WELL-APPROXIMABLE POINTS FOR JULIA SETS WITH PARABOLIC AND CRITICAL POINTS

BERND O. STRATMANN, MARIUSZ URBAŃSKI, AND MICHEL ZINSMEISTER

ABSTRACT. In this paper we consider rational functions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with parabolic and critical points contained in their Julia sets $J(f)$ such that $\sum_{n=1}^{\infty} |(f^n)'(f(c))|^{-1} < \infty$ for each critical point $c \in J(f)$. We calculate the Hausdorff dimensions of subsets of $J(f)$ consisting of elements z for which $\inf\{\text{dist}(f^n(z), \text{Crit}(f)) : n \geq 0\} > 0$ and which are well-approximable by backward iterates of the parabolic periodic points of f.

1. Introduction and Statement of Main Results

In [S1] [S2] [SU1] [SU2] [SU3] we developed a certain type of Diophantine analysis for geometrically finite Kleinian groups with parabolic elements, parabolic rational maps and tame parabolic iterated function systems. In this paper we extend this analysis to rational maps $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with parabolic points and with Julia sets $J(f)$ containing critical points such that for each of these critical points $c \in J(f)$ we have that

$$\sum_{n=1}^{\infty} |(f^n)'(f(c))|^{-1} < \infty.$$

The idea is to perform this Diophantine analysis on the set of points $z \in J(f)$ for which we have, with $\text{Crit}(f)$ denoting the set of critical points of f, that

$$\inf\{\text{dist}(f^n(z), \text{Crit}(f)) : n \geq 0\} > 0.$$

We give a further development of the techniques elaborated in the afore-mentioned papers, mainly by using results obtained in [DU] and [Pr] (see also [PU1]). In particular, a crucial quantity in our main theorem (Theorem 1.1) is the so-called dynamical dimension $\text{DD}(J(f))$ of $J(f)$, in the literature occasionally also referred to as the essential dimension, which is given by ([DU]; see also [Pr] [PU1])

$$\text{DD}(J(f)) := \sup\{\text{HD}(\mu)\}.$$

Here, the supremum is taken with respect to all f-invariant, ergodic Borel probability measures μ of positive entropy, and $\text{HD}(\mu)$ denotes the Hausdorff dimension of μ. In [DU] it was shown that $\text{DD}(J(f))$ coincides with the least possible exponent for conformal measures associated with f. Also, in [PU1] we obtained that $\text{DD}(J(f))$ coincides with the hyperbolic dimension of f in the sense of Shishikura ([Sh]).

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In order to state our main result, we first have to introduce some notation. Let Ω be the set of all parabolic periodic points of f, that is

$$\Omega := \{\omega \in \overline{\mathbb{C}} \colon f^q(\omega) = \omega \ \text{ and } \ (f^q)'(\omega) = 1 \text{ for some } \ q \geq 1\}.$$

It is well-known that Ω is a finite subset of $J(f)$. For fixed $\omega \in \Omega$ and for every pre-parabolic point $x \in \bigcup_{n \geq 0} f^{-n}(\omega)$, let $n(x)$ denote the least integer such that $f^{n(x)}(x) = \omega$. Since the set of critical points of f is finite and since critical points are not periodic, there exists a number $q(\omega) \geq 2$ such that for every pre-parabolic point x for which $n(x) \geq q(\omega)$, we have that

$$\mathrm{Crit}(f) \cap \bigcup_{n \geq 0} f^{-n}(x) = \emptyset.$$

In the following let $x = x(\omega)$ denote some fixed element of this type. Then for every $y \in \bigcup_{n \geq 0} f^{-n}(x)$ there exists a unique integer $k(y)$ such that $f^{k(y)}(y) = x$. If $B(w, r)$ denotes the closed ball centred at w of radius r, then for some fixed $\rho_0 > 0$ and every $\kappa > 0$ we define

$$B_y^\kappa := B\Big(y, (\rho_0 \, |(f^{k(y)})'(y)|^{-1})^{1+\kappa}\Big).$$

Furthermore, for $\epsilon > 0$ let

$$J_{x,\epsilon}^\kappa := \bigcap_{q \geq 1} \bigcup_{n \geq q} \bigcup_{*} B_z^\kappa,$$

where the union $\bigcup_*$ is taken with respect to all elements $z \in \bigcup_{l \geq 0} f^{-l}(x) \backslash \bigcup_{j \geq 0} f^{-j}(B(\mathrm{Crit}(f), \epsilon))$ for which $k(z) = n$. Our main interest in this paper will be focused on the sets

$$J_x^\kappa := \bigcup_{\epsilon > 0} J_{x,\epsilon}^\kappa, \ J_\omega^\kappa := \bigcup_{\{z : n(z) = q(\omega)\}} J_z^\kappa \ \text{ and } \ J^\kappa := \bigcup_{\omega \in \Omega} J_\omega^\kappa.$$

We are now in the position to state the main result of this paper. Here, we have used the common notation $p(\omega)$ to denote the number of petals of $\omega \in \Omega$.

