Rigidity of non-recurrent rational functions

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Abstract. For a natural subclass of rational NCP functions we prove that the Lipschitz conjugacy, conformal conjugacy and the same spectra of moduli of derivatives at periodic points are mutually equivalent.

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§**1. Preliminaries.** A rational function f is said to be non-recurrent (abbreviated NCP) if and only if each critical point contained in the Julia set of f is not recurrent. Some informations about NCP maps can be found for example in [CJY], [U1] and [U2]. In order to formulate our main result, Theorem 1 we need more definitions. Let $\mathrm{Crit}(f)$ denote the set of all critical points of f except for periodic ones. By $\mathrm{Sing}(f)$ denote $\overline{\bigcup_{n=0}^{\infty} f^n(Crit(f))}$. The rational function f is said to be real-analytic if the Julia set $J(f)$ is contained in a finite union of forward invariant real-analytic curves. Let h_f be the Hausdorff dimension of the Julia set of f. It is known (see [U1]) that there exists a unique h_f-conformal measure for f which is supported on the Julia set. In the sequel this measure will be denoted by m_f. It is also known (see [U2]) that there exists a σ-finite invariant measure equivalent to the conformal measure m_f. Denote one of its representatives by μ_f and the Jacobian $\frac{d\mu_f \circ f}{d\mu_f}$ by J_{μ_f}. The map f is said to be non-linear if and only if there exists a point $w \in J_f \setminus \mathrm{Sing}(f)$ and $k \geq 0$ such that the map $F(x,y) = (J_{\mu_f}(x,y), J_{\mu_f} \circ f^k(x,y))$ is locally invertible at w (if f is real-analytic already $J_{\mu_f}(x,y)$ itself is required to be locally invertible); otherwise f is said to be linear. We will not use this fact in this paper but it can be proved (comp. [Pr]) that linearity implies each of the following conditions

(a) The Jacobian J_{μ_f} is locally constant on $J(f) \setminus \mathrm{Sing}(f)$.

(b) The function $\mathrm{HD}(J(f)) \log |f'|$ is cohomologous to a locally constant function on $J(f) \setminus \mathrm{Sing}(f)$.

(c) The conformal structure on $J(f) \setminus \mathrm{Sing}(f)$ admits a conformal refinement so that f becomes affine (i.e. there exists an atlas $\{\phi_t : U_t \to \mathbb{C}\}$, $\bigcup_t U_t \supset J(f) \setminus \mathrm{Sing}(f)$, consisting of conformal injections such that the compositions $\phi_t \circ f \circ \phi_s^{-1}$ are affine.

(d) There exists a complex valued nowhere vanishing continuous function α such that for every x in a neighbourhood of x, $\mathrm{Arg} f'(x) - \mathrm{Arg}\alpha(f(x)) - \mathrm{Arg}\alpha(x)$ is locally constant.

If the Jacobian J_{μ_f} is real-analytic all these four conditions are mutually equivalent.

We call a critical value c (of an arbitrary order) exceptional if and only if every sequence $\{x_n\}$ such that $f(x_{n+1}) = x_n$ for all $n \geq 0$ and $x_0 = c$ contains a critical point. We call a rational map non-exceptional if none of its critical values is exceptional. Given two rational functions f and g we say that a homeomorphism $h : J(f) \to J(g)$ is a conjugacy on their Julia sets if $g \circ h(x) = h \circ f(x)$ for every $x \in J(f)$. Note that we do not require h to extend on the whole sphere $\overline{\mathbb{C}}$ nor on a neighbourhood of $J(f)$.

§**2. Results and proofs.** We begin with the formulation of our main result.

Theorem 1. Suppose that f and g are two non-exceptional non-linear NCP rational maps topologically conjugate on their Julia sets. Denote the conjugacy by h. Then the following four conditions are equivalent.

(1) h is real analytic

(2) h is Lipschitz continuous.

(3) For every periodic point x of f, say of period p, $|(f^p)'(x)| = |(g^p)'(h(x))|$.

(4) The measure class of m_f is transported under h to the measure class of m_g (The implication (4) $\Rightarrow$ (1) will be proved assuming that both $J(f)$ and $J(g)$ are real-analytic).

