\}_{n=0}^\infty$. More precisely, if $\{x_n\}_{n=0}^\infty \subset J(F)$ is an α-pseudoorbit, then there is a unique point $x \in J(F)$ that $\rho^{1-q}\alpha$-shadows the pseudoorbit $\{x_n\}_{n=0}^\infty$.*

Proposition 2.5. *(Closing Lemma) For every $\beta \in (0,1]$ there exists $\alpha > 0$ (the same as in Proposition 2.4) such that if $n \geq 1$ is an integer and $\{x_j\}_{j=0}^{n-1} \subset J(F)$ is such that $|x_{j+1} - F(x_j)| < \alpha$ for all $j = 0, 1, \ldots n-1$ ($x_n = x_0$), then there exist $y \in J(F)$ such that $F^n(y) = y$ and $|F^j(y) - x_j| < \beta$ for all $j = 0, 1, \ldots n-1$.*

3. Pressure, Perron-Frobenius Operators and Generalized Conformal Measures

As we promised, we keep in this section $\lambda \in \mathrm{Hyp}$. We start our investigations by dealing with the topological pressure of a large class of potentials. Let

$$\delta = \frac{1}{2}\min\left\{\frac{1}{2}, \mathrm{dist}\Big(J(F), \{F^n(\pi(0)) : n \geq 0\}\Big)\right\} \tag{3.1}$$

We know that $\delta > 0$. For every $n \geq 1$ and every $v \in J(F)$ the map

$$F_v^{-n} : B(F^n(v), 2\delta) \to P$$

is defined to be the holomorphic inverse branches of F^n defined on $B(F^n(v), 2\delta)$ and sending $F^n(v)$ to v. It follows from Proposition 2.1 and Koebe's distortion theorem that there exist constants $L > 0$ and $0 < \beta < 1$ such that for every $n \geq 0$, every $v \in J(F)$ and every $z \in B(F^n(v), \delta)$, we have

$$|(F_v^{-n})'(z)| \leq L\beta^n \tag{3.2}$$

Fix $\alpha \in (0,1]$. Given $g : J(F) \to \mathbb{C}$, let

$$v_\alpha(g) = \inf\{L \geq 0 : |g(y) - g(x)| \leq L|y-x|^\alpha \text{ for all } x, y \in J(F) \text{ with } |y-x| \leq \delta\},$$

be the α-variation of the function g, where $\delta > 0$ was defined in formula (3.1). Any function with bounded α-variation will be called α-Hölder or simply Hölder continuous if we do not want to specify the exponent of Hölder continuity. Let $C_b = C_b(J(F))$ be the space of all bounded continuous complex valued functions defined on $J(F)$ and let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty.$$

Clearly the space

$$\mathrm{H}_\alpha = \mathrm{H}_\alpha(J(F)) = \{g \in J(F) : ||g||_\alpha < \infty\}$$

endowed with the norm $||\cdot||_\alpha$ is a Banach space densely contained in C_b with respect to the $||\cdot||_\infty$ norm. Any member of the space H_α will be called a bounded Hölder continuous function with exponent α and any member of the space $\bigcup_{\alpha>0} \mathrm{H}_\alpha$ will be simply called a bounded Hölder continuous function. Given $n \geq 0$ let

$$S_n\phi(z) = \phi(z) + \phi(F(z)) + \cdots + \phi(F^{n-1}(z)).$$

Using (3.2) it is straightforward to prove the following.

Lemma 3.1. *For every $\alpha > 0$ there exists $L_\alpha > 0$ such that if $\phi : J(F) \to \mathbb{C}$ is an α-Hölder function, then*

$$|S_n\phi(F_v^{-n}(y)) - S_n\phi(F_v^{-n}(x))| \leq L_\alpha v_\alpha(\phi)|y-x|^\alpha$$

for all $n \geq 1$, all $x, y \in J(F)$ with $|x-y| \leq \delta$ and all $v \in F^{-n}(x)$.

We recall from [UZ2] that a continuous function $\phi : J(F) \to \mathbb{C}$ is dynamically Hölder with an exponent $\alpha > 0$ if there exists $c_\phi > 0$ such that

$$|\exp\Big(S_n\phi(F_v^{-n}(y))\Big) - \exp\Big(S_n\phi(F_v^{-n}(x))\Big)| \leq c_\phi \left|\exp\Big(S_n\phi(F_v^{-n}(x))\Big)\right| ||y-x|^\alpha \tag{3.3}$$

for all $n \geq 1$, all $x, y \in J(F)$ with $|x-y| \leq \delta$ and all $v \in F^{-n}(x)$. Actually we called in [UZ2] the function $\exp(\phi)$ dynamically Hölder but in this paper we prefer the above definition. We shall prove the following.

Lemma 3.2. *Every α-Hölder continuous function $\phi : J(F) \to \mathbb{C}$ is dynamically Hölder with the same exponent α. Moreover, $c_\phi \leq C_\alpha(v_\alpha(\phi))$, where $C_\alpha : [0,\infty) \to [0,\infty)$ is a non-decreasing function.*

Proof. Looking at the Taylor series expansion of the function $z \mapsto e^z$ about 0 and using Lemma 3.1, we see that there exists a required function C_α such that

$$|\exp\Big(S_n\phi(F_v^{-n}(y)) - S_n\phi(F_v^{-n}(x))\Big) - 1| \le C_\alpha(v_\alpha(\phi))|y-x|^\alpha.$$

But this means that

$$|\exp(S_n\phi(F_v^{-n}(y)) - \exp(S_n\phi(F_v^{-n}(x))| \le C_\alpha(v_\alpha(\phi))\exp\Big(S_n\phi(F_v^{-n}(x))\Big)||y-x|^\alpha$$

and we are done. ■

Given $\kappa \in I\!R$ a Hölder conditionuous function $\phi : P_+^c \to I\!R$ is called κ-tame if

$$A_\phi = \sup\{|\phi(z) + \kappa \mathrm{Re} z| : z \in P_+^c\} < +\infty.$$

A Hölder conditionuous function $\phi : P_+^c \to I\!R$ is called κ^+-tame if it is a-tame with some $a > \kappa$. Especially important will turn out to be the 1^+-tame functions. Any κ-tame function with any $\kappa \in I\!R$ is called a tame function. The above introduced tameness notions refer to any Hölder conditionuous complex-valued function if its real part satisfies the respective conditions.

Taking $n = 1$ in Lemma 3.2 and noting that each 0^+-tame function is uniformly bounded from above, we immediately get the following.

Lemma 3.3. *If $\phi : J(f) \to C\!\!\!\!I$ is a 0^+-tame or 0-tame function, then e^ϕ is a bounded Hölder continuous function with the same exponent.*

For every every $z \in J(F)$ define the lower and upper topological pressure of a tame potential ϕ respectively by

$$\underline{\mathrm{P}}_z(\phi) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x\in F^{-n}(z)} \exp(S_n\phi(x)) \text{ and } \overline{\mathrm{P}}_z(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in F^{-n}(z)} \exp(S_n\phi(x)).$$

Let us prove the following.

Lemma 3.4. *For all $w, z \in J(F)$ we have*

$$\underline{\mathrm{P}}_w(\phi) = \underline{\mathrm{P}}_z(\phi) \text{ and } \overline{\mathrm{P}}_w(t) = \overline{\mathrm{P}}_z(t).$$

Proof. It follows from Lemma 3.1 that

$$\exp\Big(S_n\phi(F_v^{-n}(y))\Big) \le \exp(L_\phi\delta^\alpha)\exp\Big(S_n\phi(F_v^{-n}(x))\Big)$$

for all $n \ge 1$, all $x, y \in J(F)$ with $|x-y| \le \delta$ and all $v \in F^{-n}(x)$. Thus, $\underline{\mathrm{P}}_y(\phi) \le \underline{\mathrm{P}}_x(\phi)$ and $\overline{\mathrm{P}}_y(\phi) \le \overline{\mathrm{P}}_x(\phi)$. But the symmetry of the argument, we also get the opposite inequalities and we are done as long as $|x-y| \le \delta$. Since any two points $w, z \in J(F)$ can be joined by a finite chain say $w = \xi_0, \xi_1, \dots, \xi_{k-1}, \xi_k = z$ of points in $J(F)$ such that $|\xi_i - \xi_{i+1}| \le \delta$ for all $i = 0, 1, \dots, k-1$, the proof is therefore finished. ■

We can therefore speak about the respective values of $\underline{\mathrm{P}}(\phi)$ and $\overline{\mathrm{P}}(\phi)$. Let

$$\mathrm{P}_z(1,\phi) := \sum_{x\in F^{-1}(z)} \exp(\phi(x)) = \sum_{k=-\infty}^{+\infty} |\exp(\phi(z_k)), \tag{3.4}$$

where z_k is the only point of the singleton $P \cap f^{-1}(\tilde{z}+2\pi ik)$ and $\tilde{z}$ is an arbitrary point in $\pi^{-1}(z)$.

Assume from now on, unless otherwise stated, that $\phi : J(F) \to I\!R$ is a 1^+-tame potential, say ϕ is κ-tame with some $\kappa > 1$. Then

$$\mathrm{P}_z(1,\phi) = \sum_{k=-\infty}^{+\infty} |\exp(\phi(z_k)) \le \sum_{k=-\infty}^{+\infty} \exp\Big(A_\phi - \kappa\mathrm{Re}(z_k)\Big) = e^{A_\phi}|\lambda|^{-\kappa} \sum_{k=-\infty}^{+\infty} |\tilde{z}+2\pi ik|^{-\kappa}. \tag{3.5}$$

Thus,

$$||\mathrm{P}(1,\phi)||_\infty = \sup_{z\in J(F)} \{\mathrm{P}_z(1,\phi)\} < \infty. \tag{3.6}$$

Observe now that for every $n \ge 1$ and every $z \in J(F)$

$$\begin{aligned}\sum_{x\in F^{-n}(z)} \exp\Big(S_n\phi(x)\Big) &= \sum_{w\in F^{-(n-1)}(z)} \sum_{x\in F^{-1}(w)} \exp\Big(S_{n-1}\phi(w)\Big)\exp(\phi(x)) \\ &= \sum_{w\in F^{-(n-1)}(z)} \exp\Big(S_{n-1}\phi(w)\Big) \sum_{x\in F^{-1}(w)} \exp(\phi(x)) \\ &\le ||\mathrm{P}(1,\phi)||_\infty \sum_{w\in F^{-(n-1)}(z)} \exp\Big(S_{n-1}\phi(w)\Big).\end{aligned} \tag{3.7}$$

Therefore, we obtain by induction that $\sum_{x\in F^{-n}(z)} \exp\Big(S_n\phi(x)\Big) \le ||\mathrm{P}(1,\phi)||_\infty^n$ and consequently

$$\overline{\mathrm{P}}(\phi) = \overline{\mathrm{P}}_z(\phi) \le \log||\mathrm{P}(1,\phi)||_\infty. \tag{3.8}$$

A Borel measure m_ϕ is called (ϕ,α)-conformal, $\alpha \ge 0$, if for any Borel set $A \subset P$ on which F is injective, we have

$$m_\phi(F(A)) = \int_A \alpha\exp(\mathrm{P}(-\phi))dm_\phi$$

Since the proof of the existence of conformal measures was technically and conceptually considerably easier to carry out working with a strip S rather then the cylinder P and since the strip $S \subset C\!\!\!\!I$ was needed exclusively for the proof of the existence of conformal measures, we placed it in the appendix. In addition in the special case of potentials of the form $A+\kappa\mathrm{Re}$ it was carried out in [UZ2]. As an immediate consequence of Theorem 16.3 from this appendix, we get the following.

Theorem 3.5. *There exist $\alpha_\phi \ge 0$ and a (ϕ, α_ϕ)-conformal measure m_ϕ for the map $F : J(F) \to J(F)$ determined by the condition that $m_\phi(\pi(A)) = m(A)$ for every Borel set A such that $\pi|_A$ is 1-to-1, where m is the measure produced in Theorem 16.3. In addition $m_\phi(J(F)) = 1$.*

Recall that $C_b = C_b(J(F))$ denotes the Banach space of all bounded continuous complex-valued functions on $J(F)$. The formulas (3.5) and (3.6) enable us to define the Perron-Frobenius operator $\mathcal{L}_\phi : C_b \to C_b$ by the formula

$$\mathcal{L}_\phi g(z) = \sum_{x \in F^{-1}(z)} \exp(\phi(x))g(x) = \sum_{k=-\infty}^{+\infty} \exp(\phi(z_k))g(z_k), \tag{3.9}$$

where, we recall from (3.4), that z_k is the only point of the singleton $\pi\Big(f^{-1}(\tilde{z} + 2\pi ik)\Big)$ and $\tilde{z}$ is an arbitrary point from $\pi^{-1}(z)$ and. It immediately follows from (3.9) and from (3.5) that

$$|\mathcal{L}_\phi g(z)| \le ||\mathcal{L}_\phi 1\!\!1||_\infty ||g||_\infty \text{ and } \lim_{\mathrm{Re}z \to +\infty} \mathcal{L}_\phi 1\!\!1(z) = 0. \tag{3.10}$$

Notice also that $\mathcal{L}_\phi : C_b \to C_b$ is a bounded operator and its norm is equal to $||\mathrm{P}(1,t)||_\infty$. Consider the dual operator $\mathcal{L}_\phi^* : C_b^* \to C_b^*$ given by the formula $\mathcal{L}_\phi^*\mu(g) = \mu(\mathcal{L}_\phi g)$. A straighforward calculation (see Proposition 2.2 in [DU1] for example, where the finiteness of the partition can be replaced by its countability) shows the following.

Proposition 3.6. *If ϕ is 1^+-tame, then $\mathcal{L}_\phi^* m_\phi = \alpha_\phi m_\phi$.*

Define now

$$\hat{\mathcal{L}}_\phi = \alpha_\phi^{-1}\mathcal{L}.$$

For every $x \in I\!\!R$ let

$$P_x = \{z \in J(F) : \mathrm{Re}(z) \le x\}.$$

Notice that there exist a positive integer q_x and a positive real $r \le \delta$ such that any two points $w, z \in P_x$ can be joined by a ξ-chain of points $w = \xi_0, \xi_1, \dots, \xi_{k-1}, \xi_{q_x} = z$ in P_x such that the union

$$U_{w,z} := B(\xi_0, 2r) \cup B(\xi_1, 2r) \cup \ldots \cup B(\xi_{q_x}, 2r)$$

is simply conttected. It therefore follows from Lemma 3.1 that there exists a constant $K_{\phi,x} \ge 1$ such that if $F_*^{-n} : U_{w,z} \to P$ is a holomorphic branch of F^{-n}, then

$$\frac{\exp\Big(S_n\phi\Big(F_*^{-n}(w)\Big)\Big)}{\exp\Big(S_n\phi\Big(F_*^{-n}(z)\Big)\Big)} \le K_{\phi,x}.$$

Consequently

$$K_{\phi,x} \le \frac{\hat{\mathcal{L}}_\phi^n(1\!\!1)(w)}{\hat{\mathcal{L}}_\phi^n(1\!\!1)(z)} \le K_{\phi,x}. \tag{3.11}$$

Following closely the appropriate argument from [UZ2], we shall prove the following.

Lemma 3.7. $Q_\phi := \sup_n\{||\hat{\mathcal{L}}_\phi^n(1\!\!1)||_\infty\} < \infty$.

Proof. Fix $x \geq 0$ so large that for every $w \in P_x^c$

$$\alpha_\phi^{-1}\sum_{k=-\infty}^{+\infty}\exp(\phi(w_k)) \leq 1, \tag{3.12}$$

We shall prove by induction that for every $n \geq 0$,

$$||\hat{\mathcal{L}}_\phi^n(1\!\!1)||_\infty \leq \frac{K_{\phi,x}}{m_\phi(P_x)}.$$

And indeed, for $n = 0$ this estimate is immediate. So, suppose that it holds for some $n \geq 0$ and let $\hat{z}_{n+1} \in P$ be such a point that $\hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)(\hat{z}_{n+1}) = ||\hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)||_\infty$ (such a point exists due to (3.10)). If $\hat{z}_{n+1} \in P_x$, then using (3.11) we obtain

$$1 = \int \hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)dm \geq \int_{P_x}\hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)dm \geq K_{\phi,x}^{-t}||\hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)||_\infty m(P_x)$$

and consequently $||\hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)||_\infty \leq K_{\phi,x}^t\Big(m(P_x)\Big)^{-1}$. If $\hat{z}_{n+1} \notin P_x$, then it follows from (3.12) and the inductive assumption that

$$\begin{aligned}
||\hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)||_\infty = \hat{\mathcal{L}}_\phi^{n+1}(1\!\!1)(\hat{z}_{n+1}) &= \sum_{k=-\infty}^{+\infty}\hat{\mathcal{L}}_\phi^n(1\!\!1)\Big((\hat{z}_{n+1})_k\Big)\alpha_\phi^{-1}|\hat{\hat{z}}_{n+1} + 2\pi ik|^{-t} \\
&\leq \sum_{k=-\infty}^{+\infty}\alpha_\phi^{-1}||\hat{\mathcal{L}}_\phi^n(1\!\!1)||_\infty|\hat{\hat{z}}_{n+1} + 2\pi ik|^{-t} \\
&\leq K_{\phi,x}^t\Big(m(P_x)\Big)^{-1}\alpha_\phi^{-1}\sum_{k=-\infty}^{+\infty}|\hat{\hat{z}}_{n+1} + 2\pi ik|^{-t} \leq K_{\phi,x}^t\Big(m(P_x)\Big)^{-1}.
\end{aligned}$$

We are done. ■

Lemma 3.8. *There exists $x_0 \geq 0$ such that for every $x \geq x_0$*

$$\inf_{n\geq 0}\sup_{z\in P_x}\{\hat{\mathcal{L}}_\phi^n(1\!\!1)(z)\} \geq \frac{1}{4}.$$

Proof. Let Q_ϕ come from Lemma 3.7. Let x_0 be so large that $m(P_{x_0}^c) \leq 1/(4Q_\phi)$. Suppose for the contrary that $\mathcal{L}_0^n(1\!\!1)(z) < 1/4$ for some $n \geq 0$ and all $z \in P_{x_0}$. Then

$$1 = \int \hat{\mathcal{L}}_\phi^n(1\!\!1)dm = \int_{P_{x_0}}\hat{\mathcal{L}}_\phi^n(1\!\!1)dm + \int_{P_{x_0}^c}\hat{\mathcal{L}}_\phi^n(1\!\!1)dm \leq \frac{1}{4}m(P_{x_0}) + Qm(P_{x_0}^c) \leq \frac{1}{4} + Q_\phi\frac{1}{4Q_\phi} = \frac{1}{2}.$$

This contradiction finishes the proof. ■

As an immediate consequence of this lemma and (3.11) we get the following.

Lemma 3.9. *For every $x \geq x_0$ we have*

$$\inf_{n\geq 0} \inf_{z\in P_x} \{\hat{\mathcal{L}}_\phi^n(1\!\!1)(z)\} \geq \frac{1}{4}\Big(\max\{K_{\phi,x}, K_{x_0}\}\Big)^{-t}.$$

We shall prove the following.

Proposition 3.10. *If ϕ is 1^+-tame, then $\underline{\mathrm{P}}(\phi) = \overline{\mathrm{P}}(\phi) = \log \alpha_\phi$.*

Proof. It follows from Lemma 3.7 that $\mathcal{L}_\phi^n(1\!\!1)(z) \leq Q_\phi \alpha_\phi^n$ for every $z \in P$. Hence

$$\overline{\mathrm{P}}(\phi) = \overline{\mathrm{P}}_z(\phi) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\phi^n(1\!\!1)(z) \leq \log \alpha_\phi.$$

In view of Lemma 3.9, $\mathcal{L}_\phi^n(1\!\!1)(x_0) \geq \frac{1}{4} K_{\phi,x_0}^{-t} \alpha_\phi^n$ and therefore

$$\underline{\mathrm{P}}(\phi) = \underline{\mathrm{P}}_z(\phi) = \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\phi^n(1\!\!1)(z) \geq \log \alpha_\phi.$$

We are done. ■

Denote the common value of $\underline{\mathrm{P}}(\phi)$ and $\overline{\mathrm{P}}(\phi)$ by $\mathrm{P}(\phi)$.

Let $\{z_n\}_{n=1}^\infty$ be a sequence of points in the cylinder P. We say that $\lim_{n\to\infty} z_n = \infty$ if and only if $\lim_{n\to\infty} \mathrm{Re}(z_n) = +\infty$.

Let

$$I_\infty(F) = \{z \in J(F) : \lim_{n\to\infty} F^n(z) = \infty\},$$

i.e. $I_\infty(F)$ is the set of points escaping to infinity under forward iterates of F. Analogously define

$$I_\infty(f) = \{z \in J(f) : \lim_{n\to\infty} f^n(z) = \infty\}.$$

Denote

$$J_r(F) = J(F) \setminus I_\infty(F) \quad \text{and} \quad J_r(f) = J(f) \setminus I_\infty(f)$$

and notice that

$$I_\infty(f) = \pi^{-1}(I_\infty(F)).$$

Let m_ϕ be the $(\phi, e^{\mathrm{P}(\phi)})$-conformal measure constructed in Theorem 3.5 (due to Proposition 3.10 $\alpha_\phi = e^{\mathrm{P}(\phi)}$). This theorem and this proposition along with Lemma 3.1 lead to the following.

Lemma 3.11. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then there is a constant $C_\phi \geq 1$ such that for every $z \in J(F)$, every $v \in F^{-n}(z)$ and every set*

$B \subset B(z,\delta)$, we have that

$$C_\phi^{-1}\exp\Big(S_n\phi\Big(F_v^{-n}(w)\Big)-\mathrm{P}(\phi)n\Big)m_\phi(B) \le$$
$$m_\phi\Big(F_v^{-n}\Big(B\Big)\Big) = \int_{B(z,\delta)}\exp\Big(S_n\phi\Big(F_v^{-n}(w)\Big)-\mathrm{P}(\phi)n\Big)dm_\phi(w)$$
$$\le C_\phi\exp\Big(S_n\phi\Big(F_v^{-n}(w)\Big)-\mathrm{P}(\phi)n\Big)m_\phi(B).$$

We shall now prove the following.

Proposition 3.12. *There exists $M>0$ such that for m_ϕ-a.e. $x\in J(F)$*
$$\liminf_{n\to\infty}\mathrm{Re}(F^n(x))\le M.$$
In particular, $m_\phi(I_\infty(F))=0$ or equivalently $m_\phi(J_r(F))=1$.

Proof. Fix $M>0$. Let $B\subset P_M^c$ be an arbitrary Borel set. We shall estimate from above the measure $m_\phi(F^{-1}(B))$. We have
$$m_\phi(F^{-1}(B))\le m_\phi(F^{-1}(B))=\sum_{k\in\mathbf{Z}}m_\phi(\{x: f(x)\in B+2k\pi i\}).$$
If $x\in P$ and $f(x)\in\tilde B+2k\pi i$, ($\tilde B\subset I\!R\times[-\pi,\pi]$ projects down onto B), then $e^x\in\lambda^{-1}(\tilde B+2k\pi i)$. Hence, $\mathrm{Re}x=-\log|\lambda|+\log|y+2k\pi i|$ with some $y\in\tilde B$. Since ϕ is κ-tame, we therefore get
$$\exp(\phi(x))\le\exp(A_\phi-\kappa\mathrm{Re}x)=e^{A_\phi}|\lambda|^\kappa|y+2k\pi i|^{-\kappa}\le C_1e^{A_\phi}|\lambda|^\kappa(M+2k\pi i)^{-\kappa}$$
with some universal constant $C_1>0$. Thus
$$\begin{aligned}m_\phi(F^{-1}(B))&\le 2\sum_{k=0}^{\infty}m_\phi(B)C_1e^{A_\phi}e^{-\mathrm{P}(\phi)}(M+2k\pi i)^{-\kappa}\\&\le C_2|\lambda|^\kappa\exp\Big(A_\phi-\mathrm{P}(\phi)\Big)(\kappa-1)^{-1}M^{1-\kappa}m_\phi(B)\end{aligned}\tag{3.13}$$
with some universal constant $C_2>0$. Since $B\cap F^{-1}(B)\cap\ldots\cap F^{-(n-1)}(B)\subset P_M^c$, we therefore get for every $n\ge1$ that
$$\begin{aligned}&m_\phi\Big(B\cap F^{-1}(B)\cap\cdots\cap F^{-n}(B)\Big)\le\\&\quad\le m_\phi\Big(F^{-1}(B)\cap\cdots\cap F^{-n}(B)\Big)\\&\quad=m_\phi\Big(F^{-1}\Big(B\cap F^{-1}(B)\cap\ldots\cap F^{-(n-1)}(B)\Big)\Big)\\&\quad\le C_2|\lambda|^\kappa\exp\Big(A_\phi-\mathrm{P}(\phi)\Big)(\kappa-1)^{-1}M^{1-\kappa}\Big(B\cap\ldots\cap F^{-(n-1)}(B)\Big).\end{aligned}$$
Therefore we obtain by induction that
$$m_\phi\Big(B\cap F^{-1}(B)\cap\cdots\cap F^{-n}(B)\Big)\le\Big(C_2|\lambda|^\kappa\exp\Big(A_\phi-\mathrm{P}(\phi)\Big)(\kappa-1)^{-1}M^{1-\kappa}\Big)^n m_\phi(B)$$

This implies that for all M large enough

$$m_\phi(\bigcap_{n=0}^{\infty} F^{-n}(P_M^c)) = 0$$

and consequently

$$m_\phi(\bigcup_{k=0}^{\infty} F^{-k}(\bigcap_{n=0}^{\infty} F^{-n}(P_M^c))) = 0.$$

The proof is finished. ■

Let us show now that the estimates used in Proposition 3.12 and Proposition 16.2 lead to the following.

Corollary 3.13.

$$m_\phi(P_M^c) \le 4C_4 \exp\Big(A_\phi - \mathrm{P}(\phi)\Big)|\lambda|^\kappa(\kappa-1)^{-1}e^{M(1-\kappa)}$$

for all $M \ge 0$ large enough, in fact all $M \ge \Big(2C_2|\lambda|^\kappa \exp\Big(A_\phi - \mathrm{P}(\phi)\Big)(\kappa-1)^{-1}\Big)^{\frac{1}{\kappa-1}}$, where C_4 comes from (16.6) and C_2 comes from (3.13).

Proof. In view of (3.13), we have

$$m_\phi\Big(P_M^c \cap F^{-1}(P_M^c)\Big) \le C_2|\lambda|^\kappa \exp\Big(A_\phi - \mathrm{P}(\phi)\Big)(\kappa-1)^{-1}M^{1-\kappa}m_\phi(P_M^c),$$

and from (formula (16.6) with m_N replaced by m_ϕ and $u(\lambda,\phi)$ replaced by $\mathrm{P}(\phi)$, we have that

$$m_\phi\Big(P_M^c \cap F^{-1}(P_M)\Big) \le 2C_4|\lambda|^\kappa \exp\Big(A_\phi - \mathrm{P}(\phi)\Big)(\kappa-1)^{-1}e^{M(1-\kappa)},$$

where the universal constant $C_4 > 0$ comes from (16.6). The two sets $P_M^c \cap F^{-1}(P_M^c)$ and $P_M^c \cap F^{-1}(P_M)$ cover the whole set P_M^c. The first inequality says that for all M so large as required in the assumptions of our corollary, the first set covers no more than one half of the measure of P_M^c. Thus,

$$m_\phi(P_M^c) \le 4C_4|\lambda|^\kappa \exp\Big(A_\phi - \mathrm{P}(\phi)\Big)(\kappa-1)^{-1}e^{M(1-\kappa)},$$

and the proof is complete. ■

Theorem 3.14. *The $(\phi, e^{\mathrm{P}(\phi)})$-conformal measure $m = m_\phi$ is a unique (ϕ,β)-conformal measure for F. In addition it is ergodic with respect to each iterate of F.*

Proof. Fix $j \ge 1$. Suppose that ν is a $(S_j\phi, \beta^j)$-conformal measure for F^j with some $\beta > 0$. The same proof as in the case of the measure m shows that $\nu(I_\infty(F)) = 0$. Let $J_{r,N}(F)$ be the subset of $J_r(F)$ defined as follows: $z \in J_{r,N}(F)$ if and only if the trajectory of z under F^j has an accumulation point in $\{z \in J(F) : \mathrm{Re} z < N\}$. Obviously, $\bigcup_N J_{r,N}(F) = J_r(F)$ and by Proposition 3.12 there exists $M' > 0$ such that $\nu(J_{r,M'}(F)) = m(J_{r,M'}(F)) = 1$. Fix $z \in J_{r,N}(F)$ and let recall that $\delta \le \mathrm{dist}\Big(\{J(F), \pi\Big(\{f^n(0)\}_{n\ge 0}\Big)\Big)/2$. Then there exist $y \in J(F)$ such that $\mathrm{Re}(y) < N$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y = \lim_{k\to\infty} F^{jn_k}(z)$.

Considering for k large enough the sets $F_z^{-jn_k}(B(y,\delta))$ and $F_z^{-jn_k}(B(y,\delta/K))$, where $F_z^{-jn_k}$ is the holomorphic inverse branch of F^{n_k} defined on $B(y,2\delta)$ and sending $F^{jn_k}(z)$ to z, using conformality of measures m and ν along with Lemma 3.1 we easily deduce that

$$B_N(\nu)^{-1}\beta^{-jn_k}\exp\Big(S_{jn_k}\phi(z) \le \nu\Big(B_N(z,c|(F^{jn_k})'(z)|^{-1})\Big) \le B_N(\nu)\beta^{-jn_k}\exp\Big(S_{jn_k}\phi(z) \tag{3.14}$$

for all $k\ge 1$ large enough, where $c \asymp 1$ and $B_N(\nu)$ is some constant depending on ν and N. Let M be fixed as above. Fix now E, an arbitrary bounded Borel set contained in $J_r(F)$ and let $E' = E\cap J_{r,M'}(F)$. Since m is regular, for every $x\in E'$ there exists a radius $r(x)\in(0,\varepsilon)$ of the form from (3.14) (and the corresponding number $n(x)=n_k(x)$ for an appropriate k) such that

$$m\Big(\bigcup_{x\in E'} B(x,r(x))\setminus E'\Big) \le \varepsilon. \tag{3.15}$$

Now by the Besicovič theorem (see [G]) we can choose a countable subcover $\{B(x_i,r(x_i))\}_{i=1}^\infty$ with $r(x_i)\le\varepsilon$ and $jn(x_i)\ge\varepsilon^{-1}$, from the cover $\{B(x,r(x))\}_{x\in E'}$ of E', of multiplicity bounded by some constant $C\ge 1$, independent of the cover. Therefore, assuming $e^{\mathrm{P}(\phi)}<\beta$ and using (3.14) along with (3.15), we obtain

$$\begin{aligned}
\nu(E)=\nu(E') &\le \sum_{i=1}^\infty \nu(B(x_i,r(x_i))) \le B_M(\nu)\sum_{i=1}^\infty \beta^{-jn(x_i)}\exp\Big(S_{jn(x_i)}\phi(x_i)\Big)\\
&\le B_M(\nu)B_M(m)\sum_{i=1}^\infty m(B(x_i,r(x_i)))\beta^{-jn(x_i)}e^{\mathrm{P}(\phi)jn(x_i)}\\
&\le B_M(\nu)B_M(m)Cm\Big(\bigcup_{i=1}^\infty B(x_i,r(x_i))\Big)\Big(e^{\mathrm{P}(\phi)}\beta^{-1}\Big)^{jn(x_i)}\\
&\le B_M(\nu)B_M(m)Cm\Big(\bigcup_{i=1}^\infty B(x_i,r(x_i))\Big)\Big(e^{\mathrm{P}(\phi)}\beta^{-1}\Big)^{\varepsilon^{-1}}\\
&\le CB_M(\nu)B_M(m)\Big(e^{\mathrm{P}(\phi)}\beta^{-1}\Big)^{\varepsilon^{-1}}(\epsilon+m(E'))\\
&= CB_M(\nu)B_M(m)\Big(e^{\mathrm{P}(\phi)}\beta^{-1}\Big)^{\varepsilon^{-1}}(\epsilon+m(E)).
\end{aligned} \tag{3.16}$$

Hence letting $\varepsilon\searrow 0$ we obtain $\nu(E)=0$ and consequently $\nu(J(F))=0$ which is a contradiction. We obtain a similar contradiction assuming that $\beta<e^{\mathrm{P}(\phi)}$ and replacing in (3.16) the roles of m and ν. Thus $\beta=e^{\mathrm{P}(\phi)}$ and letting $\epsilon\searrow 0$ again, we obtain from (3.16) that $\nu(E)\le CB_M(\nu)B_M(m)m(E)$. Exchanging m and ν, we obtain $m(E)\le CB_M(\nu)B_M(m)\nu(E)$. These two conclusions along with the already mentioned fact that $m(J_r(F))=\nu(J_r(F))=1$, imply that the measures m and ν are equivalent with Radon-Nikodym derivatives bounded away from zero and infinity.

Let us now prove that any $(\phi,e^{\mathrm{P}(\phi)})$-conformal measure ν is ergodic with respect to F^j. Indeed, suppose to the contrary that $F^{-j}(G)=G$ for some Borel set $G\subset J(F)$ with $0<$

$\nu(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(F)\setminus G}$

$$\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(F)\setminus G}(B) = \frac{\nu(B \cap J(F) \setminus G)}{\nu(J(F) \setminus G)}$$

would be $(S_j\phi, e^{j\mathrm{P}(\phi)})$-conformal for F^j and mutually singular. This contradiction finishes the proof. ■

4. **Perron-Frobenius Operators and Their Finer Properties**

As in the previous section we fix the map $f = f_\lambda$ and $F = F_\lambda$, and we assume that $f : \mathbb{C} \to \mathbb{C}$ has an attracting periodic orbit, i.e. that $\lambda \in \mathrm{Hyp}$. Let us recall that given a 1^+-tame potential $\phi : J(F) \to \mathbb{C}$, the corresponding Perron-Frobenius operator was defined by formula (3.9). In view of Lemma 3.2 the potential ϕ is dynamically Hölder, and in the terminology from [UZ2], formula (3.6) means that the potential ϕ (strictly speaking e^ϕ) is summable. Thus, Lemma 4.1 from [UZ2] includes the following.

Lemma 4.1. *If $\phi : J(F) \to \mathbb{C}$ is a 1^+-tame potential with a Hölder exponent $\alpha > 0$ then $\mathcal{L}_\phi(\mathrm{H}_\alpha) \subset \mathrm{H}_\alpha$. If, in addition, $\phi(J(F)) \subset [0, \infty)$ and $\sup_{n\ge1}\{||\mathcal{L}_\phi^n(1\!\!1)||_\infty\} < \infty$, then there exists a constant $c_1 > 0$ such that*

$$||\mathcal{L}_\phi^n g||_\alpha \le \frac{1}{2}||g||_\alpha + c_1||g||_\infty$$

for all $n \ge 1$ large enough and every $g \in \mathrm{H}_\alpha$.

In order to apply the theorem of Ionescu-Tulcea and Marinescu we also need the following. Since a 1^+-tame potential is dynamically Hölder, invoking Lemma 3.7, and the second part of (3.10) (which in the terminology from [UZ2] means that the potential ϕ is rapidly decreasing), the following fact is contained in Lemma 4.2 from [UZ2]

Lemma 4.2. *Suppose that $\phi : J(F) \to (0, \infty)$ is a 1^+-tame potential. If B is a bounded subset of H_α (with the $||\cdot||_\alpha$ norm), then $\mathcal{L}_\phi(B)$ is a pre-compact subset of C_b (with the $||\cdot||_\infty$ norm).*

Combining now Lemma 4.1 and Lemma 4.2, we see that the assumptions of Theorem 1.5 in [IM] are satisfied with Banach spaces $\mathrm{H}_\alpha \subset C_b$ and the bounded operator $\mathcal{L}_\phi : C_b \to C_b$ preserves H_α. It gives us the following.

Theorem 4.3. *If $\phi : J(F) \to (0, \infty)$ is a 1^+-tame potential, then there exist finite numbers $\gamma_1, \dots, \gamma_p \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, finitely many bounded finitely dimensional operators*

$Q_1, \dots, Q_p : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and an operator $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ such that

$$\mathcal{L}_\phi^n = \sum_{i=1}^{p} \gamma_i^n Q_i + S^n$$

for all $n \geq 1$,

$$Q_i^2 = Q_i,\ Q_i \circ Q_j = 0, (i \neq j),\ Q_i \circ S = S \circ Q_i = 0$$

and

$$||S^n||_\alpha \leq C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \geq 1$. In particular all numbers $\gamma_1, \dots, \gamma_p$ are isolated eigenvalues of the operator $\mathcal{L}_\phi : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and this operator is quasi-compact.

