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FINER GEOMETRIC RIGIDITY OF LIMIT SETS OF CONFORMAL IFS

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ABSTRACT. We consider infinite conformal iterated function systems in the phase space $\mathbb{R}^d$ with $d \geq 3$. Let J be the limit set of such a system. Under a mild technical assumption which is always satisfied if the system is finite, we prove that either the Hausdorff dimension of J exceeds the topological dimension k of the closure of J or else the closure of J is a proper compact subset of either a geometric sphere or an affine subspace of dimension k. A similar dichotomy holds for conformal expanding repellers.

1. Introduction and preliminaries

In this paper we explore the finer geometric structure of limit sets J of infinite conformal iterated function systems. Under a natural easily verifiable technical condition (always satisfied if the system is finite), we demonstrate the following dichotomy. Either the Hausdorff dimension of J exceeds the topological dimension k of the closure of J or else the closure of J is a proper compact subset of either a geometric sphere or an affine subspace of dimension k (in addition, if any one of our conformal contracting mappings is a similarity, then the latter case holds). This is a strengthening of the following weaker dichotomy proven in [MMU]: if $\overline{J}$ is connected, then either the Hausdorff dimension of J exceeds 1 or else $\overline{J}$ is a proper compact segment of either a geometric circle or a straight line. The picture is now much clearer. We would like to add that so far, up to our knowledge all the similar works concerned the same dichotomy as that produced in [MMU] (see for ex. ([Bo], [MU2], [Ma], [Pr], [Ru], [Su], [U1], [UV], [Z1], [Z2]) most of which dealing with the plane case $d = 2$ and applying the Riemann Mapping Theorem. Our approach is based on an extensive use of the concept of rectifiablity.

To start the preliminaries concerning conformal iterated function systems, let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact set X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n \geq 1} I^n$, the space of finite words, and for $\tau \in I^n$, $n \geq 1$, let $\phi_\tau = \phi_{\tau_1} \circ \phi_{\tau_2} \circ \cdots \circ \phi_{\tau_n}$. Let $I^\infty = \{\{\tau_n\}_{n=1}^\infty\}$ be the set of all infinite sequences of elements of I. If $\tau \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of τ, we denote by $\tau|_n$ the word

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$\tau_1\tau_2\ldots\tau_n$. Since given $\tau \in I^\infty$, the diameters of the compact sets $\phi_{\tau|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\tau|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\tau)$, we define the coding map

$$\pi : I^\infty \to X.$$

The main object in the theory of iterated function systems is the limit set defined as follows.

$$J = \pi(I^\infty) = \bigcup_{\tau\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\tau|n}(X)$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems. Let $S(\infty)$ be the set of limit points of all sequences $x_i \in \phi_i(X)$, $i \in I'$, where I' ranges over all infinite subsets of I. In [MU1] the following has been proved

Proposition 1.1. *If* $\lim_{i\in I} \operatorname{diam}(\phi_i(X)) = 0$, *then* $\overline{J} = J \cup \bigcup_{\omega\in I^*} \phi_\omega(S(\infty))$.

An iterated function system S is said to be *conformal* if $X \subset I\!R^d$ for some $d \geq 1$ and the following conditions are satisfied.

- **(1a):** Open Set Condition (OSC). $\phi_i(\mathrm{Int}X) \cap \phi_j(\mathrm{Int}X) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.
- **(1b):** There exists an open connected set V such that $X \subset V \subset I\!R^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V. (Note that for $d = 1$ this just means that all the maps ϕ_i, $i \in I$, are C^1 monotone diffeomorphisms, for $d \geq 2$ the words *conformal* mean holomorphic or antiholomorphic, and for $d \geq 3$, the maps ϕ_i, $i \in I$ are Möbius transformations. The proof of the last statement can be found in [BP] and [Va] for example, where it is called Liouville's theorem)
- **(1c):** (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, u, \alpha) \subset \mathrm{Int}(X)$ with vertex x, the symmetry axis determined by vector $u \in I\!R^d$ of length l and a central angle of Lebesgue measure α. Here $\mathrm{Con}(x, u, \alpha, l) = \{y : 0 < (y - x, u) \leq \cos\alpha||y - x|| \leq l\}$.
- **(1d):** (Bounded Distortion Property (BDP)). There exists $K \geq 1$ such that

  $$|\phi'_\tau(y)| \leq K|\phi'_\tau(x)|$$

  for every $\tau \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\tau(x)|$ means the norm of the derivative.