In addition if one of the conditions (1)-(4) is satisfied and moreover there is a real-analytic extension of h onto a neighbourhood of $J(f)$ then h is conformal.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To see that (2) $\Rightarrow$ (3) suppose that $x \in J(f)$ is a periodic point of period p and $|(f^p)'(x)| \neq |(g^p)'(h(x))|$. Without loosing generality we may assume that $|(f^p)'(x)| < |(g^p)'(h(x))|$. Fix $|(f^p)'(x)| < \mu < \lambda < |(g^p)'(h(x))|$. Let U be a neighbourhood of x such that both inverse branches $f_x^{-p} : U \to U$ of f^p and $g_{h(x)}^{-p} : h(U) \to h(U)$ of g^p sending respectively x to x and $h(x)$ to $h(x)$ are well defined. Taking U sufficiently small we may assume that $|f_x^{-pn}(z) - x| \geq \mu^{-n}$ and $|g_{h(w)}^{-pn}(w) - h(x)| \leq \lambda^{-n}$ for all $n \geq 1$, $z \in U$ and $w \in h(U)$. Hence

$$\frac{|f_x^{-pn}(z) - x|}{|h(f_x^{-pn}(z)) - h(x)|} \geq \frac{\mu^{-pn}}{\lambda^{-pn}} = \left(\frac{\lambda}{\mu}\right)^{pn} \to \infty$$

if $n \to \infty$. So, the implication (2) $\Rightarrow$ (3) is proved. In order to show that (3) $\Rightarrow$ (4) we need first the following version of the closing lemma.

Lemma 2. Fix $t > 0$. Then for all $0 < \rho_2 < t$ there exist $\rho_1 > 0$ and an integer $n_1 \geq 1$ such that if $f^n(x) \in J(f) \setminus B(\mathrm{Sing}(f), t)$, if $f^n(x) \in B(x, \rho_1)$, then there exists $y \in J(f)$ such that $f^n(y) = y$ and $|f^j(y) - f^j(x)| < \rho_2$ for all $0 \leq j \leq n - n_1$ and $|y - f^n(x)| < \rho_2$.

Proof. It follows from Lemma 1.1 of [Pr] that

$$\lim_{n\to\infty} \sup\{\mathrm{diam}(P_n)\} = 0,$$

where P_n range over all connected components of $f^{-n}(B(z, \rho_2))$, $z \in J(f) \setminus B(\mathrm{Sing}(f), t)$. Take n_1 so large that $\mathrm{diam}(P_n) < \rho_2/2$ for all $n \geq n_1$. Take $\rho_1 < \rho_2/2$. Let B_n, $n \geq n_1$, be the connected component of $f^{-n}(B(f^n(x), \rho_2)$ containing x. Let $f_x^{-n} : B(f^n(x), \rho_2) \to B_n$ be the holomorphic inverse branch of f^n sending $f^n(x)$ to x. We then have

$$f_x^{-n}\big(B(f^n(x), \rho_2)\big) \subset B(x, \rho_2/2) \subset B\left(f^n(x), \rho_1 + \frac{\rho_2}{2}\right) \subset B(f^n(x), \rho_2).$$

Hence by the Brouwer fixed point theorem there exists $y \in B(f^n(x), \rho_2)$ such that $f_x^{-n}(y) = y$ which implies that $f^n(y) = y$. Since $|f_x^{-j}(f^n(x)) - f_x^{-j}(y)| = |f_x^{-j}(f^n(x)) - f_x^{-j}(f^n(y))| \leq \rho_2/2 \leq \rho_2$ for all $j \geq n_1$, the lemma is proved. ♣

By topological exactness of f the set of transitive points is dense in $J(f)$. Choose one such a point, say x. For every $z \in J(f)$ define

$$\eta(z) = \log|g'(h(z))| - \log|f'(z)|$$

and for every $n \geq 1$ set

$$u(f^n(x)) = \sum_{j=0}^{n-1} \eta(f^j(x)). \tag{1}$$

Our next technical result is the following.

Lemma 3. For every $t > 0$ the function u restricted to the set $(J(f) \setminus B(\mathrm{Sing}(f), t)) \cap \{f^n(x) : n \geq 0\}$ is uniformly continuous.