The proof of the next result is the same as that of Theorem 4.4 in [UZ2]; one only needs to replace $\hat{\mathcal{L}}_t$ by $\hat{\mathcal{L}}_\phi$.

Theorem 4.4. *If $\phi : J(F) \to (0,\infty)$ is a 1^+-tame potential, then we have the following.*

(a) *The number 1 is a simple isolated eigenvalue of the operator $\hat{\mathcal{L}}_\phi : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$.*
(b) *The eigenspace of the eigenvalue 1 is generated by a nowhere vanishing function $g_\phi \in \mathrm{H}_\alpha$ such that $\int g_\phi dm_\phi = 1$ and $\lim_{\mathrm{Re}z \to +\infty} g_\phi(z) = 0$.*
(c) *The number 1 is the only eigenvalue of modulus 1.*
(d) *With $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ as in Theorem 4.3, we have*

$$\hat{\mathcal{L}}_\phi = Q_1 + S,$$

where $Q_1 : \mathrm{H}_\alpha \to \mathbb{C}g_\phi$ is a projector on the eigenspace $\mathbb{C}g_\phi$, $Q_1 \circ S = S \circ Q_1 = 0$ and

$$||S^n||_\alpha \leq C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \geq 1$.

5. **Invariant Measures**

The results of this section are formal (some of them immediate, some of them deeper) consequences of Theorem 4.4. Their exposition (without proofs) is the same as in [UZ2]. The following theorem immediately follows from Theorem 4.4, Proposition 3.12 and Theorem 3.14.

Theorem 5.1. *If $\phi : J(F) \to (0,\infty)$ is a 1^+-tame potential, then the measure $\mu = \mu_\phi = g_\phi m_\phi$ is F-invariant, ergodic with respect to each iterate of F and equivalent to the measure m_ϕ. In particular $\mu(J_r(F)) = 1$.*

The measure μ_ϕ is called the invariant Gibbs state of the potential ϕ. Due to Theorem 4.4 the F-invariant measure μ_ϕ has much finer stochastic properties than ergodicity of all iterates of F. Here these follow.

Theorem 5.2. *The dynamical system (F, μ_ϕ) is metrically exact i.e., its Rokhlin natural extension is a K-system.*

The proof of this fact is the same as the proof of Corollary 37 in [DU2]. The next two theorems are standard consequences of Theorem 4.4 (see [DPU] and [PU] for example). Let g_1 and g_2 be real square-μ integrable functions on $J_r(F)$. For every positive integer n the n-th corelation of the pair g_1, g_2, is the number

$$C_n(g_1, g_2) := |\int g_1 \cdot (g_2 \circ F^n)\, d\mu_\phi - \int g_1\, d\mu \int g_2\, d\mu_\phi|.$$

provided the above integrals exist. Notice that due to the F-invariance of μ we can also write

$$C_n(g_1, g_2) = |\int (g_1 - Eg_1)\Big((g_2 - Eg_2) \circ F^n\Big)\, d\mu_\phi|,$$

where we write $Eg = \int g\, d\mu_\phi$. We have the following.

Theorem 5.3. *There exist $C \geq 1$ and $\rho < 1$ such that for all $g_1 \in \mathrm{H}_\alpha(P), g_2 \in L^1(m_\phi)$*

$$C_n(g_1, g_2) \leq C\rho^n \|g_1 - Eg_1\|_\alpha \|g_2 - Eg_2\|_{L^1_{m_\phi}}.$$

Let $g : J_r(F) \to R$ be a square-integrable function. The limit

$$\sigma^2(g) = \lim_{n\to\infty} \frac{1}{n} \int \Big(\sum_{j=0}^{n-1} g \circ F^j - nEg\Big)^2 d\mu_\phi$$

is called asymptotic variance or dispersion, provided it exists.

Theorem 5.4. *If $g \in \mathrm{H}_\alpha(\mathrm{P})$, $\alpha \in (0, 1]$, then $\sigma^2(g)$ exists and, if $\sigma^2(g) > 0$, then the sequence of random variables $\{g \circ F^n\}_{n=0}^\infty$ with respect to the probability measure μ_ϕ satisfies the Central Limit Theorem, i.e.*

$$\mu_\phi\left(\left\{x \in J_r(F) : \frac{\sum_{j=0}^{n-1} g \circ F^j - nEg}{\sqrt{n}} < r\right\}\right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-t^2/2\sigma^2}\, dt.$$

6. **Variational Principle**

This section follows closely the reasoning included in [UZ3]. We present it here for the sake of completeness and the convenience of the reader.

Lemma 6.1. *If μ is a Borel probability measure on the real line $I\!R$, then for every $\delta > 0$, every $\alpha > 0$, and every integer $n \in \mathbb{Z}$ there exists $x_n \in [\delta n, \delta(n+1))$ such that*

$$\sum_{n=-\infty}^{+\infty} \sum_{k=0}^{\infty} \mu\Big([x_n - e^{-\alpha k}, x_n + e^{-\alpha k}]\Big) < \infty.$$

Proof. For every $x \in I\!R$ and every $k \in \mathbb{Z}$ write

$$I_{x,k} = [x - e^{-\alpha k}, x + e^{-\alpha k})$$

and note that $\int_{-\infty}^{+\infty} \mu(I_{x,k}) dx = 2e^{-\alpha k}$. Let

$$g(x) = \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{\infty} \mu(I_{x+\delta n,k}).$$

Then

$$\int_0^\delta g(x) dx = \sum_{k=0}^{\infty} \sum_{n=-\infty}^{+\infty} \int_{\delta n}^{\delta(n+1)} \mu(I_{y,k}) dy = \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \mu(I_{y,k}) dy = \sum_{k=0}^{\infty} 2e^{-\alpha k} < +\infty.$$

Hence there is $x \in (0, \delta)$ such that $g(x) < +\infty$, and taking $x_n = x + \delta n$ completes the proof.
■

Lemma 6.2. *If μ is a Borel probability measure on the cylinder Q, then for every $\delta > 0$ there exists a partition α of Q by rectangles with all sides of length $\le \delta$ parallel to the coordinate axes such that for every $\beta > 0$*

$$\sum_{n=0}^{\infty} \sum_{A \in \alpha} \mu\Big(B(\partial A, e^{-\beta n})\Big) < \infty.$$

Proof. Let $p_1 : Q \to I\!R$ be the orthogonal projection onto the x-axis and let $p_2 : Q \to I\!R$ be the orthogonal projection onto the circle $I\!R/2\pi\mathbb{Z}$. In view of Lemma 6.1 there are points $[x_n \in \left[\frac{\delta}{2}n, \frac{\delta}{2}(n+1)\right] \subset I\!R$, $n \in \mathbb{Z}$, $y_1 < y_2 < \ldots < y_q \in I\!R/2\pi\mathbb{Z}$ such that $|y_{i+1} - y_i| < \delta$ and

$$\sum_{n=-\infty}^{+\infty} \sum_{k=0}^{\infty} \mu \circ p_1^{-1}\Big([x_n - e^{-\beta k}, x_n + e^{-\beta k}]\Big) < \infty \text{ and } \sum_{j=1}^{q} \sum_{k=0}^{\infty} \mu \circ p_2^{-1}\Big([y_j - e^{-\beta k}, y_j + e^{-\beta k}]\Big) < \infty. \tag{6.1}$$

Let α be the partition formed by all the rectangles $[x_n, x_{n+1}] \times [y_j, y_{j+1(mod q)}] \subset Q$, $n \in \mathbb{Z}$, $j = 1, 2, \ldots, q$. Then

$$\partial\alpha = \bigcup_{n=-\infty}^{+\infty} \bigcup_{j=1}^{q} \{x_n\} \times S^1) \cup (I\!R \times \{y_j\})$$

and

$$B\Big(\partial\alpha, e^{-\beta k}\Big) = \left(\bigcup_{n=-\infty}^{+\infty} B(x_n, e^{-\beta k}) \times S^1\right) \cup \left(\bigcup_{j=1}^{q} I\!R \times B(y_j, e^{-\beta k})\right).$$

Hence, due to (6.1)

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{A\in\alpha}\mu\Big(B(\partial A,e^{-\beta k})\Big) &\le \sum_{n=-\infty}^{+\infty}\mu\Big(B(x_n,e^{-\beta k})\times S^1\Big)+\sum_{j=1}^{q}\mu\Big(I\!R\times B(y_j,e^{-\beta k})\Big)\\ &\le \sum_{n=-\infty}^{+\infty}\mu\circ p_1^{-1}\Big(B(x_n,e^{-\beta k})\Big)+\sum_{j=1}^{q}\mu\circ p_2^{-1}\Big(B(y_j,e^{-\beta k})\Big). < \infty\end{aligned}$$

We are done. ■

Note that by Lemma 8.3, whose proof is independent of our present considertions, the integral $\int \psi d\mu_\phi$ is a finite number for every 1^+-tame function ϕ and every tame function ψ. The main result of this section is the following variational principle.

Theorem 6.3. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{C}$ is a 1^+-tame potential, then the invariant measure μ_ϕ is an equilibrium state of the potential ϕ, that is*

$$\mathrm{P}(\phi) = \sup\{h_\mu(F) + \int \phi d\mu\}$$

where the supremum is taken over all Borel probability F-invariant ergodic measures μ with $\int \phi d\mu > -\infty$, and

$$\mathrm{P}(\phi) = h_{\mu_\phi} + \int \phi d\mu_\phi.$$

Proof. We shall show first that if μ is a Borel probability F-invariant ergodic measure on $J(F)$ with $\int \phi d\mu > -\infty$, then

$$\mathrm{P}(\phi) \ge \mathrm{h}_\mu + \int \phi d\mu.$$

Let α be the partition constructed in Lemma 6.2 with diameter $\le \delta$. We shall demonstrate that there exists a Borel set $Z \subset J(F)$ such that $\mu(Z) = 1$ and

$$\mu_\phi(\alpha_n(x)) \le \mu(\alpha_n(x)) \tag{6.2}$$

for all $x \in Z$ and all $n \ge 1$ large enough (depending on x). If μ and μ_ϕ are not mutually singular, then $\mu = \mu_\phi$ since both measures are ergodic, and (6.2) becomes obvious. So, we may assume that μ and μ_ϕ are mutually singular. This means that there exists a Borel set $Y \subset J(F)$ such that $\mu_\phi(Y) = 0$ and $\mu(Y) = 1$. Seeking contradiction suppose now that there exists a compact set $S \subset Y$ with the following two properties.

(a) $\mu(S) > 0$.

(b) For every $x \in S$ there exists an unbounded increasing sequence $\{n_j(x)\}_{j=1}^{\infty}$ such that

$$\mu_\phi(\alpha_{n_j(x)}(x) > \mu(\alpha_{n_j(x)}(x)$$

for all $j \ge 1$.

Since the measure μ_ϕ is regular and $\mu_\phi(S) = 0$, there exists $\varepsilon > 0$ such that $\mu_\phi(B(S,\varepsilon)) < \mu(S)/2$. Since for every $x \in J(F)$ and every $n \geq 0$, $\alpha_n(x) \subset F_x^{-n}(B(F^n(x),\delta))$, looking at (3.2) we see that

$$\lim_{n\to\infty} \mathrm{diam}(\alpha_n(x)) = 0.$$

Fix $\varepsilon > 0$. For every $x \in S$ there thus exists $j(x) \geq 1$ such that $\mathrm{diam}\Big(\alpha_{n_{j(x)}}(x)\Big) < \varepsilon$. Since any two elements of the family $\Big\{\alpha_{n_{j(x)}}(x)\Big\}_{x\in S}$ are either disjoint or one is contained in the other, we can choose countably many points $\{x_k\}_{k=1}^\infty \subset S$ such that all the sets $A_k = \alpha_{n_{j(x_k)}}(x_k)$, $k \geq 1$, are mutually disjoint and $\bigcup_{k\geq 1} A_k \supset S$. Using the property (b) we then get

$$\mu(S) \leq \mu\left(\bigcup_{k=1}^\infty A_k\right) = \sum_{k=1}^\infty \mu(A_k) < \sum_{k=1}^\infty \mu_\phi(A_k) = \mu_\phi\left(\bigcup_{k=1}^\infty A_k\right) \leq \mu_\phi(B(S,\varepsilon)) < \frac{1}{2}\mu(S).$$

This contradiction finishes the proof of (6.2). In view of Lemma 6.2 we get for every $\beta > 0$ that

$$\sum_{n=1}^\infty \sum_{A\in\alpha} \mu\Big(F^{-n}\Big(B(\partial A, e^{-\beta n})\Big)\Big) = \sum_{n=1}^\infty \sum_{A\in\alpha} \mu\Big(B(\partial A, e^{-\beta n})\Big) < \infty.$$

Therefore $\mu(Z_1) = 1$, where Z_1 is the set of all those $z \in Z$ that $F^n(z) \in B(\partial A, e^{-\beta n})$ for finitely many n's only. Fix $z \in Z_1$ and let $q \geq 1$ be such that $F^n(z) \notin B(\partial A, e^{-\beta n})$ for all $n \geq q$. For every $k \in \{0,1,2,\ldots,n\}$, put $F_*^{-k} := F_{F^{n-k}(z)}^{-k}$. Let $p \geq 0$ be an arbitrary real number such that $C^{-1}e^{-\kappa p} \leq e^{-\beta(n-p)}$. This equivalently means that $\log C + \kappa p \geq \beta n - \beta p$ or $(\kappa+\beta)p \geq \beta n - \log C$, and finally this means that $p \geq \frac{\beta}{\kappa+\beta}n - \frac{\log C}{\kappa+\beta}$. So, putting

$$p = p_n = E\left(\frac{\beta}{\kappa+\beta}n\right) + 1,$$

we have

$$C^{-1}e^{-\kappa k} \leq e^{-\beta(n-k)}$$

for all $k \geq p$. This implies that $\alpha(F^{n-k}(z)) \supset F_*^{-k}\Big(B(F^n(z),\delta)\Big)$ for all $p \leq k \leq n-q$. Hence

$$F^{-(n-k-q)}\Big(\alpha(F^{n-k}(z))\Big) \supset F_q^{-(n-k-q)} \circ F_*^{-k}\Big(B(F^n(z),\delta)\Big) = F_*^{-(n-q)}\Big(B(F^n(z),\delta)\Big),$$

where $F_q^{-(n-k-q)} : B(F^{n-k}(z), 2\delta) \to \mathbb{C}$ is the holomorphic inverse branch of F^{n-k-q} sending $F^{n-k}(z)$ to $F^q(z)$. Thus,

$$\alpha_{n-q-p}(z) = \bigcap_{k=p}^{n-q} F^{-(n-k-q)}\Big(\alpha(F^{n-k}(z))\Big) \supset F_*^{-(n-q)}\Big(B(F^n(z),\delta)\Big).$$

Hence, utilizing Lemma 3.11, we obtain

$$\begin{aligned}\mu_\phi\Big(\alpha_{n-q-p}(z)\Big) &\geq \mu_\phi\Big(F_*^{-(n-q)}\Big(B(F^n(z),\delta)\Big)\Big)\\ &\succeq \psi(F^q(z))\exp\Big(S_{n-q}\phi(F^q(z)) - \mathrm{P}(\phi)n\Big)m_\phi\Big(B(F^n(z),\delta)\Big)\\ &\asymp \exp\Big(S_n\phi(z) - \mathrm{P}(\phi)n\Big)m_\phi\Big(B(F^n(z),\delta)\Big),\end{aligned}$$

where $\psi = d\mu_\phi/dm_\phi$ and the comparability constant, call it C_1 appearing in the above formula depends on z but is independent of n. Consequently

$$\frac{1}{n}\log\Big(\mu_\phi\Big(\alpha_{n-q-p}(z)\Big)\Big) \geq \frac{\log(C_1)}{n} + \frac{1}{n}S_n\phi(z) - \mathrm{P}(\phi) + \frac{1}{n}\log\Big(m_\phi\Big(B(F^n(z),\delta)\Big)\Big). \tag{6.3}$$

In view of Birkhoff's ergodic theorem and the Breiman-McMillan-Shanon theorem, there exists an F-invariant Borel set $Z_2 \subset Z_1$ such that $\mu(Z_2) = 1$, and for all $z \in Z_2$

$$\liminf_{n\to\infty} \mathrm{Re}(F^n(z)) < \infty, \ \lim_{n\to\infty}\frac{1}{n}S_n\phi(z) = \int \phi d\mu, \lim_{n\to\infty}\frac{1}{n}\log\Big(\mu(\alpha_n(z))\Big) = -\mathrm{h}_\mu. \tag{6.4}$$

Fix $z \in Z_2$. There thus exists an unbounded increasing sequence $\{n_j\}_{j=1}^\infty$ of positive integers such that $\mathrm{Re}(F^{n_j}(z)) \leq M$ for some $M > 0$ and all $j \geq 1$. Hence, there exists $T > 0$ such that $m_\phi\Big(B(F^{n_j}(z),\delta)\Big) \geq T$ for all $j \geq 1$. It therefore follows from (6.3) and (6.4) that for every $z \in Z_2$,

$$\liminf_{j\to\infty}\frac{1}{n_j}\log\Big(\mu_\phi\Big(\alpha_{n_j-q-p_{n_j}}(F^q(z))\Big)\Big) \geq \int \phi d\mu - \mathrm{P}(\phi).$$

Combining this along with (6.2) and the last part of (6.4), we get

$$\begin{aligned}
-\mathrm{h}_\mu &\geq \liminf_{j\to\infty}\left(\frac{n_j}{n_j-q-p_{n_j}}\cdot\frac{1}{n_j}\log\Big(\mu_\phi\Big(\alpha_{n_j-q-p_{n_j}}(F^q(z))\Big)\Big)\right)\\
&\geq \liminf_{j\to\infty}\left(\frac{n_j}{n_j-q-p_{n_j}}\right)\Big(\int\phi d\mu - \mathrm{P}(\phi)\Big).
\end{aligned}$$

But it follows from the definition of the sequence $\{p_n\}_{n=1}^\infty$ that

$$\liminf_{n\to\infty}\frac{n}{n-q-p_n} = \liminf_{n\to\infty}\left(\frac{1}{1-\frac{q}{n}-\frac{p_n}{n}}\right) = \lim_{n\to\infty}\left(\frac{1}{1-\frac{p_n}{n}}\right) = \frac{1}{1-\frac{\beta}{\kappa+\beta}} = \frac{\kappa+\beta}{\kappa}.$$

Hence

$$\int\phi d\mu - \mathrm{P}(\phi) \leq -\mathrm{h}_\mu\frac{\kappa}{\kappa+\beta}.$$

Letting now $\beta \searrow 0$, we finally obtain that

$$\mathrm{P}(\phi) \geq \mathrm{h}_\mu + \int\phi d\mu.$$

We shall now prove the easier part of our theorem, that

$$\mathrm{P}(\phi) \leq \mathrm{h}_{mu_\phi} + \int\phi d\mu_\phi.$$

Indeed, let α be the same partition as in the first part of the proof. Since F restricted to each atom of the partition α is 1-to-1, it follows that $\alpha_n(x) \subset F_x^{-n}\Big(B(F^n(x),\delta)\Big)$ for all $x \in J(F)$ and all $n \geq 0$. Applying Birkhoff's ergodic theorem and the Breiman-McMillan-Shanon theorem for the F-invariant measure μ_ϕ and utilizing Theorem 3.11 along with Theorem 5.1, we therefore get for μ_ϕ-a.e. $x \in J(F)$ that

$$\begin{aligned}
-\mathrm{h}_{mu_\phi} &\le \liminf_{n\to\infty} \frac{1}{n} \log \mu_\phi\Big(F_x^{-n}\Big(B(F^n(x),\delta)\Big)\Big) \preceq \liminf_{n\to\infty} \frac{1}{n}\Big(\log(2g_\phi(x)) + S_n\phi(x) - \mathrm{P}(\phi)n\Big) \\
&= \lim_{n\to\infty} \frac{1}{n} S_n\phi(x) - \mathrm{P}(\phi) = \int \phi d\mu_\phi - \mathrm{P}(\phi).
\end{aligned}$$

Thus $\mathrm{P}(\phi) \le \mathrm{h}_{mu_\phi} + \int \phi d\mu_\phi$ and we are done. ■

7. Analyticity of the Pressure Function

In this section we establish analyticity of the pressure function. Since each 1^+-tame function $\phi : J(F) \to \mathbb{C}$ is summable in the terminology of [UZ2], since, by Lemma 3.3, e^ϕ is a bounded Hölder continuous with the same exponent, and since each Hölder continuous function is dynamically Hölder with the same exponent (see Lemma 3.2), the following proposition follows immediately from Theorem 7.6 in [UZ2] by applying Hartogs theorem.

Proposition 7.1. *Suppose that G is an open connected subset of $\mathbb{C}^n$, $n \ge 1$, and that $\phi_\sigma : J(F) \to \mathbb{C}$, $\sigma \in G$, is a family of Hölder continuous functions with a positive exponent $\alpha > 0$ such that the following assumptions are satisfied.*

(a) *For every $\sigma \in G$, ϕ_σ is a 1^+-tame function.*
(b) *The function $\sigma \mapsto e^{\phi_\sigma} \in \mathrm{H}_\alpha$ ($\sigma \in G$) is continuous with respect to each variable separately.*
(c) *The family $\{c_{\phi_\sigma}\}_{\sigma \in G}$ is bounded.*
(d) *The function $\sigma \mapsto \phi_\sigma(z) \in \mathbb{C}$, $\sigma \in G$, is holomorphic for every $z \in J(F)$.*
(e) $\forall(\sigma_2 \in G)\, \exists(r > 0)\, \exists(\sigma_1 \in G)$

$$\sup\{||\exp\Big(\phi_\sigma - \phi_{\sigma_1}\Big)||_\infty : \sigma \in \overline{B(\sigma_2, r)}\} < \infty.$$

Then the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$, $\sigma \in G$, is holomorphic.

Let us now derive some more special consequences of this general fact. Let $\phi, \psi : J(F) \to \mathbb{R}$ be arbitrary two tame functions. Consider the set

$$\Sigma_1(\phi,\psi) := \{q \in \mathbb{C} : \mathrm{Re}(q)\kappa(\phi) + \kappa(\psi) > 1\}.$$

We shall prove the following.

Proposition 7.2. *If $\phi, \psi : J(F) \to \mathbb{R}$ are arbitrary two tame functions, then the function $q \mapsto \mathcal{L}_{q\phi+\psi}$, $q \in \Sigma_1(\phi,\psi)$, is holomorphic.*

Our approach to prove this proposition is to apply Proposition 7.1 with $n = 1$. All the functions $q\phi + \psi$ are obviously Hölder continuous with the same exponent, say α. We first shall check condition (b) which we single out as a separate lemma.

Lemma 7.3. *The function $q \mapsto \exp(q\phi + \psi) \in \mathrm{H}_\alpha$, $q \in \Sigma_1(\phi, \psi)$, is continuous.*

Proof. Fix $q_0 \in \Sigma_1(\phi, \psi)$. Since $\Sigma_1(\phi, \psi)$ is an open subset of $\mathbb{C}$, there exists $R > 0$ such that $\overline{B}(q_0, R) \subset \Sigma_1(\phi, \psi)$. Now notice that if $\psi(z) \ge 0$, then

$$|\exp\Big(q\phi(z) + \psi(z)\Big)| \le |\exp\Big((q_0 + R)\phi(z) + \psi(z)\Big)|$$

for all $z \in J(F)$ and all $q \in B(q_0, R)$. On the other hand, if $\psi(z) \le 0$, then

$$|\exp\Big(q\phi(z) + \psi(z)\Big)| \le |\exp\Big((q_0 - R_1)\phi(z) + \psi(z)\Big)|$$

for all $z \in J(F)$ and all $q \in B(q_0, R)$. Since $q_0 + R, q_0 - R \in \Sigma_1(\phi, \psi)$, both functions $(q_0 + R)\phi + \psi$ and $(q_0 - R)\phi + \psi$ are 1^+-tame. In particular

$$\lim_{z\to+\infty}\Big(\mathrm{Re}\Big((q_0 + R)\phi(z) + \psi(z)\Big)\Big) = -\infty \text{ and } \lim_{z\to+\infty}\Big(\mathrm{Re}\Big((q_0 - R)\phi(z) + \psi(z)\Big)\Big) = -\infty.$$

Thus

$$\lim_{z\to+\infty} \max\Big\{|\exp\Big((q_0 + R)\phi(z) + \psi(z)\Big)|, |\exp\Big((q_0 - R)\phi(z) + \psi(z)\Big)|\Big\} = 0.$$

Therefore, for every $\varepsilon > 0$ there exists $T > 0$ such that

$$\max\Big\{|\exp\Big((q_0 + R)\phi(z) + \psi(z)\Big)|, |\exp\Big((q_0 - R)\phi(z) + \psi(z)\Big)|\Big\} < \varepsilon/2$$

for all $z \in J(F)^c_T$. Thus

$$\begin{aligned} |\exp\Big(q_2\phi(z) + \psi(z)\Big) - \exp\Big(q_1\phi(z) + \psi(z)\Big)| &\le \\ &\le |\exp\Big(q_2\phi(z) + \psi(z)\Big)| + |\exp\Big(q_1\phi(z) + \psi(z)\Big)| \\ &\le 2\max\Big\{|\exp\Big((q_0 + R)\phi(z) + \psi(z)\Big)|, |\exp\Big((q_0 - R)\phi(z) + \psi(z)\Big)|\Big\} \\ &< \varepsilon \end{aligned} \tag{7.1}$$

for all $q_1, q_2 \in B(q_0, R)$ and all $z \in J(F)^c_T$. Since $J(F)_T$ is a compact set, there obviously exists $\delta_1 > 0$ so small that if $z \in J(F)_T$, $q_1, q_2 \in B(q_0, R)$, and $|q_2 - q_1| \le \delta_1$, then $\exp\Big(q_2\phi(z) + \psi(z)\Big) - \exp\Big(q_1\phi(z) + \psi(z)\Big)| < \varepsilon$. Looking at this and (7.1), we see that

$$||\exp\Big(q_2\phi(z) + \psi(z)\Big) - \exp\Big(q_1\phi(z) + \psi(z)\Big)||_\infty < \varepsilon/2 \tag{7.2}$$

for all $q_1, q_2 \in B(q_0, R)$ with $|q_2 - q_1| \le \delta_1$. Fix now an arbitrary point $q_1 \in B(q_0, R)$ and take $\delta_2 \in (0, \delta_1)$ so small that $B(q_1, \delta_2) \subset B(q_0, R)$. For all $q_2 \in B(q_1, \delta_2)$ and all $w, z \in J(F)$, we have

$$\Big(\exp(q_2\phi+\psi)-\exp(q_1\phi+\psi)\Big)(w)-\Big(\exp(q_2\phi+\psi)-\exp(q_1\phi+\psi)\Big)(z)=$$
$$=\exp(q_2\phi(w)+\psi(w))-\exp(q_1\phi(w)+\psi(w))-\exp(q_2\phi(z)+\psi(z))+\exp(q_1\phi(z)+\psi(z)).$$

Let $\alpha>0$ be a Hölder exponent of ϕ and ψ and let $H>0$ be their (common) Hölder constant. For every $z\in J(F)$ consider an auxiliary function $W_z:\mathbb{D}_{\mathbb{C}^3}(q_1,\phi(z),\psi(z);\delta_2,H\delta^\alpha,H\delta^\alpha)\to\mathbb{C}$ given by the formula

$$W_z(q,x,y)=\exp(q_2x+y)-\exp(q_1x+y)-\exp(q_2\phi(z)+\psi(z))+\exp(q_1\phi(z)+\psi(z)).$$

Since the domain of q_2 is contained in a compact subset of $\Sigma_1(\phi,\psi)$, we easily deduce that

$$s=\sup\{\mathrm{Re}(q_2)\phi(\xi)+\psi(\xi):q_2\in B(q_1,\delta_2),\xi\in J(F)\}<\infty\}.$$

In particular

$$|\exp(q_2\phi(z)+\psi(z))|,|\exp(q_1\phi(z)+\psi(z))|\le e^s. \tag{7.3}$$

If now $(q_2,x,y)\in\mathbb{D}_{\mathbb{C}^3}(q_1,\phi(z),\psi(z);\delta_2,H\delta^\alpha,H\delta^\alpha)\to\mathbb{C}$, then

$$\begin{aligned}|\exp(q_2x+y)|&=\exp(\mathrm{Re}(q_2x+y))\le\exp\Big(\mathrm{Re}(q_2\phi(z)+\psi(z))+|(q_2x+y)-(q_2\phi(z)+\psi(z))|\Big)\\&\le e^s\exp\Big(|q_2||x-\phi(z)|\Big)\exp(|y-\psi(z)|)\\&\le e^s\exp\Big((|q_1|+\delta_2)H^2\delta^{2\alpha}\Big).\end{aligned}$$

This and (7.3) give that for all $z\in J(F)$ and all $(q_2,x,y)\in\mathbb{D}_{\mathbb{C}^3}(q_1,\phi(z),\psi(z);\delta_2,H\delta^\alpha,H\delta^\alpha)$

$$|W_z(q_2,x,y)|\le M,$$

where $M=2e^s\Big(\exp\Big((|q_1|+\delta_2)H^2\delta^{2\alpha}\Big)+1\Big)$. Let $r=\min\{\delta_2,H\delta^\alpha\}$. It follows from Cauchy's estimates that

$$|D^\alpha W_z(q_1,\phi(z),\psi(z))|\le\frac{M\alpha!}{r^\alpha}$$

for every multiindex α (do not confuse this multiindex with the Hölder exponent α). Hence

$$W_z(q_2,x,y)=\sum_{\alpha_1,\alpha_2,\alpha_3}a_{\alpha_1,\alpha_2,\alpha_3}(q_2-q_1)^{\alpha_1}(x-\phi(z))^{\alpha_2}(y-\psi(z))^{\alpha_3},$$

where

$$|a_{\alpha_1,\alpha_2,\alpha_3}|\le\frac{M}{r^{\alpha_1+\alpha_2+\alpha_3}}. \tag{7.4}$$

Since $W_z(q_1,x,y)\equiv0$, we get $D^{\alpha_2\alpha_3}W_z(q_1,\phi(z),\psi(z))=0$ for all $\alpha_2,\alpha_3\ge0$. A direct calculation shows that $D^{\alpha_1}W_z(q_1,\phi(z),\psi(z))=0$ for every index $\alpha_1\ge0$. As a conclusion

$$W_z(q_2,x,y)=(q_2-q_1)\sum_{\alpha_1\ge1}\sum_{\substack{\alpha_2,\alpha_3\\ \alpha_2+\alpha_3\ge1}}a_{\alpha_1,\alpha_2,\alpha_3}(q_2-q_1)^{\alpha_1-1}(x-\phi(z))^{\alpha_2}(y-\psi(z))^{\alpha_3}. \tag{7.5}$$

If $|x-\phi(z)|,|y-\psi(z)|<r/2$, then

$$|((x-\phi(z))^{\alpha_2}(y-\psi(z))^{\alpha_3}|\le\Big(|x-\phi(z)|+|y-\psi(z)|\Big)(r/2)^{\alpha_2+\alpha_3-1}.$$

It therefore follows from (7.5) and (7.4) that

$$\begin{aligned}
|W_z(q_2,x,y)| &\le \\
&\le |q_2-q_1|\sum_{\alpha_1\ge1}\sum_{\substack{\alpha_2,\alpha_3\\ \alpha_2+\alpha_3\ge1}} Mr^{-(\alpha_1+\alpha_2+\alpha_3)}\left(\frac{r}{2}\right)^{\alpha_1-1}\Big(|x-\phi(z)|+|y-\psi(z)|\Big)\left(\frac{r}{2}\right)^{\alpha_2+\alpha_3-1} \\
&\le M|q_2-q_1|\Big(|x-\phi(z)|+|y-\psi(z)|\Big)\sum_{\alpha_1\ge1}\sum_{\substack{\alpha_2,\alpha_3\\ \alpha_2+\alpha_3\ge1}} r^{-2}4\cdot 2^{-(\alpha_1+\alpha_2+\alpha_3)} \\
&\le 4Mr^{-2}|q_2-q_1|\Big(|x-\phi(z)|+|y-\psi(z)|\Big)
\end{aligned}$$

for all $z\in J(F)$ and all $(q_2,x,y)\in I\!\!D_{\mathbb{C}^3}(q_1,\phi(z),\psi(z);r/2,r/2,r/2)$. Thus

$$\begin{aligned}
&\Big|\Big(\exp(q_2\phi+\psi)-\exp(q_1\phi+\psi)\Big)(w)-\Big(\exp(q_2\phi+\psi)-\exp(q_1\phi+\psi)\Big)(z)\Big| = \\
&\qquad = |W_z(q_2,\phi(w),\psi(w))| \le 4Mr^{-2}|q_2-q_1|\Big(|\phi(w)-\phi(z)|+|\psi(w)-\psi(z)|\Big) \\
&\qquad \le 8HMr^{-2}|q_2-q_1||w-z|^\alpha
\end{aligned}$$

for all $q_2\in B(q_1,r/2)$ and all $w,z\in J(F)$ with $|w-z|\le\left(\frac{r}{2H}\right)^{1/\alpha}$. This means that $\exp(q_2\phi+\psi)-\exp(q_1\phi+\psi)$ is a Hölder continuous function with a Hölder exponent α and the Hölder constant bounded above by $8HMr^{-2}|q_2-q_1|$. Combining this with (7.2), we conclude that the function $q_2\mapsto\exp(q_2\phi+\psi)\in \mathrm{H}_\alpha$ is countinuous at the point q_1. The proof is complete. ■

Thus the condition (b) of Proposition 7.1 is verified. Condition (a) is satisfied due to our definition of the set $\Sigma_1(\phi,\psi)$. Keep the the same notation as in the proof of Lemma 7.3. It follows from Lemma 3.2 that condition (c) is satisfied if q ranges over a bounded set; in particular if $q\in B(q_0,R)\subset\Sigma_1(\phi,\psi)$. The validity of the condition (d) that the function $q\mapsto\exp(q\phi(z)+\psi(z))$, $q\in\mathbb{C}$, is holomorphic for every $z\in J(F)$, is obvious. In order to check (e) notice that

$$\Big|\exp\Big((q\phi(z)+\psi(z))-(q_1\phi(z)+\psi(z))\Big)\Big| = \Big|\exp\Big((q-q_1)\phi(z)\Big)\Big| = \exp\Big(\phi(z)(\mathrm{Re}q-\mathrm{Re}q_1)\Big)$$

for all $q,q_1\in\mathbb{C}$ and all $z\in J(F)$. Fix now $q_2\in G:=B_{\mathbb{C}}(q_0,R_1)$ and $r>0$ so small that there exists a point $q_1\in G$ such that $\mathrm{Re}(q)>\mathrm{Re}(q_1)$ for all $q\in B(q_2,r)\subset B(q_0,R)$. We therefore obtain that for all $q\in B(q_2,r)$ and all $z\in J(F)$.

$$\Big|\exp\Big((q\phi(z)+\psi(z))-(q_1\phi(z)+\psi(z))\Big)\Big| \le \exp\Big(2R_1\sup\{\phi(\xi):\xi\in J(F)\}\Big).$$

The verification of condition (e) is thus finished and the proof Proposition 7.2 is complete. ■

Let us now derive some consequences of this proposition. We start with the following easy but useful fact resulting immediately from Hölder's inequality.

Lemma 7.4. *If ϕ and ψ are arbitrary tame functions, then the function $q \mapsto \mathrm{P}(q\phi+\psi)$, $q \in \Sigma_1(\phi,\psi)\cap I\!R$, is convex.*

For every $q \in \Sigma_1(\phi,\psi)\cap I\!R$ let $Q_{1,q} : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ be the projection operator associated to the operator $\hat{\mathcal{L}}_{q\phi+\psi}$ via Theorem 4.4. Let

$$S_q = \hat{\mathcal{L}}_{q\phi+\psi} - Q_{1,q}$$

be the difference operator appearing in Theorem 4.4 and let

$$g_q = g_{q\phi+\psi}$$

be the eigenfunction of $\mathcal{L}_{q\phi+\psi}$ (fixed point of $\hat{\mathcal{L}}_{q\phi+\psi}$) also appearing in Theorem 4.4. Usinng heavily Theorem 7.2 and the perturbation theory for linear operators (see [Ka] for its account), we shall prove the following.