Under these assumptions in was shown in [MU1] that the hypothesis of Proposition 1.1 holds and we can change the order of the union and intersection operations to obtain:

$$J = \pi(I^\infty) = \bigcap_{n\geq 1} \bigcup_{|\tau|=n} \phi_\tau(X).$$

Let us first collect some geometric consequences of (BDP). We have for all words $\tau \in I^*$ and all convex subsets C of V

$$\operatorname{diam}(\phi_\tau(C)) \le ||\phi_\tau'||\operatorname{diam}(C) \tag{1.1}$$

and

$$\operatorname{diam}(\phi_\tau(V)) \le D||\phi_\tau'||, \tag{1.2}$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \ge 1$ is a universal constant. Moreover,

$$\operatorname{diam}(\phi_\tau(J)) \ge D^{-1}||\phi_\tau'|| \tag{1.3}$$

and

$$\phi_\tau(B(x,r)) \supset B(\phi_\tau(x), K^{-1}||\phi_\tau'||r), \tag{1.4}$$

for every $x \in X$, every $0 < r \le \operatorname{dist}(X, \partial V)$, and every word $\tau \in I^*$.

From now on throughout the entire paper we assume that $d \ge 2$. By H^t we will denote the t-dimensional Hausdorff measure, by HD the Hausdorff dimension and by TD the topological dimension (we will only deal with subsets of $I\!R^d$ so all Hausdorff and topological dimensions are finite). The main result of our paper is the following.

Theorem 1.2. *If $d \ge 3$, $S = \{\phi_i\}_{i\in I}$ is a conformal IFS and $\mathrm{HD}(S(\infty)) < \mathrm{HD}(J)$, then either*

- **(a):** $\mathrm{HD}(J) > \mathrm{TD}(\overline{J})$ *or*
- **(b):** $\overline{J}$ *is a proper compact subset of either a geometric sphere of dimension* $\mathrm{TD}(\overline{J})$ *or a* $\mathrm{TD}(\overline{J})$*-dimensional affine hyperspace, both contained in $I\!R^d$.*

In addition, if any one of the maps ϕ_i is a similarity mapping, then the latter case holds.

Since in the finite case the set $S(\infty)$ is empty, we get immediately the following.

Corollary 1.3. *If $d \ge 3$, $S = \{\phi_i\}_{i\in I}$ is a finite conformal IFS, then either*

- **(a):** $\mathrm{HD}(J) > \mathrm{TD}(\overline{J})$ *or*
- **(b):** $\overline{J}$ *is a proper compact subset of either a geometric sphere of dimension* $\mathrm{TD}(\overline{J})$ *or a* $\mathrm{TD}(\overline{J})$*-dimensional affine hyperspace, both contained in $I\!R^d$.*

In addition, if any one of the maps ϕ_i is a similarity mapping, then the latter case holds.

Remark 1.4. *Put $k = \mathrm{TD}(\overline{J})$. Since a compact subset of a k-dimensional sphere or hyperspace G has topological dimension k if and only if its interior in G is not empty, we see that in the second alternative of Theorem 1.2 and Corollary 1.3, $\overline{J}$ contains an open ball in the appropriate sphere or hyperspace and, for dynamical reasons, it turns out that there is an open subset Ω of that sphere or hyperspace such that $\overline{J} = \overline{\Omega}$.*

With essentially the same methods as those employed in the proof of Theorem 1.2 one can prove the following.

Theorem 1.5. *If $F : Y \to Y$ is a conformal expanding repeller, then the same dichotomy holds as in Corollary 1.3. Only one geometric sphere or an affine hyperspace is to be replaced by a finite union of such sets. In the case when Y is connected also this finite union reduces to either one sphere or one hyperspace.*

Let us now recall from [MU1] that a Borel probability measure m is said to be t-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \neq j$. It has been proved in [MU1] that if a t-conformal measure exists, then $t = h$, the Hausdorff dimension of the limit set J_S of S and this measure is unique. The system S is called regular if a conformal measure exists. Let $\sigma : \Sigma \to \Sigma$ be the left shift transformation (cutting out the first coordinate), $\sigma(\{x_n\}_{n=1}^\infty) = (\{x_n\}_{n=2}^\infty)$. In [MU1] we have proved the following.