Proof. Fix $0 < \rho_2 < t$ and choose ρ_1 and n_1 according to Lemma 2. Consider two points $f^n(x)$ and $f^m(x)$ in $J(f) \setminus B(\mathrm{Sing}(f), t)$ such that $n \geq m$ and $|f^n(x) - f^m(x)| < \rho_1$. Then in view of Lemma 2 there exists a point $y \in J(f)$ such that $f^{n-m}(f^m(y)) = f^m(y)$, $|f^{m+j}(x) - f^{m+j}(y)| < \rho_2$ for all $j = 0, 1, n - m - n_1$, and $|f^n(x) - f^n(y)| < \rho_2$. Since by the assumption $\sum_{j=m}^{n-1} \eta(f^j(y)) = 0$, we therefore get

$$
\begin{aligned}
u(f^n(x)) - u(f^m(x)) &= \sum_{j=m}^{n-1} \eta(f^j(x)) = \sum_{j=m}^{n-1} \big(\eta(f^j(x)) - \eta(f^j(y))\big) \\
&= \sum_{j=m}^{n-1} \log|g'(h(f^j(x)))| - \log|g'(h(f^j(y)))| - \big(\log|(f'(f^j(x))| - \log|f'(f^j(y))|\big) \\
&= \log\left|\frac{(g^{n-m})'(h(g^m(x)))}{(g^{n-m})'(h(g^m(y)))}\right| - \log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|.
\end{aligned}
$$

We want $u(f^n(x))$ and $u(f^m(x))$ to be close one to each other if $f^n(x)$ and $f^m(x)$ are. For this it suffices to know that each term $\log\left|\frac{(g^{n-m})'(h(g^m(x)))}{(g^{n-m})'(h(g^m(y)))}\right|$ and $\log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|$ is small. But for the latter term this follows from the Koebe distortion theorem since $|f^n(x) - f^n(y)| < \rho_2$ and the inverse branch $f^{-(n-m)}_{f^m(x)}$ sending $f^n(x)$ to $f^m(x)$ is defined on $B(f^n(x), t)$. A similar argument works for the second term. The proof is finished. ♣

Consequently the function u extends continuously to each set $J(f) \setminus B(\mathrm{Sing}(f), t)$, $t > 0$, and therefore to the set $J(f) \setminus \mathrm{Sing}(f)$.

Lemma 4. The functions $\log|f'(z)|$ and $\log|g'(h(z))|$ are cohomologous in the class of continuous functions on $(J(f) \setminus \mathrm{Sing}(f)) \cap f^{-1}(J(f) \setminus \mathrm{Sing}(f))$. More precisely, $\log|g'(h(z))| - \log|f'(z)| = u(f(z)) - u(z)$ for all $z \in (J(f) \setminus \mathrm{Sing}(f)) \cap f^{-1}(J(f) \setminus \mathrm{Sing}(f))$.

Proof. Subtracting (1) from (1) written for $n-1$ we get $\eta(f^n(x)) = u(f^n(x)) - u(f^{n-1}(x))$. Since the set $\{f^n(x) : n \geq 1\}$ is dense in $(J(f) \setminus \mathrm{Sing}(f)) \cap f^{-1}(J(f) \setminus \mathrm{Sing}(f))$ and all the functions η, u, $u \circ f$ are continuous in this set, the lemma is proved. ♣

Lemma 5. Let μ and ν be Borel probability measures on Y, a bounded subset of a Euclidean space. Suppose that there are a constant $M > 0$ and for every point $x \in Y$ a decreasing to zero sequence $\{r_j(x) : j \geq 1\}$ of positive radii such that for all $j \geq 1$ and all $x \in Y$

$$\mu(B(x, r_j(x)) \leq M\nu(B(x, r_j(x)).$$

Then the measure μ is absolutely continuous with respect to ν and the Radon-Nikodym derivative $d\mu/d\nu \leq CM$, where C is a universal constant depending only on the dimension of the Euclidean space under consideration.