Lemma 7.5. *If ϕ and ψ are arbitrary tame functions, then all the four functions $q \mapsto \mathrm{P}(q\phi+\psi), Q_{1,q}, S_q, g_q,\ \ q \in \Sigma_1(\phi,\psi)\cap I\!R$, are real-analytic.*

Proof. Fix $q_0 \in \Sigma_1(\phi,\psi)\cap I\!R$. Applying now Theorem 7.2, the perturbation theory for linear operators (see [Ka]), Theorem 4.4, and Proposition 3.10, we see that there exist $R_1 > 0$ (so small that $B_{\mathbb{C}}(q_0,R_1) \subset \Sigma_1(\phi,\psi)$) and three holomorphic functions $\gamma : B_{\mathbb{C}}(q_0,R_1) \to \mathbb{C}$, $Q : B_{\mathbb{C}}(q_0,R_1) \to L(\mathrm{H}_\alpha)$ and $g : B_{\mathbb{C}}(q_0,R_1) \to \mathrm{H}_\alpha$ such that $\gamma(q_0) = e^{\mathrm{P}(q_0\phi+\psi)}$ and $g(q_0) = g_{q_0\phi+\psi}$, for every $q \in B_{\mathbb{C}}(q_0,R_1)$ the number $\gamma(q)$ is a simple isolated eigenvalue of the operator $\mathcal{L}_{q\phi+\psi}$ with the remainder part of the spectrum uniformly separated from $\gamma(q)$, $g(q)$ is its normalized eigenfunnction, and $Q(q) : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ is the projection operator corresponding to the eigenvalue $\gamma(q)$. In particular there exist $0 < R_2 \le R_1$ and $\eta > 0$ such that

$$\Sigma_1(\mathcal{L}_{q\phi+\psi}) \cap B\Big(\exp\Big(\mathrm{P}(q_0\phi+\psi)\Big),\eta\Big) = \{\gamma(q)\} \tag{7.6}$$

for all $q \in B_{\mathbb{C}}(q_0),R_2)$. In view of Lemma 7.4 there thus exists $R_3 \in (0,R_2]$ such that $\mathrm{P}(q\phi+\psi) = \log(\gamma(q))$ for all $B_{I\!R}(q_0,R_3)$. Consequently also $Q_{1,q} = \exp\Big(-(\mathrm{P}(q\phi+\psi)\Big)Q(q)$ for all $q \in B_{I\!R}(q_0,R_3)$ and $g(q) = g_{q\phi+\psi}$. The proof is now completed by noting that $S_q = \exp\Big(-(\mathrm{P}(q\phi+\psi)\Big)\mathcal{L}_{q\phi+\psi} - Q_{1,q}$. ■

Put

$$\Sigma_2(\phi,\psi) = \{(q,t) \in \mathbb{C}\times\mathbb{C} : \mathrm{Re}(q)\kappa + \mathrm{Re}(t)\gamma > 1\}\,.$$

We will also need the following, strictly speaking stronger, result.

Lemma 7.6. *If ϕ and ψ are arbitrary two tame functions, then all the four functions $(q,t) \mapsto \mathrm{P}(q\phi+t\psi), Q_{1,(q,t)}, S_{q,t}, g_{q,t}$, where $(q,t) \in \Sigma_2(\phi,\psi)$, (the objects $Q_{1,(q,t)}, S_{q,t}, g_{q,t}$ have obvious meaning) are real-analytic.*

Proof. It follows from Proposition 7.2 and Hartogs theorem that the function $(q,t) \mapsto \mathcal{L}_{q\phi+t\psi}$, $q \in \Sigma_2(\phi,\psi)$, is holomorphic. From now on the proof goes with obvious modifications exactly as the proof of Lemma 7.5. ■

Lemma 7.7. *For every $q_0 \in \Sigma_1(\phi,\psi) \cap I\!R$ there exist $\eta > 0$, $C > 0$ and $\theta \in (0,1)$ such that $\overline{B}(q_0,\eta) \subset \Sigma_1(\phi,\psi)$,*

$$||S_q^n||_\alpha \le C\theta^n, \ ||\hat{\mathcal{L}}_{q\phi+\psi}^n||_\alpha \le C \text{ and } ||g_{q\phi+\psi}||_\alpha \le C.$$

for all $q \in B_{I\!R}(q_0,\eta)$ and all $n \ge 0$.

Proof. It follows from Theorem 4.4(d) that there exists $u \ge 1$ such that $||S_{q_0}^u||_\alpha \le 1/8$. Hence, in view of Lemma 7.5, there exists $\eta > 0$ so small that $\overline{B}_{I\!R}(q_0,\eta) \subset \Sigma_1(\phi,\psi) \cap I\!R$ and $||S_q^u||_\alpha \le 1/4$ for all $q \in B_{I\!R}(q_0,\eta)$. Using again Lemma 7.5 we see that $||S_q||_\alpha \le M$ for all $q \in B_{I\!R}(q_0,\eta)$ and some $M \ge 1$. Hence $||S_q^j||_\alpha \le M^u$ for all $q \in B_{I\!R}(q_0,\eta)$ all $j = 0,1,\dots,u-1$. A starightforward induction shows now that there exists a constant $C_1 > 0$ such that $||S_q^n||_\alpha \le C_1(1/2)^{n/u}$ for all $n \ge 0$. Taking $\eta > 0$ sufficiently small, it follows immediately from Lemma 7.5 that $||Q_{1,q}||_\alpha \le C_2$ for some $C_2 > 0$ and all $q \in B(q_0,\eta)$. Hence $||\hat{\mathcal{L}}_{q\phi+\psi}^n||_\alpha \le ||Q_{1,q}||_\alpha + ||S_q^n||_\alpha \le C_2 + C_1$. Taking $C = C_1 + C_2$, we are therefore done. ■

As a consequence of this lemma and Lemma 7.5, we shall prove the following strenghtening of Theorem 5.3.

Corollary 7.8. *Fix $q_0 \in \Sigma_1(\phi,\psi) \cap I\!R$ and let η, θ and C come from Lemma 7.7. If $u \in \mathrm{H}_\alpha$, $q \in (q_0-\delta, q_0+\delta)$, and $v \in L^1_{m_{q\phi+\psi}}$, then for all $n \ge 0$*

$$C_{q,n}(u,v) \le 2C^2(1+C)\theta^n ||u||_\alpha ||v||_{L^1_{m_{q\phi+\psi}}},$$

where $C_{q,n}$ is the corellation function with respect to the measure $\mu_{q\phi+\psi}$.

Proof. Write $\mu_q = \mu_{q\phi+\psi}$, $m_q = m_{q\phi+\psi}$, $g_q = g_{q\phi+\psi}$, $U = u - \mu_q(u) = u - \int g_q u\, dm_q$ and $V = v - \mu_q(v) = v - \int g_q v\, dm_q$. Using then Theorem 4.4 and Lemma 7.7, we obtain for every $n \ge 0$ that

$$\begin{aligned}
C_{q,n}(u,v) &= \left|\int U \cdot (V \circ F^n)\, d\mu_q\right| = \left|\int U g_q \cdot (V \circ F^n)\, dm_q\right| = \left|\hat{\mathcal{L}}_q^n\Big(U g_q \cdot (V \circ F^n)\Big) dm_q\right| \\
&= \left|\int V \cdot \mathcal{L}_q^n(U g_q)\, dm_q\right| = \left|\int V S_q^n(U g_q)\, dm_q\right| \le \int |V||S_q^n(U g_q)|\, dm_q \\
&\le ||S_q^n(U g_q)||_\infty \int |V|\, dm_q \le ||S_q^n(U g_q)||_\alpha ||V||_{L^1_{m_q}} \\
&\le ||S_q^n||_\alpha ||U g_q||_\alpha (1 + ||g_q||_\infty) ||v||_{L^1_{m_q}} \le C\theta^n 2||u|||_a ||g_q|||_a (1+C) ||v||_{L^1_{m_q}} \\
&\le 2C^2(1+C)\theta^n ||u|||_a ||v||_{L^1_{m_q}}
\end{aligned}$$

We are done. ■

8. **Derivatives of the Pressure function**

In this section we derive the formulas for the first and second derivatives of the pressure function. Throughout the entire section $\phi : J(F) \to I\!R$, a 1^+-tame function, and $\psi : J(F) \to I\!R$, a β-tame function, $\beta \in I\!R$, are fixed. We start with the following.

Lemma 8.1. *Suppose that $\phi : J(F) \to I\!R$ is a 1^+-tame function, $\psi : J(F) \to I\!R$ is a β-tame function, $\beta \in I\!R$, and that $\zeta : J(F) \to I\!R$ is a tame function. Denote their common Hölder exponent by α. Then there exists $\eta > 0$ and $\Gamma(\zeta) > 0$ such that if $|t| < \eta$ and $G \in \mathrm{H}_\alpha$, then $\hat{\mathcal{L}}_{\phi+t\psi}(\zeta G) \in \mathrm{H}_\alpha$ and moreover $||\hat{\mathcal{L}}_{\phi+t\psi}(\zeta G)||_\alpha \le \Gamma(\zeta)||G||_\alpha$. In addition $\Gamma(1\!\!1) \le C$ and $\Gamma(a\zeta + b\omega) \le |a|\Gamma(\zeta) + |b|\Gamma(\omega)$ for all $a, b \in I\!R$ and all tame functions ω with the Hölder exponent α.*

Proof. Let $\kappa = \kappa(\phi)$. Take $\eta \in (0, 1]$ so small that $\rho := \kappa - \eta|\beta| > 1$. Put for every $t \in I\!R$, $\hat{\mathcal{L}}_t = \hat{\mathcal{L}}_{\phi+t\psi}$. Consider $G \in \mathrm{H}_\alpha$. We shall show first that for all $t \in (-\eta, \eta)$, the norm $||\hat{\mathcal{L}}_t(\zeta G)||_\infty$ is bounded above by a constant depending only on ζ (actually also on ϕ and ψ but, we recall, that we treat these functions as fixed in this section). Fix an arbitrary $\gamma \ge 0$ sach that $|\kappa(\zeta)| \le \gamma$. Using Lemma 7.7, for every $t \in (-\eta, \eta)$ and every $z \in J(F)$ we have that

$$
\begin{aligned}
|\hat{\mathcal{L}}_t(\zeta G)(z)| &\le \sum_{k=-\infty}^{\infty} \exp\Big(\phi(z_k) + t\psi(z_k)\Big)|\zeta(z_k)||G(z_k)| \\
&\le \sum_{k=-\infty}^{\infty} \exp\Big(A_\phi + |t|A_\psi - \kappa\mathrm{Re}(z_k) + |\beta||t|\mathrm{Re}(z_k)\Big)||G||_\infty(A_\zeta + \gamma\mathrm{Re}(z_k)) \\
||G||_\alpha &\le \exp\Big(A_\phi + A_\psi\Big) \sum_{k=-\infty}^{\infty} \exp\Big(-(\kappa - |\beta||t|)\mathrm{Re}(z_k)\Big)(A_\zeta + \gamma\mathrm{Re}(z_k)) \qquad (8.1) \\
&\le ||G||_\alpha \exp\Big(A_\phi + A_\psi\Big) \sum_{k=-\infty}^{\infty} \exp\Big(-\rho\mathrm{Re}(z_k)\Big)(A_\zeta + \gamma\mathrm{Re}(z_k)),
\end{aligned}
$$

where z_k is the only point of the singleton $P \cap f^{-1}(\tilde{z} + 2\pi ik)$ and $\tilde{z}$ is an arbitrary point in $\pi^{-1}(z)$. Then $\mathrm{Re}(z_k) = \log|\tilde{z} + 2\pi ik| - \log|\lambda|$ and, since $\inf\{\mathrm{Re}(J(F))\} > 0$, we can continue (8.1), with appropriately large $M_1 > 0$, as follows

$$
\begin{aligned}
|\hat{\mathcal{L}}_t(\zeta G)(z)| &\le ||G||_\alpha \exp\Big(A_\phi + A_\psi\Big) \sum_{k=-\infty}^{\infty} |\lambda|^\rho |\tilde{z} + 2\pi ik|^{-\rho}(\gamma + M_1)\log|\tilde{z} + 2\pi ik| \\
&\le ||G||_\alpha \exp\Big(A_\phi + A_\psi\Big)\lambda|^\rho(\gamma + M_1) \sum_{k=-\infty}^{\infty} |\tilde{z} + 2\pi ik|^{-\rho}\log|\tilde{z} + 2\pi ik| \qquad (8.2) \\
&\le T||G||_\alpha
\end{aligned}
$$

for some $T < \infty$ independent of z since $\rho > 1$ and $\inf\{\mathrm{Re}(J(F))\} > 0$. Passing to the estimate of the α-variation of $\hat{\mathcal{L}}_t(\zeta G)$, observe that if $x, y \in J(F)$ and $|x-y| \le \delta$, then

$$\begin{aligned}|\zeta(y)G(y) - \zeta(x)G(x)| &\le |\zeta(y)||G(y) - G(x)| + |G(x)||\zeta(y) - zeta(x)| \\ &\le |\zeta(y)|||G||_\alpha |y-x|^\alpha + ||G||_\alpha v_\alpha(\zeta)|y-x|^\alpha \\ &= ||G||_\alpha \Big(|\zeta(y)| + v_\alpha(\zeta)\Big)|y-x|^\alpha.\end{aligned}$$

Now, by Lemma 3.2, the function $\phi + t\psi$ is dynamically Hölder, and it easily follows from the proof of this lemma that $M = \sup\{c_{\phi+t\psi} : |t| \le \eta\} < \infty$. Threfore, using the formula above,(8.2), the calculations involved in (8.1) and in (8.2), and Lemma 2.1, we get

$$\begin{aligned}|\hat{\mathcal{L}}_t(\zeta G)(y) - \hat{\mathcal{L}}_t(\zeta G)(x)| = \\ &= \Big\| \sum_{k=-\infty}^{\infty} \exp\Big(\phi(y_k) t\psi(y_k)\Big)|\zeta(y_k)||G(z_k)| - \exp\Big(\phi(x_k) + t\psi(x_k)\Big)|\zeta(x_k)||G(x_k)|\Big\| \\ &\le \sum_{k=-\infty}^{\infty} |\zeta(y_k)||G(z_k)\Big| \exp\Big(\phi(y_k) + t\psi(y_k)\Big) - \exp\Big(\phi(x_k) + t\psi(x_k)\Big)\Big| + \\ &+ \sum_{k=-\infty}^{\infty} \exp\Big(\phi(x_k) + t\psi(x_k)\Big)|\zeta(y_k)G(y_k) - \zeta(x_k)G(x_k)| \\ &\le \sum_{k=-\infty}^{\infty} (A_\zeta + \gamma \mathrm{Re}(z_k))||G||_\alpha M \exp\Big(\phi(y_k) + t\psi(y_k)\Big)|y-x|^\alpha + \\ &+ \sum_{k=-\infty}^{\infty} \exp\Big(-\rho \mathrm{Re}(z_k)\Big)||G||_\alpha \Big(|\zeta(x_k)| + v_\alpha(\zeta)\Big)|y_k - x_k|^\alpha \\ &\le M\tau \sum_{k=-\infty}^{\infty} (A_\zeta + \gamma \mathrm{Re}(y_k)) \exp\Big(-\rho \mathrm{Re}(y_k)\Big)|y-x|^\alpha + \\ &+ c^{-1}||G||_\alpha \sum_{k=-\infty}^{\infty} \exp\Big(-\rho \mathrm{Re}(z_k)\Big)\Big(A_\zeta + \gamma \mathrm{Re}(x_k) + ||G||_\alpha v_\alpha(\zeta)\Big)|y-x|^\alpha \\ &\le |y-x|^\alpha \Big(M\tau \sum_{k=-\infty}^{\infty} (\gamma + M_1) \log|\tilde{y} + 2\pi ik||\lambda|^\rho |\tilde{y} + 2\pi ik|^{-\rho} + \\ &+ c^{-1}||G||_\alpha \sum_{k=-\infty}^{\infty} |\lambda|^\rho |\tilde{x} + 2\pi ik|^{-\rho} + (\gamma + M_1)\log|\tilde{x} + 2\pi ik|\Big) \\ &\le ||G||_\alpha |\lambda|^\rho (\gamma + M_1)|y-x|^\alpha M \sum_{k=-\infty}^{\infty} |\tilde{y} + 2\pi ik|^{-\rho} \log|\tilde{y} + 2\pi ik| + \\ &+ c^{-1} \sum_{k=-\infty}^{\infty} |\tilde{x} + 2\pi ik|^{-\rho} \log|\tilde{x} + 2\pi ik| \\ &\le |\lambda|^\rho (\gamma + M_1)(M + c^{-1})T||G||_\alpha |y-x|^\alpha,\end{aligned}$$

where $M_1 > 0$ is assumed to be so large that $A_\zeta + \gamma\mathrm{Re}(w) + v_\alpha(\zeta) \le (\gamma + M_1)\mathrm{Re} w$ for all $w \in J(F)$. Combining this and (8.2), the proof of the first part of our lemma is complete. The second part follows immediately from Lemma 7.7. The last part is an immediate consequence of of linearity of the operator $\hat{\mathcal{L}}_t$. ■

Remark 8.2. *Notice that if ζ_1 and ζ_2 are two α-Hölder functions, then for all $x, y \in J(F)$ with $|y - x| \le \delta$, we have*

$$\begin{aligned}|\zeta_1\zeta_2(y) - \zeta_1\zeta_2(x)| &= |\zeta_1(y)(\zeta_2(y) - \zeta_2(x)) + \zeta_2(x)(\zeta_1(y) - \zeta_1(x))| \\ &\le |\zeta_1(y)||\zeta_2(y) - \zeta_2(x)| + |\zeta_2(x)||\zeta_1(y) - \zeta_1(x)| \\ &\le |\zeta_1(y)|v_\alpha(\zeta_2)|y-x|^\alpha + |\zeta_2(x)|v_\alpha(\zeta_1)|y-x|^\alpha \\ &\le 2\max\{v_\alpha(\zeta_1), v_\alpha(\zeta_2)\}\max\{|\zeta_2(x)|, |\zeta_1(y)|\}|y-x|^\alpha.\end{aligned}$$

Therefore, the proof of Lemma 8.1 goes through with obvious modifications with ζ replaced by the product of any two tame functions.

Before we formulate the next lemma, observe that the absolute value of a tame function is is also tame (in fact if ρ is tame, then $\kappa(|\rho|) = |\kappa(\rho)|$), and consequently, the absolute value of the product of two tame functions is a product of two tame functions.

Lemma 8.3. *Assume that ρ is either a tame function or a product of two tame functions. With the assumptions and notation as in Lemma 8.1, we have that*

$$\int |\rho| dm_{\phi+t\psi} \le \Gamma(|\rho|)$$

for all $t \in (-\eta, \eta)$.

Proof Indeed, in view Lemma 8.1

$$\int |\rho| dm_{\phi+t\psi} = \int \hat{\mathcal{L}}_{\phi+t\psi}(|\rho|) dm_{\phi+t\psi} \le ||\hat{\mathcal{L}}_{\phi+t\psi}(|\rho|)||_\infty \le ||\hat{\mathcal{L}}_{\phi+t\psi}(|\rho|)||_\alpha \le \Gamma(|\rho|).$$

We are done. ■

We are now able to establish the following strengthening of Corollary 7.8.

Lemma 8.4. *Fix $t_0 \in \Sigma_1(\psi, \phi)$ and let η, θ, and C come from Lemma 7.7. If each function ζ, ρ is either a tame function or a product of two tame functions and if $s \in (-\eta, \eta)$, then for all $n \ge 1$,*

$$C_{s,n}(\zeta, \rho) \le C^2(1 + C)\Gamma(|\rho|)(\Gamma(\zeta) + C^2\Gamma(|\zeta|))\theta^{n-1},$$

where $C_{s,n}$ is the corellation function with respect to the measure $\mu_{\phi+s\psi}$.

Proof. We use the obvious notation μ_s, m_s, and g_s. We put $\overline{\zeta} = \zeta - \mu_s(\zeta)$ and $\overline{\rho} = \rho - \mu_s(\rho)$. Notice that by Lemma 8.3 and Lemma 7.7, we have

$$\begin{aligned}\int |\overline{\rho}| dm_s &\le \int (|\rho| + |\mu_s(\rho|) dm_s \le \int |\rho| dm_s + \int |\rho| g_s dm_s \le \Gamma(|\rho|) + C \int |\rho| dm_s \\ &\le (1 + C)\Gamma(|\rho|). \end{aligned} \tag{8.3}$$

In view of Lemma 8.1, the estimate $|\mu_s(\zeta)| \le C\Gamma(|\zeta|)$ obtained in the computation of the previous formula

$$\Gamma(\overline{\zeta}) \le \Gamma(\zeta) + |\mu_s(\zeta)|\Gamma(1\!\!1) \le \Gamma(\zeta) + C^2\Gamma(|\zeta|). \tag{8.4}$$

Proceeding now exactly as in the proof of Corollary 7.8, utilising (8.3), Lemma 7.7, Lemma 8.1, Remark 8.2, and (8.4)

$$\begin{aligned} C_{s,n}(\zeta, \rho) &= \left| \int \overline{\rho} \hat{\mathcal{L}}_s^n (\overline{\zeta} g_s) dm_s \right| = \left| \int \overline{\rho} \hat{\mathcal{L}}_s^{n-1} \Big(\hat{\mathcal{L}}_s (\overline{\zeta} g_s) \Big) dm_s \right| = \left| \int \overline{\rho} S_s^{n-1} \Big(\hat{\mathcal{L}}_s (\overline{\zeta} g_s) \Big) dm_s \right| \\ &\le \int |\overline{\rho}| ||S_s^{n-1} \Big(\hat{\mathcal{L}}_s (\overline{\zeta} g_s) \Big)||_\alpha = ||S_s^{n-1} \Big(\hat{\mathcal{L}}_s (\overline{\zeta} g_s) \Big)||_\alpha \int |\overline{\rho}| dm_s \\ &\le ||S_s^{n-1}||_\alpha ||\hat{\mathcal{L}}_s \Big(\overline{\zeta} g_s \Big)||_\alpha (1 + C)\Gamma(|\rho|) \le C(1 + C)\Gamma(|\rho|)\theta^{n-1} ||\hat{\mathcal{L}}_s \Big(\overline{\zeta} g_s \Big)||_\alpha \\ &\le C(1 + C)\Gamma(|\rho|)\theta^{n-1} \Gamma(\overline{\zeta}) ||g_s||_\alpha \le C^2(1 + C)\Gamma(|\rho|)(\Gamma(\zeta) + C^2\Gamma(|\zeta|))\theta^{n-1} \end{aligned}$$

We are done. ■

Lemma 8.5. *Assume the same as in Lemma 8.1. Fix $x \in J(F)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \frac{\sum_{y \in F^{-n}(x)} S_n\zeta(y) \exp\Big(S_n(\phi + t\psi)(y) \Big)}{\sum_{y \in F^{-n}(x)} \exp\Big(S_n(\phi + t\psi)(y) \Big)} = \int \zeta dm_{\phi + t\psi}$$

uniformly with respect to all $t \in (-\eta, \eta)$, where η comes from Lemma 8.1.

Proof. Put for every $t \in I\!\!R$,

$$\mathcal{L}_t = \mathcal{L}_{\phi + t\psi} \text{ and } \hat{\mathcal{L}}_t = \hat{\mathcal{L}}_{\phi + t\psi}$$

and

$$g_t = g_{\phi + t\psi} = Q_{1,t}(1\!\!1)$$

Observe that

$$\frac{1}{n} \sum_{j=0}^{n-1} \frac{\hat{\mathcal{L}}_t^n (\zeta \circ F^j)(x)}{\hat{\mathcal{L}}_t^n (1\!\!1)(x)} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{\hat{\mathcal{L}}_t^{n-j} \Big(\zeta \hat{\mathcal{L}}_t^j (1\!\!1) \Big)(x)}{\hat{\mathcal{L}}_t^n (1\!\!1)(x)} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{\hat{\mathcal{L}}_t^{n-j-1} \Big(\hat{\mathcal{L}}_t \Big(\zeta \hat{\mathcal{L}}_t^j (1\!\!1) \Big) \Big)(x)}{\hat{\mathcal{L}}_t^n (1\!\!1)(x)}. \tag{8.5}$$

Therefore

$$
\begin{aligned}
\frac{1}{n}\sum_{j=0}^{n-1}\frac{\hat{\mathcal{L}}_t^n(\zeta\circ F^j)(x)}{\hat{\mathcal{L}}_t^n(1\!\!1)(x)} - \int \zeta dm_{\phi+t\psi} = \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\frac{\hat{\mathcal{L}}_t^{n-j-1}\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big)(x)}{g_t(x)} + \\
&+ \frac{1}{n}\sum_{j=0}^{n-1}\frac{\hat{\mathcal{L}}_t^{n-j-1}\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big)(x)\Big(Q_{1,t}(1\!\!1)(x)-\hat{\mathcal{L}}_t^n(1\!\!1)(x)\Big)g_t(x)}{\hat{\mathcal{L}}_t^n(1\!\!1)(x)} - \\
&- \frac{1}{n}\sum_{j=0}^{n-1}\int \hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)dm_{\phi+t\psi} + \\
&+ \left(\frac{1}{n}\sum_{j=0}^{n-1}\int \hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)dm_{\phi+t\psi} - \int \zeta dm_{\phi+t\psi}\right) \qquad (8.6) \\
&= g_t^{-1}(x)\frac{1}{n}\sum_{j=0}^{n-1}\Big(\hat{\mathcal{L}}_t^{n-j-1}\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big)(x) - \\
&- g_t(x)\int \hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)dm_{\phi+t\psi}\Big) + \\
&+ \frac{1}{n}\frac{\sum_{j=0}^{n-1}\Big(\hat{\mathcal{L}}_t^{n-j-1}\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big)(x)\Big(Q_{1,t}(1\!\!1)(x)-\hat{\mathcal{L}}_t^n(1\!\!1)(x)\Big)\Big)}{g_t(x)\hat{\mathcal{L}}_t^n(1\!\!1)(x)} + \\
&+ \frac{1}{n}\sum_{j=0}^{n-1}\int \zeta\hat{\mathcal{L}}_t^j(1\!\!1)dm_{\phi+t\psi} - \int \zeta dm_{\phi+t\psi}.
\end{aligned}
$$

It immediately follows from Lemma 7.7 and Lemma 8.1 (with G being of the form $\hat{\mathcal{L}}_t^j(1\!\!1)$) that the α-Hölder norm (and so the supremum norm as well) of the second summand converges to zero uniformly with respect to $t\in(-\eta,\eta)$. Dealing with the first summand notice that, in view of Lemma 8.1, we get that

$$
\begin{aligned}
||\hat{\mathcal{L}}_t^{n-j-1}\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big) - g_t\int \hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)dm_{\phi+t\psi}||_\alpha = \\
&= ||\hat{\mathcal{L}}_t^{n-j-1}\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big) - Q_{1,t}\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big)||_\alpha \\
&= ||\Big(\hat{\mathcal{L}}_t^{n-j-1} - Q_{1,t}\Big)\Big(\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)\Big)||_\alpha \\
&\le ||\hat{\mathcal{L}}_t^{n-j-1} - Q_{1,t}||_\alpha||\hat{\mathcal{L}}_t\Big(\zeta\hat{\mathcal{L}}_t^j(1\!\!1)\Big)||_\alpha \le C(\zeta,(1+C))||\hat{\mathcal{L}}_t^{n-j-1} - Q_{1,t}||_\alpha
\end{aligned}
$$

Therefore, applying Lemma 7.7, we see that the first summand in the last part of (8.6) converges to zero uniformly with respect to $t\in(-\eta,\eta)$. It follows immediately from Lemma 8.3 and Lemma 7.7 so does the third summand in the last part of (8.6). We are done. ■

Theorem 8.6. *Suppose that $\phi : J(F) \to I\!\!R$ is a 1^+-tame function and $\psi : J(F) \to I\!\!R$ is a tame function. Then*

$$\frac{d}{dt}\Big|_{t=0} \mathrm{P}(\phi + t\psi) = \int \psi d\mu_\phi.$$

Proof. Fix $x \in J(F)$. Put

$$\mathrm{P}_n(t) = \frac{1}{n} \log \sum_{y \in F^{-n}(x)} \exp\Big(S_n(\phi + t\psi)(y)\Big).$$

Then

$$\frac{d\mathrm{P}_n}{dt} = \frac{1}{n} \frac{\sum_{y \in F^{-n}(x)} S_n\psi(y) \exp\Big(S_n(\phi + t\psi)(y)\Big)}{\sum_{y \in F^{-n}(x)} \exp\Big(S_n(\phi + t\psi)(y)\Big)},$$

and, in view of Lemma 8.5, $\frac{d\mathrm{P}_n}{dt}$ converges uniformly with respect to $t \in (-\eta, \eta)$ to $\int \psi d\mu_{\phi+t\psi}$. Since, in addition, $\lim_{n\to\infty} \mathrm{P}_n(t) = \mathrm{P}(\phi + t\psi)$, we conclude that $\frac{d\mathrm{P}}{dt} = \int \psi d\mu_{\phi+t\psi}$ for all $t \in (-\eta, \eta)$. Taking $t = 0$ we are therefore done. ■

We are now in position to prove the following.

Theorem 8.7. *Suppose that $\phi : J(F) \to I\!\!R$ is a 1^+-tame function and $\psi, \zeta : J(F) \to I\!\!R$ are tame functions. Then*

$$\frac{\partial^2}{\partial s \partial t}\Big|_{(0,0)} \mathrm{P}(\phi + s\psi + t\zeta) = \sigma^2(\psi, \zeta),$$

where

$$\begin{aligned}\sigma^2(\psi,\zeta) &= \lim_{n\to\infty} \frac{1}{n} \int S_n\Big(\psi - \mu_\phi(\psi)\Big) S_n\Big(\zeta - \mu_\phi(\zeta)\Big) d\mu_\phi \\ &= \int (\psi - \mu_\phi(\psi))(\zeta - \mu_\phi(\zeta)) d\mu_\phi + \sum_{k=1}^{\infty} \int (\psi - \mu_\phi(\psi))(\zeta - \mu_\phi(\zeta)) \circ F^k d\mu_\phi + \\ &\qquad \sum_{k=1}^{\infty} \int (\zeta - \mu_\phi(\zeta))(\psi - \mu_\phi(\psi)) \circ F^k d\mu_\phi\end{aligned}$$

(if $\psi = \zeta$ we simply write $\sigma^2(\psi)$ for $\sigma^2(\psi, \psi)$)

Proof. Put

$$\hat{\mathcal{L}}_s = \hat{\mathcal{L}}_{\phi + s\psi}.$$

The symbols m_s, μ_s and g_s have also the corresponding obvious meaning. It follows from the proof of Theorem 8.6 that

$$\frac{\partial^2}{\partial s \partial t}\Big|_{t=0} \mathrm{P}(\phi + s\psi + t\zeta) = \frac{d}{ds} \lim_{n\to\infty} \frac{1}{n} \frac{\sum_{y \in F^{-n}(x)} S_n\zeta(y) \exp\Big(S_n(\phi + s\psi)(y)\Big)}{\sum_{y \in F^{-n}(x)} \exp\Big(S_n(\phi + s\psi)(y)\Big)} \tag{8.7}$$

for all s sufficiently small in absolute value. Fix $x \in J(F)$, $n \geq 1$ and abbreviate the notation $\sum_{y \in F^{-n}(x)}$ to $\sum_y$. We have this

$$
\begin{aligned}
\Delta_n(s) &:= \frac{d}{ds}\left(\frac{\sum_y S_n\zeta(y)\exp\big(S_n(\phi+s\psi)(y)\big)}{\sum_y \exp\big(S_n(\phi+s\psi)(y)\big)}\right) \\
&= \frac{\sum_y S_n\psi(y)S_n\zeta(y)\exp\big(S_n(\phi+s\psi)(y)\big)}{\sum_y \exp\big(S_n(\phi+s\psi)(y)\big)} - \\
&- \frac{\Big(\sum_y S_n\psi(y)\exp\big(S_n(\phi+s\psi)(y)\big)\Big)\sum_y S_n\zeta(y)\exp\big(S_n(\phi+s\psi)(y)\big)}{\Big(\sum_y \exp\big(S_n(\phi+s\psi)(y)\big)\Big)^2} \\
&= \frac{\hat{\mathcal{L}}_s^n\big(S_n\psi S_n\zeta\big)(x)}{\hat{\mathcal{L}}_s^n(1\!\!1)(x)} - \frac{\hat{\mathcal{L}}_s^n\psi(x)\hat{\mathcal{L}}_s^n\zeta(x)}{\hat{\mathcal{L}}_s^n(1\!\!1)(x)\hat{\mathcal{L}}_s^n(1\!\!1)(x)} = \int S_n\psi S_n\zeta\, d\mu_{s,n} - \int S_n\psi\, d\mu_{s,n}\int S_n\zeta\, d\mu_{s,n} \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int\Big(\psi\circ F^i - \mu_{s,n}(\psi\circ F^i)\Big)\Big(\zeta\circ F^i - \mu_{s,n}(\zeta\circ F^i)\Big)d\mu_{s,n},
\end{aligned}
$$

where

$$
\mu_{s,n} = \frac{\sum_y \delta_y \exp\big(S_n(\phi+s\psi)(y)\big)}{\sum_y \exp\big(S_n(\phi+s\psi)(y)\big)}
$$

and δ_y is the Dirac measure supported at y. In order to simplify notation put $\psi_i = \psi\circ F^i$, $\zeta_j = \zeta\circ F^j$ and

$$
K_{i,j} = \int\Big(\psi_i - \mu_{s,n}(\psi_i)\Big)\Big(\zeta_j - \mu_{s,n}(\zeta_j)\Big)d\mu_{s,n} = \frac{\hat{\mathcal{L}}_s^n\Big(\big(\psi_i - \mu_{s,n}(\psi_i)\big)\big(\zeta_j - \mu_{s,n}(\zeta_j)\big)\Big)}{\hat{\mathcal{L}}_s^n(1\!\!1)(x)}.
$$

Fix now $0 \leq i \leq j \leq n-1$. Then

$$
K_{i,j} = \frac{\hat{\mathcal{L}}_s^{n-j-1}\Big(\hat{\mathcal{L}}_s\Big(\hat{\mathcal{L}}_s^{j-i}\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta - \mu_{s,n}(\zeta_j))\Big)\Big)(x))}{\hat{\mathcal{L}}_s^n(1\!\!1)(x)}.
$$

If $j > i$, it follows from Lemma 7.7 and Lemma 8.1 that for every $s \in (-\eta, \eta)$, we have

$$
\begin{aligned}
||\hat{\mathcal{L}}_s^{j-i}\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1) - g_s \int (\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1) dm_s||_\alpha = \\
&= ||\hat{\mathcal{L}}_s^{j-i-1}\Big(\hat{\mathcal{L}}_s\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^j(1\!\!1)\Big) - g_s \int \hat{\mathcal{L}}_s\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^j(1\!\!1)\Big) dm_s||_\alpha \\
&= ||S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)\Big)\Big)||_\alpha \\
&\le C\theta^{j-i-1}||\hat{\mathcal{L}}_s\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)\Big)||_\alpha \qquad (8.8) \\
&\le C\theta^{j-i-1}\Big(||\hat{\mathcal{L}}_s\Big((\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big)||_\alpha + \mu_{s,n}(\psi_i)||\hat{\mathcal{L}}_s^{i+1}(1\!\!1)\Big)||_\alpha\Big) \\
&\le C\theta^{j-i-1}\Big(C\Gamma(\psi) + C|\mu_{s,n}(\psi_i)|\Big) = C^2\theta^{j-i-1}\Big(\Gamma(\psi) + |\mu_{s,n}(\psi_i)|\Big)
\end{aligned}
$$

In view of Theorem 4.4 there exists $n_1 \ge 1$ such that for all $n \ge 1$, $\hat{\mathcal{L}}_s^n(1\!\!1)(x) \ge g_s(x)/2$, and in view of Lemma 7.5, $g_s(x) \ge g(x)/2$ assuming $\eta > 0$ to be small enough. Denote $g_0(x)$ by g. Using Lemma 7.7 and Lemma 8.1, we then get

$$
\begin{aligned}
|\mu_{s,n}(\psi_i))| &= \frac{|\hat{\mathcal{L}}_s^n\psi_i(x)|}{|\hat{\mathcal{L}}_s^n 1\!\!1(x)|} = \frac{|\hat{\mathcal{L}}_s^{n-i}\hat{\mathcal{L}}_s^i(\psi \circ F^i)(x)|}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} = \frac{|\hat{\mathcal{L}}_s^{n-i}\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big)(x)|}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} \\
&= \frac{|\hat{\mathcal{L}}_s^{n-i-1}\Big(\hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big)\Big)(x)|}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} \le \frac{C\Gamma(\psi)\hat{\mathcal{L}}_s^{n-i-1}(1\!\!1)(x)}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} \le \frac{C^2\Gamma(\psi)}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} \qquad (8.9) \\
&\le \frac{4C^2\Gamma(\psi)}{g}.
\end{aligned}
$$