Theorem 1.6. *If the system S is regular and m is the h-conformal measure, then*

- **(a):** *There exists a unique Borel probability measure $\tilde{m}$ on I^∞ such that $\tilde{m} \circ \pi^{-1} = m$. The measure $\tilde{m}$ is positive on non-empty open subsets of I^∞.*
- **(b):** *There exists a unique σ-invariant probability measure $\tilde{\mu}$ absolutely continuous with respect to $\tilde{m}$. Moreover, $\tilde{\mu}$ is equivalent with $\tilde{m}$, $\sup\left\{|\log\left(\frac{d\tilde{\mu}}{d\tilde{m}}\right)|\right\} < \infty$ and the dynamical system $\sigma : I^\infty \to I^\infty$ is ergodic with respect to the measure $\tilde{\mu}$.*

2. Rectifiability and tangents

A set $Q \subset I\!R^d$ is called k-rectifiable if $\mathrm{H}^k(Q) > 0$ and there exist Lipschitz maps $g_i : I\!R^k \to I\!R^d$, $i = 1, 2, \ldots$ such that

$$\mathrm{H}^k\left(Q \setminus \bigcup_{i=1}^\infty g_i(I\!R^k)\right) = 0.$$

A set $T \subset I\!R^d$ is called purely k-unrectifiable if and only if $\mathrm{H}^k(Q \cap T) = 0$ for every k-rectifiable set Q.

It follows from Theorem 15.19 in [Ma2] that for H^k-a.e. point z in a k-rectifiable set $Q \subset I\!R^d$ there is a unique approximate tangent k-plane for Q at z. This tangent plane will be denoted in the sequel by $T_z Q \in G(d, k)$. We recall that $G(d, k)$ is the Grassmannian manifold of all

k-dimensional linear subspaces of $I\!R^d$ and that the existence of a tangent k-plane T_zQ for Q at z implies that, for every $0 < s < 1$,

$$\lim_{r\to 0}\frac{1}{r^k}H^k\left(Q\cap B(z,r)\setminus Z(z,T_zQ,s)\right)=0\ \ ,$$

where

$$Z(z,V,s)=\{x\in I\!R^d\ ;\ \ d(x-z,V)<s|x-z|\}\quad .$$

The space $G(d,k)$ has a natural measure $\gamma_{d,k}$ (see Section 3.9 in [Ma2] for its definition and basic properties). Given $V\in G(d,k)$ we denote by $P_V : I\!R^d \to V$ the orthogonal projection from $I\!R^d$ onto V.

The following lemma is crucial since it gives rectifiability of the limit set provided the topological and Hausdorff dimension coincide.

Lemma 2.1. *If $S=\{\phi_i\}_{i\in I}$ is a conformal IFS and $\mathrm{H}^{\mathrm{TD}(\overline{J})}(\overline{J_S})=\mathrm{H}^{\mathrm{TD}(\overline{J})}(J)>0$, $\mathrm{HD}(\overline{J})=\mathrm{TD}(\overline{J})$, then the system S is regular, $m=\frac{\mathrm{H}^k}{\mathrm{H}^k(J)}|_J$ is the k-conformal measure on J and the closure $\overline{J_S}$ is $\mathrm{TD}(\overline{J_S})$-rectifiable.*

Proof. Put $k=\mathrm{TD}(\overline{J})$. Since $\mathrm{H}^k(J)>0$ and since $\mathrm{HD}(J)=k$, we conclude from Theorem 4.16 in [MU1] that the system S is regular and, using Lemma 4.2 in [MU1], we deduce that $\mathrm{H}^k(J)<\infty$ and $m=\frac{\mathrm{H}^k}{\mathrm{H}^k(J)}|_J$ is the k-conformal measure on J. It follows from Federer's theorem on p. 545 in [Fe] that the integralgeometric measure $\mathcal{I}_1^k(\overline{J})>0$. Since (see [Ma2], p. 86)

$$\mathcal{I}_1^k(\overline{J})=\int_{G(d,k)}\int_V \mathrm{H}^0(\overline{J}\cap \mathrm{P}_V^{-1}(a))d\mathrm{H}^k(a)d\gamma_{d,k}(V),$$

we therefore conclude that there exists a Borel set $G\subset G(d,k)$ with $\gamma_{d,k}(G)>0$ such that $\mathrm{H}^0(\overline{J}\cap \mathrm{P}_V^{-1}(a))>0$ for every $V\in G$ and all a in some set $W_V\subset V$ with $\mathrm{H}^k(W_V)>0$. In particular $\mathrm{P}_V(\overline{J})\supset W_V$ and therefore $\mathrm{H}^k(\mathrm{P}_V(\overline{J}))>0$ for all $V\in G$. Hence, it follows from Theorem 18.1(2) on p. 250 in [Ma2] that $\overline{J}$ is not purely k-unrectifiable. Therefore, combining Theorems 17.6 (Notice that although this is not indicated in Matillas's book, we need to know that $\mathrm{H}^k(\overline{J})>0$ for this theorem to make actually sense), Theorem 6.2(1) in [Ma2] and the fact that $\mathrm{H}^k(\overline{J})=\mathrm{H}^k(J)>0$, we conclude that $\Theta^k(\overline{J},x)=1$ for all x in some set $F\subset J$ with $\mathrm{H}^k(F)>0$, where the density functions Θ^k as well as Θ_*^k and Θ^{*k} were defined on p. 89 in [Ma2]. Fix now $x\in J$. It follows from the distortion property (1d) that for all $i\in I$ and all $r>0$ small enough