Proof. Consider a Borel set $E \subset Y$ and fix $\varepsilon > 0$. Since $\lim_{j\to\infty} r_j(x) = 0$ and since ν is regular, for every $x \in E$ there exists a radius $r(x)$ being of the form $r_j(x)$ such that $\nu(\bigcup_{x\in E} B(x, r(x)) \setminus E) < \varepsilon$. Now by the Besicovič theorem (see [Gu]) we can choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^\infty$ from the cover $\{B(x, r(x))\}_{x\in E}$ of E, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore we obtain

$$\begin{aligned}
\mu(E) &\leq \sum_{i=1}^\infty \mu(B(x_i, r(x_i))) \leq M \sum_{i=1}^\infty \nu(B(x_i, r(x_i))) \\
&\leq MC\nu\big(\bigcup_{i=1}^\infty B(x_i, r(x_i))\big) \\
&\leq MC(\varepsilon + \nu(E)).
\end{aligned}$$

Letting $\varepsilon \searrow 0$ we obtain $\nu(E) \leq MC\nu(E)$. So μ is absolutely continuous with respect to ν with Radon–Nikodym derivative bounded by MC. ♣

By Lemma 5.2 of [U1] Sing(f) is nowhere dense in $J(f)$ and therefore there exist $y \in J(f)$ and $\beta_f > 0$ such that $B(y, 2\beta_f) \cap \mathrm{Sing}(f) = \emptyset$. Let T_f be the set of transitive points of the map $f : J(f) \to J(f)$. Fix $0 < t < \beta_f$ and $z \in T_f \setminus B(\mathrm{Sing}(f), t)$. Then there exists an infinite sequence $n_j = n_j(z)$ such that $f^{n_j}(z) \in B(y, \beta_f)$ for all $j \geq 1$. Applying now Lemma 4 and Lemma 3 we see that there exists a constant $Q_t \geq 1$ such that

$$Q_t^{-1} \leq \frac{|(g^{n_j})'h((z))|}{|(f^{n_j})'(z)|} \leq Q_t. \tag{2}$$

By the uniform continuity of h there exists $\gamma \leq \beta_f$ such that $h(J(f) \cap B(x, \gamma/2) \subset B(h(x), \beta_g/2)$ for all $x \in J(f)$. By the choice of the sequence $n_j = n_j(z)$ for every $j \geq 1$ there exists an inverse branch $f_z^{-n_j} : B(f_z^{n_j}(z), \beta_f) \to \overline{\mathbb{C}}$ sending $f^{n_j}(z)$ to z. Set $r_j(z) = \frac{1}{8}|(f_z^{-n_j})'(f^{n_j}(z)|\gamma$. Then by the Koebe Distortion Theorem

$$B(z, r_j(z)) \supset f_z^{-n_j}(B(f^{n_j}(z), K^{-1}\gamma/8),$$

where K is a universal constant, and therefore

$$\begin{aligned}
m_f(B(z, r_j(z))) &\geq K^{-h}|(f_z^{-n_j})'(f^{n_j}(z)|^{h_f} m(B(f^{n_j}(z), K^{-1}\gamma/8) \\
&\geq MK^{-h_f}|(f_z^{-n_j})'(f^{n_j}(z)|^{h_f},
\end{aligned} \tag{3}$$

where $M = \inf\{m_f(B(x, K^{-1}\gamma/8) : x \in J(f)\} > 0$. Similarly by the $\frac{1}{4}$ - Koebe Distortion Theorem $B(z, r_j(z)) \subset f_z^{-n_j}(B(f^{n_j}(z), \gamma/2)$. Hence

$$\begin{aligned}
h\big(J(f) \cap B(z, r_j(z))\big) &\subset h\big(f_z^{-n_j}(B(f^{n_j}(z), \gamma/2) \cap J(f)\big) \\
&\subset g_{h(z)}^{-n_j}\big(h(B(f^{n_j}(z), \gamma/2) \cap J(f))\big) \subset g_{h(z)}^{-n_j}\big(B(hf^{n_j}(z), \beta_g/2))\big) \\
&= g_{h(z)}^{-n_j}\big(B(g^{n_j}(h(z)), \beta_g/2))\big)
\end{aligned}$$