It therefore follows from (8.8) that

$$
||\hat{\mathcal{L}}_s^{j-i}\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1) - g_s \int (\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1) dm_s||_\alpha \le C^2\Gamma(\psi)(1 + 4C^2g^{-1})\theta^{j-i-1}. \qquad (8.10)
$$

Now

$$
\begin{aligned}
&|\int (\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1) dm_s| = |\int \hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big) dm_s - \mu_{s,n}(\psi_i)| = \\
&= \left|\int \hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big) dm_s - \frac{\hat{\mathcal{L}}_s^n\psi_i(x)}{\hat{\mathcal{L}}_s^n 1\!\!1(x)}\right| = \frac{\left|\hat{\mathcal{L}}_s^n 1\!\!1(x) \int \hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big) dm_s - \hat{\mathcal{L}}_s^n\psi_i(x)\right|}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} \\
&= \frac{\left|\hat{\mathcal{L}}_s^n 1\!\!1(x) \int \hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big) dm_s - \hat{\mathcal{L}}_s^{n-i}\hat{\mathcal{L}}_s^i(\psi_i)(x)\right|}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} \\
&= \frac{\left|\hat{\mathcal{L}}_s^n 1\!\!1(x) \int \hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big) dm_s - \hat{\mathcal{L}}_s^{n-i}\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big)(x)\right|}{\hat{\mathcal{L}}_s^n 1\!\!1(x)} \\
&= \frac{\left|\Big(\hat{\mathcal{L}}_s^n 1\!\!1(x) - g_s(x)\Big) \int \hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big) dm_s + \Big(g_s(x) \int \hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big) dm_s - \hat{\mathcal{L}}_s^{n-i-1}\Big(\hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(1\!\!1)\Big)\Big)(x)\Big)\right|}{\hat{\mathcal{L}}_s^n 1\!\!1(x)}
\end{aligned}
$$

Assuming now that $n \geq n_1$, it therefore follows from Lemma 7.7 and Lemma 8.1 that

$$
\begin{aligned}
|\int (\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(\mathbb{1})dm_s| &\leq \frac{C\Gamma(\psi)|S_s^n(\mathbb{1})(x)| + |S_s^{n-i-1}\Big(\hat{\mathcal{L}}_s\Big(\psi\hat{\mathcal{L}}_s^i(\mathbb{1})\Big)\Big)(x)|}{g/4} \\
&\leq \frac{C^2\Gamma(\psi)\theta^n + C\theta^{n-i-1}C\Gamma(\psi)}{g/4} \\
&\leq 8C^2\Gamma(\psi)g^{-1}\theta^{n-i-1}.
\end{aligned}
\tag{8.11}
$$

Assuming $\eta > 0$ to be small enough, we have $\hat{g} = \sup\{||g_s||_\infty : s \in (-\eta, \eta)\} < \infty$. Combining now (8.11) and (8.10), we get for every $j = 0, 1, \dots, n-1$ that

$$
\begin{aligned}
\left\|\sum_{i=0}^{j-1} \hat{\mathcal{L}}_s^{j-i}\Big((\psi - \mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(\mathbb{1})\right\|_\alpha &\leq \Big(C^2\Gamma(\psi)(1 + 4C^2g^{-1}) + 8C^3\Gamma(\psi)g^{-1}\Big)\sum_{i=0}^{j-1}\theta^{j-i-1} \\
&\leq C^2\theta^{-1}\Gamma(\psi)\Big(1 + 4C^2g^{-1}) + 8Cg^{-1}\Big)\sum_{i=0}^{\infty}\theta^i \\
&:= C_1 < \infty.
\end{aligned}
\tag{8.12}
$$

It therefore follows from Lemma 7.7, Lemma 8.1, (8.12) and (8.9) (with ψ_i replaced by ζ_j), that

$$
\begin{aligned}
&\Big\|\hat{\mathcal{L}}_s^{n-j}\Big(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\Big) - \\
&\qquad\qquad - g_s\int\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))dm_s\Big\|_\alpha \\
&\quad= \left\|\hat{\mathcal{L}}_s^{n-j-1}\left(\hat{\mathcal{L}}_s\left(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\right)\right) - \right.\\
&\qquad\qquad \left. - g_s\int\hat{\mathcal{L}}_s\left(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\right)dm_s\right\|_\alpha \\
&\quad\le \left\|S_s^{n-j-1}\left(\hat{\mathcal{L}}_s\left(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\right)\right)\right\|_\alpha \\
&\quad\le C\theta^{n-j-1}\left\|\hat{\mathcal{L}}_s\left(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\right)\right\|_\alpha \\
&\quad\le C\theta^{n-j-1}\Bigg(\left\|\hat{\mathcal{L}}_s\left(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)\zeta\right)\right\|_\alpha + \\
&\qquad\qquad + |\mu_{s,n}(\zeta_j)|\left\|\hat{\mathcal{L}}_s\left(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)\right)\right\|_\alpha\Bigg) \\
&\quad\le C\theta^{n-j-1}\Big(\Gamma(\zeta)C_1 + |\mu_{s,n}(\zeta_j)|C_1\Big) \\
&\quad\le C_2\theta^{n-j-1},
\end{aligned}
\tag{8.13}
$$

where $C_2 = CC_1\Big(\Gamma(\zeta)+4C^2\Gamma(\zeta)g^{-1}\Big)$. Now

$$
\begin{aligned}
&\int\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))dm_s = \\
&\qquad = \int\hat{\mathcal{L}}_s^{n-j}\Big(\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^{i}(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\Big)dm_s \\
&\qquad = \int\hat{\mathcal{L}}_s^{n}\Big((\psi_i-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))\Big)dm_s \\
&\qquad = \int(\psi_i-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))dm_s
\end{aligned}
$$

Combining this and (8.13), we get

$$
\begin{aligned}
&\Big|\sum_{i=0}^{j-1}K_{i,j}-\sum_{i=0}^{j-1}\int(\psi_i-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))dm_s\Big| = \\
&=\Big(\hat{\mathcal{L}}_s^n1\!\!1(x)\Big)^{-1}\Big|\hat{\mathcal{L}}_s^{n-j}\Big(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\Big)(x)- \\
&\qquad -g_s(x)\int\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))dm_s+ \\
&\qquad +\Big(g_s(x)-\hat{\mathcal{L}}_s^n1\!\!1(x)\Big)\int\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big((\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))dm_s\Big| \\
&\le\Big(\hat{\mathcal{L}}_s^n1\!\!1(x)\Big)^{-1}\left(C_2\theta^{n-j-1}+C\theta^n\left|\int\hat{\mathcal{L}}_s\Big(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big(\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\Big)dm_s\right|\right) \\
&\le 4g^{-1}\left(C_2\theta^{n-j-1}+C\theta^n\left\|\hat{\mathcal{L}}_s\Big(\sum_{i=0}^{j-1}\hat{\mathcal{L}}_s^{j-i}\Big(\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)\Big)(\zeta-\mu_{s,n}(\zeta_j))\Big)\right\|_\alpha\right) \\
&\le 4g^{-1}\Big(C_2\theta^{n-j-1}+C\theta^nC_2\Big)\le 8C_2Cg^{-1}\theta^{n-j-1}.
\end{aligned}
\tag{8.14}
$$

Let us now deal with the case when $i=j$. It follows from Remark 8.2, Lemma 7.7, Lemma 8.1 and (8.9) that

$$
\begin{aligned}
&||\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)||_\alpha= \\
&=||\hat{\mathcal{L}}_s\Big(\psi\zeta\hat{\mathcal{L}}_s^j(1\!\!1)\Big)-\mu_{s,n}(\zeta_j)\hat{\mathcal{L}}_s\Big(\psi\mathcal{L}_s^j(1\!\!1)\Big)-\mu_{s,n}(\psi_j)\hat{\mathcal{L}}_s\Big(\zeta\mathcal{L}_s^j(1\!\!1)\Big)+\mu_{s,n}(\psi_j)\mu_{s,n}(\zeta_j)\hat{\mathcal{L}}_s^{j+1}(1\!\!1)||_\alpha \\
&\le||\hat{\mathcal{L}}_s\Big(\psi\zeta\hat{\mathcal{L}}_s^j(1\!\!1)\Big)||_\alpha+|\mu_{s,n}(\zeta_j)|||\hat{\mathcal{L}}_s\Big(\psi\mathcal{L}_s^j(1\!\!1)\Big)||_\alpha+|\mu_{s,n}(\psi_j)|||\hat{\mathcal{L}}_s\Big(\zeta\mathcal{L}_s^j(1\!\!1)\Big)||_\alpha+ \\
&\qquad\qquad +|\mu_{s,n}(\psi_j)||\mu_{s,n}(\zeta_j)|||\hat{\mathcal{L}}_s^{j+1}(1\!\!1)||_\alpha \\
&\le C\Gamma(\psi\zeta)+4g^{-1}C\Gamma(\zeta)CC\Gamma(\psi)C+4g^{-1}\Gamma(\psi)CC\Gamma(\zeta)C+8g^{-2}C^3\Gamma(\zeta)C\Gamma(\psi)C \\
&=C\Big(\Gamma(\psi\zeta)+C^3\Gamma(\zeta)\Gamma(\psi)\Big(C+4g^{-1}+C^28g^{-2}
\end{aligned}
$$

Denote this last constant by C_3. We then have

$$
\begin{aligned}
\Big|\int\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)dm_s\Big|&\le||\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)||_\infty \\
&\le||\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)||_\alpha \\
&\le C_3.
\end{aligned}
$$

Applying now Lemma 7.7, we therefore get

$$\begin{aligned}
&||\hat{\mathcal{L}}_s^{n-j}\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)-\hat{\mathcal{L}}_s^n(1\!\!1)\int(\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))dm_s||_\alpha\\
&\quad\le ||\hat{\mathcal{L}}_s^{n-j-1}\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)-\\
&\qquad\qquad -g_s\int\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)dm_s+\\
&\qquad\qquad +\Big(g_s-\hat{\mathcal{L}}_s^n(1\!\!1)\Big)\int\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)dm_s||_\alpha\\
&\quad=||S_s^{n-j-1}\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)+\\
&\qquad\qquad +S_s^n\int\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)dm_s||_\alpha\\
&\quad\le C\theta^{n-j-1}||\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)||_\alpha+\\
&\qquad\qquad +C\theta^n|\int\hat{\mathcal{L}}_s\Big((\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))\Big)dm_s|\\
&\quad\le C\theta^{n-j-1}C_3+C\theta^nC_3\le 2CC_3\theta^{n-j-1}.
\end{aligned}$$

But

$$\begin{aligned}
\int(\psi-\mu_{s,n}(\psi_j))\hat{\mathcal{L}}_s^j(1\!\!1)(\zeta-\mu_{s,n}(\zeta_j))dm_s&=\int\hat{\mathcal{L}}_s^j\Big((\psi_j-\mu_{s,n}(\psi_j))(\zeta_j-\mu_{s,n}(\zeta_j))\Big)dm_s\\
&=\int(\psi_j-\mu_{s,n}(\psi_j))(\zeta_j-\mu_{s,n}(\zeta_j))dm_s
\end{aligned}$$

and consequently

$$\begin{aligned}
|K_{j,j}-\int(\psi_j-\mu_{s,n}(\psi_j))(\zeta_j-\mu_{s,n}(\zeta_j))dm_s|&\le 2CC_3\Big(\hat{\mathcal{L}}_s^n(1\!\!1)(x)\Big)^{-1}\theta^{n-j-1}\\
&\le 8CC_3g^{-1}\theta^{n-j-1}.
\end{aligned}$$

Combining this and (8.14), we get

$$\left|\sum_{i=0}^j K_{i,j}-\sum_{i=0}^j\int(\psi_i-\mu_{s,n}(\psi_i))(\zeta_i-\mu_{s,n}(\zeta_i))dm_s\right|\le 8Cg^{-1}(C_2+C_3)\theta^{n-j-1}.$$

Hence

$$\begin{aligned}
&\Big|\Delta_n(s)-\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int(\psi_i-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))dm_s\Big| = \\
&=\Big|\sum_{j=0}^{n-1}\Big(\sum_{i=0}^{j}K_{i,j}-\sum_{i=0}^{j}\int(\psi_i-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))dm_s\Big)+ \\
&\qquad+\sum_{k=0}^{n-1}\Big(\sum_{l=0}^{k-1}K_{k,l}-\sum_{i=0}^{j}\int(\psi_k-\mu_{s,n}(\psi_k))(\zeta_l-\mu_{s,n}(\zeta_l))dm_s\Big)\Big| \qquad (8.15)\\
&\le\sum_{j=0}^{n-1}8Cg^{-1}(C_2+C_3)\theta^{n-j-1}\le\sum_{u=0}^{\infty}8Cg^{-1}(C_2+C_3)\theta^{u} \\
&=8Cg^{-1}(C_2+C_3)(1-\theta)^{-1}<\infty.
\end{aligned}$$

Utilizing now the formula $ab-cd=(a-c)b+c(b-d)$, we get

$$\begin{aligned}
&|\int(\psi_i-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))dm_s-\int(\psi_i-m_s(\psi_i))(\zeta_j-m_s(\zeta_j))dm_s| = \\
&=|\int\Big((\psi_i-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))-(\psi_i-m_s(\psi_i))(\zeta_j-m_s(\zeta_j))\Big)dm_s| \\
&=|\int\Big((m_s(\psi_i)-\mu_{s,n}(\psi_i))(\zeta_j-\mu_{s,n}(\zeta_j))+(\psi_i-m_s(\psi_i))(m_s(\zeta_j)-\mu_{s,n}(\zeta_j)\Big)dm_s| \\
&=|(m_s(\psi_i)-\mu_{s,n}(\psi_i))(m_s(\zeta_j)-\mu_{s,n}(\zeta_j))+(m_s(\psi_i)-m_s(\psi_i))(m_s(\zeta_j)-\mu_{s,n}(\zeta_j)| \qquad (8.16)\\
&=|(m_s(\psi_i)-\mu_{s,n}(\psi_i))(m_s(\zeta_j)-\mu_{s,n}(\zeta_j))| \\
&=|m_s(\psi_i)-\mu_{s,n}(\psi_i)|\cdot|m_s(\zeta_j)-\mu_{s,n}(\zeta_j)|.
\end{aligned}$$

Since

$$m_s(\psi_i)-\mu_{s,n}(\psi_i)=\int(\psi_i-\mu_{s,n}(\psi_i))dm_s=\int\hat{\mathcal{L}}_s^i\Big(\psi_i-\mu_{s,n}(\psi_i)\Big)dm_s=\int(\psi-\mu_{s,n}(\psi_i))\hat{\mathcal{L}}_s^i(1\!\!1)dm_s,$$

it follows from (8.11) that

$$|m_s(\psi_i)-\mu_{s,n}(\psi_i)|\le 8C(g\theta)^{-1}\Gamma(\psi)\theta^{n-i}.$$

Similarly

$$|m_s(\zeta_j)-\mu_{s,n}(\zeta_j)|\le 8C(g\theta)^{-1}|Ga(\zeta)\theta^{n-j}.$$

Combining these last two estimates along with (8.15) and (8.16), we get

$$\begin{aligned}
&\Big|\Delta_n(s)-\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int(\psi_i-m_s(\psi_i))(\zeta_j-m_s(\zeta_j))\Big)dm_s\Big|\le \\
&\le 8Cg^{-1}(C_2+C_3)(1-\theta)^{-1}+(8C^2(g\theta)^{-1})^2\Gamma(\psi)\Gamma(\zeta)\sum_{i,j=0}^{n-1}\theta^{n-i}\theta^{n-j} \qquad (8.17)\\
&\le 8Cg^{-1}(C_2+C_3)(1-\theta)^{-1}+(8C^2(g\theta)^{-1})^2(1-\theta)^{-2}\Gamma(\psi)\Gamma(\zeta):E_1.
\end{aligned}$$

Since for every $k \geq 0$ and every tame function $\omega : J(F) \to I\!R$, we have

$$m_s(\omega_k) = m_s\Big(\hat{\mathcal{L}}_s^k(\omega_k)\Big) = m_s\Big(\omega\hat{\mathcal{L}}_s^k(1\!\!1)\Big) = m_s\Big(\omega g + \omega S_s^k(1\!\!1)\Big) = \mu_s(\omega) + m_s\Big(\omega S_s^k(1\!\!1)\Big), \tag{8.18}$$

it follows from Lemma 8.3 and Lemma 7.7 that

$$|m_s(\omega_k) - \mu_s(\omega)| \leq ||S_s^k(1\!\!1)||_\infty m_s(|\omega|) \leq ||S_s^k(1\!\!1)||_\alpha \Gamma(|\omega|) \leq C\Gamma(|\omega|)\theta^k \tag{8.19}$$

for all $s \in (-\eta, \eta)$. We also have

$$\begin{aligned}\int(\omega_k - \mu_s(\omega_k))dm_s &= \int \hat{\mathcal{L}}_s^k(1\!\!1)(\omega - \mu_s(\omega))dm_s = \int(g_s + S_s^k(1\!\!1))(\omega - \mu_s(\omega))dm_s \\ &= \int \omega g_s dm_s - \mu_s(\omega)\int g_s dm_s + \int S_s^k(1\!\!1)(\omega - \mu_s(\omega))dm_s \\ &= \mu_s(\omega) - \mu_s(\omega) + \int S_s^k(1\!\!1)(\omega - \mu_s(\omega))dm_s \\ &= \int S_s^k(1\!\!1)(\omega - \mu_s(\omega))dm_s\end{aligned}$$

Hence, using Lemma 7.7 and Lemma 8.3, we obtain

$$\begin{aligned}|\int(\omega_k - \mu_s(\omega_k))dm_s| &\leq \int |S_s^k(1\!\!1)||\omega - \mu_s(\omega)|dm_s \leq \int ||S_s^k(1\!\!1)||_\alpha(|\omega| + |\mu_s(\omega)|)dm_s \\ &\leq C\theta^k \int(|\omega| + Cm_s(|\omega|))dm_s = C(1+C)m_s(|\omega|)\theta^k \\ &\leq C(1+C)\Gamma(|\omega|)\theta^k.\end{aligned}$$

Therefore, utilizing (8.18), (8.19), Lemma 7.7 and Lemma 8.3, for every $0 \leq i \leq j$, we get that

$$\begin{aligned}&\Big|\int\Big(\psi_i - \mu_s(\psi_i))(\zeta_j - \mu_s(\zeta_j)) - (\psi_i - m_s(\psi_i))(\zeta_j - m_s(\zeta_j)\Big)dm_s\Big| \leq \\ &\quad = \Big|\int\Big((m_s(\psi_i) - \mu_s(\psi_i))(\zeta_j - \mu_s(\zeta_j)) + (\psi_i - m_s(\psi_i))(m_s(\zeta_j) - \mu_s(\zeta_j))\Big)dm_s\Big| \\ &\quad \leq \Big|\int(\zeta_j - \mu_s(\zeta_j))(m_s(\psi_i) - \mu_s(\psi_i))dm_s\Big| + \int |\psi_i - m_s(\psi_i)||m_s(\zeta_j) - \mu_s(\zeta_j)|dm_s\Big| \\ &\quad \leq \Big|\int(\zeta_j - \mu_s(\zeta_j))m_s(\psi S_s^i(1\!\!1))dm_s\Big| + \int |\psi_i - m_s(\psi_i)|dm_s C(1+C)\Gamma(|\zeta|)\theta^j \\ &\quad \leq |m_s(\psi S_s^i(1\!\!1))\int(\zeta_j - \mu_s(\zeta_j))dm_s| + 2m_s(|\psi_i|)C(1+C)\Gamma(|\zeta|)\theta^j \\ &\quad = |m_s(\psi S_s^i(1\!\!1))| \cdot |m_s(\zeta_j - \mu_s(\zeta_j))| + 2m_s(|\psi|\hat{\mathcal{L}}_s^i(1\!\!1))C(1+C)\Gamma(|\zeta|)\theta^j \\ &\quad \leq C\theta^i m_s(|\psi|)C\Gamma(|\zeta|)\theta^j + 2C^2(1+C)\Gamma(|\psi|)\Gamma(|\zeta|)\theta^j \\ &\quad \leq C^2\Gamma(|\psi|)\Gamma(|\zeta|)\theta^i\theta^j + 2C^2(1+C)\Gamma(|\psi|)\Gamma(|\zeta|)\theta^j \\ &\quad \leq C^2(1+2+2C)\Gamma(|\psi|)\Gamma(|\zeta|)\theta^j.\end{aligned}$$

Hence, putting

$$E_2 = 2C^2(1+2+2C)\Gamma(|\psi|)\Gamma(|\zeta|)\sum_{j=0}^\infty j\theta^j < \infty$$

$$\Big|\sum_{i,j=0}^{n-1}\int\Big(\psi_i-\mu_s(\psi_i))(\zeta_j-\mu_s(\zeta_j)\Big)dm_s - \sum_{i,j=0}^{n-1}\int\Big(\psi_i-m_s(\psi_i))(\zeta_j-m_s(\zeta_j)\Big)dm_s\Big| \le E_2 \tag{8.20}$$

for all $s\in(-\eta,\eta)$ and all $n\ge 1$. Put $\overline{\psi}=\psi-\mu_s(\psi)$ and $\overline{\zeta}=\zeta-\mu_s(\zeta)$. For all $0\le i<j$ we then have

$$\begin{aligned}
\int \overline{\psi}_i\overline{\zeta}_j dm_s &= \int\Big(\overline{\psi}\,\overline{\zeta}_{j-i}\Big)\circ F^i dm_s = \int \overline{\psi}\,\overline{\zeta}_{j-i}\hat{\mathcal{L}}_s^i(1\!\!1)dm_s = \int \overline{\psi}\,\overline{\zeta}_{j-i}(g_s+S_s^i(1\!\!1))dm_s \\
&= \int \overline{\psi}\,\overline{\zeta}_{j-i}g_s dm_s + \int \overline{\psi}\,\overline{\zeta}_{j-i}S_s^i(1\!\!1)dm_s = \int \overline{\psi}\,\overline{\zeta}_{j-i}d\mu_s + \int \hat{\mathcal{L}}_s^{j-i}\Big(\overline{\psi}\,\overline{\zeta}_{j-i}S_s^i(1\!\!1)\Big)dm_s \\
&= \int \overline{\psi}_i\overline{\zeta}_j d\mu_s + \int \overline{\zeta}\hat{\mathcal{L}}_s^{j-i}\Big(\overline{\psi}S_s^i(1\!\!1)\Big)dm_s = \int \overline{\psi}_i\overline{\zeta}_j d\mu_s + \int \overline{\zeta}\hat{\mathcal{L}}_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)dm_s \\
&= \int \overline{\psi}_i\overline{\zeta}_j d\mu_s + \int \overline{\zeta}_{j-i}m_s\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)g_s dm_s + \int \overline{\zeta}S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)dm_s \\
&= \int \overline{\psi}_i\overline{\zeta}_j d\mu_s + m_s\Big(\overline{\psi}S_s^i(1\!\!1))\Big)\int \overline{\zeta}_{j-i}d\mu_s + \int \overline{\zeta}S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)dm_s \\
&= \int \overline{\psi}_i\overline{\zeta}_j d\mu_s + m_s\Big(\overline{\psi}S_s^i(1\!\!1))\Big)\int \overline{\zeta}d\mu_s + \int \overline{\zeta}S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)dm_s \\
&= \int \overline{\psi}_i\overline{\zeta}_j d\mu_s + \int \overline{\zeta}S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)dm_s
\end{aligned}$$

Let us now estimate the absolute value of the second summand. Using Lemma 7.7, Lemma 8.1 and Lemma 8.3, we obtain

$$\begin{aligned}
\Big|\int \overline{\zeta}S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)dm_s\Big| &\le \int |\overline{\zeta}||S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)|dm_s \\
&\le \int |\overline{\zeta}|||S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)||_\infty dm_s \\
&\le ||S_s^{j-i-1}\Big(\hat{\mathcal{L}}_s(\overline{\psi}S_s^i(1\!\!1))\Big)||_\alpha \int |\overline{\zeta}|dm_s \\
&\le C\theta^{j-i-1}||\hat{\mathcal{L}}_s\Big(\overline{\psi}S_s^i(1\!\!1)\Big)||_\alpha \int(|\zeta|+|\mu_s(\zeta)|)dm_s \\
&\le C\theta^{j-i-1}\Gamma(\hat{\psi})||S_s^i(1\!\!1)\Big)||_\alpha(\Gamma(|\zeta|)+C\Gamma(|\zeta|)) \\
&\le C^2(1+C)\Gamma(\hat{\psi})\Gamma(|\zeta|)\theta^{-1}\theta^j.
\end{aligned}$$

Hence, if $0\le i<j$, then

$$\Big|\int \overline{\psi}_i\overline{\zeta}_j dm_s - \int \overline{\psi}_i\overline{\zeta}_j d\mu_s\Big| \le C^2(1+C)\theta^{-1}\Gamma(\hat{\psi})\Gamma(|\zeta|)\theta^j. \tag{8.21}$$

Now, for every $j \geq 0$ we have

$$\begin{aligned}\int \overline{\psi}_j \overline{\zeta}_j dm_s &= \int \hat{\mathcal{L}}_s^i(1\!\!1)\overline{\psi}\,\overline{\zeta} dm_s = \int \overline{\psi}\,\overline{\zeta}(g_s + S_s^j(1\!\!1))dm_s = \int \overline{\psi}\,\overline{\zeta} g_s dm_s + \int \overline{\psi}\,\overline{\zeta} S_s^j(1\!\!1)dm_s \\ &= \int \overline{\psi}_j \overline{\zeta}_j d\mu_s + \int \hat{\mathcal{L}}_s\Big(\overline{\psi}\,\overline{\zeta} S_s^j(1\!\!1)\Big)dm_s\end{aligned}$$

Hence, utilising Lemma 8.1, Remark 8.2 and Lemma 7.7, we obtain

$$\begin{aligned}|\int \overline{\psi}_j \overline{\zeta}_j dm_s - \int \overline{\psi}_j \overline{\zeta}_j d\mu_s| &= |\int \hat{\mathcal{L}}_s\Big(\overline{\psi}\,\overline{\zeta} S_s^j(1\!\!1)\Big)dm_s| \leq ||\hat{\mathcal{L}}_s\Big(\overline{\psi}\,\overline{\zeta} S_s^j(1\!\!1)\Big)||_\alpha \\ &\leq \Gamma(\overline{\psi}\,\overline{\zeta})||S_s^j(1\!\!1)||_\alpha \leq C\Gamma(\overline{\psi}\,\overline{\zeta})\theta^j. \end{aligned} \tag{8.22}$$

Now, by Lemma 8.1, Lemma 8.3 and Lemma 7.7, we get

$$\begin{aligned}\Gamma(\overline{\psi}\,\overline{\zeta}) &= \Gamma\Big(\psi\zeta - \mu_s(\psi)\zeta - \mu_s(\zeta)\psi + \mu_s(\psi)\mu_s(\zeta)\Big) \\ &\leq \Gamma\Big(\psi\zeta\Big) + |\mu_s(\psi)|\Gamma(\zeta) + |\mu_s(\zeta)|\Gamma(\psi) + |\mu_s(\psi)\mu_s(\zeta)|\Gamma(1\!\!1) \\ &\leq \Gamma\Big(\psi\zeta\Big) + ||g_s||_\infty m_s(|\psi|)\Gamma(\zeta) + ||g_s||_\infty m_s(|\zeta|)\Gamma(\psi) + C||g_s||_\infty^2 m_s(|\psi|)m_s(|\zeta|) \\ &\leq \Gamma\Big(\psi\zeta\Big) + ||g_s||_\alpha \Gamma(|\psi|)\Gamma(\zeta) + ||g_s||_\alpha \Gamma(|\zeta|)\Gamma(\psi) + C||g_s||_\alpha^2 \Gamma(|\psi|)\Gamma(|\zeta|) \\ &\leq \Gamma\Big(\psi\zeta\Big) + C\Gamma(|\psi|)\Gamma(\zeta) + C\Gamma(|\zeta|)\Gamma(\psi) + C^3\Gamma(|\psi|)\Gamma(|\zeta|).\end{aligned}$$

We can therefore conclude (8.22) by writing

$$\Big|\int \overline{\psi}_j \overline{\zeta}_j dm_s - \int \overline{\psi}_j \overline{\zeta}_j d\mu_s\Big| \leq C(\Gamma\Big(\psi\zeta\Big) + C\Gamma(|\psi|)\Gamma(\zeta) + C\Gamma(|\zeta|)\Gamma(\psi) + C^3\Gamma(|\psi|)\Gamma(|\zeta|))\theta^j.$$

Denoting by E_3 the maximum of coefficients of θ^j appearing in this inequality and in (8.21), we get that

$$\left|\sum_{i,j=0}^{n-1}\Big(\int \overline{\psi}_i \overline{\zeta}_j dm_s - \int \overline{\psi}_i \overline{\zeta}_j d\mu_s\Big)\right| \leq E_3 \sum_{k=0}^{\infty}(k+1)\theta^k := E_4.$$

Combing this, (8.17) and (8.20), we obtain for all $n \geq 1$ and all $s \in (-\eta, \eta)$ that

$$\left|\Delta_n(s) - \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int(\psi_i - \mu_s(\psi_i))(\zeta_j - \mu_s(\zeta_j))d\mu_s\right| E_1 + E_2 + E_4. \tag{8.23}$$

Denote again $\psi - \mu(\psi)$ by $\overline{\psi}$ and $\zeta - \mu_s(\zeta)$ by $\overline{\zeta}$. We then have

$$\begin{aligned}
\frac{1}{n}\int \sum_{i,j=0}^{n-1} \overline{\psi}_i\overline{\zeta}_j d\mu_s &= \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\int \overline{\psi}_i\overline{\zeta}_j d\mu_s + \frac{1}{n}\sum_{i=0}^{n-1}\int \overline{\psi}_i\overline{\zeta}_i d\mu_s \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\int \overline{\psi}\,\overline{\zeta} d\mu_s + \frac{1}{n}\sum_{k=1}^{n-1}(n-1-k)\int \overline{\psi}\,\overline{\zeta}_k d\mu_s + \frac{1}{n}\sum_{k=1}^{n-1}(n-1-k)\int \overline{\zeta\psi}_k d\mu_s \\
&= \int \overline{\psi}\,\overline{\zeta} d\mu_s + \sum_{k=1}^{\infty}\int \overline{\psi}\,\overline{\zeta}_k d\mu_s - \frac{1}{n}\sum_{k=n}^{\infty}\int \overline{\psi}\,\overline{\zeta}_k d\mu_s - \frac{1}{n}\sum_{k=1}^{n-1}(k+1)\int \overline{\psi}\,\overline{\zeta}_k d\mu_s + \\
&\qquad + \sum_{k=1}^{\infty}\int \overline{\zeta\psi}_k d\mu_s - \frac{1}{n}\sum_{k=n}^{\infty}\int \overline{\zeta\psi}_k d\mu_s - \frac{1}{n}\sum_{k=1}^{n-1}(k+1)\int \overline{\zeta\psi}_k d\mu_s \\
&= \int \overline{\psi}\,\overline{\zeta} d\mu_s + \sum_{k=1}^{\infty} C_{s,k}(\psi,\zeta) + \sum_{k=1}^{\infty} C_{s,k}(\zeta,\psi) - \frac{1}{n}\sum_{k=n}^{\infty}\Big(C_{s,k}(\psi,\zeta) + C_{s,k}(\zeta,\psi)\Big) - \\
&\qquad - \frac{1}{n}\sum_{k=1}^{n-1}(k+1)\Big(C_{s,k}(\psi,\zeta) + C_{s,k}(\zeta,\psi)\Big).
\end{aligned}$$

It now immediately follows from Lemma 8.4 that all the series appearing in the last part of this formula are uniformly convergent with respect to $s \in (-\eta,\eta)$, that the second two summands are uniformly bounded with respect to $s \in (-\eta,\eta)$, and the last two terms converge to 0 when $n \to \infty$ uniformly ith respect to $s \in (-\eta,\eta)$. Combining this with (8.23), we see that $\frac{1}{n}\Delta_n(s)$ converges to

$$\int \overline{\psi}\,\overline{\zeta} d\mu_s + \sum_{k=1}^{\infty} C_{s,k}(\psi,\zeta) + \sum_{k=1}^{\infty} C_{s,k}(\zeta,\psi)$$

uniformly ith respect to $s \in (-\eta,\eta)$. Applying now (8.7) completes the proof. ■

9. **Cohomologies**

Let $\mathcal{F}$ be any class of real-valued functions defined on $J(F)$. Two functions $\phi,\psi : J(F) \to I\!R$ are said to be cohomologous in the class of function $\mathcal{F}$ is there exists a function $u \in \mathcal{F}$ such that

$$\phi - \psi = u - u \circ F.$$

We start with the following.

Theorem 9.1. *If $\phi,\psi : J(F) \to I\!R$ are two arbitrary 1^+-tame functions, then the following conditions are equivalent:*

1. *$\mu_\phi = \mu_\psi$.*
2. *There exists a constant R such that for each $n \ge 1$, if $F^n(z) = z$ $(z \in J(F))$, then*

$$S_n\phi(z) - S_n\psi(z) = nR.$$

(3) *The difference* $\psi - \phi$ *is cohomologous to a constant* R *in the class of bounded Hölder continuous functions.*
(4) *The difference* $\psi - \phi$ *is cohomologous to a constant in the class of bounded continuous functions.*
(5) *The difference* $\psi - \phi$ *is cohomologous to a constant in the class of all functions defined everywhere in* $J(F)$.
(6) *There exist constants* S *and* T *such that for every* $z \in J(F)$ *and every* $n \geq 1$

$$|S_n\phi(z) - S_n\psi(z)| \leq T.$$

If these conditions are satisfied, then $R = S = \mathrm{P}(\phi) - \mathrm{P}(\psi)$.