$$\mathrm{H}^k(\overline{J}\cap B(\phi_i(x),|\phi_i'(x)|r))\geq \mathrm{H}^k(\phi_i(J\cap B(x,K_r^{-1}r)))\geq K_r^{-k}|\phi_i'(x)|^h\mathrm{H}^k(J\cap B(x,K_r^{-1}r)),$$

where K_r is such that $\lim_{r\to 0}K_r=1$. Hence

$$\frac{\mathrm{H}^k(\overline{J}\cap B(\phi_i(x),|\phi_i'(x)|r))}{(2|\phi_i'(x)|r)^k}\geq \frac{K_r^{-k}|\phi_i'(x)|^k}{(2|\phi_i'(x)|r)^k}\mathrm{H}^k(J\cap B(x,K_r^{-1}r))=K_r^{-2k}\frac{\mathrm{H}^k(J\cap B(x,K_r^{-1}r))}{(2K_r^{-1}r)^k}.$$

and letting $r \searrow 0$ we conclude that

$$\Theta_*^k(\overline{J}, \phi_i(x)) \geq \Theta_*^k(\overline{J}, x). \tag{2.1}$$

Let $\tilde{m}$ be the lift of the conformal measure m to the coding space I^∞ and let $\tilde{\mu}$ be its shift-invariant version produced in Theorem 1.6. Since by this theorem the dynamical system $(\sigma, \tilde{\mu})$ is ergodic, it therefore follows from Birkhoff's Ergodic Theorem and (2.1) that the function $\omega \mapsto \Theta_*^k(\overline{J}, \pi(\omega))$ is constant $\tilde{\mu}$-a.e.. Since $\tilde{\mu}(\pi^{-1}(F)) > 0$, we therefore conclude that $\Theta_*^k(\overline{J}, \pi(\omega)) = 1$ for $\tilde{\mu}$-a.e. $\omega \in I^\infty$. Thus $\Theta_*^k(\overline{J}, x) = 1$ for H^k-a.e. $x \in \overline{J}$. Combining this with Theorem 6.2(1) in [Ma2] we see that $\Theta^k(\overline{J}, x)$ exists and is equal to 1 for H^k-a.e. $x \in \overline{J}$. Invoking now Theorem 17.6(1) in [Ma2] finishes the proof. ■

3. **Proof of Theorem 1.2**

Put $k = \mathrm{TD}(\overline{J_S})$ and suppose that $\mathrm{HD}(J) \leq k$. Since $\mathrm{HD}(S(\infty)) < \mathrm{HD}(J)$, using Proposition 1.1, we conclude that $\mathrm{HD}(\overline{J}) = \mathrm{HD}(J) \leq k$. Hence, $\mathrm{HD}(\overline{J}) = k$ and $\mathrm{H}^k(J) = \mathrm{H}^k(\overline{J}) > 0$. Thus the assumptions of Lemma 2.1 are satisfied. In view of this Lemma the set $\overline{J}$ is k-rectifiable. By Theorem 15.19 in [Ma2] this equivalently means that for H^k-almost all $z \in J$ there is a unique approximate tangent k-plane $T_z\overline{J}$ for $\overline{J}$ at z.

We fix now such a point, say $z_0 = \pi(\omega) \in J$, $\omega \in I^\infty$, and make the following renormalisation. Set $\lambda_n = |\phi_{\omega|_n}(\pi(\sigma^n(\omega)))|^{-1}$ and define then

$$\beta_n(z) = \lambda_n(z - z_0).$$

It follows from the Bounded Distortion Property (1d) that each mapping $\beta_n \circ \phi_{\omega|_n} : X \to I\!R^d$ is locally Lipschitz continuous with the Lipschitz constant K and from (1.2) that $\beta_n \circ \phi_{\omega|_n}(X) \subset B(0, DK)$. Therefore, the Ascoli-Arzela theorem applies and there exists an increasing to infinity sequence of positive integers $\{n_j\}_{j=1}^\infty$ such that the sequence $\psi_j : X \to I\!R^d$ converges uniformly to a continuous function $\Psi : X \to I\!R^d$, where $\psi_j = \beta_{n_j} \circ \phi_{\omega|_{n_j}}$. The limit function $\Psi : X \to X$ is conformal. We shall prove the following

Claim 3.1. $\Psi(\overline{J}) \subset T_{z_0}\overline{J}$.