Therefore using (3)

$$
\begin{aligned}
m_g \circ h(B(z,r_j(z))) &\le m_g\big(g_{h(z)}^{-n_j}\big(B(g^{n_j}(h(z)),\beta_g/2))\big)\big) \\
&\le K^{h_g}|(g_{h(z)}^{-n_j})'(g^{n_j}(h(z)))|^{h_g} m_g\big(B(g^{n_j}(h(z)),\beta_g/2))\big)\big) \\
&\le K^{h_g}|(g^{n_j})'(h(z))|^{-h_g} \le K^{h_g}Q_t^{h_g}|(f^{n_j})'(z)|^{-h_f+(h_f-h_g)} \\
&\le M^{-1}K^{h_f+h_g}Q_t^{h_g} m_f(B(z,r_j(z))|(f^{n_j})'(z)|^{h_f-h_g}
\end{aligned}
$$

If $h_f - h_g < 0$ then it would follow from Lemma 5 and the fact that $\lim_{n\to\infty}|(f^{n_j})'(z)| = \infty$ that $m_g \circ h(T_f \setminus B(\mathrm{Sing}(f),t)) = 0$. Letting $t \to 0$ it would imply that $m_g(h(T_f)) = 0$. But since h is a conjugacy between f and g, $h(T_f) = T_g$. Hence $m_g(T_g) = 0$ which would contradict Lemma 7.3 of [U1] saying that the set of transitive points is of measure 1. Hence $m_g \circ h(B(z,r_j(z))) \le M^{-1}K^{2h_f}Q_t^{h_g} m_f B(z,r_j(z))$. Therefore letting $t \to 0$ and applying Lemma 5 we conclude that $m_g \circ h|_{T_f}$ is absolutely continuous with respect to $m_f|_{T_f}$. Take now A, an arbitrary Borel set contained in $J(f)$ such that $m_f(A) = 0$. Write $A = (A \cap T_f) \cup (A \cap T_f^c)$. Then $m_f(A \cap T_f) = 0$ and consequently $m_g \circ h(A \cap T_f) = 0$. Since h is a topological conjugacy between f and g, $h(A \cap T_f^c) \subset h(T_f^c) = T_g^c$. Hence, by Lemma 7.3 of [U1], $m_g(h(A \cap T_f^c)) \le m_g(T_g^c) = 0$. Thus finally $m_g \circ h(A) = 0$ and $m_g \circ h$ is absolutely continuous with respect m_f. By symmetry $m_f \circ h^{-1}$ is absolutely continuous with respect to m_g and consequently the measures $m_g \circ h$ and m_f are equivalent. The proof of the implication (3) $\Rightarrow$ (4) is finished. ♣

As we already mentioned in the introduction it is known (see [U2]) that there exists a σ-finite invariant measure equivalent with the conformal measure m_f. Denote one of its representatives by μ_f. Before proving the implication (4) $\Rightarrow$ (1) we will demonstrate the following.

Lemma 6. If $J(f)$ is real analytic (contained in a finite union of forward invariant real-analytic curves), then the Radon-Nikodyn derivative $d\mu_f/dm_f$ is a real analytic function on $J(f) \setminus \mathrm{Sing}(f)$.

Proof. It follows from [U2] that there exists a set X_0 such that up to a multiplicative constant the measure μ_f is given by the formula

$$\mu_f(F) = \lim_{n\to\infty} \frac{\sum_{k=0}^n m \circ f^{-k}(F)}{\sum_{k=0}^n m \circ f^{-k}(X_0)}.$$

A straightforward computation shows then that

$$\frac{d\mu_f}{dm_f}(x) = \lim_{n\to\infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{y \in f^{-k}(x)} |(f^k)'(y)|^{-h_f},$$

where $a_n = \sum_{k=0}^n m \circ f^{-k}(X_0)$. Taking an appropriate atlas we may assume that $J(f)$ is contained in a real interval. Then for every $x \in J(f) \setminus \mathrm{Sing}(f)$, $k \ge 1$, and $y \in f^{-k}(x)$ there exists $\nu(k,y) \in \{1,-1\}$ such that for sufficiently small $r > 0$

$$|(f_y^{-k})'(z)|^{h_f} = (\nu(k,y)(f_y^{-k})'(z))^{h_f}$$

for all $z \in J(f) \cap B(x,r)$, where $f_y^{-k} : B(x,r) \to \mathbb{C}$ are the inverse branches of f^k sending x to y. Consider the following sequence of analytic functions

$$g_n(z) = \frac{1}{a_n} \sum_{k=0}^{n} \sum_{y \in f^{-k}(x)} \big(\nu(k,y)((f_y^{-k})'(z)\big)^{-h_f},\ z \in B(x,r).$$