Proof. (1) $\Rightarrow$ (2). It follows from Theorem 4.4 and Lemma 3.11 that there exists a constant $C_z \geq 1$ (remember that $F^n(z) = z$) such that for every $k \geq 1$

$$C_z^{-1}\exp\Big(kS_n\phi(z) - \mathrm{P}(\phi)kn\Big) \leq \mu_\phi\Big(F_z^{-kn}(B(z,\delta))\Big) \leq C_z\exp\Big(kS_n\phi(z) - \mathrm{P}(\phi)kn\Big)$$

and

$$C_z^{-1}\exp\Big(kS_n\psi(z) - \mathrm{P}(\psi)kn\Big) \leq \mu_\psi\Big(F_z^{-kn}(B(z,\delta))\Big) \leq C_z\exp\Big(kS_n\psi(z) - \mathrm{P}(\psi)kn\Big).$$

Since $\mu_\phi = \mu_\psi$, this gives that

$$C_z^{-2} \leq \frac{\exp\Big(kS_n\phi(z) - \mathrm{P}(\phi)kn)\Big)}{\exp\Big(kS_n\psi(z) - \mathrm{P}(\psi)kn)\Big)} \leq C_z^2$$

or equivalently

$$C_z^{-2} \leq \exp\Big(k\Big((S_n\phi(z) - S_n\psi(z)) - (\mathrm{P}(\phi) - \mathrm{P}(\psi))n\Big)\Big) \leq C_z^2$$

and

$$-2\log C_z \leq k\Big((S_n\phi(z) - S_n\psi(z)) - (\mathrm{P}(\phi) - \mathrm{P}(\psi))n\Big) \leq 2\log C_z.$$

Therefore, letting $k \nearrow \infty$, we conclude that $S_n\phi(z) - S_n\psi(z) = (\mathrm{P}(\phi) - \mathrm{P}(\psi))n$. Thus, putting $R = \mathrm{P}(f) - \mathrm{P}(g)$ completes the proof of the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Define

$$\eta = \phi - \psi - R.$$

Since the measure μ_ϕ is ergodic and positive on non-empty open sets, the set of transitive points of F has a full measure μ_ϕ. Fix a transitive point $w \in J(F)$ and put

$$\Gamma = \{F^k(w) : k \geq 1\}.$$

Define the function $u : \Gamma \to I\!R$ by setting

$$u(F^k(w)) = \sum_{j=0}^{k-1}\eta(F^j(w)).$$

Note that the function u is well defined since the points $F^k(w)$, $k \geq 1$, are mutually distinct. It immediately follows from Proposition 2.2 and equality $\phi(w_n) - \psi(w_n) = R$, where w_n, $n \geq 1$, are the points produced in this proposition. Consequently

$$||\phi - \psi||_\infty \leq A_\phi + A_\psi. \tag{9.1}$$

Let $\alpha > 0$ be ascribed to $\beta = 1$ according to Proposition 2.5. Fix $k \geq 1$. By Lemma 2.3, $w \in F^p(B(F^k(w), \alpha))$ for some $0 \leq p \leq q(w, \alpha)$. Hence, there is a point $\xi \in B(F^k(w), \alpha)$ such that $F^p(\xi) = w$. Observe now that the sequence $(w, F(w), \ldots, F^{k-1}(w), \xi, F(\xi), \ldots, F^{p-1}(\xi))$ satisfies the assumptions of Proposition 2.5, and consequently, there exists $y \in J(F)$ which 1-shadows this sequence and that $F^{k+p}(y) = w$. Since F restricted to any Ball of radius 1 is 1-to-1, we deduce that for every $0 \leq j \leq k$, we have $F^j(y) = F_{F^j(w)}^{-(k-j)}(F^k(y))$, where $F_{F^j(w)}^{-(k-j)} : B(F^k(w), 2) \to \mathbb{C}$ is the unique holomorphic inverse branch of $F^{-(k-j)}$ defined on $B(F^k(w), 2)$ and sending $F^k(w)$ to $F^j(w)$. Utilizing Proposition 2.1 (F is an expanding map on its Julia set), we get that

$$|F^j(y) - F^j(w)| \leq c\xi^{-(k-j)}|y - w| \leq c\xi^{-(k-j)}, \tag{9.2}$$

where the numbers c and ξ come from Proposition 2.1. Since both functions ϕ and ψ are locally Hölder continuous, there thus exist two universal constants $H > 0$ and $\zeta \in (0, 1)$ such that

$$|\phi(F^j(w)) - \phi(F^j(y))|, |\psi(F^j(w)) - \psi(F^j(y))| \leq Hc^\zeta\xi^{-\zeta(k-j)} \tag{9.3}$$

for all $0 \leq j \leq k$. Therefore, using our assumption (2) and (9.1), we get that

$$\begin{aligned}
|u(F^k(w))| &= \left|\sum_{j=0}^{k-1}\eta(F^j(w)) - \sum_{j=0}^{k+p-1}\eta(F^j(y))\right| \\
&= \left|\sum_{j=0}^{k-1}\Big(\phi(F^j(w)) - \psi(F^j(w)) - \Big(\phi(F^j(y)) - \psi(F^j(y))\Big)\Big) - \sum_{j=k}^{k+p-1}\eta(F^j(y))\right| \\
&= \left|\sum_{j=0}^{k-1}\Big(\phi(F^j(w)) - \phi(F^j(y))\Big) - \sum_{j=0}^{k-1}\Big(\psi(F^j(w)) - \psi(F^j(y))\Big) - \sum_{j=k}^{k+p-1}\eta(F^j(y))\right| \\
&\leq \sum_{j=0}^{k-1}|\phi(F^j(w)) - \phi(F^j(y))| + \sum_{j=0}^{k-1}|\psi(F^j(w)) - \psi(F^j(y))| + \sum_{j=k}^{k+p-1}|\eta(F^j(y))| \qquad (9.4) \\
&\leq \sum_{j=0}^{k-1}2Hc^\zeta\xi^{-\zeta(k-j)} + p||\phi - \psi - R||_\infty \\
&\leq 2Hc^\zeta\sum_{i=1}^{\infty}\xi^{-\zeta i} + (A_\phi + A_\psi + |R|)q(w, \alpha) \\
&= 2Hc^\zeta(\xi^\zeta - 1)^{-1} + (A_\phi + A_\psi + |R|)q(w, \alpha) < \infty.
\end{aligned}$$

This means that the function $u : \Gamma \to I\!R$ is uniformly bounded. Let us now show that u is locally Hölder continuous. So, suppose that $|F^k(w) - F^l(w)| < \delta$ for some $0 \le k < l$. In view of Proposition 2.5 there exists a point $y \in J(F)$ such that $F^{l-k}(y) = y$ and $|F^j(y) - F^j(F^k(w))| \le \rho^{1-q}|F^k(w) - F^l(w)|$ for all $j = 0, 1, \dots, l-k$. Using (9.2) with w replaced by $F^k(w)$, we get more. Namely

$$|F^j(y) - F^j(F^k(w))| \le c\xi^{l-k-j}|y - F^k(w)| \le c\rho^{1-q}\xi^{l-k-j}|F^k(w) - F^l(w)|,$$

and by the same argument as that leading to (9.3) we therefore get

$$|\phi(F^j(y)) - \phi(F^j(F^k(w)))|, |\psi(F^j(y)) - \psi(F^j(F^k(w)))| \le H(c\rho^{1-q})^\zeta \xi^{\zeta(l-k-j)}|F^k(w) - F^l(w)|^\zeta$$

for all $j = 0, 1, \dots, l-k$. Hence, using our assumption (2), we obtain

$$\begin{aligned} |u(F^l(w)) - u(F^k(w))| &= \left|\sum_{j=k}^{l-1}\eta(F^j(y))\right| = \left|\sum_{j=0}^{l-k-1}\Big(\eta(-F^j(F^k(w)) - \eta(F^j(y))\Big)\right| \\ &\le \sum_{j=0}^{l-k-1}\Big(|\phi(F^j(F^k(w))) - \phi(F^j(y))| + |\psi(F^j(F^k(w))) - \psi(F^j(y))|\Big) \\ &\le \sum_{j=0}^{l-k-1} 2H(c\rho^{1-q})^\zeta \xi^{\zeta(l-k-j)}|F^k(w) - F^l(w)|^\zeta. \end{aligned} \tag{9.5}$$

In particular u is uniformly continuous on Γ. Since Γ is a dense subset of $J(F)$, we therefore conclude that u has a unique continuous extension on $J(F)$. Moreover, it follows from (9.4) and (9.5) that $u : J(F) \to I\!R$ is a bounded locally Hölder continuous function. Since $\phi(z) - \psi(z) - R = u(z) - u(F(z))$ for all $z \in \Gamma$, this equality continues to hold for all $z \in J(F)$. The proof of the implication (2)$\Rightarrow$(3) is complete.

The implications (3)$\Rightarrow$(4) and (4)$\Rightarrow$(6) are obvious.

(6)$\Rightarrow$(1). Fix $z \in J_{r,M}$, where the sets $J_{4,N}$ were defined at the beginning of the proof of Theorem 3.14 and $M > 0$ comes from Proposition 3.12. There the exists an unbounded increasing sequence $\{n_k\}_{k=1}^\infty$ such that $\mathrm{Re}(F^{n_k}(z) \le 2M$ for all $k \ge 1$. Using Lemma 3.11 along with $\frac{1}{4}$-Koebe's distortion theorem and the standard version of Koebe's distortion theorem, we get that

$$\begin{aligned} m_\phi\left(B\Big(z, \frac{1}{4}\delta|(F^{n_k})'(z)|^{-1}\Big)\right) &\le m_\phi\left(F_z^{-n_k}\left(B\Big(F^{n_k}(z), \delta\Big)\right)\right) \\ &\le C_\phi \exp\Big(S_{n_k}\phi(z) - \mathrm{P}(\phi)n_k\Big) m_\phi\Big(B\Big(F^{n_k}(z), \delta\Big)\Big) \\ &\le C_\phi \exp\Big(S_{n_k}\phi(z) - \mathrm{P}(\phi)n_k\Big) \end{aligned}$$

and

$$\begin{aligned} m_\phi\left(B\Big(z,\frac{1}{4}\delta|(F^{n_k})'(z)|^{-1}\Big)\right) &\ge m_\phi\left(F_z^{-n_k}\left(B\Big(F^{n_k}(z),\frac{1}{4}K^{-1}\delta\Big)\right)\right) \\ &\ge C_\phi^{-1}\exp\Big(S_{n_k}\phi(z)-\mathrm{P}(\phi)n_k\Big)m_\phi\Big(B\Big(F^{n_k}(z),\frac{1}{4}K^{-1}\delta\Big)\Big) \\ &\ge C_\phi^{-1}Z_\phi\exp\Big(S_{n_k}\phi(z)-\mathrm{P}(\phi)n_k\Big), \end{aligned} \tag{9.6}$$

where $Z_\phi=\inf\{m_\phi\Big(B(w,(4K)^{-1}\delta)\Big):w\in J(F)\cap P_{2M}\}$ is positive since the set $J(F)\cap P_{2M}$ is compact and the topological support of m_ϕ is equal to the Julia set $J(F)$. Obviously, analogous inequalities hold for the potential ψ. Using (6) we therefore get

$$\begin{aligned} (C_\phi C_\psi e^T)^{-1}Z_\phi\exp\Big((\mathrm{P}(\psi)-\mathrm{P}(\phi)+S)n_k\Big) &\le \\ &\le \frac{m_\phi\left(B\Big(z,4^{-1}\delta|(F^{n_k})'(z)|^{-1}\Big)\right)}{m_\psi\left(B\Big(z,4^{-1}\delta|(F^{n_k})'(z)|^{-1}\Big)\right)} \\ &\le C_\phi C_\psi e^T Z_\psi^{-1}\exp\Big((\mathrm{P}(\psi)-\mathrm{P}(\phi)+S)n_k\Big). \end{aligned} \tag{9.7}$$

Suppose that $S\ne\mathrm{P}(\phi)-\mathrm{P}(\psi)$. Without loss of generality we mayy assume that $S<\mathrm{P}(\phi)-\mathrm{P}(\psi)$. But then, using the right-hand side of (9.7), we conclude that $m_\phi\Big(J_{r,M}\Big)=0$. This contraditcs Proposition 3.12 and shows that $S=\mathrm{P}(\phi)-\mathrm{P}(\psi)$. Then (9.7) implies that the measures m_ϕ and m_ψ are equivalent on $J_{r,M}$. Since $m_\phi\Big(J_{r,M}\Big)=_\psi\Big(J_{r,M}\Big)=1$, these two measures are equivalent as considered on $J(F)$. Since the measures μ_ϕ and μ_ψ are ergodic and equivalent repectively to m_ϕ and m_ψ, we conclude that $\mu_\phi=\mu_\psi$. Thus the proof of the implication (6)⇒(1) is complete. Observe now that obviously (4)⇒(5) and (5)⇒(1), and therefore the entire proof of Theorem9.1 is complete. ■

Proposition 9.2. *If $\phi:P_+^c\to I\!R$ is a 1^+-tame function and $\psi:P_+^c\to I\!R$ is a tame function, then $\sigma^2_{\mu_\phi}(\psi)=0$ if and only if ψ is cohomologous to a constant function in the class of bounded Hölder continuous functions on $J(F)$.*

Proof. Let

$$\tilde J=\{\{\omega_n\}_{n=0}^\infty\in J(F)^\infty: f(x_{n+1})=x_n \text{ for all } n\ge 0\}$$

be the Rokhlin natural extension of $J(F)$. Notice that the map $\tilde F:\tilde J\to\tilde J$ given by the formula

$$\tilde F\Big(\{\omega_n\}_{n=0}^\infty\Big)=\{f(\omega_n)\}_{n=0}^\infty$$

is a homeomorphism. For every $n\ge 0$ let $\pi_n:\tilde J\to J(F)$ be the projection given by the formula

$$\pi_n\Big(\{\omega_n\}_{n=0}^\infty\Big)=\omega_n.$$

It is well-know that there exists a unique Borel probability $\tilde{f}$-invariant measure $\tilde{\mu}_\phi$ on $\tilde{J}$ such that $\tilde{\mu}_\phi \circ \pi_n^{-1} = \mu_\phi$ for all $n \geq 0$. Fix $\xi \in J(F)$ and $\omega \in \pi_0^{-1}(\xi)$. Put

$$J_\omega = \{\tau \in \pi_0^{-1}\Big(B(\xi,\delta) \cap J(F)\Big) : \tau_n = F_{\omega_n}^{-n}(\tau_0)\}.$$

Denote by π_ω the map $\pi_0|_{J_\omega}$. Notice that

$$\pi_\omega : J_\omega \to B(\xi,\delta) \cap J(F)$$

is a homeomorphism and its inverse map

$$\pi_\omega^{-1} : B(\xi,\delta) \cap J(F) \to J_\omega$$

is given by the formula

$$\pi_\omega^{-1}(z) = \{F_{\omega_n}^{-n}(z)\}_{n=0}^\infty.$$

Notice that the family $\{J_\omega : \omega \in \pi_0^{-1}(\xi)\}$ is a partition of $\pi_0^{-1}(B(\xi,\delta) \cap J(F))$ and let $\{\mu_\omega : \omega \in \pi_0^{-1}(\xi)\}$ be the canonical system of conditional measures of $\tilde{\mu}_\phi$ associated with this partition. For every $\omega \in \pi_0^{-1}(\xi)$ and every $n \geq 0$, let

$$J_{\omega_n} = \pi_n^{-1}\Big(F_{\omega_n}^{-n}(B(\xi,\delta) \cap J(F)\Big).$$

Then $J_\omega = \bigcap_{n\geq 0} J_{\omega_n}$. For every Borel set $B \subset B(\xi,\delta) \cap J(F)$ and $\omega \in \pi_0^{-1}(\xi)\}$, we then have

$$\mu_\omega\Big(\pi_\omega^{-1}(B)\Big) = \lim_{n\to\infty} \frac{\tilde{\mu}_\phi\Big(\pi_0^{-1}(B) \cap J_{\omega_n}\Big)}{\tilde{\mu}_\phi\Big(J_{\omega_n}\Big)}.$$

It now follows from Theorem 4.4 and Lemma 3.11, that

$$\begin{aligned}\tilde{\mu}_\phi\Big(\pi_0^{-1}(B) \cap J_{\omega_n}\Big) &= \tilde{\mu}_\phi\Big(\pi_0^{-1}(B) \cap \pi_n^{-1}\Big(F_{\omega_n}^{-n}(B(\xi,\delta) \cap J(F))\Big)\Big) = \tilde{\mu}_\phi \circ \pi_n^{-1}\Big(F_{\omega_n}^{-n}(B)\Big) \\ &= \mu_\phi\Big(F_{\omega_n}^{-n}(B)\Big) \asymp g_\phi\Big(F_{\omega_n}^{-n}(\xi)\Big)m_\phi\Big(F_{\omega_n}^{-n}(B)\Big) \\ &\asymp g_\phi\Big(F_{\omega_n}^{-n}(\xi)\Big)m_\phi(B)\exp\Big(S_n\phi(F_{\omega_n}^{-n}((\xi) - \mathrm{P}(\phi)n\Big).\end{aligned}$$

Combining the last two formulas (the latter with B replaced by $B(\xi,\delta) \cap J(F)$) we get that

$$\mu_\omega\Big(\pi_\omega^{-1}(B)\Big) \asymp \frac{m_\phi(B)}{m_\phi(B(\xi,\delta) \cap J(F))} \asymp m_\phi(B). \tag{9.8}$$

Now assume without loss of generality that $\int \psi d\mu_\phi = 0$. Since $\sigma^2_{\mu_\phi}(\psi) = 0$, it therefore follows from [IL] that there exists $u \in L_2(\tilde{\mu}_\phi)$ such that

$$\psi = u - u \circ \tilde{F} \tag{9.9}$$

$\tilde{\mu}_\phi$ a.e. Our aim is to show that u has a Hölder continuous version of order $s > 0$ being a common Hölder exponent of ϕ and ψ. We first extend ψ and u on the Rokhlin natural extension $\tilde{J}$ of $J(F)$ by declaring

$$\psi(\omega) = \psi(\omega_0) \ \text{ and } \ u(\omega) = u(\omega_0).$$

wherever $u(\omega_0)$ is defined. The cohomological equation (9.9) remains satisfied since

$$u(\omega) - u \circ \tilde{F}(\omega) = u(\omega_0) - u(\omega_1) = u(\omega_0) - u \circ F(\omega_0) = \psi(\omega_0) = \psi(\omega). \tag{9.10}$$

Fix $\xi \in J(F)$. In view of Luzin's theorem there exists a compact set $D \subset \tilde{J}$ such that $\tilde{\mu}_\phi(D) > 1/2$ and the function $u|_D$ is continuous. In view of Birkhoff's ergodic theorem there exists a Borel set $B = B_\xi \subset \pi_0^{-1}(B(\xi,\delta) \cap J(F))$ such that $\tilde{\mu}_\phi(B) = \mu_\phi(B(\xi,\delta) \cap J(F))$, for every $x \in B$, $\tilde{F}^{-n}(x)$ visits D with the asymptotic frequence $> 1/2$, u is well-defined on $\bigcup_{n \in \boldsymbol{Z}} \tilde{F}^{-n}(B)$ and (9.9) holds on $\bigcup_{n \in \boldsymbol{Z}} \sigma^{-n}(B)$. By the definition of conditional measures and by (9.8), there exists a Borel set $G \subset \pi_0^{-1}(B(\xi,\delta) \cap J(F))$ such that $\tilde{\mu}_\phi(G) = \mu_\phi(B(\xi,\delta) \cap J(F))$, for all $\omega \in G$ we have $\mu_\omega(B \cap J_\omega) = \mu_\omega(J_\omega)$ and $\mathrm{supp}(\mu_\omega) = J_\omega$. In particular, for every $\omega \in G$ the set $B \cap J_\omega$ is dense in J_ω. Fix one $\omega = \omega_\xi \in G$ and consider two arbitrary elements $\rho, \tau \in B \cap J_\omega$. Then there exists an unbounded increasing sequence $\{n_j\}$ such that $\tilde{F}^{-n_j}(\rho), \tilde{F}^{-n_j}(\tau) \in D$ for all $j \geq 1$. Using (9.9) we get

$$\begin{aligned} |u(\rho) - u(\tau)| &= \left| u(\tilde{F}^{-n_j}(\rho)) - \sum_{k=1}^{n_j} \psi(\tilde{F}^{-k}(\rho)) - \left(u(\tilde{F}^{-n_j}(\tau)) - \sum_{k=1}^{n_j} \psi(\tilde{F}^{-k}(\rho)) \right) \right| \\ &\leq |u(\tilde{F}^{-n_j}(\rho)) - u(\tilde{F}^{-n_j}(\tau))| + \sum_{k=1}^{n_j} |\psi(\tilde{F}^{-k}(\rho)) - u(\tilde{F}^{-k}(\tau))|. \end{aligned} \tag{9.11}$$

Now, since $\lim_{j \to \infty} \mathrm{dist}(\tilde{F}^{-n_j}(\rho), \tilde{F}^{-n_j}(\tau)) = 0$, since both $\tilde{F}^{-n_j}(\rho)$ and $\tilde{F}^{-n_j}(\tau)$ belong to D and since $u|_D$ is uniformly continuous (as D is compact), we conclude that

$$\lim_{j \to \infty} |u(\tilde{F}^{-n_j}(\rho)) - u(\tilde{F}^{-n_j}(\tau))| = 0.$$

Since ψ is Hölder continuous of order s, using 3.2, we get

$$\begin{aligned} \sum_{k=1}^{n_j} |\psi(\tilde{F}^{-k}(\rho)) - u(\tilde{F}^{-k}(\tau))| &= \sum_{k=1}^{n_j} |\psi(\rho_k) - \psi(\tau_k)| = \sum_{k=1}^{n_j} \left|\psi\Big(F_{\omega_k}^{-k}(\rho_0)\Big) - \psi\Big(F_{\omega_k}^{-k}(\tau_0)\Big)\right| \\ &\leq \sum_{k=1}^{n_j} C_1 |F_{\omega_k}^{-k}(\rho_0) - F_{\omega_k}^{-k}(\tau_0)|^s \leq C_1 \sum_{k=1}^{n_j} L^s \beta^{sk} |\rho_0 - \tau_0|^s \\ &\leq \frac{C_1 L^s}{1 - \beta^s} |\rho_0 - \tau_0|^s \end{aligned}$$

Therefore, it follows from (9.11) that

$$|u(\rho) - u(\tau)| \leq \frac{C_1 L^s}{1 - \beta^s} |\rho_0 - \tau_0|^s = \frac{C_1 L^s}{1 - \beta^s} |\pi_0(\rho) - \pi_0(\tau)|^s.$$

Thus, for all $x, y \in B(\xi,\delta) \cap J(F) \cap \pi_\omega(B)$, we have

$$|u(y) - u(x)| = |u(\pi_\omega^{-1}(y)) - u(\pi_\omega^{-1}(x))| \leq \frac{C_1 L^s}{1 - \beta^s} |\pi_0(\pi_\omega^{-1}(y)) - \pi_0(\pi_\omega^{-1}(x))|^s = \frac{C_1 L^s}{1 - \beta^s} ||y - x|^s.$$

Since $\pi_\omega : J_\omega \to B(\xi,\delta)\cap J(F)$ is a homeomorphism and since $B\cap J_\omega$ is dense in J_ω, the set $B(\xi,\delta)\cap J(F)\cap\pi_\omega(B)$ is dense in $B(\xi,\delta)\cap J(F)$. It therefore follows from the above formula that u restricted to $B(\xi,\delta)\cap J(F)\cap\pi_\omega(B)$ has a Hölder continuous extension to u_ξ on the ball $B(\xi,\delta)\cap J(F)$ with the Hölder exponent s and the Hölder constant independent of ξ. Take now arbitrary teo elements $\xi_1,\xi_2\in J(F)$ such that the intersection $U = B(\xi_1,\delta)\cap J(F)\cap B(\xi_2,\delta)\cap J(F)$ is not empty. Using (9.8) we see that $\mu_\phi(U) = \mu_\phi\Big(\pi_{\omega_{\xi_1}}(B_{\xi_1})\cap\pi_{\omega_{\xi_2}}(B_{\xi_2})\Big)$, and therefore this intersection, denote it by W, is dense in U. Since $u_{\xi_1}|_{B_{\xi_1}} = u = u_{xi_2}|_{B_{\xi_2}}$, we thus conclude that u_{ξ_1} and u_{ξ_2} coincide on U. We have therfore defined a Hölder continuous $\overline{U} : J(F)\to I\!R$, which agrees with u on a (dense) set of full measure μ_ϕ. In particular, the equation $\psi = \overline{u} - \overline{u}\circ F$ holds on a dense subset of $J(F)$. Since all functions involved in this equation are continuous, we conclude that the equation $\psi = \overline{u} - \overline{u}\circ F$ continues to hold throughout the whole Julia set $J(F)$. Invoking now the fact that in Theorem 9.1 condition (5) implies (3), completes the proof. ■

10. **Auxiliary properties of the pressure function**

We prove in this section two auxiliary results. The first is this.

Lemma 10.1. *If ϕ and ψ are tame functions and $\kappa(\phi) > 0$, then*

$$\lim_{q\searrow\frac{1-\kappa(\psi)}{\kappa(\phi)}} \mathrm{P}(q\phi+\psi) = +\infty.$$

Proof. Put $\alpha = \kappa(\phi)$ and $\gamma = \kappa(\psi)$. Take $w > \inf\{J(F)\}$ and observe that if $w_n \in P\cap f^{-1}(w+2\pi in)$, $n\ge 1$, then

$$\sup\{\mathrm{Re}\Big(F_{w_n}^{-1}(P_w)\Big)\} \le \log|w+2\pi in| - \log|\lambda| \le \log(w+2\pi in) - \log|\lambda|.$$

Therefore, if $\log(w+2\pi in) - \log|\lambda| \le w$, then $F_{w_n}^{-1}(P_w)\subset P_w$. Assuming now that w is large enough, this inclusion therefore holds fo all $n\le e^{w/2}$. Now, for any $z\in P_w$ and

$q \in \left(\frac{1-\gamma}{\alpha}, 2\frac{1-\gamma}{\alpha}\right)$, we have

$$\begin{aligned}
\sum_{k=1}^{e^{w/2}} \exp\Big(q\phi(z_k)+\psi(z_k)\Big) &\ge \sum_{k=1}^{e^{w/2}} \exp\Big(q(-A_\phi - \alpha\mathrm{Re}(z_k)) - A_\psi - \gamma\mathrm{Re}(z_k)\Big) \\
&= \exp\Big(-(qA_\phi + A_\psi)\Big) \sum_{k=1}^{e^{w/2}} \exp\Big(-(q\alpha+\gamma)\mathrm{Re}(z_k)\Big) \\
&=\ge \exp\Big(-(qA_\phi + A_\psi)\Big) \sum_{k=1}^{e^{w/2}} |\lambda|^{-1}|z+2k\pi i|^{-(q\alpha+\gamma)} \\
&\ge |\lambda|^{-1}| \exp\Big(-(qA_\phi + A_\psi)\Big) \sum_{k=1}^{e^{w/2}} \Big(w+2(k+1)\pi\Big)^{-(q\alpha+\gamma)} \\
&\ge -C_1 + C_2(q\alpha+\gamma-1)(w+4\pi)^{1-(q\alpha+\gamma)}
\end{aligned}$$

where $C_1, C_2 > 0$ are universal constant, $z_k = F_{w_k}^{-1}(z)$, and the last inequality was written replacing in the standard way the sum by an integral. Therefore we get by an obvious induction that

$$\mathrm{P}(q\phi+\psi) \ge \log\Big(-C_1 + C_2(q\alpha+\gamma-1)^{-1}(w+4\pi)^{1-(q\alpha+\gamma)}\Big)$$

Since the right-hand side of this inequality diverges to $+\infty$ when $q \searrow \frac{1-\gamma}{\alpha}$, we are done. ■

Now fix a 1^+-tame potential $\phi : P_+ \to I\!\!R$. Subtracting $\mathrm{P}(\phi)$ from ϕ, we can assume without loss of generality that $\mathrm{P}(\phi) = 0$ and call ϕ normalized. Consider the two-parameter family of potentials $\phi_{q,t} : P_+ \to I\!\!R$, $q, t \in I\!\!R$, defined as follows.

$$\phi_{q,t} = -t\log|F'| + q\phi.$$

Note that $\phi_{q,t}$ is a $t + q\kappa(\phi)$-tame function. We shall prove the following.

Lemma 10.2. *For every $q \in I\!\!R$ there exists a unique $t = T(q) \in I\!\!R$ such that $\mathrm{P}(\phi_{q,t}) = 0$. In addition, $(q, T(q)) \in \Sigma_2(\phi, -\log|F'|)$.*

Proof. Since the function $q\phi$ is $q\kappa(\phi)$-tame and $-\log|F'|$ is 1-tame, Lemma 10.1 applies to give

$$\lim_{t\searrow 1-q\kappa(\phi)} = +\infty. \tag{10.1}$$

Since, by Lemma 7.5 the function $t \mapsto \mathrm{P}(\phi_{q,t})$, $t > 1 - q\kappa(\phi)$, is differentiable, and since by Theorem 8.6 followed by Proposition 2.1,

$$\frac{\partial \mathrm{P}}{\partial t} = -\int \log|F'|d\mu_{q\phi - t\log|F'|} \le -\log\xi < 0,$$

we conclude that the function $t \mapsto \mathrm{P}(\phi_{q,t})$, $t > 1 - q\kappa(\phi)$, is strictly decreasing and that $\lim_{t\to+\infty} \mathrm{P}(\phi_{q,t}) = -\infty$. Combining these three properties of $\mathrm{P}(t)$ along with (10.1), our lemma follows. ■

For every $q \in I\!R$ put

$$m_q = m_{q\phi - T(q)\log|F'|} \quad \text{and} \quad \mu_q = \mu_{q\phi - T(q)\log|F'|}. \tag{10.2}$$

We can now pass to deal with geometric properties of conformal measures m_ϕ and Gibbs states μ_ϕ. We start with the following.

11. **Hausdorff Dimension of Gibbs States**

By HD(A) we denote the Hausdorff dimension of the set A, which is a subset of a metric space. Given a Borel probability measure μ on a metric space, we define for a point x in this space, the number (called the local dimension at the point x)

$$d_\mu(x) = \lim_{r \to 0} \log \mu(B(x,r)) \log r$$

provided that the limit exists. Recall that HD(μ), the Hausdorff dimension of the measures μ is the infimum of the numbers HD(Y) taken over all Borel sets Y such that $\mu(Y) = 1$. If $d_\mu(x)$ is constant a.e. (denote it by d_μ), then HD$(\mu) = d_\mu$. The proof of the following only result of this section is taken from [Ur]. We include it here for the sake of completeness and convenience of the reader. If μ is a Borel probability F-invariant measure on $J(F)$, then the number

$$\chi_\mu = \int \log|F'| d\mu$$

is called the Lyapunov exponent of the map F with respect to the measure μ. Because of Proposition 2.1, this exponent is always positive and because of Lemma 8.3 it is finite for every Gibbs measure μ_ϕ corresponding to a 1^+-tame potential ϕ.

Theorem 11.1. *If $f : \mathbb{C} \to \mathbb{C}$ is a hyperbolic exponential map and $\phi : J(F) \to \mathbb{C}$ is a 1^+-tame potential, then for μ_ϕ-a.e. $z \in J(F)$, the local dimension $d_\mu(x)$ exists and is equal to $h_{\mu_\phi}/\chi_{\mu_\phi}$. In particular*

$$\mathrm{HD}(\mu_\phi) = \frac{h_{\mu_\phi}}{\chi_{\mu_\phi}}.$$

Proof. In view of Birkhoff's ergodic theorem there exists a Borel set $X \subset J(F)$ such that $\mu_\phi(X) = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} \log|(F^n)'(x)| = \chi_{\mu_\phi} \text{ and } \lim_{n\to\infty} \frac{1}{n} S_n\phi(x) = \int \phi d\mu_\phi \tag{11.1}$$

for every $x \in X$. Fix $x \in X$ and $\varepsilon > 0$. There then exists $k \geq 1$ such that

$$\left|\frac{1}{n} \log|(F^n)'(x)| - \chi_{\mu_\phi}\right| < \varepsilon \tag{11.2}$$

for every $n \ge k$. Fix $r \in (0, \delta)$ and let $n = n(r) \ge 0$ be the largest integer such that

$$B(x, r) \subset F_x^{-n}\Big(B(F^n(x), \delta)\Big). \tag{11.3}$$

Then $B(x, r)$ is not contained in $F_x^{-(n+1)}(B(F^{n+1}(x), \delta))$ and it follows from the $\frac{1}{4}$-Koebe's distortion theorem that

$$r \ge \frac{1}{4}\delta|(F^{n+1})'(x)|^{-1}. \tag{11.4}$$

Taking $r > 0$ sufficiently small, we may assume that $n \ge k$. Applying Theorem 3.11 and utilizing (11.3) along with Lemma 3.1, we get that

$$\begin{aligned} m_\phi(B(x, r)) &\le \int_{B(F^n(x),\delta)} \exp\Big(S_n\phi \circ F_x^{-n} - \mathrm{P}(\phi)n\Big) dm_\phi \\ &\le T\exp\Big(S_n\phi(x) - \mathrm{P}(\phi)n\Big) m_\phi\Big(B(F^n(x), \delta)\Big) \le T\exp\Big(S_n\phi(x) - \mathrm{P}(\phi)n\Big). \end{aligned}$$

Applying now (11.4) and (11.2), we obtain

$$\begin{aligned} \frac{\log m_\phi(B(x, r))}{\log r} &\ge \frac{\log T + S_n\phi(x) - \mathrm{P}(\phi)n}{\log r} \ge \frac{\log T + S_n\phi(x) - \mathrm{P}(\phi)n}{\log\delta - \log 4 - \log|(F^{n+1})'(x)|} \\ &\ge \frac{\log T + S_n\phi(x) - \mathrm{P}(\phi)n}{\log\delta - \log 4 - (\chi_{\mu_\phi} - \varepsilon)(n+1)}. \end{aligned}$$

Dividing now the numerator and the denominator of the last quotient by $n = n(r)$, letting $r \searrow 0$ (which implies that $n(r) \to \infty$) and using the second part of (11.1), we therefore get that

$$\liminf_{r\to 0} \frac{\log\Big(m_\phi(B(z, r)\Big)}{\log r} \ge \frac{-\int \phi d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}.$$

Since, by Theorem 4.4(b), the measures μ_ϕ and m_ϕ are equivalent with positive continuous Radon-Nikodym derivatives , we obtain for all $x \in$ that

$$\liminf_{r\to 0} \frac{\log\Big(\mu_\phi(B(z, r)\Big)}{\log r}(x) \ge \frac{-\int \phi d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}. \tag{11.5}$$

For every $s > 0$, let $J_s = \{z \in J(F) : |\mathrm{Re}(z)| \le s\}$. Take M so large that $\mu_\phi(J_M) > 0$. Since the measure m_ϕ is positive on non-empty open subsets of $J(F)$, we get that

$$W := \inf\{m_\phi(B(z, \delta) : z \in J_M\} > 0.$$

In view of ergodicity of the measure μ_ϕ and Birkhoff's ergodic theorem, there exists a Borel set $Y \subset X$ such that $\mu_\phi(Y) = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} S_n\Big(1\!\!1_{J_M}\Big)(x) = \mu_\phi(J_M) > 0$$

for all $x \in Y$. In particular, if $\{n_j\}_{j=1}^\infty$ is the unbounded increasing sequence of all integers $n \geq 1$ such that $F^n(x) \in J_M$, then

$$\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1. \tag{11.6}$$

Keep $x \in Y$ and let $l \geq 0$ be the least integer such that

$$B(x,r) \supset F_x^{-i}\Big(B(F^i(x),\delta)\Big)$$

for all $i \geq l$. Taking $r > 0$ small enough, we may assume that $l > \max\{k, n_1\}$. There then exists a unique $j \geq 2$ such that

$$n_{j-1} < l \leq n_j. \tag{11.7}$$

Also $F_x^{-(l-1)}\Big(B(F^{l-1}(x),\delta)\Big)$ is not contained in $B(x,r)$, and it therefore follows from Koebe's distortion theorem that

$$r \leq K\delta |(F^{l-1})'(x)|^{-1}. \tag{11.8}$$

It follows from the definition of l and formula (11.7) along with Lemma 3.1 that

$$\begin{aligned} m_\phi(B(x,r)) &\geq m_\phi\Big(F_x^{-n_j}\Big(B(F^{n_j}(x),\delta)\Big)\Big) = \int_{B(F^{n_j}(x),\delta)} \exp\Big(S_{n_j}\phi\circ F_x^{-n_j} - \mathrm{P}(\phi)n_j\Big)dm_\phi \\ &\geq T^{-1}\exp\Big(S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j\Big)m_\phi\Big(B(F^{n_j}(x),\delta)\Big) \\ &\geq WT^{-1}\exp\Big(S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j\Big). \end{aligned}$$

Applying now (11.2), (11.4) and (11.7), we obtain

$$\begin{aligned} \frac{\log m_\phi(B(x,r))}{\log r} &\leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log r} \leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log(K\delta) - \log|(F^{l-1})'(x)|} \\ &\leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log(K\delta) - (\chi_{\mu_\phi}+\varepsilon)(l-1)} \leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log(K\delta) - (\chi_{\mu_\phi}+\varepsilon)n_{j-1}}. \end{aligned}$$

Dividing now the numerator and the denominator of the last quotient by n_{j-1} letting $r \searrow 0$ (which implies that $n_{j-1} \to \infty$) and using the second part of (11.1) along with (11.6), we therefore get that

$$\limsup_{r\to 0} \frac{\log\Big(m_\phi(B(z,r))\Big)}{\log r} \leq \frac{-\int \phi\, d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}.$$

Since, by Theorem 4.4, the measures μ_ϕ and m_ϕ are equivalent with positive continuous Radon-Nikodym derivatives, we obtain for all $x \in Y$ that

$$\limsup_{r\to 0} \frac{\log\Big(\mu_\phi(B(z,r))\Big)}{\log r} \leq \frac{-\int \phi\, d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}.$$

Since, by Theorem 6.3, $\mathrm{P}(\phi) - \int \phi\, d\mu_\phi = \mathrm{h}_{\mu_\phi}$, combining this inequality with (11.5), completes the proof. ■

12. **Multifractal Analysis of 1^+-Tame Functions**

First we define the set $J_{rr}(F) \subset J_r(F)$ suitable for multifractal analysis on balls. Given an integer $s > \inf\big(\text{Re}J(F)\big)$ and a point $z \in J(F)$ let $\{t_s^n(z)\}_{n=1}^\infty$ denote the sequence of consequtive visits of the point z to P_s under the action of F, i.e. this sequence is srictly increasing (perhaps finite, perhaps empty), $F^{t_s^n(z)}(z) \in P_q$ for all $n \ge 1$ and $F^j(z) \notin P_s$ for all $t_s^n(z) < j < t_s^{n+1}(z)$. Now, define the sets

$$M_s = \left\{ z \in J_r(F) : \lim_{n\to\infty} \frac{\log\left|\left(F^{t_s^{n+1}(z)-t_s^n(z)}\right)'\left(F^{t_s^n(z)}(z)\right)\right|}{|\log\left|\left(F^{t_s^n(z)}\right)'(z)\right|} = 0 \ \text{ and } \ \lim_{n\to\infty}\frac{t_s^{n+1}(z)}{t_s^n(z)} = 1 \right\}$$

and

$$J_{rr}(F) = \bigcup_s M_s,$$

where the union is taken over all integers $s > \inf\big(\text{Re}J(F)\big)$. Observe that if $z \in M_s$, then for every $i \ge 1$,

$$\lim_{n\to\infty} \frac{\log\left|\left(F^{t_s^{n+i}(z)-t_s^n(z)}\right)'\left(F^{t_s^n(z)}(z)\right)\right|}{|\log\left|\left(F^{t_s^n(z)}\right)'(z)\right|} = 0 \ \text{ and } \ \lim_{n\to\infty}\frac{t_s^{n+i}(z)}{t_s^n(z)} = 1. \tag{12.1}$$

Now let us record the fact that the set $J_{rr}(F)$ is dynamically significant; precisely, we have the following.