Proof. Suppose on the contrary that the claim does not hold. Then there exists an open bounded set $\Omega \subset \Psi(\overline{J})$ such that

$$\eta = \mathrm{dist}(\Omega, I\!R^k) > 0\ . \tag{3.1}$$

Since Ω is an open subset of $\overline{J}$, we get $\mathrm{H}^k(\Omega) > 0$. Put $U = \Psi^{-1}(\Omega)$ and $U_j = f_j(U)$. Then

$$0 < \mathrm{H}^k(\Omega) = \int_U |\Psi'|^k d\mathrm{H}^k = \lim_{j\to\infty} \int_U |\psi_j'|^k d\mathrm{H}^k \lim_{j\to\infty} \mathrm{H}^k(\psi_j(U)) = \lim_{j\to\infty} \mathrm{H}^k(\beta_{n_j}(U_j)).$$

Hence, there exists $\tau > 0$ and $j_0 \geq 1$ such that

$$0 < \tau \leq \mathrm{H}^k(\beta_{n_j}(U_j)) = \lambda_{n_j}^k \mathrm{H}^k(U_j) \tag{3.2}$$

for all $j \geq j_0$. Due to (3.1) and the boundednes of Ω we can choose $0 < s < 1$ such that $Z(0, I\!R^k, 2s) \cap \Omega = \emptyset$. Consider the cones $\mathcal{Z} = Z(0, T_{z_0}\overline{J}, s)$ and $\mathcal{Z}_j = Z(z_0, T_{z_0}\overline{J}, s) = \beta_{n_j}^{-1}(\mathcal{Z})$. Fix also a ball $B = B(0, R)$ such that $\Omega \subset B(0, R/2)$ and set $B_j = \beta_{n_j}^{-1}(B) = B(z_0, R\lambda_{n_j}^{-1})$. Since $T_{z_0}\overline{J}$ is an approximate tangent k-plane of $\overline{J}$ at z_0, we have

$$\lim_{j\to\infty} \lambda_{n_j}^k \mathrm{H}^k(\overline{J} \cap B_j \setminus \mathcal{Z}_j) = \lim_{j\to\infty} \left(R\lambda_{n_j}^{-1}\right)^{-k} \mathrm{H}^k(\overline{J} \cap B_j \setminus \mathcal{Z}_j) = 0. \tag{3.3}$$

But, if j is sufficiently large, then $\beta_{n_j}(U_j) = \Psi_j \circ \Psi^{-1}(\Omega) \subset B \setminus \mathcal{Z}$ and therefore $U_j \subset \overline{J} \cap B_j \setminus \mathcal{Z}_j$. It then follows from (3.2) that

$$\lambda_{n_j}^k \mathrm{H}^k(\overline{J} \cap B_j \setminus \mathcal{Z}_j) \geq \lambda_{n_j}^k \mathrm{H}^k(U_j) \geq \tau > 0$$

and this contradicts (3.3). We thus proved the claim and therefore the "smooth or fractal" dichotomy announced in Theorem 1.2.

We are left to show that if one of the maps ϕ_i is a similarity map ($\lambda_i A_i + a_i$, $0 < \lambda_i < 1$), then $\overline{J}$ is contained in a k-dimensional hyperspace of $I\!R^d$. And indeed, suppose on the contrary that $\overline{J} \subset Q$, a geometric sphere in $I\!R^d$. Since $\phi_i(Q) = \lambda_i A_i(Q) + a_i$ is a geometric sphere of dimension k and the sphere $Q \cap \phi_i(Q)$ contains the k-dimensional set $\overline{J}$, this intersection is a k-dimensional sphere, and therefore equal to both Q and $\phi_i(Q)$. This contradicts the fact that $\phi_i : Q \to Q$ is a strict contraction with a Lipschitz constant equal to λ_i. We are done. ■

In the case of a conformal expanding repeller F appearing in Theorem 1.5 the proof of Lemma 2.1 requires only minor obvious modifications. In the proof of Claim 3.1 one replaces X by balls of some sufficiently small radii and the maps $\phi_{\omega|_{n_j}}$ by appropriate inverse branches of forward iterates of F. We thus get that each point of Y has a neighbourhood being an open subset of either a k-dimensional geometric sphere or hyperspace. Compactness of Y proves the first part of the theorem. If Y is connected this finite union clearly reduces to one sphere or a hyperspace.

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