There is no problem here with raising to the h_fth power since $B(x,r)$, the domain of $\nu(k,y)(f_y^{-k})'$ is simply connected. Fix $w \in J(f) \cap B(x,r/2)$ such that $g_n(w)$ converges. By the Koebe Distortion Theorem for every $z \in B(x,r/2)$, every $k \geq 1$ and every $y \in f^{-k}(x)$ we have $|(f_y^{-k})'(z)| \leq K|(f_y^{-k})'(w)|$. Hence $|g_n(z)| \leq |g_n(w)| = g_n(w)$ and therefore the functions $\{g_n|_{B(x,r/2)}\}_{n \geq 1}$ form a normal family in the sense of Montel. Since $g_n(z)$ converges for μ_f a.e.$z \in J \cap B(x,r/2)$, it follows that g_n converges to an analytic function g on $B(x,r/2)$. In particular $g|_{J \cap B(x,r/2)}$ is real-analytic and we are done. ♣

As an immediate consequence of this lemma and the following computation

$$\frac{d\mu_f \circ f}{d\mu_f} = \frac{d\mu_f \circ f}{dm_f \circ f} \cdot \frac{dm_f \circ f}{dm_f} \cdot \frac{dm_f}{d\mu_f} = \frac{d\mu_f}{dm_f} \circ f \cdot |f'|^h \cdot \frac{dm_f}{d\mu_f}$$

we get the following.

Corollary 7. If J_f is real-analytic, then Jacobian $J_{\mu_f} = \frac{d\mu_f \circ f}{d\mu_f}$ is real-analytic.

Coming back to the situation of two non-linear maps f and g we will provide the rest of the proof in the general setting assuming that the Jacobian $J_{\mu_f} = \frac{d\mu_f \circ f}{d\mu_f}$ is real-analytic. So, we know that there exists a point $w \in J_g \setminus \mathrm{Sing}(g)$ and $k \geq 0$ such that the map $G(x,y) = (J_{\mu_g}(x,y), J_{\mu_g} \circ g^k(x,y))$ is locally invertible at w (if g is real-analytic already $J_{\mu_g}(x,y)$ itself is locally invertible). It is straight forward to check that $G \circ h = F$, where $F(x,y) = (J_{\mu_f}(x,y), J_{\mu_f} \circ f^k(x,y))$. Hence, around w, say in a neighbourhood U, $G^{-1} \circ F$ is an analytic extension of h. Recall that we called a critical value c (of an arbitrary order) exceptional if and only if every sequence $\{x_n\}$ such that $f(x_{n+1}) = x_n$ for all $n \geq 0$ and $x_0 = c$ contains a critical point. Since each point z of $J(f)$ is not an exceptional critical value, there exists $k \geq 1$ and $b \in f^{-k}(c)$ such that $\{f^j(b) : 0 \leq j \leq k-1\} \cup \bigcup_{n \geq 0} f^{-n}(b)$ contains no critical points. Hence by the topological exactness there exists a neighbourhood V of z such that there exists an inverse branch $f_\nu^{-n} : V \to \mathbb{C}$ of f^n with $f_\nu^{-n}(V) \subset U$. Hence $h|_V \cap J(f) = g^n \circ h \circ f_\nu^{-n}|_{V \cap J(f)}$ and $g^n \circ h \circ f_\nu^{-n}|_{V \cap J(f)}$ has a real-analytic extension $g^n \circ (G^{-1} \circ F) \circ f_\nu^{-n}$ to V. The proof of the implication (4) $\Rightarrow$ (1) is finished. ♣

The last part of the theorem can be proved similarly as Lemma 7.2.7 in [Pr]. ♣

Remark 8. In the literature there exist a large number of papers devoted to various kinds of rigidity theorems. Let us list here only those which seem to be the closest to what is contained in this paper: [Co], [Mc], [Pr], [SS] and [Su].

Remark 9. It follows from [FU] that if the Julia set of a real-analytic rational function is connected, then, in fact, it is equal to a geometric circle or an interval.

Remark 10. Notice that the Lattes examples provide counterexamples to to Theorem 1 if non-exceptionality is relaxed.

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