Proposition 12.1. *If mu is a Borel probability F-invariant ergodic measure on $J(F)$ with finite Lyapunov exponent (in particular if $\mu = \mu_\phi$, where ϕ is a 1^+-potential), then $\mu(J_{rr}(F)) = 1$.*

Proof. Since

$$\left(F^{t_s^{n+1}(z)-t_s^n(z)}\right)'\left(F^{t_s^n(z)}(z)\right) = \left(F^{t_s^{n+1}(z)}\right)'(z)/\left(F^{t_s^n(z)}\right)'(z),$$

and since, by Birkhoff's Ergodic Theorem, $\lim_{n\to\infty}\big(t_s^{n+i}(z)/t_s^n(z)\big) = 1$ for all $s > \inf\big(\text{Re}J(F)\big)$ and μ-a.e. $z \in J_r(F)$, the proof is concluded by applying Birkhoff's Ergodic Theorem to the integrable function $\log|F'|$. ■

Given a 1^+-tame function $\phi : P_+^c \to I\!R$, and a real number $\alpha \ge 0$, we define the following two sets.

$$\mathcal{K}_\phi(\alpha) = \left\{ z \in J_r(F) : \lim_{n\to\infty} \frac{\mathrm{P}(\phi)n - S_n\phi(z)}{\log|(F^n)'(z)|} = \alpha \right\}$$

and

$$D_\phi(\alpha) = \left\{ z \in J_{rr}(F) : \lim_{r\to 0} \frac{\log \mu_\phi(B(z,r))}{\log r} = \alpha \right\}.$$

The first relation between these two sets is provided by the following.

Proposition 12.2. *For every $\alpha \geq 0$, we have that*

$$D_\phi(\alpha) = \mathcal{K}_\phi(\alpha) \cap J_{rr}(F).$$

Proof. We are to prove that if $z \in J_{rr}(F)$, then

$$\lim_{j\to\infty} \frac{\mathrm{P}(\phi)j - S_j\phi(z)}{\log |(F^j)'(z)|} = \alpha \Leftrightarrow \lim_{r\to 0} \frac{\log \mu_\phi(B(z,r))}{\log r} = \alpha. \tag{12.2}$$

And indeed, take $z \in J_{rr}(F)$ and assume that the left-hand side limit of (12.2) is equal to α. By the definition of $J_{rr}(F)$ there exists an integer $s > \inf(\mathrm{Re}J(F))$ such that $z \in M_s$. Fix $\in (0,1)$ small enough and let $n = n(r) \geq 1$ be the largest integer such that

$$r\left|\left(F^{t_s^n(z)}(z)\right)'(z)\right| \leq \delta/4. \tag{12.3}$$

Then

$$r\left|\left(F^{t_s^{n+q}(z)}(z)\right)'(z)\right| > \delta/4, \tag{12.4}$$

where the integer $q \geq 1$ comes from (2.2). Put

$$t_s^n(z) = t_s^n \;\text{ and } t_s^{n+q}(z) = t_s^{n+q}.$$

It follows from (12.4) and Koebe's Distortion Theorem that

$$F_z^{-t_s^{n+q}}\left(B\left(F^{t_s^{n+q}}(z), (4K)^{-1}\delta\right)\right) \subset B(z,r)$$

and from (12.4) along with Koebe's $\frac{1}{4}$-Distortion Theorem, that

$$F_z^{-t_s^n}\left(B\left(F^{t_s^n}(z), \delta\right)\right) \supset B(z,r).$$

Put

$$\psi = \phi - \mathrm{P}(\phi).$$

Applying Lemma 3.11 we therefore get that

$$\begin{aligned} m_\phi(B(z,r)) &\leq m_\phi\left(F_z^{-t_s^n}\left(B\left(F^{t_s^n}(z),\delta\right)\right)\right) \leq C_\phi \exp\left(S_{t_s^n}\psi(z)\right) m_\phi\left(B\left(F^{t_s^n}(z),\delta\right)\right) \\ &\leq C_\phi \exp\left(S_{t_s^n}\psi(z)\right) \end{aligned}$$

and

$$\begin{aligned} m_\phi(B(z,r)) &\geq m_\phi\left(F_z^{-t_s^{n+q}}\left(B\left(F^{t_s^{n+q}}(z),(4K)^{-1}\delta\right)\right)\right) \\ &\geq C_\phi^{-1} \exp\left(S_{t_s^{n+q}}\psi(z)\right) m_\phi\left(B\left(F^{t_s^{n+q}}(z),(4K)^{-1}\delta\right)\right) \\ &\geq TC_\phi^{-1} \exp\left(S_{t_s^{n+q}}\psi(z)\right), \end{aligned}$$

where $T=\inf\{m_\phi(B(w,(4K)^{-1}\delta\big):w\in J(F)\cap P_s\}>0$. Using these two estimates and both (12.3) and (12.4), we obtain

$$\begin{aligned}\frac{\log(m_\phi(B(z,r)))}{\log r}&\ge\frac{\log C_\phi+S_{t_s^n}\psi(z)}{\log r}\ge\frac{\log C_\phi+S_{t_s^n}\psi(z)}{\log(\delta/4)-\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|}\\&=\frac{\log C_\phi+S_{t_s^n}\psi(z)}{\log(\delta/4)-\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|-\log\left|\left(F^{t_s^{n+q}-t_s^n}\right)'\left(F^{t_s^n}(z)\right)\right|}.\end{aligned}\tag{12.5}$$

and

$$\begin{aligned}\frac{\log(m_\phi(B(z,r)))}{\log r}&\le\frac{\log T-\log C_\phi+S_{t_s^{n+q}}\psi(z)}{\log r}\le\frac{\log T-\log C_\phi+S_{t_s^{n+q}}\psi(z)}{\log(\delta/4)-\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|}\\&=\frac{\log T-\log C_\phi+S_{t_s^{n+q}}\psi(z)}{\log(\delta/4)-\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|+\log\left|\left(F^{t_s^{n+q}-t_s^n}\right)'\left(F^{t_s^n}(z)\right)\right|}.\end{aligned}\tag{12.6}$$

Dividing the numerators and the denominators of the right-hand sides of (12.5) and (12.6) respectively by $\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|$ and $\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|$, and noting also that $\lim_{r\to0}t_s^{n(r)}=\lim_{r\to0}t_s^{n(r)+q}=+\infty$, we thus get using (12.1) respectively that

$$\liminf_{r\to0}\frac{\log(m_\phi(B(z,r)))}{\log r}\ge\alpha\quad\text{and}\quad\limsup_{r\to0}\frac{\log(m_\phi(B(z,r)))}{\log r}\le\alpha.$$

Since obviously, $\lim_{r\to0}\frac{\log(\mu_\phi(B(z,r)))}{\log r}=\lim_{r\to0}\frac{\log(m_\phi(B(z,r)))}{\log r}$, we are done with the proof of the implication $\Rightarrow$ in (12.2).

In order to prove the other implication in (12.2) keep $z\in J_{rr}(F)$, i.e. in some M_s, and assume that the right-hand side limit in (12.2) is equal to α. Fix $j\ge t_s^1$. There then exists a unique $n=n(j)\ge1$ such that

$$t_s^n\le j<t_s^{n+q}.$$

Since ψ is a 1^+-tame function and since F is expanding (by Proposition 2.1), we get with some constants $A_1>0$ and $A_2>0$, that

$$\begin{aligned}\frac{-S_j(\psi)(z)}{\log|(F^j)'(z)|}&\le\frac{-S_{t_s^{n+q}}\psi(z)}{-A_1+\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|}\le\frac{-S_{t_s^n}\psi(z)+A_2(t_s^{n+q}-t_s^n)}{-A_1+\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|-\log\left|\left(F^{t_s^{n+q}-t_s^n}\right)'\left(F^{t_s^n}(z)\right)\right|}\\&=\frac{\frac{-S_{t_s^n}\psi(z)}{\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|}+A_2\frac{t_s^{n+q}-t_s^n}{\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|}}{1-\frac{\log\left|\left(F^{t_s^{n+q}-t_s^n}\right)'\left(F^{t_s^n}(z)\right)\right|}{\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|}-\frac{A_1}{\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|}}.\end{aligned}$$

In view of Proposition 2.1, $\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right| \ge \log C + t_s^{n+q}\log\zeta$. Since $z \in M_s$, we therefore obtain that

$$\limsup_{j\to\infty}\frac{-S_j\psi(z)}{\log|(F^j)'(z)|} \le \limsup_{n\to\infty}\frac{-S_{t_s^n}\psi(z)}{\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|}. \tag{12.7}$$

Similarly we get that

$$\begin{aligned}\frac{-S_j(\psi)(z)}{\log|(F^j)'(z)|} &\ge \frac{-S_{t_s^n}\psi(z)}{A_1+\log\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right|}\\ &\ge \frac{-S_{t_s^{n+q}}\psi(z) - A_2(t_s^{n+q}-t_s^n)}{A_1+\log\left|\left(F^{t_s^n}(z)\right)'(z)\right| + \log\left|\left(F^{t_s^{n+q}-t_s^n}\right)'\left(F^{t_s^n}(z)\right)\right|}\\ &= \frac{\frac{-S_{t_s^{n+q}}\psi(z)}{\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|} - A_2\frac{t_s^{n+q}-t_s^n}{\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|}}{1+\frac{\log\left|\left(F^{t_s^{n+q}-t_s^n}\right)'\left(F^{t_s^n}(z)\right)\right|}{\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|}+\frac{A_1}{\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|}}.\end{aligned}$$

In view of Proposition 2.1, $\log\left|\left(F^{t_s^n}(z)\right)'(z)\right| \ge \log C + t_s^n\log\zeta$. Since $z \in M_s$, we therefore obtain that

$$\limsup_{j\to\infty}\frac{-S_j\psi(z)}{\log|(F^j)'(z)|} \le \limsup_{n\to\infty}\frac{S_{t_s^{n+q}}\psi(z)}{\log\left|\left(F^{t_s^n}(z)\right)'(z)\right|}. \tag{12.8}$$

Keep $j \ge 1$ and $n = n(j) \ge 1$ as above. Put

$$r = r_j = (\delta/4)\left|\left(F^{t_s^n}(z)\right)'(z)\right|. \tag{12.9}$$

Then, in view of (2.2 and inequality $t_s^{n+q} - t_s^n \ge q$, we get that

$$r_j\left|\left(F^{t_s^{n+q}}(z)\right)'(z)\right| = r_j\left|\left(F^{t_s^n}(z)\right)'(z)\right|\cdot\left|\left(F^{t_s^{n+q}-t_s^n}\right)'\left(F^{t_s^n}(z)\right)\right| \ge 2r_j\left|\left(F^{t_s^n}(z)\right)'(z)\right| > \delta/4. \tag{12.10}$$

Since when deriving formulas (12.5) and (12.6) we used only (12.3) and (12.4) and not at all the knowledge that n is the largest integer for which (12.3) is satisfied, starting with (12.9 and (12.10, we conclude that (12.5) and (12.6) are true with $n = n(j)$ and $r = r_j$Therefore, employing also (12.7) and (12.8), we get that

$$\lim_{j\to\infty}\frac{-S_j\psi(z)}{\log|(F^j)'(z)|} = \lim_{r\to 0}\frac{\log(m_\phi(B(z,r)))}{\log r} = \lim_{r\to 0}\frac{\log(\mu_\phi(B(z,r)))}{\log r} = \alpha.$$

We are done. ■

We set

$$k^+_{\mu_\phi} = \mathrm{HD}(\mathcal{K}_\phi(\alpha)) \text{ and } k^-_{\mu_\phi} = \mathrm{HD}(D_\phi(\alpha)).$$

By Proposition 12.2, we have

$$k^-_{\mu_\phi} \le k^+_{\mu_\phi}. \tag{12.11}$$

Let k be a strictly convex map on an interval I, hence $k'' > 0$ for all $x \in I$. The Legendre transform of k is the function g of a new variable p defined by

$$g(p) = \max_I \{px - k(x)\}$$

everywhere where the maximum exists. it can be proved that the domain of g is either a point, an interval or a semi line. It is also easy to show that g is strictly convex and that the Legendre transform is involutive. We then say that the functions f and g form a Legendre transform pair. The following theorem (see [Ro]) gives a useful characterization of a Legendre transform pair.

Theorem 12.3. *Two strictly convex functions k and g form a Legendre transform pair if and only if $g(-k'(q)) = k(q) - qk'(q)$.*

We are now in position to formulate our main theorem in this section.

Theorem 12.4. *Suppose that $\phi : P^c_+ \to I\!R$ is a 1^+-tame function. Then the functions $k^-_{\mu_\phi}$ and $k^+_{\mu_\phi}$ coincide in their natural domain (α_1, α_2) produced in item (c). Denote their common value by k_{μ_ϕ}. Then the following statements are true.*

(a) *The pointwise dimension $d_{\mu_\phi}(x)$ exists for μ_ϕ-almost every $x \in J(F)$ and*

$$d_{\mu_\phi}(x) = \frac{\mathrm{P}(\phi) - \int \phi d\mu_\phi}{\int \log|F'| d\mu_\phi}.$$

(b) *The function $q \mapsto T(q)$, $q \in I\!R$ is real-analytic, $T(0) = \mathrm{HD}(J_r(F))$, $T'(q) < 0$ and $T''(q) \ge 0$. In addition T'' vanishes in one point if and only if it vanishes at all points if and only if $\mu_\phi = \mu_{-h\log|F'|}$ if and only if ϕ and $-h\log|F'|$ are cohomologous modulo constant in the class of all (bounded) Hölder continuous functions.*

(c) *The function $-T'(q)$ attains values in an interval (α_1, α_2) where $0 \le \alpha_1 \le \alpha_2 < \infty$.*

(d) *For every $q \in I\!R$, $k_{\mu_\phi}(-T'(q)) = T(q) - qT'(q)$.*

(f) *The function k_{μ_ϕ} is real-analytic throughout its whole domain (α_1, α_2).*

(f) *If $\mu_\phi \ne \mu_{-h\log|F'|}$, then the functions $k_{\mu_\phi}(\alpha)$ and $T(q)$ form a Legendre transform pair.*

Proof. The statement in (a) is an immediate consequence of Theorem 11.1 and the second part of Theorem 6.3. In the remainder of this proof we may assume without loss of generality that the potential ϕ is normalized, i.e. that $\mathrm{P}(\phi)=0$. Since, by Theorem 8.6 followed by Proposition 2.1,

$$\frac{\partial \mathrm{P}}{\partial t} = -\int \log|F'| d\mu_{q\phi - t\log|F'|} \le -\log\xi < 0,$$

applying Lemma 10.2 and Lemma 7.6, we infer that the function $q \mapsto T(q)$, $q \in I\!R$, is real-analytic. Differentiating the equation $\mathrm{P}(\phi_{q,t})=0$ and using Theorem 8.6 again, we obtain

$$0 = \frac{\partial \mathrm{P}}{\partial t}T'(q) + \frac{\partial \mathrm{P}}{\partial q} = -T'(q)\int \log|F'| d\mu_q + \int \phi d\mu_q. \tag{12.12}$$

Therefore

$$T'(q) = \frac{\int \phi d\mu_q}{\int \log|F'| d\mu_q} < 0. \tag{12.13}$$

The equality $T(0)=h=\mathrm{HD}(J_r(F))$ is just Theorem 6.2 from [UZ2]. Let us now show that the function $T: I\!R \to I\!R$ is convex, i.e. that $T''(q) \ge 0$ for all $q \in I\!R$. Differentiating the first part of (12.12), we obtain

$$\begin{aligned} T''(q) &= -\frac{T'(q)^2 \frac{\partial^2 \mathrm{P}}{\partial t^2} + 2T'(q)\frac{\partial^2 \mathrm{P}(\phi_{q,t})}{\partial q \partial t} + \frac{\partial^2 \mathrm{P}}{\partial q^2}}{\frac{\partial \mathrm{P}}{\partial t}} \\ &= \frac{T'(q)^2 \frac{\partial^2 \mathrm{P}}{\partial t^2} + 2T'(q)\frac{\partial^2 \mathrm{P}}{\partial q \partial t} + \frac{\partial^2 \mathrm{P}}{\partial q^2}}{\chi_{\mu_q}}, \end{aligned} \tag{12.14}$$

where $\chi_{\mu_q} = \int \log|F'| d\mu_q$ is the characteristic Lyapunov exponent of the measure μ_q. Invoking Theorem 8.7 we see that

$$\frac{\partial^2 \mathrm{P}}{\partial t^2} = \sigma^2_{\mu_q}(-\log|F'|),\ \frac{\partial^2 \mathrm{P}}{\partial q \partial t} = \sigma^2_{\mu_q}(-\log|F'|, \phi)\ \frac{\partial^2 \mathrm{P}}{\partial q^2} = \sigma^2_{\mu_q}(\phi),$$

Using these formulas a straightforward but lengthy calculation, based on Theorem 8.7, shows that

$$T'(q)^2 \frac{\partial^2 \mathrm{P}}{\partial t^2} + 2T'(q)\frac{\partial^2 \mathrm{P}}{\partial q \partial t} + \frac{\partial^2 \mathrm{P}}{\partial q^2} = \sigma^2_{\mu_q}(-T'(q)\log|F'| + \phi). \tag{12.15}$$

Since $\sigma^2_{\mu_q}(-T'(q)\log|F'|+\phi) \ge 0$ (and $\chi_{\mu_q} > 0$), we conclude that $T''(q) \ge 0$. Passing to the proof of the second part of item (b), notice that, in view of (12.14) and (12.15), $T''(q)=0$ if and only if $\sigma^2_{\mu_q}(-T'(q)\log|F'|+\phi)=0$, which, in view of Proposition 9.2, equivalently means that the function $-T'(q)\log|F'|+\phi$ is cohomologous to a constant, say a, in the class of bounded Hölder continuous functions on $J(F)$. It then follows from (12.13) that $a = \int a d\mu_q = \int(-T'(q)\log|F'|+\phi)d\mu_q = 0$. Therefore $T'(q)\log|F'|$ is cohomologous to ϕ in the class of bounded Hölder continuous functions, and consequently, as $\mathrm{P}(\phi)=0$, also $\mathrm{P}(T'(q)\log|F'|)=0$. It then follows from Theorem 6.2 in [UZ2] that $T'(q)=-h$. Hence, ϕ and $-h\log|F'|$ are cohomologous in the class of bounded Hölder continuous functions, which due to Theorem 9.1 completes the proof of item (b).

Turning to the proof of item (c) notice first that

$$\int -\phi d\mu_q \leq \int \Big(A_\phi + \kappa \mathrm{Re}(z)\Big) d\mu_q(z) = \int \Big(A_\phi + \kappa(\log|F'(z)| - \log|\lambda|)\Big) d\mu_q(z)$$
$$= A_\phi - \kappa \log|\lambda| + \kappa \int \log|F'(z)| d\mu_q(z).$$

Therefore, using (12.13) and Proposition 2.1, we get that

$$-T'(q) \leq \frac{A_\phi - \kappa \log|\lambda|}{\int \log|F'(z)| d\mu_q(z)} + \kappa \leq \xi^{-1}(A_\phi - \kappa \log|\lambda|) + \kappa.$$

Hence, the item (c) is proved.

In order to prove (d) and equality $k^-_{\mu_\phi} = k^+_{\mu_\phi}$, we first give the estimate of the function $k^-_{\mu_\phi}(-T'(q))$ from below. By Birkhoff's Ergodic Theorem and Proposition 12.1 there exists a Borel set $X \subset J_{rr}(F)$ such that $\mu_q(X) = 1$, and such that for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \log|(F^n)'(x)| = \int \log|F'| d\mu_q \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} S_n\phi(x) = \int \phi d\mu_q.$$

Hence, using (12.13), we obtain

$$\lim_{n\to\infty} \frac{-S_n\phi(x)}{\log|(f^n)'(x)|} = -\frac{\int \phi d\mu_q}{\int \log|f'| d\mu_q} = -T'(q),$$

which implies that $X \subset \mathcal{K}_\phi(-T'(q)) \cap J_{rr}(F) = D_\phi(-T'(q))$, where this last equality sign was written due to Proposition 12.2. Thus, using Theorem 11.1, Theorem 6.3, the fact that $\mathrm{P}(\phi_{q,T(q)}) = 0$, and (12.13), we get that

$$\begin{aligned} k^-_{\mu_\phi}(-T'(q)) &= \mathrm{HD}(D_\phi(-T'(q))) \geq \mathrm{HD}(X) \geq \mathrm{HD}(\mu_q) = \frac{\mathrm{h}_{\mu_q}(f)}{\chi_{\mu_q}} \\ &= \frac{T(q)\chi_{\mu_q} - q\int \phi d\mu_q}{\chi_{\mu_q}} = T(q) - q\frac{\int \phi d\mu_q}{\chi_{\mu_q}} = T(q) - qT'(q). \end{aligned} \tag{12.16}$$

This gives the required lower bound for $k^-_{\mu_\phi}$. For the upper bound of $k^+_{\mu_\phi}$ let us fix an element $x \in \mathcal{K}_\phi(-T'(q))$. Since $x \in J_r(F)$, there exist $M > 0$ and an unbounded increasing sequence $\{k_n\}_{n=1}^\infty$ such that $\mathrm{Re}\Big(F^{k_n}(x)\Big) \leq M$ for all $n \geq 1$. The estimate (9.6) with ϕ replaced by $\phi_{q,T(q)}$ gives us that

$$m_q\left(B\Big(x, \delta|(F^{k_n})'(x)|^{-1}\Big)\right) \geq C\exp\Big(S_{k_n}(-T(q)\log|F'| + q\phi)(x)\Big)$$

with some constant C independent of x and n. Hence,

$$\begin{aligned} \liminf_{r\to 0} \frac{\log m_q(B(x,r))}{\log r} &\leq \liminf_{n\to\infty} \frac{\log m_q\left(B\Big(x, \delta|(F^{k_n})'(x)|^{-1}\Big)\right)}{\log\Big(\delta|(F^{k_n})'(x)|^{-1}\Big)} \\ &\leq \lim_{n\to\infty} \frac{-T(q)\log|(F^{k_n})'(x)| + qS_{k_n}\phi(x)}{-\log|(F^{k_n})'(x)|} = T(q) - qT'(q). \end{aligned}$$

This shows that $k^+_{\mu_\phi}(-T'(q)) = \mathrm{HD}(\mathcal{K}_\phi(-T'(q))) \le T(q) - qT'(q)$. Putting this together with (12.16) and (12.11), we see that $k^+_{\mu_\phi} = k^-_{\mu_\phi} = T(q) - qT'(q)$. So, the equality $k^+_{\mu_\phi} = k^-_{\mu_\phi}$ is established and also the proof of item (d) is complete.

If now $\mu_\phi \ne \mu_{-h\log|F'|}$, then, due to (b), $T(q)$ is strictly convex and due to (d) and Theorem 12.3 the part (f) is proved. This also shows that the function $-T'$ is strictly decreasing. Let (α_1, α_2) denote its range (comp. (c)) and let $\beta : (\alpha_1, \alpha_2) \to \mathbb{R}$ be the function inverse to $-T'(q)$ which is also real-analytic. Then it follows from item (d) that the function $f_{\mu_\phi}(\alpha) = T(\beta(a)) - \alpha\beta(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$ is also real-analytic, and the part (e) is proved. Thus the proof of Theorem 12.4 is complete. ■

13. **Quasiconformal Conjugacies**

In this section we employ an idea from Section 9 of [KU] to present a different approach than that from [UZ2] to study quasiconformal conjugacies. In particular the arguments and calculations take on substantially simplified forms.

We start with the following simple result.

Lemma 13.1. *If* $\lambda_0 \in \mathrm{Hyp}$, $\lim_{n\to\infty} \lambda_n = \lambda_0$, *if* $z_n \in J(F_{\lambda_n})$ *for all* $n \ge 1$, *and if* $\lim_{n\to\infty} z_n = z$ *for some* $z \in \mathbb{C}$, *then* $z \in J(F_{\lambda_0})$.

Proof. Since $\lambda_0 \in \mathrm{Hyp}$, the map F_{λ_0} has an attracting periodic orbit. Since each periodic orbit is structurally stable, there exists a neighborhood U of this orbit, which is contained in the basin of immediate attraction to an attracting periodic orbit of F_λ for all $\lambda \in \mathbb{C}$ sufficiently close to λ_0. Now Suppose on the contrary that z is in the Fatou set of F_{λ_0}. Then there exists $k \ge 1$ so large that $F^k_{\lambda_0}(z) \in U$. Then, by continuity, $F^k_{\lambda_n}(z) \in U$ for all $n \ge 1$ large enough. Hence $F^k_{\lambda_n}(z)$, and consequently also z, belong to the Fatou set of F_{λ_n} for all $n \ge 1$ large enough. This contradition finishes the proof. ■

We will need the following improvment of Proposition 2.1, which was formulated in [UZ2] without actual proof.

Lemma 13.2. *If* $\lambda_0 \in \mathrm{Hyp}$, *then there exist constants* $r_1 > 0$, $c > 0$ *and* $\gamma > 1$ *such that*

$$|(f^n_\lambda)'(z)| \ge c\gamma^n$$

for all $\lambda \in B(\lambda_0, r_1)$, *all* $z \in J(f_\lambda)$ *and all* $n \ge 1$.

Proof. Obviously an equivalent formulation of this lemma is when f_λ is replaced by F_λ and $J(f_\lambda)$ by $J(F_\lambda)$. For every $\lambda \in \mathbb{C} \setminus \{0\}$ and every $k \ge 1$ let

$$A_k(\lambda) = \{z \in \mathbb{C} : |(F^k_\lambda)'(z)| > 2\}.$$

There obviously exists $M > 0$ so large that

$$P_M^c \subset A_1(\lambda) \tag{13.1}$$

for all $\lambda\imath\mathbb{C}\setminus\{0\}$. It follows from Proposition 2.1 that there exists $q \geq 1$ such that

$$J(F_{\lambda_0}) \subset A_1(\lambda_0) \cup A_2(\lambda_0) \cup \ldots \cup A_q(\lambda_0). \tag{13.2}$$

We shall show that

$$J(F_\lambda) \subset A_1(\lambda) \cup A_2(\lambda) \cup \ldots \cup A_q(\lambda) \tag{13.3}$$

for all $\lambda \in \mathbb{C}$ sufficiently close to λ_0, say for $\lambda \in B(\lambda_0, r')$. Indeed, suppose on the contrary that there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ converging to λ_0 and a sequence $\{z_n\}_{n=1}^\infty$ ($z_n \in J(F_{\lambda_n})$) such that

$$|(F_{\lambda_n}^j)'(z_n)| \leq 2 \tag{13.4}$$

for all $n \geq 1$ and all $j = 1, 2, \ldots, q$. By (13.1), $z_n \in P_M^c$ for all $n \geq 1$. Since in addition $\mathrm{Re} z_n \leq 0$ for all $n \geq 1$, passing to a subsequence, we may assume without loss of generality that the sequence $\{z_n\}_{n=1}^\infty$ converges to a point $z \in \mathbb{C}$. It therefore follows from Lemma 13.1 that

$$z \in J(F_{\lambda_0}). \tag{13.5}$$

Since due to Hartogs's theorem, each map $(\lambda, z) \mapsto F_\lambda^n(z)$, $n \geq 1$, is an analytic function from $\mathbb{C}^2 \to \mathbb{C}$, using (13.4) we obtain

$$|(F_\lambda^j)'(z)| = \lim_{n\to\infty} |(F_{\lambda_n}^j)'(z_n)| \leq 2$$

for all $j = 1, 2, \ldots, q$. Since $z \in J(F_{\lambda_0})$ (see (13.5)), these inequalities (for $j = 1, 2, \ldots, q$) contradict (13.2), and the proof of formula (13.3) is finished. Since the set $\mathrm{Re}(J(F_\lambda))$ is bounded below, it therefore follows from Lemma 13.1 that there exists $0 < r'' \leq r'$ so small such that $\inf\{\inf(\mathrm{Re}(J(F_\lambda))) : \lambda \in B(\lambda_0, r')\} > -\infty$. Therefore

$$u := \min\{1, \inf\{|F_\lambda'(z)| : \lambda \in B(\lambda_0, r'), z \in J(F_\lambda)\}\} > 0.$$

Fix $\lambda \in B(\lambda_0, r'')$, $z \in J(F_\lambda)$ and $n \geq 1$. Using the finite cover $\{A_1(\lambda), A_2(\lambda), \ldots, A_q(\lambda)\}$ of $J(F)$ and (13.3), we can divide the sequence $z, F_\lambda(z), \ldots, F_\lambda^n(z)$ into blocks of length $\leq q$ such that the modulus of the derivative of the composition along each (possibly except for the last one) such a block is larger than 2. This gives that $|(F_\lambda^n)'(z)| > 2^{E(n/q)}u^q$ and we are done. ■

It is well known (see [EL]) that all hyperbolic exponential functions are structurally stable. In fact we have the following.

Theorem 13.3. *Fix $\lambda_0 \in \mathrm{Hyp}$. Then for every $\lambda \in \mathbb{C}$ sufficiently close to λ_0 there exists a quasiconformal homeomorphism conjugating f_λ and f_{λ_0} (i.e. $f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$). These conjugating homeomorphisms $h_\lambda : \mathbb{C} \to \mathbb{C}$ can be chosen so that the following properties are satisfied.*

(a) *For every* $z \in \mathbb{C}$, *the map* $\lambda \mapsto h_\lambda(z)$ *is holomorphic.*
(b) *The mapping* $(\lambda, z) \to h_\lambda(z)$ *is continuous.*
(c) *The dilation of the maps* h_λ *converges to* 1 *when* $\lambda \to \lambda_0$.

For every $z \in \mathbb{C}$ we will write $h'_\lambda(z)$ to denote the derivative of the holomorphic function $\lambda \mapsto h_\lambda(z)$. Also, in order to use a more convenient notation in the proof of the next proposition, we will frequently write $F(\lambda, z)$ for $F_\lambda(z)$ and more generally $F^n(\lambda, z)$ for $F^n_\lambda(z)$. The operators D_1 and D_2 denote then the partial derivatives respectively with respect to the first variable (λ) and the second variable (z). We shall now prove the following.

Proposition 13.4. *For every* $\lambda_0 \in \mathrm{Hyp}$ *there exists* $\rho > 0$ *such that* $B(\lambda_0, \rho) \subset \mathrm{Hyp}$ *and*

$$T := \sup\{|h'_\lambda(z)| : \lambda \in B(\lambda_0, \rho), z \in J(f_{\lambda_0})\} < \infty.$$

Proof. Consider a periodic point $z \in J(f_{\lambda_0})$ of f_{λ_0}. Fix a period $n \geq 1$ of z. This means that

$$f^n(\lambda, h_\lambda(z)) = h_\lambda(z)$$

for all $\lambda \in B(\lambda_0, r\rho)$ with some $0 < \rho \leq r_1$, where $r_1 > 0$ comes from Lemma 13.2. Differentiating this equation with respect to the (first) variable λ, we get

$$D_1 f^n(\lambda, h_\lambda(z)) + D_2 f^n(\lambda, h_\lambda(z)) h'_\lambda(z) = h'_\lambda(z)$$

or equivalently

$$h'_\lambda(z) = \frac{D_1 f^n(\lambda, h_\lambda(z))}{1 - D_2 f^n(\lambda, h_\lambda(z))}.$$

It follows from Lemma 13.2 that if $n \geq 1$, kept to be a period of z, is large (independently of λ) enough, then

$$|h'_\lambda(z)| \leq 2 \frac{|D_1 f^n(\lambda, h_\lambda(z))|}{|D_2 f^n(\lambda, h_\lambda(z))|}. \tag{13.6}$$

So, all we need to do is to get a satisfactory upper bound for the right-hand side of this inequality. We have for all $\lambda \in B(\lambda_0, \rho)$, all $w \in \mathbb{C}$ and all $n \geq 1$, that

$$\begin{aligned} D_1 f^n(\lambda, w) &= D_1\Big(f(\lambda, f^{n-1}(\lambda, w))\Big) \\ &= D_1 f(\lambda, f^{n-1}(\lambda, w)) + D_2 f(\lambda, f^{n-1}(\lambda, w)) D_1 f^{n-1}(\lambda, w) \\ &= 1 + D_2 f(\lambda, f^{n-1}(\lambda, w)) D_1 f^{n-1}(\lambda, w). \end{aligned}$$

Therefore, using the formula $D_1 f(\lambda, z) = \lambda^{-1} D_2(\lambda, z)$, we get by induction that

$$D_1 f^n(\lambda, w) = \frac{1}{\lambda} \sum_{j=1}^{n} D_2 f^j(\lambda, f^{n-j}(\lambda, w)).$$

Hence, assuming that $\rho \le |\lambda_0|/2$, it follows from (13.6) and Lemma 13.2 that

$$\begin{aligned} |h'_\lambda(z)| &\le \frac{2}{|\lambda|}\sum_{j=1}^{n}|D_2f^{n-j}(\lambda,h_\lambda(z))|^{-1} = \frac{2}{|\lambda|}\sum_{j=0}^{n-1}|D_2f^{j}(\lambda,h_\lambda(z))|^{-1} \\ &\le \frac{4}{|\lambda_0|}\sum_{j=0}^{\infty}|(f_\lambda^j)'(h_\lambda(z))|^{-1} \le 2c^{-1}\sum_{j=0}^{\infty}\gamma^{-j} = \frac{2\gamma}{c(\gamma-1)} < \infty \end{aligned} \tag{13.7}$$

for all $\lambda \in \mathbb{C}$ sufficiently close to λ_0, say for $\lambda \in B(\lambda_0, r)$. Now fix an arbitrary point $w \in J(f_{\lambda_0})$. Since periodic points of f are dense in $J(f_{\lambda_0})$, there exists a sequence $\{w_n\}_{n=1}^\infty \subset J(f_{\lambda_0})$ consisting of periodic points of f_{λ_0} such that $\lim_{n\to\infty} w_n = w$. It then follows from condition (b) of Theorem 13.3 that the sequence $\{\lambda \mapsto h_\lambda(w_n)\}_{n=1}^\infty$ converges to the function $\lambda \mapsto h_\lambda(w)$ uniformly on all compact subsets of $B(\lambda_0, \rho)$. Since the item (a) of Theorem 13.3 says that all the functions $\lambda \mapsto h_\lambda(\xi)$, $\xi \in J(f_{\lambda_0})$, are analytic, it follows from (13.7) that

$$|h'_\lambda(w)| = \lim_{n\to\infty}|h'_\lambda(w_n)| \le \frac{2\gamma}{c(\gamma-1)}.$$

We are done. ■

Given $K, \alpha > 0$ and $\lambda \in \mathbb{C} \setminus \{0\}$ we say that a map $h : \mathbb{C} \to \mathbb{C}$ is (K, α, λ)-Hölder continuous if

$$|h(x) - h(y)| \le K|x-y|^\alpha$$

for all $z \in J(f_\lambda)$ and all $x, y \in B(z, 2^{-10})$.

Proposition 13.5. *For every $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}\lambda_0 > 1$ there exists $\hat{K}_{\lambda_0} \ge 1$ such that if $\lambda \in \mathbb{C}$ is sufficiently close to λ_0, then the conjugating homeomorphism $h_\lambda : \mathbb{C} \to \mathbb{C}$ is $(\hat{K}_{\lambda_0}, 1/q_\lambda, \lambda_0)$-Hölder continuous, where q_λ is the quasiconformality constant of h_λ.*

Proof. In view of Theorem 13.3, we may assume $\rho > 0$ produced in Proposition 13.4 to be so small that $q_\lambda < 2$ for all $\lambda \in B(\lambda_0, \rho)$. Fix $x \in J(F_{\lambda_0})$. Let $G = h_\lambda(B(x, 1)))$. Since $h_\lambda : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, G is an open connected simply connected set. Let $R : B(0, 1) \to G$ be its conformal representation such that $R(0) = h_\lambda(x)$. Since the Julia set of a hyperbolic exponential function is the closer of Devaney's hairs, there exists a connected subset of the Julia set $J(f_{\lambda_0})$ joining x and the boundary $\partial B(x, 1)$ of $B(x, 1)$. Since $J(f_\lambda) = h_\lambda\big(J(f_{\lambda_0})\big)$, there thus exists a connected subset of $J(f_\lambda)$ joining $h_\lambda(x)$ and ∂G. Consequently, there exists $z \in \partial B(0, 1/2)$ such that $R(z) \in J(f_\lambda)$. Hence, $R(B(0, 1/2))$ does not contain any ball centered at $h_\lambda(x)$ and with radius $> |R(z) - h_\lambda(x)|$. But writing $R(z) = h_\lambda(w)$, where

$w \in B(x,1) \cap J(F_{\lambda_0})$ as $R(z) \in G \cap J(f_\lambda)$, it follows from Proposition 13.4 that

$$\begin{aligned}|R(z) - h_\lambda(x)| &= |h_\lambda(w) - h_\lambda(x)| \le |h_\lambda(w) - w| + |w - x| + |x - h_\lambda(x)| \\ &= |h_\lambda(w) - h_{\lambda_0}(w)| + 1 + |h_{\lambda_0}(x) - h_\lambda(x)| \\ &= \left|\int_{\lambda_0}^{\lambda} h'_\gamma(w)d\gamma\right| + 1 + \left|\int_{\lambda_0}^{\lambda} h'_\gamma(x)d\gamma\right| \\ &\le \int_{\lambda_0}^{\lambda} |h'_\gamma(w)||d\gamma| + 1 + \int_{\lambda_0}^{\lambda} |h'_\gamma(x)||d\gamma| \\ &\le 1 + 2T|\lambda - \lambda_0| \le 1 + 2rT.\end{aligned}$$

Therefore $R(B(0,1/2))$ does not contain the ball $B(h_\lambda(x), 2(1+2rT))$. So, Koebe's $\frac{1}{4}$-distortion theorem implies that

$$|R'(0)| \le 16(1+2rT). \tag{13.8}$$

The map $g = R^{-1} \circ h_\lambda : B(x,1) \to B(0,1)$ is a quasiconformal homeomorphism between two disks of radius 1. Hence, by Mori's theorem, $|g(z_1) - g(z_2)| \le 16|z_1 - z_2|^{1/q_\lambda}$ for all $z_, z_2 \in B(x,1)$. In particular, for every $z \in B(x,1)$,

$$|g(z)| = |g(z) - g(x)| \le 16|z - x|^{1/q_\lambda}.$$

This implies that if $|z-x| \le 2^{-10}$, then $|g(z)| \le 16\left(2^{-10}\right)^{1/q_\lambda} \le 16\left(2^{-10}\right)^{1/2} = 1/2$. Therefore, if $|z_1 - x|, |z_2 - x| \le 2^{-10}$, then using (13.8), we obtain

$$\begin{aligned}|h_\lambda(z_2) - h_\lambda(z_1)| &= |R(g(z_2)) - R(g(z_2))| \le K|R'(0)||g(z_2) - g(z_1)| \\ &\le 16K|R'(0)| \cdot |z_2 - z_1|^{1/q_\lambda} \le 16K(1+2rT)|z_2 - z_1|^{1/q_\lambda}\end{aligned}$$

where K is the Koebe distortion constant corresponding to the scale 1/2. We are done. ■

We end this section with the following simple observation.

Proposition 13.6. *For every $\lambda_0 \in \mathrm{Hyp}$, every λ sufficiently close to λ_0 and every $z \in \mathbb{C}$, we have $h_\lambda(z + 2\pi i) = h_\lambda(z) + 2\pi i$.*

Proof. Since h_λ conjugates f_λ and f_{λ_0}, we have

$$\lambda e^{h_\lambda(z)} = h_\lambda(\lambda_0 e^z) = h_\lambda(\lambda_0 e^{z+2\pi i}) = \lambda e^{h_\lambda(z+2\pi i)}.$$

Hence $h_\lambda(z + 2\pi i) - h_\lambda(z) \in 2\pi i\boldsymbol{Z}$ and, since the function $z \mapsto h_\lambda(z + 2\pi i) - h_\lambda(z)$ is continuous, there exists $k(\lambda) \in \boldsymbol{Z}$ such that $h_\lambda(z + 2\pi) - h_\lambda(z) = 2\pi i k(\lambda)$ for all $z \in \mathbb{C}$. Since the function $\lambda \mapsto h_\lambda(2\pi i) - h_\lambda(0) = 2\pi i k(\lambda)$ is holomorphic, we conclude that $k(\lambda)$ is a constant independent of λ. Since $h_{\lambda_0}(2\pi i) - h_{\lambda_0}(0) = 2\pi i - 0 = 2\pi i$, the proof is complete. ■

In particular, h_λ induces the map of the cylinder P onto itself and we reserve for it the same symbol h_λ.

14. Finer Analytic Properties of Perron-Frobenius Operators and Topological Pressure

For every $s \in I\!R$ let
$$\mathrm{Hyp}(s) = \{\lambda \in \mathrm{Hyp} : J(f_\lambda) \subset \mathbb{C}_s^c\}.$$
Now fix $a \in I\!R$ and $\lambda_0 \in \mathrm{Hyp}(a)$. Since by Proposition 13.5, $h_\lambda : J(F_{\lambda_0}) \to J(F_\lambda)$ is Hölder continuous with the Hölder exponent $\alpha(\lambda)$ depending on λ and since $\alpha(\lambda)$ converges to 1 as $\lambda \to \lambda_0$, we get that for every $r > 0$ sufficiently small that
$$\alpha = \inf\{\alpha(\lambda) : \lambda \in B(\lambda_0, r)\} > 0. \tag{14.1}$$
Fix two tame harmonic functions $\phi, \psi : P_a^c \to I\!R\}$. For every $\lambda \in B(\lambda_0, r)$ and all $(\lambda, q, t) \in B(\lambda_0, r) \times (\Sigma_2(\phi, \psi) \cap I\!R^2)$ let $\mathcal{L}^0_{\lambda,q,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$ be the operator induced by the potential $q\phi \circ h_\lambda + t\psi \circ h_\lambda : J(F_{\lambda_0}) \to I\!R$, i.e.
$$\mathcal{L}^0_{\lambda,q,t} g(z) = \sum_{x \in F_{\lambda_0}^{-1}(z)} \exp\Big(q\phi \circ h_\lambda + t\psi \circ h_\lambda\Big) g(x).$$
Our aim is to use Corollary 7.1. However, the potential $q\phi\lambda \circ h_\lambda + t\psi \circ h_\lambda$ does not depend on $(\lambda, q, t) \in (\lambda, q, t) \in B(\lambda_0, r) \times (\Sigma_2(\phi, \psi) \cap I\!R^2)$ in a holomorphic way. For this reason, we have to embed λ into $\mathbb{C}^2$ and q into $\mathbb{C}$ as well as t into $\mathbb{C}$. We embed the complex plane $\mathbb{C}$ into $\mathbb{C}^2$ by the formula $x + iy \mapsto (x, y) \in \mathbb{C}^2$. So, $\lambda \in \mathbb{C} = I\!R^2$ may be treated as an element of $\mathbb{C}^2$. Fix
$$f = f_{\lambda_0} \text{ and } F = F_{\lambda_0}.$$
The main result of this section is this.

Proposition 14.1. *Fix $a \in I\!R$, $\lambda_0 \in \mathrm{Hyp}(a)$, two tame harmonic functions $\phi, \psi : P_a^c \to I\!R$, and $(q_0, t_0) \in \Sigma_2(\phi, \psi) \cap I\!R^2$. There then exist $R > 0$ and a holomorphic function*
$$L : \mathbb{D}_{\mathbb{C}^4}((\lambda_0, q_0, t_0), R) \to L(\mathrm{H}_\alpha(J(F(\lambda_0)))$$
(λ_0 is treated here as elements of $\mathbb{C}^2$, q_0 and t_0 as elements of $\mathbb{C}$) and α comes from (14.1) with r replaced by R) such that for every $(\lambda, q, t) \in B_{\mathbb{C}}(\lambda_0, R) \times B_{I\!R}(q_0, R) \times B_{I\!R}(t_0, R)$,
$$L(\lambda, q, t) = \mathcal{L}^0_{\lambda,q,t}. \tag{14.2}$$

The proof of this proposition is long and technically involved. For the purpose of its readability we split it in a sequence of lemmas culminating in an application of Proposition 7.1. We start with the following.

Lemma 14.2. *If $g : \mathbb{C}_a^c \to I\!R$ is a bounded harmonic function and $\hat{g} : \mathbb{C}_a^c \to \mathbb{C}$ is a holomorphic function whose real part is equal to g, then for every $b > a$, $\sup\{|\hat{g}'(z)| : z \in \mathbb{C}_b^c\} < \infty$. In particular $\hat{g}|_{\mathbb{C}_b^c} : \mathbb{C}_b^c \to \mathbb{C}$ is Lipschitz continuous.*

Proof. Since $g : \mathbb{C}_a^c \to \mathbb{R}$ is bounded, there exists $A \ge 0$ such that $-A \le g(z) \le A$ for all $z \in \mathbb{C}_a^c$. Consider the function $G(z) = \exp(\hat{g}(z))$, $z \in \mathbb{C}_a^c$. Then

$$e^{-A} \le |G(z)| \le e^A \tag{14.3}$$

for all $z \in \mathbb{C}_a^c$. Now put $r : b - a > 0$. The for every $z \in \mathbb{C}_b^c$, $B(z, r) \subset \mathbb{C}_a^c$, and it follows from Cauchy's Estimate that

$$|G'(z)| \le \frac{e^A}{r}. \tag{14.4}$$

Since $G'(z) = G(z)\hat{g}'(z)$, we get $\hat{g}'(z) = G'(z)/G(z)$, and applying (14.4) along with (14.3), we obtain that for all $z \in \mathbb{C}_b^c$

$$|\hat{g}'(z)| = \frac{|G'(z}{|G(z)|} \le \frac{e^{2A}}{r}.$$

We are done. ■

Now fix an α-tame harmonic potential $U : P_a^c \to \mathbb{R}$. Then the function $U \circ \pi : \mathbb{C}_a^c \to \mathbb{R}$ is also harmonic, and let $W :: \mathbb{C}_a^c \to \mathbb{R}$ be its harmonic conjugate. Then the function $z \mapsto W(z + 2\pi i)$ is conjugate to the function $z \mapsto U \circ \pi(z + 2\pi i)$, and, since for every $z \in \mathbb{C}_a^c$. $U \circ \pi(z + 2\pi i) = U \circ \pi(z)$, we see that the function $z \mapsto U \circ \pi(z + 2\pi i)$ is conjugate to the function $z \mapsto U \circ \pi(z)$. Thus, there exists $c \in \mathbb{C}$ such that

$$W(z + 2\pi i) - W(z) = c \tag{14.5}$$

for all $z \in \mathbb{C}_a^c$. Put

$$\widehat{U \circ \pi} := U \circ \pi + iW : \mathbb{C}_a^c \to \mathbb{C}.$$

Since by Lemma 13.6, $h_\lambda(z + 2\pi i) = h_\lambda(z) + 2\pi i$, using (14.5) we get that

$$\begin{aligned}
&\Big(\widehat{U \circ \pi} \circ h_\lambda(z + 2\pi i) - \widehat{U \circ \pi}(z + 2\pi i)\Big) - \Big(\widehat{U \circ \pi} \circ h_\lambda(z) - \widehat{U \circ \pi}(z)\Big) = \\
&\quad = \widehat{U \circ \pi}(h_\lambda(z) + 2\pi i) - \widehat{U \circ \pi}(z + 2\pi i)\Big) - \widehat{U \circ \pi} \circ h_\lambda(z) + \widehat{U \circ \pi}(z)\Big) \\
&\quad = U \circ \pi(h_\lambda(z) + 2\pi i) + iW(h_\lambda(z) + 2\pi i) - U \circ \pi(z + 2\pi i) - iW(z + 2\pi i) - \\
&\qquad - U \circ \pi(h_\lambda(z)) - iW(h_\lambda(z)) + U \circ \pi(z) + iW(z) \\
&\quad = U \circ \pi(h_\lambda(z)) + iW(h_\lambda(z)) + ic - U \circ \pi(z) - iW(z) - ic - \\
&\qquad - U \circ \pi(h_\lambda(z)) - iW(h_\lambda(z)) + U \circ \pi(z) + iW(z) \\
&\quad = 0.
\end{aligned}$$

Therefore, for every $z \in P_a^c$, the map $\Omega_z^U : \mathrm{Hyp}(a) \to \mathbb{C}$ given by the formula

$$\Omega_z^U(\lambda) = \widehat{U \circ \pi}(h_\lambda(\tilde{z})) - \widehat{U \circ \pi}(\tilde{z}) \tag{14.6}$$

is well-defined independently of $\tilde{z} \in \pi^{-1}(z)$. Now fix $r > 0$ so small that the ball $B(\lambda_0, 3r)$ is contained in $\mathrm{Hyp}(a)$. Then fix $b > a$ so close to a that $J(f_\lambda) \subset \mathbb{C}_b^c$ for all $\lambda \in B(\lambda_0, 2r)$. Since the harmonic function $U \circ \pi + \kappa(U)\mathrm{Re} : \mathbb{C}_a^c \to \mathbb{R}$ is bounded, it follows from Lemma 14.2 that the function $\widehat{U \circ \pi} + \kappa(U)\mathrm{Id} : \mathbb{C}_b^c \to \mathbb{C}$ is Lipschitz continuous. Since $\kappa(U)\mathrm{Id}$ is obviously

Lipschitz continuous (with a Lipschitz constant $\kappa(U)$, we therefore conclude that the function $\widehat{U\circ\pi}:\mathbb{C}_b^c\to\mathbb{C}$ is Lipschitz continuous. Denote its Lipschiz constant by T_U. Let $M_1<\infty$ be the product of the supremum appearing in Proposition 13.4 over the ball $B(\lambda_0,r)$ and the number r. Fix $\lambda\in B(\lambda_0,r)$, $z\in J(F_\lambda)$ and $\tilde z\in\pi^{-1}(z)$. Using Proposition 13.4 and (14.6) we then get that

$$\begin{aligned}|\Omega_z^U(\lambda)| &= |\widehat{U\circ\pi}(h_\lambda(\tilde z))-\widehat{U\circ\pi}(\tilde z)|\le T_U|h_\lambda(\tilde z)-\tilde z| = T_U|h_\lambda(\tilde z)-h_{\lambda_0}(\tilde z)|\\ &\le T_UM_1r^{-1}|\lambda-\lambda_0|\le T_UM_1. \end{aligned}\tag{14.7}$$

Now fix $z,w\in J(F_\lambda)$ with $|w-z|\le\delta$ and then fix $\tilde z\in\pi^{-1}(z)$ and $\tilde w\in\pi^{-1}(w)$ with $|\tilde w-\tilde z|=|w-z|$. using again Lipschitz continuity of the function $\widehat{U\circ\pi}:\mathbb{C}_b^c\to\mathbb{C}$, Proposition 13.4 and (14.6) along with (14.1), we get that

$$\begin{aligned}|\Omega_w^U(\lambda)-\Omega_z^U(\lambda)| &= \left|\widehat{U\circ\pi}(h_\lambda(\tilde w))-\widehat{U\circ\pi}(\tilde w)-\widehat{U\circ\pi}(h_\lambda(\tilde z))+\widehat{U\circ\pi}(\tilde z)\right|\\ &= \left|\left(\widehat{U\circ\pi}(h_\lambda(\tilde w))-\widehat{U\circ\pi}(h_\lambda(\tilde z))\right)+\left(\widehat{U\circ\pi}(\tilde z)-\widehat{U\circ\pi}(\tilde w)\right)\right|\\ &\le \left|\widehat{U\circ\pi}(h_\lambda(\tilde w))-\widehat{U\circ\pi}(h_\lambda(\tilde z))\right|+\left|\widehat{U\circ\pi}(\tilde z)-\widehat{U\circ\pi}(\tilde w)\right|\\ &\le T_U|h_\lambda(\tilde w)-h_\lambda(\tilde z)|+T_U|\tilde w-\tilde z|\\ &\le T_UC|\tilde w-\tilde z|^\alpha+T_U|\tilde w-\tilde z|\\ &\le T_U(C+1)|\tilde w-\tilde z|^\alpha = T_U(C+1)|w-z|^\alpha\\ &\le 2CT_U|w-z|^\alpha. \end{aligned}\tag{14.8}$$

with an appropriate constant $C>0$ coming from Proposition 13.5. Combining this and (14.7), we have thus proved the following.

Lemma 14.3. *Ror every $\lambda\in B(\lambda_0,r)$ the function $z\mapsto\Omega_z^U(\lambda)$, $z\in J(F)$, belongs to H_α and its Hölder norm is bounded above by $T_U(2C+M_1)$.*

Since the function $\Omega_z^U:B(\lambda_0,r)\to\mathbb{C}$ is holomorphic and since $\Omega_z^U(\lambda_0)=0$, this function is uniquely represented as a power series

$$\Omega_z^U(\lambda)=\sum_{n=1}^\infty a_n(z)(\lambda-\lambda_0)^n.$$

By Cauchy's inequalities and (14.7), we get that

$$|a_n(z)|\le\frac{T_UM_1}{r^n}\tag{14.9}$$

for all $n \geq 0$. For every $\lambda = x + iy \in B(\lambda_0, r) \subset \mathbb{C}$, we have

$$\begin{aligned} \mathrm{Re}\Omega_z^U(\lambda) &= \mathrm{Re}\left(\sum_{n=1}^{\infty} a_n(z)\Big((x - \mathrm{Re}(\lambda_0)) + (y - \mathrm{Im}(\lambda_0))i\Big)^n\right) \\ &= \sum_{p,q=0}^{\infty} c_{p,q}(z)\Big(x - \mathrm{Re}(\lambda_0)\Big)^p\Big(y - \mathrm{Im}(\lambda_0)\Big)^q, \end{aligned} \tag{14.10}$$

where $c_{p,q}(z) = a_{p+q}(z)\binom{p+q}{q} i^q$. Due to (14.9)

$$|c_{p,q}(z)| \leq |a_{p+q}(z)| \cdot 2^{p+q} \leq T_U M_1 2^{p+q} r^{-(p+q)} \tag{14.11}$$

Hence, $\mathrm{Re}\Omega_z^U$ extends by the same power series expansion

$$\sum_{p,q=0}^{\infty} c_{p,q}(z)\Big(x - \mathrm{Re}(\lambda_0))^p\Big(y - \mathrm{Im}(\lambda_0)\Big)^q$$

to a holomorphic function on the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$. We denote this extension by the same symbol $\mathrm{Re}\Omega_z^U$ and we have

$$|\mathrm{Re}\Omega_z^U(\lambda)| \leq \sum_{p,q=0}^{\infty} M_1 2^{-(p+q)} = 4T_U M_1 \tag{14.12}$$

on $\mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$. We now shall prove the following.

Lemma 14.4. *For every $\lambda \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$ the function $z \mapsto \mathrm{Re}\Omega_z^U(\lambda)$, $z \in J(F)$, is in H_α.*

Proof. Applying Lemma 14.3 and using Cauchy's inequalities, we conclude that

$$|a_n(z) - a_n(w)| \leq T_U(2C + M_1)\left(\frac{4}{r}\right)^n |z - w|^\alpha$$

for all $z, w \in J(F)$ with $|z - w| \leq \delta$. Therefore,

$$|c_{p,q}(z) - c_{p,q}(w)| \leq T_U(2C + M_1) \cdot 2^{p+q}\left(\frac{4}{r}\right)^{p+q} |z - w|^\alpha = T_U(2C + M_1)\left(\frac{8}{r}\right)^{p+q} |z - w|^\alpha \tag{14.13}$$

Hence,

$$\begin{aligned} |\mathrm{Re}\Omega_z^U(\lambda) - \mathrm{Re}\Omega_w^U(\lambda)| &\leq T_U(2C + M_1)\sum_{p,q=0}^{\infty}\left(\frac{8}{r}\right)^{p+q}\left(\frac{r}{16}\right)^p\left(\frac{r}{16}\right)^q |z - w|^\alpha \\ &= 2T_U(2C + M_1)|z - w|^\alpha \end{aligned} \tag{14.14}$$

for every $\lambda \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16))$ and all $z, w \in J(F)$ with $|z - w| \leq \delta$. Hence, using (14.12), we see that the function $z \mapsto \mathrm{Re}\log\Omega_z^U(\lambda)$, $z \in J(F)$, is in H_α for every $\lambda \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$. ■

In view of Lemma 14.4, the function $z \mapsto q\mathrm{Re}\Omega_z^U(\lambda)$, $z \in J(F)$, is in H_α for every $q \in \mathbb{C}$ and every $\lambda \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$. We shall prove the following.

Lemma 14.5. *For every $\rho > 0$ all the functions $\lambda \mapsto q\mathrm{Re}\Omega^U_{(\cdot)}(\lambda) \in \mathrm{H}_a$, $\lambda \in I\!\!D_{\mathbb{C}^2}(\lambda_0, r/16)$, $q \in B_{\mathbb{C}}(q_0, \rho)$, are Lipschitz continuous with Lipschitz constants independent of q (but depending on ρ).*

Proof. It obviously suffices to show that the function $\lambda \mapsto \mathrm{Re}\Omega^U_{(\cdot)}(\lambda) \in \mathrm{H}_a$ is Lipschitz continuous. In order to do it, fix $\lambda = (\lambda_x, \lambda_y), \lambda' = (\lambda'_x, \lambda'_y) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ In view of (14.10) we have for all $z \in J(F)$ that

$$\begin{aligned}
|\mathrm{Re}\Omega^U_z(\lambda') - \mathrm{Re}\Omega^U_z(\lambda)| &= \\
&= \sum_{p,q=0}^{\infty} c_{p,q}(z) \Big((\lambda'_x - \mathrm{Re}\lambda_0)^p(\lambda'_y - \mathrm{Im}\lambda_0)^q - (\lambda_x - \mathrm{Re}\lambda_0)^p(\lambda_y - \mathrm{Im}\lambda_0)^q\Big)
\end{aligned} \tag{14.15}$$

Put $a_x = \lambda'_x - \mathrm{Re}\lambda_0$, $a_y = \lambda'_y - \mathrm{Im}\lambda_0$, $b_x = \lambda_x - \mathrm{Re}\lambda_0$ and $b_y = \lambda_y - \mathrm{Im}\lambda_0$. We then have

$$\begin{aligned}
|a_x^p a_y^q - b_x^p b_y^q| &= |a_x^p(a_y^q - b_y^q) + b_y^q(a_x^p - b_x^p)| \\
&\le |a_x^p||a_y - b_y| \sum_{i=0}^{q-1} |a_y|^i |b_y|^{q-1-i} + |b_y^q||a_x - b_x| \sum_{i=0}^{p-1} |a_x|^i |b_x|^{p-1-i} \\
&\le \left(q\left(\frac{r}{16}\right)^p \left(\frac{r}{16}\right)^{q-1} + p\left(\frac{r}{16}\right)^q \left(\frac{r}{16}\right)^{p-1}\right) ||\lambda' - \lambda|| \qquad (14.16) \\
&\le \frac{16}{r}(p+q)\left(\frac{r}{16}\right)^p \left(\frac{r}{16}\right)^q ||\lambda' - \lambda||
\end{aligned}$$

Combining this, (14.15) and (14.11), we obtain

$$\begin{aligned}
|\mathrm{Re}\Omega^U_z(\lambda') - \mathrm{Re}\Omega^U_z(\lambda)| &\le \frac{16}{r}||\lambda' - \lambda|| T_U M_1 \sum_{p,q=0}^{\infty} (p+q) 8^{-(p+q)} \\
&= \frac{16 C_1}{r}||\lambda' - \lambda||,
\end{aligned}$$

(where $C_1 = T_U M_1 \sum_{p,q=0}^{\infty}(p+q)8^{-(p+q)}$ is finite) for all $\lambda, \lambda' \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$. In other words

$$||\mathrm{Re}\Omega^U_{(\cdot)}(\lambda') - \mathrm{Re}\Omega^U_{(\cdot)}(\lambda)||_\infty \le 16 C_1 r^{-1} ||\lambda' - \lambda||. \tag{14.17}$$

Now fix $z, w \in J(F)$ with $|z - w| \le \delta$. It follows from (14.13) and (14.16) that

$$\begin{aligned}
&\Big|\mathrm{Re}\Omega^U_w(\lambda') - \mathrm{Re}\Omega^U_z(\lambda) - (\mathrm{Re}\Omega^U_z(\lambda') - \mathrm{Re}\Omega^U_z(\lambda))\Big| = \\
&\quad = \left|\sum_{p,q=0}^{\infty} (c_{p,q}(w) - c_{p,q}(z)) \Big((\lambda'_x - \mathrm{Re}\lambda_0)^p(\lambda'_y - \mathrm{Im}\lambda_0)^q - (\lambda_x - \mathrm{Re}\lambda_0)^p(\lambda_y - \mathrm{Im}\lambda_0)^q\Big)\right| \\
&\quad \le 16 T_U (2C + M_1) r^{-1} |z - w|^\alpha ||\lambda' - \lambda|| \sum_{p,q=0}^{\infty} (p+q) 2^{-(p+q)} \\
&\quad \le 16 C_2 T_U (2C + M_1) r^{-1} ||\lambda' - \lambda|| |z - w|^\alpha,
\end{aligned}$$

where $C_2 = \sum_{p,q=0}^{\infty}(p+q)2^{-(p+q)}$ is finite. Thus,

$$v_\alpha\Big(-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda) - (-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda'))\Big) \le 16C_2T_U(2C+M_1)r^{-1}||\lambda'-\lambda||.$$

Combining this and (14.17) we get for all $\lambda, \lambda' \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ that

$$||\mathrm{Re}\Omega^U_{(\cdot)}(\lambda') - \mathrm{Re}\Omega^U_{(\cdot)}(\lambda)||_\alpha \le 16C_1r^{-1}(1+C_2T_U(2C+M_1))||\lambda'-\lambda||.$$

We are done. ■

As an immediate consequence of this lemma we get the following.

Lemma 14.6. *The mapping $(\lambda, q) \mapsto q\mathrm{Re}\Omega^U_{(\cdot)}(\lambda) \in \mathrm{H}_a$ is continuous in the domain $\mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times \mathbb{C}$.*

Our next lemma is an immediate consequence of Lemma 7.3.

Lemma 14.7. *If $V : P_a^c \to \mathbb{R}$ is another tame function, then for all $(q,t) \in \Sigma_2(U,V)$ the function $z \mapsto \exp\Big(qU(z)+tV(z)\Big)$, $z \in J(F)$, is in H_α and, the function $(q,t) \mapsto \exp\Big(qU(\cdot)+tV(\cdot)\Big)$, $(q,t) \in \Sigma_2(U,V)$, is continuous with respect to the variable q and t.*

Let now, as in the beginning of this section, $\phi, \psi : P_a^c : \mathbb{R}$ be two tame functions. Let λ_0, q_0, t_0 come from the assumptions of Proposition 14.1. We shall check that with sufficiently small $R > 0$, the family of potentials $\zeta_{(\lambda,q,t)} : (\lambda, q, t) \in \mathbb{D}_{\mathbb{C}^4}(\lambda_0, q_0, t_0), R)\}$ defined by the formula

$$\zeta_{(\lambda,q,t)}(z) = q\mathrm{Re}\Omega^\phi_z(\lambda) + q\phi(z) + t\mathrm{Re}\Omega^\psi_z(\lambda) + t\psi(z), \; z \in J(F), \tag{14.18}$$

satisfies the assumptions of Proposition 7.1.

Lemma 14.8. *The potentials $\zeta_{(\lambda,q,t)} : (\lambda, q, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times \Sigma_2(\phi, \psi)$ satisfy the assumption (a) of Proposition 7.1.*

Proof. In view of Lemma 14.4, for all $\lambda \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and all $q, t \in \mathbb{C}$, both functions $z \to q\mathrm{Re}\Omega^\phi_z(\lambda)$ and $z \to t\mathrm{Re}\Omega^\psi_z(\lambda)$ belong to H_α. Since $q\phi$ and $t\psi$ are α-Hölder continuous, we see that the function $\zeta_{(\lambda,q,t)}$ is α-Hölder continuous, and since $(q,t) \in \Sigma_2(\phi, \psi)$, we see that $\mathrm{Re}\zeta_{(\lambda,q,t)}$ is a 1^+-tame function. We are done. ■

As an immediate consequence of Lemma 14.6, Lemma 14.7 and continuity of the exponential function $\exp : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$, we get the following.

Lemma 14.9. *The function $(\lambda, q, t) \mapsto \exp\Big(\zeta_{(\lambda,q,t)}\Big) \in \mathrm{H}_\alpha$, $(\lambda, q, t) : (\lambda, q, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times \Sigma_2(\phi, \psi)$ is continuous with respect to each variable separately, i.e. the assumption (b) of Proposition 7.1 is satisfied.*

Lemma 14.10. *If $R>0$ is so small that $\mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)\subset\Sigma_2(\phi,\psi)$, then*
$$\sup\Big\{c_{\zeta_{(\lambda,q,t)}}:(\lambda,q,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times\mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)\Big\}<\infty.$$
This means that the assumption (c) of of Proposition 7.1 is satisfied.

Proof. It follows from (14.18) and (14.14) that
$$v_\alpha\Big(\zeta_{(\lambda,q,t)}\Big)\le C_\phi|q|+C_\psi|t|\le C_\phi(|q_0|+R)+C_\psi(|t_0|+R)$$
for all $(\lambda,q,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times\mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)$. Applying now Lemma 3.2 completes the proof. ■

Since both functions $\lambda\mapsto\mathrm{Re}\Omega_z^\phi(\lambda)$ and $\lambda\mapsto\mathrm{Re}\Omega_z^\psi(\lambda)$ are holomorphic in the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)$ for every $z\in J(F)$, as an immediate consequence of (14.18), we get this.

Lemma 14.11. *The function $(\lambda,q,t)\mapsto\zeta_{(\lambda,q,t)}(z)$, $(\lambda,q,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)\times\mathbb{C}^2$, is holomorphic for every $z\in J(F)$. This means that the assumption (d) of of Proposition 7.1 is satisfied.*

Lemma 14.12. *The assumption (e) of of Proposition 7.1 is satisfied for the potential $\zeta_{(\lambda,q,t)}(z)$ with the set $G:=\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times\mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)$.*

Proof. Fix arbitrary $(\lambda_2,q_2,t_2)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times\mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)$. Take $\gamma>0$ so small that $\mathbb{D}_{\mathbb{C}^4}((\lambda_2,q_2,t_2),\gamma)\subset G:=\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times\mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)$. Then fix arbitrary $\lambda_1\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)$ and $(q_1,t_1)\in\mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)$ such that
$$\mathrm{Re}(q_1)\in\begin{cases}(\mathrm{Re}(q_2)+\gamma,+\infty) & if\kappa(\phi)\ge 0\\(-\infty,\mathrm{Re}(q_2)+\gamma) & if\kappa(\phi)\ge 0\end{cases}\ \text{ and }\ \mathrm{Re}(t_1)\in\begin{cases}(\mathrm{Re}(t_2)+\gamma,+\infty) & if\kappa(\psi)\le 0\\(-\infty,\mathrm{Re}(t_2)+\gamma) & if\kappa(\psi)\ge 0\end{cases}\tag{14.19}$$
Then for every $(\lambda,q,t)\in\mathbb{D}_{\mathbb{C}^4}((\lambda_2,q_2,t_2),\gamma)$, we have
$$\begin{aligned}&\Big|\exp\Big(\zeta_{(\lambda,q,t)}(z)-\zeta_{(\lambda_1,q_1,t_1)}(z)\Big)\Big|=\\&=\Big|\exp\Big(q_1\Big(\mathrm{Re}\Omega_z^\phi(\lambda)-\mathrm{Re}\Omega_z^\phi(\lambda_1)\Big)\Big)\exp\Big((q-q_1)\mathrm{Re}\Omega_z^\phi(\lambda)\Big)\exp\Big(t_1\Big(\mathrm{Re}\Omega_z^\psi(\lambda)-\mathrm{Re}\Omega_z^\psi(\lambda_1)\Big)\Big)\cdot\\&\quad\cdot\exp\Big((t-t_1)\mathrm{Re}\Omega_z^\psi(\lambda)\Big)\exp\Big((q-q_1)\phi(z)\Big)\exp\Big((t-t_1)\psi(z)\Big)\Big|.\end{aligned}\tag{14.20}$$
Using (14.12), we can estimate
$$\begin{aligned}&\Big|\exp\Big(q_1\Big(\mathrm{Re}\Omega_z^\phi(\lambda)-\mathrm{Re}\Omega_z^\phi(\lambda_1)\Big)\Big)\exp\Big(t_1\Big(\mathrm{Re}\Omega_z^\psi(\lambda)-\mathrm{Re}\Omega_z^\psi(\lambda_1)\Big)\Big)\Big|=\\&=\exp\Big(\mathrm{Re}\Big(q_1\Big(\mathrm{Re}\Omega_z^\phi(\lambda)-\mathrm{Re}\Omega_z^\phi(\lambda_1)\Big)\Big)\Big)\exp\Big(\mathrm{Re}\Big(t_1\Big(\mathrm{Re}\Omega_z^\psi(\lambda)-\mathrm{Re}\Omega_z^\psi(\lambda_1)\Big)\Big)\Big)\\&\le\exp\left(\Big|\Big(q_1\Big(\mathrm{Re}\Omega_z^\phi(\lambda)-\mathrm{Re}\Omega_z^\phi(\lambda_1)\Big)\Big|\right)\exp\left(\Big|\Big(t_1\Big(\mathrm{Re}\Omega_z^\psi(\lambda)-\mathrm{Re}\Omega_z^\psi(\lambda_1)\Big)\Big|\right)\\&\le\exp\Big(|q_1|\Big(|\mathrm{Re}\Omega_z^\phi(\lambda)|+|\mathrm{Re}\Omega_z^\phi(\lambda_1)|\Big)\Big)\exp\Big(|t_1|\Big(|\mathrm{Re}\Omega_z^\psi(\lambda)|+|\mathrm{Re}\Omega_z^\psi(\lambda_1)|\Big)\Big)\\&\le\exp\Big(8M_1(T_\phi|q_1|+T_\psi|t_1|\Big)\end{aligned}\tag{14.21}$$

and

$$\begin{aligned}
\Big|\exp\Big((q-q_1)\mathrm{Re}\Omega_z^\phi(\lambda)\Big)\exp\Big((t-t_1)\mathrm{Re}\Omega_z^\psi(\lambda)\Big)\Big| &= \\
&= \exp\Big(\mathrm{Re}\Big((q-q_1)\mathrm{Re}\Omega_z^\phi(\lambda)\Big)\Big)\exp\Big(\mathrm{Re}\Big((t-t_1)\mathrm{Re}\Omega_z^\psi(\lambda)\Big)\Big) \\
&\le \exp\Big(|q-q_1||\mathrm{Re}\Omega_z^\phi(\lambda)|\Big)\exp\Big(|t-t_1||\mathrm{Re}\Omega_z^\psi(\lambda)|\Big) \qquad (14.22)\\
&\le \exp\Big(8RM_1(T_\phi+T_\psi)\Big).
\end{aligned}$$

Putting $I=\inf\{\mathrm{Re}z : z\in J(F)\}$, in view of (14.19 we get

$$\begin{aligned}
\Big|\exp\Big((q-q_1)\phi(z)\Big)\exp\Big((t-t_1)\psi(z)\Big)\Big| &= \exp\Big(\mathrm{Re}(q-q_1)\phi(z)\Big)\exp\Big(\mathrm{Re}(t-t_1)\psi(z)\Big)\\
&= \exp\Big(\phi(z)+\kappa(\phi)\Big)^{\mathrm{Re}(q-q_1)}|e^z|^{\kappa(\phi)\mathrm{Re}(q-q_1)} = \exp\Big(\psi(z)+\kappa(\psi)\Big)^{\mathrm{Re}(t-t_1)}|e^z|^{\kappa(\psi)\mathrm{Re}(t-t_1)}\\
&\le \exp(A_\phi|q-q_1|)\exp\Big(I\kappa(\phi)\mathrm{Re}(q-q_1)\Big)\exp(A_\psi|t-1_1|)\exp\Big(I\kappa(\psi)\mathrm{Re}(t-t_1)\Big)\\
&\le \exp(2RA_\phi)\max\{1,\exp(-2I\kappa(\phi)R)\}\exp(2RA_\psi)\max\{1,\exp(-2I\kappa(\psi)R)\}\\
&= \exp\Big(2R(A_\phi)+A_\psi)\Big)\exp\Big(-2R\min\{0,I\}(\kappa(\phi)+\kappa(\psi)\Big)
\end{aligned}$$

Combining now formulas (14.19)-(14.22) and the last estimate, we get that the number $\Big|\exp\Big(\zeta_{(\lambda,q,t)}(z)-\zeta_{(\lambda_1,q_1,t_1)}(z)\Big)\Big|$ is bounded above by a constant independent of $(\lambda,q,t)\in \mathbb{D}_{\mathbb{C}^4}((\lambda_2,q_2,t_2),\gamma)$. We are done. ■

The proof of Proposition 14.1 follows now from Lemmas 14.8-14.11 applied together with Proposition 7.1 and from the observation that the functions $\zeta_{(\lambda,q,t)}$, $(\lambda,q,t)\in G=\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times \mathbb{D}_{\mathbb{C}^2}((q_0,t_0),R)$, for parameters (λ,q,t) in the set $B_{\mathbb{C}}(\lambda_0,r/16)\times B_{\mathbb{R}}(q_0,R)\times B_{\mathbb{R}}(t_0,R)$, coincide with $q\phi\lambda\circ h_\lambda+t\psi\circ h_\lambda$. ■

Given a 1^+-tame potential $\phi: P_a^c\to\mathbb{R}$ and $\lambda\in\mathrm{Hyp}(a)$, we denote by $\mathrm{P}_\lambda(\phi)$ the topological pressure of ϕ with respect to the dynamical system $F_\lambda: J(F_\lambda)\to J(F_\lambda)$. In order to derive from Proposition 14.1 analytic consequences for topological pressure, we start with the following.

Lemma 14.13. *If ϕ,ψ are tame and $\kappa(\phi),\kappa(\psi)>0$, then the function $(\lambda,q,t)\mapsto \mathrm{P}_\lambda(q\phi+t\psi)$, $(\lambda,q,t)\in\mathrm{Hyp}(a)\times\Sigma_2(\phi,\psi)\cap\mathbb{R}^2$, is continuous.*

Proof. Fix $(\lambda_0,q_0,t_0)\in\mathrm{Hyp}(a)\times\Sigma_2(\phi,\psi)\cap\mathbb{R}^2$ and then $r>0$ so small that $B(\lambda_0,r)\subset\mathrm{Hyp}(a)$ and for every $\lambda\in B(\lambda_0,r)$ f_λ, f_{λ_0} are quasiconformally conjugate via the homeomophism $h_\lambda:\mathbb{C}\to\mathbb{C}$ described in +++Section 8. Then h_λ conjugates also F_λ and F_{λ_0}. Fix $\varepsilon>0$. In view of Lemma 7.6, for every $\lambda\in\mathrm{Hyp}(a)$, the function $(q,t)\mapsto \mathrm{P}_\lambda(q\phi+t\psi)$, $(q,t)\in\Sigma_2(\phi,\psi)\cap\mathbb{R}^2$, is continuous. There thus exists $\delta_1>0$ such that if $(q,t)\in(q_0-\delta_1,q_0+\delta_1)\times(t_0-\delta_1,t_0+\delta_1)$, then

$$|\mathrm{P}_{\lambda_0}(q\phi+t\psi)-\mathrm{P}_{\lambda_0}(q_0\phi+t_0\psi)|<\varepsilon/2. \qquad (14.23)$$

Since ϕ and ψ are Hölder continuous, these are uniformly continuous and therefore there exists $r>0$ so small that if $|w-z|\le r$, then

$$|\phi(w)-\phi(z)|\le \frac{\varepsilon/4}{|q_0|+\delta_1} \ \text{ and } \ |\psi(w)-\psi(z)|\le \frac{\varepsilon/4}{|t_0|+\delta_1}. \tag{14.24}$$

In view of Proposition 13.4 there exists $\delta_2\in(0,\min\{\delta_1,r\})$ so small that if $|\lambda-\lambda_0|<\delta_2$ then $|h_\lambda(z)-z|<r$ for all $z\in J(F_{\lambda_0})$. Fix now $z\in J(F_{\lambda_0})$. Take $n\ge 1$ and $x\in F_{\lambda_0}^{-n}(z)$. Since h_λ conjugates F_λ and F_{λ_0}, $h_\lambda(F_{\lambda_0}^{-n}(z))=F_\lambda^{-n}(z)$. Also, for every $0\le i\le n$ and every $x\in F_{\lambda_0}^{-n}(z)$, $h_\lambda(F_{\lambda_0}^i(x))=F_\lambda^i(h_\lambda)(x))$ and, therefore $|F_{\lambda_0}^i(x)-F_\lambda^i(h_\lambda(x))|\le r$. Hence, using (14.24), we get for all $\lambda\in B(\lambda_0,\delta_2)$, all $(q,t)\in(q_0-\delta_1,q_0+\delta_1)\times(t_0-\delta_1,t_0+\delta_1)$, and all $x\in F_{\lambda_0}^{-n}(z)$, that

$$\begin{aligned}
&\frac{\exp\Big(S_n^\lambda(q\phi+t\psi)(h_\lambda(x))\Big)}{\exp\Big(S_n^{\lambda_0}(q\phi+t\psi)(x)\Big)}\\
&=\exp\left(\sum_{j=0}^{n-1}(q\phi+t\psi)(F_\lambda^j(h_\lambda(x)))-(q\phi+t\psi)(F_{\lambda_0}^j(x))\right)\\
&\le\exp\left(\sum_{j=0}^{n-1}\Big|(q\phi+t\psi)(F_\lambda^j(h_\lambda(x)))-(q\phi+t\psi)(F_{\lambda_0}^j(x))\Big|\right)\\
&\le\exp\left(\sum_{j=0}^{n-1}|q||\phi(F_\lambda^j(h_\lambda(x)))-\phi(F_{\lambda_0}^j(x))|+|t||\psi(F_\lambda^j(h_\lambda(x)))-\psi(F_{\lambda_0}^j(x))|\right)\\
&\le\exp\left((|q_0|+\delta_1)\sum_{j=0}^{n-1}|\phi(F_\lambda^j(h_\lambda(x)))-\phi(F_{\lambda_0}^j(x))|+(|t_0|+\delta_1)\sum_{j=0}^{n-1}|\psi(F_\lambda^j(h_\lambda(x)))-\psi(F_{\lambda_0}^j(x))|\right)\\
&\le\exp\left(\frac{\varepsilon}{4}n+\frac{\varepsilon}{4}n\right)=\exp\left(\frac{\varepsilon}{2}n\right).
\end{aligned}$$

Since $h_\lambda: F_{\lambda_0}^{-n}(z)\to F_\lambda^{-n}(h_\lambda(z))$ is a bijection, we therefore conclude that

$$\frac{\sum_{x\in F_\lambda^{-n}(h_\lambda(z))}\exp\Big(S_n^\lambda(q\phi+t\psi)(h_\lambda(x))\Big)}{\sum_{x\in F_{\lambda_0}^{-n}(z)}\exp\Big(S_n^{\lambda_0}(q\phi+t\psi)(x)\Big)}\in\left(\exp\left(-\frac{\varepsilon}{2}n\right),\exp\left(\frac{\varepsilon}{2}n\right)\right)$$

and from this,

$$-\frac{\varepsilon}{2}n\le\frac{1}{n}\log\sum_{x\in F_\lambda^{-n}(h_\lambda(z))}\exp\Big(S_n^\lambda(q\phi+t\psi)(h_\lambda(x))\Big)-\frac{1}{n}\sum_{x\in F_{\lambda_0}^{-n}(z)}\sum_{x\in F_{\lambda_0}^{-n}(z)}\exp\Big(S_n^{\lambda_0}(q\phi+t\psi)(x)\Big)\le\frac{\varepsilon}{2}n.$$

So, $|\mathrm{P}_\lambda(q\phi+t\psi)-\mathrm{P}_{\lambda_0}(q\phi+t\psi)|\le(\varepsilon/2)n$ for all $(\lambda,q,t)\in B(\lambda_0,\delta_1)\times(q_0-\delta_1,q_0+\delta_1)\times(t_0-\delta_1,t_0+\delta_1)$. Combining this with (14.23), we see that $|\mathrm{P}_l(q\phi+t\psi)-\mathrm{P}_{\lambda_0}(q_0\phi+t_0\psi)|<\varepsilon$ for all $(\lambda,q,t)\in B(\lambda_0,\delta_1)\times(q_0-\delta_1,q_0+\delta_1)\times(t_0-\delta_1,t_0+\delta_1)$. The continuity of the function $(\lambda,q,t)\mapsto\mathrm{P}_\lambda(q\phi+t\psi)$, $(\lambda,q,t)\in\mathrm{Hyp}(a)\times\Sigma_2(\phi,\psi)\cap I\!R^2$, is established. ■

The result we will need in the sequel is contained in the following consequence of Proposition 14.1 and Lemma 14.13.

Corollary 14.14. *If ϕ, ψ are tame and $\kappa(\phi), \kappa(\psi) > 0$, then the function $(\lambda, q, t) \mapsto \mathrm{P}_\lambda(q\phi + t\psi)$, $(\lambda, q, t) \in \mathrm{Hyp}(a) \times \Sigma_2(\phi, \psi) \cap I\!R^2$, is real-analytic.*

Proof. Recall from the proof of Proposition 14.1 that for every $(\lambda, q, t) \in \mathrm{Hyp}(a) \times \Sigma_2(\phi, \psi) \cap I\!R^2$, $\mathcal{L}^0_{\lambda,q,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$ is the operator induced by the weight function $\exp\Big(q\phi\lambda \circ h_\lambda + t\psi \circ h_\lambda\Big)$, i.e.

$$\mathcal{L}^0_{\lambda,q,t} g(z) = \sum_{x \in F^{-1}_{\lambda_0}(z)} \exp\Big(q\phi\lambda \circ h_\lambda(x) + t\psi \circ h_\lambda(x)\Big) g(x).$$

Fix $(\lambda_0, q_0, t_0) \in \mathrm{Hyp}(a) \times \Sigma_2(\phi, \psi) \cap I\!R^2$. Since by Proposition 13.5, $h_\lambda : J(F_{\lambda_0}) \to J(F_\lambda)$ is Hölder continuous with the Hölder exponent $\alpha(\lambda)$ depending on λ but converging to 1 as $\lambda \to \lambda_0$, we get that for every $r > 0$ sufficiently small

$$\alpha = \inf\{\alpha_\lambda : \lambda \in B(\lambda_0, r)\} > 0.$$

Proposition 14.1 says that there exist $R \in (0, r)$ and a holomorphic function

$$L : \mathbb{D}_{\mathbb{C}^4}((\lambda_0, q_0, t_0), R) \to L(\mathrm{H}_\alpha(J(F\lambda_0))$$

(λ_0 is treated here as elements of $\mathbb{C}^2$ and t_0 as an element of $\mathbb{C}$) such that for every $(\lambda, q, t) \in B_{\mathbb{C}}(\lambda_0, R) \times B_{I\!R}(q_0, R) \times B_{I\!R}(t_0, R)$

$$L(\lambda, q, t) = \mathcal{L}^0_{\lambda,q,t}. \tag{14.25}$$

Now, in view of Theorem 4.4 and Proposition 3.10, $\exp\Big(\mathrm{P}_{\lambda_0}(q\phi + t\psi)\Big)$, $(q, t) \in B_{I\!R}(q_0, R) \times B_{I\!R}(t_0, R)$, is a simple isolated eigenvalue of the operator $L(\lambda_0, q, t) = \mathcal{L}^0_{\lambda_0,q,t} : \mathrm{H}_\alpha(J(F_{\lambda_0}) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$. Applying now the perturbation theory for linear operators (see [Ka]), we see that there exists $0 < R_1 \le R$ and a holomorphic function $\gamma : \mathbb{D}_{\mathbb{C}^4}((\lambda_0, q_0, t_0), R_1) \to \mathbb{C}$ such that $\gamma(\lambda_0, q_0, t_0) = \exp\Big(\mathrm{P}_{\lambda_0}(q_0\phi + t_0\psi)\Big)$ and for every $(\lambda, q, t) \in \mathbb{D}_{\mathbb{C}^4}((\lambda_0, q_0, t_0), R_1)$ the number $\gamma(\lambda, q, t)$ is a simple isolated eigenvalue of $L(\lambda, t)$ with the remainder part of the spectrum uniformly separated from $\gamma(\lambda, t)$. In particular there exist $0 < R_2 \le R_1$ and $\kappa > 0$ such that

$$\sigma(L(\lambda, q, t)) \cap B\Big(\exp\Big(\mathrm{P}_{\lambda_0}(q_0\phi + t_0\psi)\Big), \kappa\Big) = \{\gamma(\lambda, q, t)\} \tag{14.26}$$

for all $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^4}((\lambda_0, q_0, t_0), R_2)$. Consider now for each $(\lambda, t) \in B(\lambda_0, t_0) \times (t_0 - R, t_0 + R)$ the operator $\mathcal{L}_{\lambda,t} : \mathrm{H}_1(J(F_\lambda)) \to \mathrm{H}_1(J(F_\lambda))$ (see Lemma 4.1) given by the formula

$$\mathcal{L}_{\lambda,q,t} g(z) = \sum_{x \in F^{-1}_\lambda(z)} \exp\Big(q\phi(x) + t\psi(x)\Big) g(x).$$

It is easy to see that the map $T_\lambda : C_b(J(F_\lambda)) \to C_b(J(F_{\lambda_0}))$ defined by the formula $T_\lambda(g) = g \circ h_\lambda$ establishes a bounded linear conjugacy between $\mathcal{L}_{\lambda,q,t} : C_b(J(F_\lambda)) \to C_b(J(F_\lambda))$ and

$\mathcal{L}^0_{\lambda,q,t} : C_b(J(F_{\lambda_0})) \to C_b(J(F_{\lambda_0}))$. Since the map $h_\lambda : J(F_{\lambda_0}) \to J(F_\lambda)$ is Hölder continuous with the Hölder exponent α, we obtain that

$$T_\lambda(\mathrm{H}_1(J(F_\lambda))) \subset H_\alpha(J(F_{\lambda_0})).$$

Hence $\exp\Big(P_\lambda(q\phi + t\psi)\Big)$ is an eigenvalue of the operator

$$\mathcal{L}^0_{\lambda,q,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$$

and, by Lemma 14.13, $\exp\Big(P_\lambda(q\phi + t\psi)\Big) \in B\Big(\exp\Big(P_{\lambda_0}(q_0\phi + t_0\psi)\Big), \kappa\Big)$ for all $(\lambda, q, t) \in B_{\mathbb{C}}(\lambda_0, R_2 \times B_{\mathbb{R}}(q_0, R_2) \times B_{\mathbb{R}}(t_0, R_2)$, where $R_2 \in (0, \mathbb{R}_1)$ is sufficiently small. Combining this, (14.25) and (14.26) we see that $\gamma(\lambda, q, t) = \exp\Big(\mathrm{P}_\lambda(q\phi + t\psi)\Big)$ for λ, q, t as above. Therefore the function $(\lambda, q, t) \mapsto \mathrm{P}_\lambda(q\phi + t\psi)$, $(\lambda, q, t) \in B_{\mathbb{C}}(\lambda_0, R_2 \times B_{\mathbb{R}}(q_0, R_2) \times B_{\mathbb{R}}(t_0, R_2)$, is real-analytic. ■

15. **Refined Multifractal Analysis**

In this section, similarly as in the previous one, we are especially interested in the dependence on λ objects introduced in the earlier sections, particularly the multifractal function k_{μ_ϕ}. Throughout the entire section we fix $a \in \mathbb{C}$ and a harmonic 1^+-tame potential $\phi : P^c_a \to \mathbb{R}$. We then consider the three-parameter family of potentials $\phi^\lambda_{q,t} : P^c_a \to \mathbb{R}$, $\lambda \in \mathrm{Hyp}(a)$, $q, t \in \mathbb{R}$, defined as follows.

$$\phi^\lambda_{q,t} = -t \log |F'_\lambda| + q\phi.$$

Notice that if we put $\psi_{q,t} = -t \log \circ \mathrm{Re} + q\phi$, then $\phi^\lambda_{q,t} = \psi_{q,t} - t \log |\lambda|$ and $\mathrm{P}_\lambda(\phi^\lambda_{q,t}) = \mathrm{P}_\lambda(\psi_{q,t} - t \log |\lambda|$. So, in view of Corollary 14.14, the function $(\lambda, q, t) \mapsto \mathrm{P}_\lambda(\phi^\lambda_{q,t})$, $(\lambda, q, t) \in \mathrm{Hyp}(a) \times \Sigma_2(\phi, \psi) \cap \mathbb{R}^2$, is real-analytic. In view of Lemma 10.2, for every pair $(\lambda, q) \in \mathrm{Hyp}(a) \times \mathbb{R}$ there exists a unique $T_\lambda(q) \in \mathbb{R}$ such that $\mathrm{P}_\lambda(\phi^\lambda_{q,T_\lambda(q)}) = 0$ and $(q, T(q)) \in \Sigma_2(\phi, -\log |F'|)$. Therefore, applying the Implicite Function Theorem and (12.13), we get the following.

Proposition 15.1. *The function* $(\lambda, q) \mapsto T_\lambda(q)$, $(\lambda, q) \in \mathrm{Hyp}(a) \times \mathbb{R}$, *is real-analytic.*

In view of Theorem 12.4, for every $\lambda \in \mathrm{Hyp}(a)$, the range of the function $q \mapsto -T'_\lambda(q)$, $q \in \mathbb{R}$, is an open interval $(\alpha_1(\lambda), \alpha_2(\lambda)$ with $0 \le \alpha_1(\lambda) \le \alpha_2(\lambda) < +\infty$. As an immediate consequence of Proposition 15.1, we get the following.

Lemma 15.2. *The functions* $\lambda \mapsto \alpha_1(\lambda)$ *and* $\lambda \mapsto \alpha_2(\lambda)$, $\lambda \in \mathrm{Hyp}(a)$, *are respectively upper and lower semi-continuous.*

In turn, as an immediate consequence of this lemma, we get the following.

Proposition 15.3. *If G is an open subset of* $\mathrm{Hyp}(a)$, *then the set*

$$U(a,\phi,G) = \bigcup_{\lambda\in G} \{\lambda\} \times (\alpha_1(\lambda), \alpha_2(\lambda)) \subset \mathbb{C} \times \mathbb{R}$$

is open.

We put

$$U_{a,\phi} = U(\phi, \mathrm{Hyp}(a)).$$

Given $\lambda \in \mathrm{Hyp}(a)$, let μ_ϕ^λ be the Gibbs state corresponding to the potential ϕ and the dynamical system $F_\lambda : J(F_\lambda) \to J(F_\lambda)$. We define the function $k_\phi : U_{a,\phi} \to [0,2]$ by the formula

$$k_\phi(\lambda,\alpha) = k_{\mu_\phi^\lambda}(\alpha).$$

The main theorem of this section, and in a sense a culminating point of the whole paper, is the following.

Theorem 15.4. *If $\phi : P_a^c \to \mathbb{R}$ is a harmonic 1^+-tame potential, then the function $k_\phi : U_{a,\phi} \to \mathbb{R}$ is real-analytic.*

Proof. It follows from Theorem 12.4(d) that for every $(\lambda, q) \in \mathrm{Hyp}(a) \times \mathbb{R}$,

$$k_\phi(\lambda, -T'_\lambda(q)) = T_\lambda(q) - qT'_\lambda(q).$$

Now, fix an element $(\lambda_0, \alpha_0) \in U_{a,\phi}$. Then $\alpha_0 \in (\alpha_1(\lambda_0), \alpha_2(\lambda_0))$ and, in particular, $\alpha_1(\lambda_0) < \alpha_2(\lambda_0)$. It then follows from Theorem 12.4(b) that there exists a unique $q_0 \in \mathbb{R}$ such that $\alpha_0 = -T'_{\lambda_0}(q_0)$ and $T''_{\lambda_0}(q_0) \neq 0$. Therefore, applying the Implicit Function Theorem to the real-analytic function (see Proposition 15.1) $G(\lambda,\alpha,q) = \alpha + T'_\lambda(q)$, we see that there exists a real-analytic function $\rho : V \to \mathbb{R}$ defined on an open neighborhood $V \subset U_{a,\phi}$ of (λ_0, α_0) such that $\rho(\lambda_0,\alpha_0) = q_0$ and $\alpha = -T'_\lambda(\rho(\lambda,\alpha))$ for all $(\lambda,\alpha) \in V$. Hence, $k_\phi(\lambda,\alpha) = T_\lambda(\rho(\lambda,\alpha)) + \rho(\lambda,\alpha)\alpha$ for all $(\lambda,\alpha) \in V$. Since compositions and products of real-analytic functions are real-analytic, we are done. ■

We now shall look a little bit closer at the structure of the set $U_{a,\phi}$. We start with the following trivial observation following immediately from its definition.

Proposition 15.5. *The set $U_{a,\phi}$ is vertically connected, i.e. for every $\lambda \in \mathrm{Hyp}(a)$, the set $(\{\lambda\} \times \mathbb{R}) \cap U_{a,\phi}$ is connected.*

A harmonic 1^+-tame potential ϕ is called essential over an open set $G \subset \mathrm{Hyp}(a)$ if for every $\lambda \in G$, the function ϕ is not cohomologous to $-\mathrm{HD}(J_r(F_\lambda)) \log |F'_\lambda|$ for any $\lambda \in G$ in the class of all (bounded) Hölder continuous functions. The potential ϕ is simply called essential if ϕ is essential over $\mathrm{Hyp}(a)$.

Proposition 15.6. *If a harmonic 1^+-tame potential ϕ is essential over an open set $G \subset$ Hyp(a), then the orthogonal projection of $U(a,\phi,G)$ on $\mathbb{C}$ is equal to G. If in addition G is connected, then so is $U(a,\phi,G)$.*

Proof. Let $\pi_1 : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first coordinate. It is obvious that $\pi_1(U(a,\phi,G)) \subset G$. Since ϕ is essential over G, it follows from Theorem 12.4(b) that $\alpha_1(\lambda) < \alpha_2(\lambda)$ for all $\lambda \in G$. Consequently $\pi_1(U(a,\phi,G)) \supset G$. Now, it follows from Lemma 15.2 that for every $\lambda \in G$ there exists radius $r(\lambda) > 0$ such that the set $B(\lambda, r(\lambda)) \subset G$ and $U(\phi, B(\lambda, r(\lambda))) \subset U(a,\phi,G)$ is connected. Suppose now in addition that the set $G \subset$ Hyp(a) is connected. Fix two arbitrary points $(\lambda,\alpha), (\lambda,\alpha') \in U(a,\phi,G)$. Then there exists a compact (polygonal) arc γ joining λ and λ' in G. The standard compactness argument shows that there exist finitely many points $\lambda_1, \lambda_2, \dots, \lambda_n$ on γ such that $\lambda_1 = \lambda$, $\lambda_n = \lambda'$ and $B(\lambda_i, r(\lambda_i)) \cap B(\lambda_{i+1}, r(\lambda_{i+1})) \neq \emptyset$ for all $i = 1,2,\dots,n-1$. Then all the sets $U(\phi, B(\lambda_i, r(\lambda_i))) \subset U(a,\phi,G)$, $i = 1,2,\dots,n-1$, are connected and

$$U(\phi, B(\lambda_i, r(\lambda_i))) \cap U(\phi, B(\lambda_{i+1}, r(\lambda_{i+1}))) = U\Big(\phi, B(\lambda_i, r(\lambda_i)) \cap B(\lambda_{i+1}, r(\lambda_{i+1}))\Big) \neq \emptyset$$

for all $i = 1,2,\dots,n-1$, and $(\lambda,\alpha) \in U(\phi, B(\lambda_1, r(\lambda_1)))$ and $(\lambda',\alpha') \in U(\phi, B(\lambda_n, r(\lambda_n)))$. Hence, the set $U(a,\phi,G)$ is connected and we are done. ■

Theorem 15.7. *If ϕ is an essential harmonic 1^+-tame potential, then for every open set $G \subset$ Hyp(a), the orthogonal projection of $U(a,\phi,G)$ on $\mathbb{C}$ is equal to G (in particular the prohection of $U_{a,\phi}$ is equal to* Hyp(a)*). If in addition G is connected, then so is $U(a,\phi,G)$.*

The next result provides an extremely easy to verify sufficient condition for a harmonic tame potential to be essential.

Proposition 15.8. *If ϕ is a harmonic tame potential with $\kappa(\phi) \geq 2$, then ϕ is essential.*

Proof. First notice that of ϕ and ψ are tame potentials cohomologous modulo constant, then $\kappa(\phi) = \kappa(\psi)$. Since $\kappa\Big(-\mathrm{HD}(J_r(F_\lambda)) \log|F_\lambda'|\Big) = \mathrm{HD}(J_r(F_\lambda))$ and since $\mathrm{HD}(J_r(F_\lambda)) < 2$ for all $\lambda \in Hyp$, we are done. ■

We end this section with the following.

Remark 15.9. *Given $\kappa \in \mathbb{R}$ and $a \in \mathbb{C}$, the class of κ-tame harmonic potentials is very large. Indeed, the class of all bounded harmonic functions $\psi : \mathbb{C}_a^c \to \mathbb{R}$ is very big, and for every $b > a$ the potential $\psi - \kappa\mathrm{Re} :: \mathbb{C}_b^c \to \mathbb{R}$ is tame and harmonic.*

16. Appendix: Construction of conformal measures

Let $\lambda \in$ Hyp. In what follows we rely on the description given in c[BD], we also use the notation of this paper. We assume that $z_0, z_1, \ldots, z_n = z_0$ is an attracting cycle of $f = f_\lambda$. Assume that the singular value 0 is contained in the domain A_1, the immediate basin of attraction of z_1. The Jordan domain with smooth boundary $B_{n+1} \subset A_1$ is chosen so that $0, z_1 \in B_{n+1}$, $\mathrm{cl} f^n(B_{n+1}) \subset B_{n+1}$. Then B_n is defined as $B_n = f^{-1}(B_{n+1})$. Notice that B_n contains some half-plane $\mathrm{Re} z < -M$ and $z_0 \in B_n$. The set $B_{n-1}, B_{n-2}, \ldots, B_0$ are then defined inductively as follows. B_i is a connected component of $f^{-1}(B_{i+1})$, containing z_i. Notice that for $i < n$ B_{n-i} is a simply-connected unbounded domain. The set B_0 is a complement of a union of infinitely many sets G_i. Obviously, $G_i = G_0 + 2k\pi i$) (see Fig 3. in [BD]). (Here, $G_0 + 2k\pi i$ is the image of G_0 under the map $z \to z + 2k\pi i$.) To build an appropriate dynamics, we fix one component G_0 of the complement of B_0. Since the construction is very transparent in the case of a fixed attracting point, we describe it first. So, assume that $n = 1$. Then we start with B_2. B_1 is an unbounded domain containing some half-plane, $f^{-1}(B_1)$ is connected and $B_1 \subset B_0 = f^{-1}(B_1)$. Each component G_i of the complement of B_0 is mapped by f onto $\mathbb{C} \setminus \mathrm{cl} B_1 \supset \mathrm{cl} \bigcup G_i$. Let $S = G_0$. Then we see that

$$f(S) \supset \mathrm{cl}\left(\bigcup_k (S + 2k\pi i)\right)$$

and $f^{-1}(S) \cap S$ is a union of countably many disjoint unbounded simply-connected domains ("fingers"), each of them being mapped by f univalently onto $S + 2k\pi i$ for some $k \in \boldsymbol{Z}$. We define $\hat{F} : f^{-1}(S) \cap S \to S$ as $\hat{F} = \pi \circ f$ where π is the natural projection $\pi : \bigcup G_i \to G_0$.

In the general case of periodic attracting orbit the construction is slightly more complicated. Again, let G_0 be a component of the complement of B_0. Let

$$S = G_0 \setminus \pi^{-1}(\bigcup_{i=1}^{n-1} B_i),$$

where π is the natural projection $\pi : \bigcup G_i \to G_0$. So,

$$f(S) \supset \bigcup_k (S + 2k\pi i)$$

and, actually, modifying the set S slightly, we can require that $f(S) \supset \mathrm{cl} \bigcup_k (S + 2k\pi i)$. Again, $S \cap f^{-1}(\pi^{-1}(S))$ is a union of disjoint unbounded, simply connected domains and $\hat{F} : f^{-1}(S) \cap S \to S$ as $\hat{F} = \pi \circ f$ is defined as $\hat{F} = \pi \circ f$. Let

$$J(F) = \{z \in S : \hat{F}^n \text{ is defined for all n}\}.$$

One can easily see that

$$J(f) \cap S = J(\hat{F}).$$

In this we construct first conformal measures for the map $\hat{F} : S \cap J(f) \to S \cap J(f)$ and a κ-tame potential $\phi : J(\hat{F}) \to \mathbb{R}$ with some $\kappa > 1$. The details are presented for the sake of completeness and the convenience of the reader. A Borel measure m is called (ϕ, α)-conformal if for any Borel set $A \subset S$ on which $\hat{F}$ is injective, we have

$$m(\hat{F}(A)) = \int_A \alpha \exp(-\phi) dm$$

Let

$$\tilde{S}_M = \{z \in S : \mathrm{Re} z \leq M\}.$$

Consider the preimage $\hat{F}^{-1}(\tilde{S}_M)$. This set is a union of infinitely many topological disks Q_i and

$$\overline{Q_i} \cap \overline{Q_j} = \emptyset.$$

Now, we consider the finite family of disks Q_i^M, which are contained in $\tilde{S}_M$. In this way we obtain the finite iterated function system:

$$\{\phi_i : \tilde{S}_M \to Q_i^M\},$$

where ϕ_i are appropriate holomorphic branches of $\hat{F}^{-1}$. Let J_M be the limit set of this system and let m_M be its unique $(\phi, \exp(\mathrm{P}_M(\phi)))$-conformal measure, where

$$\mathrm{P}_M(\phi) \leq \overline{\mathrm{P}}(\phi) < \infty \tag{16.1}$$

is the standard topological pressure for the repeller $F|_{J_M} : J_M \to J_M$ and the potential ϕ.

Remark 16.1. *We have $J_M \subset J_{M+1}$ for all M large enough, say $M \geq p_\lambda$. In order to see this, take Q_i^M and let Q_i^{M+1}, be the preimage of $\tilde{P}_{M+1}$ under the same holomorphic branch $\hat{F}_*^{-1}$ of $\hat{F}^{-1}$. Then, obviously, $Q_i^{M+1} \supset Q_i^M$. Since $\hat{F}(Q_i^{M+1} \setminus Q_i^M) \subset \{z \in \tilde{P}_{M+1} : M < \mathrm{Re} z \leq M+1\}$ and since the derivative of $\hat{F}_*^{-1}$ on $\{z \in \tilde{P}_{M+1} : M < \mathrm{Re} z \leq M+1\}$ is bounded from above by $C_1 M^{-1}$, we conclude that* $\mathrm{diam}(Q_i^{M+1} \setminus Q_i^M) \leq C_2 M^{-1}$ *for some appropriate constants C_1 and C_2. Since $Q_i^M \subset \{\mathrm{Re} z \leq M\}$, this implies that*

$$Q_i^{M+1} \subset \{\mathrm{Re} z \leq M+1\}$$

for all M large enough. Hence, each $Q_i^{M+1} \supset Q_i^M$ is (see the definition) used in the construction of J_{M+1}. Thus, the corresponding limit set J_{M+1} contains J_M.

Proposition 16.2. *The sequence of measures m_M, $M \in \mathbb{N}$ is tight, i.e. for every $\varepsilon > 0$ there exists M so large that for every N*

$$m_N(\{z \in S : \mathrm{Re} z > M\}) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$, $M > 0$ and $N \geq p_\lambda$. We shall estimate separately the measure m_N of two sets, which cover $\{z \in S : \mathrm{Re} z > M\}$. First, we have

$$m_N(\{x \in J_N : \mathrm{Re} F(x) \geq M\}) = \sum_{k \in \boldsymbol{Z}} m_N(\{x \in J_N : f(x) \in [M, N] \times ([-\pi, \pi] + 2k\pi i).$$

If $x \in J_N$ and $f(x) \in [M,N] \times [-\pi,\pi] + 2k\pi i\}$, then

$$\begin{aligned}|\exp(-\phi(x))| &\geq e^{-A_\phi}\exp^\kappa(\mathrm{Re}(x)) = |\lambda|^{-\kappa}e^{-A_\phi}|f(x)|^\kappa \geq |\lambda|^{-\kappa}e^{-A_\phi}\left(\frac{1}{2}(M+\pi|k|)\right)^\kappa \\ &\geq (2|\lambda|)^{-\kappa}e^{-A_\phi}(M+\pi|k|)^\kappa,\end{aligned}$$

which gives

$$\begin{aligned}&m_N(\{x : \mathrm{Re}\hat{F}(x) \geq M\}) \leq \\ &\qquad \leq 2\sum_{k=0}^{\infty} m_N(\{x : M \leq \mathrm{Re}x \leq N\}) \cdot \exp\Big(-\mathrm{P}_N(\phi)\Big)(2|\lambda|)^\kappa e^{A_\phi}(M+\pi|k|)^{-\kappa} \\ &\qquad \leq 2(2|\lambda|)^\kappa e^{A_\phi}\exp\Big(-\mathrm{P}_N(\phi)\Big)\sum_{k=0}^{\infty}(M+\pi|k|)^{-\kappa} \\ &\qquad \leq C_3(\kappa-1)^{-1}M^{1-\kappa}(2|\lambda|)^\kappa e^{A_\phi}\exp\Big(-\mathrm{P}_N(\phi)\Big)\end{aligned} \tag{16.2}$$

with an absolute constant $C_3 > 0$. If $N \leq M$, then

$$m_N(\{z \in S : \mathrm{Re}z > M\}) = 0. \tag{16.3}$$

By Remark 16.1 the sequence $N \mapsto \mathrm{P}_N(\phi)$, $N \geq p_\lambda$, is non-decreasing, and therefore $\sup_N\{-\mathrm{P}_N(\phi)\} \leq -\mathrm{P}_{p_\lambda}(\phi) \leq -u(\lambda,\phi)$, where

$$u(\lambda,\phi) = \sup\left\{\phi|_{P_{p_\lambda}}\right\} < \infty, \tag{16.4}$$

If $N > M$, then it follows from this along with (16.2) and (16.1) that

$$m_N(\{x : \mathrm{Re}F(x) \geq M\}) \leq C_1(2|\lambda|)^\kappa(\kappa-1)^{-1}\exp\Big(A_\phi - u(\lambda,\phi)\Big)M^{1-\kappa}, \tag{16.5}$$

Keeping $N > M$ we now estimate the measure of the second set. Namely

$$m_N(\{x : M < \mathrm{Re}x < N \text{ and } \mathrm{Re}F(x) < M\}).$$

If $\mathrm{Re}x > M$, then $|f(x)| > \lambda e^M$ and therefore $|\mathrm{Im}f(x)| \geq \sqrt{\lambda^2e^{2M} - M^2}$. Hence

$$\begin{aligned}&m_N(\{x : M < \mathrm{Re}x < N \text{ and } \mathrm{Re}F(x) < M\} \leq \\ &\qquad \leq 2|\lambda|^\kappa\exp\Big(A_\phi - u(\lambda,\phi)\Big)\sum_{k\geq\sqrt{\lambda^2e^{2M}-M^2}}^{\infty} k^{-\kappa} \\ &\qquad \leq 2C_4|\lambda|^\kappa\exp\Big(A_\phi - u(\lambda,\phi)\Big)(\kappa-1)^{-1}e^{M(1-\kappa)},\end{aligned} \tag{16.6}$$

where $C_4 > 0$ is an absolute constant. Combining (16.6) along with (16.3) and (16.5) we obtain

$$m_N(\{x : \mathrm{Re}x > M\}) < \varepsilon$$

for all N and all M large enough. ■

Since, by Prop 16.2, the sequence m_N is tight, it follows from Prokhorov's Theorem that there exists a subsequence m_{N_i} weakly convergent to some limit probability measure $m = m_\phi$. This is the measure we are looking for. Recall that

$$J(\hat{F}) = S \cap J(f).$$

The proof of the following theorem requires only minor obvious modifications of the proof of Theorem 3.4 in [UZ1].

Theorem 16.3. *The measure m_ϕ is (ϕ, α_ϕ)-conformal, where $\alpha_\phi = \lim_{N\to\infty} \exp(\mathrm{P}_N(\phi))$ and $m(J(\hat{F})) = 1$.*

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