Theorem 1.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with parabolic periodic points and critical points such that for each critical point $c \in J(f)$ we have that $\sum_{n=1}^{\infty} |(f^n)'(f(c))|^{-1} < \infty$. Then for every parabolic point $\omega \in \Omega$ and each $\kappa > 0$ it holds that*

$$\mathrm{HD}(J_\omega^\kappa) = \begin{cases} \frac{\mathrm{DD}(J(f))}{1+\kappa} & \text{if } \ \kappa \geq \mathrm{DD}(J(f)) - 1 \\ \frac{\mathrm{DD}(J(f)) + \kappa p(\omega)}{1+\kappa(1+p(\omega))} & \text{if } \ \kappa \leq \mathrm{DD}(J(f)) - 1. \end{cases}$$

Hence in particular, we have with $p_{\min} := \min\{p(\omega) : \omega \in \Omega\}$ that

$$\mathrm{HD}(J^\kappa) = \begin{cases} \frac{\mathrm{DD}(J(f))}{1+\kappa} & \text{if } \kappa \geq \mathrm{DD}(J(f)) - 1 \\ \frac{\mathrm{DD}(J(f)) + \kappa p_{\min}}{1+\kappa(1+p_{\min})} & \text{if } \kappa \leq \mathrm{DD}(J(f)) - 1. \end{cases}$$

We remark that a similar type of Diophantine analysis could also be given for repelling periodic orbits rather than parabolic periodic points. In fact, in that case the arguments and also the results would be far less involved, mainly due to the lack of 'conformal fluctuation' of the conformal measure at repelling periodic orbits.

Finally, we remark that Theorem 1.1 represents a generalization of the following classical result in metrical Diophantine analysis by Jarník [Ja] and Besicovitch [Be] on well-approximable irrationals.

$$\mathrm{HD}\left(\left\{\xi \in \mathbb{R} : |\xi - p/q| < q^{-2(1+\sigma)} \text{ for infinitely many } (p,q) = 1\right\}\right) = (1+\sigma)^{-1}.$$

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2. **Proof of the Theorem 1.1**

In this section we give the proof of Theorem 1.1. It will be prepared first by giving a series of separate statements. From now on we assume that the preliminaries in Theorem 1.1 are satisfied. For $\epsilon > 0$, we define the set

$$K_\epsilon := J(f) \setminus \bigcup_{n \geq 0} f^{-n}(B(\mathrm{Crit}(f), \epsilon)).$$

Observe that by [PU2] (Appendix B, Theorem B.1), for each $\epsilon, \theta > 0$ there exists $\tau_\epsilon(\theta) > 0$ with the property that if $z \in K_\epsilon$ such that $f^n(z) \notin B(\Omega, \theta)$, then for every $0 \leq k \leq n$ we have that

$$\mathrm{diam}\Big(f^k(C_n(z, f^n(z), 2\tau_\epsilon(\theta))\Big) < \epsilon.$$

Here $C_n(z, f^n(z), 2\tau_\epsilon(\theta))$ refers to the connected component of $f^{-n}(B(f^n(z), 2\tau_\epsilon(\theta)))$ which contains z. Note that this implies in particular that $f^k(C_n(z, f^n(z), 2\tau_\epsilon(\theta))) \cap \mathrm{Crit}(f) = \emptyset$. This observation is summarized in the following lemma.

Lemma 2.1. *For every $\epsilon > 0$ and every $\theta > 0$ there exists $\tau_\epsilon(\theta) > 0$ such that if $z \in K_\epsilon$ and $f^n(z) \notin B(\Omega, \theta)$, then*

$$C_n(z, f^n(z), 2\tau_\epsilon(\theta)) \cap \mathrm{Crit}(f^n) = \emptyset.$$

The next step in the proof of Theorem 1.1 is to verify the following upper bound for the Hausdorff dimension.

Lemma 2.2. *For each $\omega \in \Omega$, $x \in \bigcup_{n \geq 0} f^{-n}(\omega)$ such that $n(x) \geq q(\omega)$, and for every $\epsilon, \kappa > 0$ we have that*

$$\mathrm{HD}(J^\kappa_{x,\epsilon}) \leq \min\left\{\frac{\mathrm{DD}(J(f))}{1+\kappa}, \frac{\mathrm{DD}(J(f)) + \kappa p(\omega)}{1 + \kappa(1 + p(\omega))}\right\}.$$

Proof. Throughout we shall always assume that

$$\theta = \mathrm{dist}(x, \Omega).$$

In view of Lemma 2.1, for every $n \geq 0$ and each $z \in f^{-n}(x) \cap K_\epsilon$ there exists a unique holomorphic branch $f_z^{-n} : B(x, 2\tau_\epsilon(\theta)) \to \mathbb{C}$ of f^{-n} which maps x to z. Since the family $\{f_z^{-n} : B(x, 2\tau_\epsilon(\theta)) \to \mathbb{C} : n \geq 0, z \in f^{-n}(x) \cap K_\epsilon\}$ is normal and since $x \in J(f)$, all the limit functions of this family are constant. Thus, it follows that

$$\lim_{n\to\infty} \sup\{|(f_z^{-n})'(y)| : y \in B(x, \tau_\epsilon(\theta))\} = 0. \tag{2.1}$$

Hence in particular, we have that

$$M := \sup\{|(f_z^{-n})'(y)| : n \geq 0,\ y \in B(x, \tau_\epsilon(\theta))\} < \infty,$$

and also, that there exists $s \geq 1$ sufficiently large such that $|(f^n)'(z)| > 2K^2$ for all $n > s$ and $z \in f^{-n}(x) \cap K_\epsilon$. Here $K \geq 1$ refers to the Koebe constant corresponding to the scale $1/2$. For every $l \geq 0$ we define

$$Z_l := \{z \in \bigcup_{n \geq 0} f^{-n}(x) \cap K_\epsilon : 2^{-(l+1)} M \leq |(f^{k(z)})'(z)|^{-1} \leq 2^{-l} M\}.$$

We claim that for each $l \geq 0$ the family

$$\Re_l := \{f_z^{-k(z)}(B(x, \tau_\epsilon(\theta))) : z \in Z_l\}$$

has multiplicity bounded above by s. In order to see this, suppose that for some $l \geq 0$ and for distinct $z, w \in Z_l$ we have that

$$f_z^{-k(z)}(B(x, \tau_\epsilon(\theta))) \cap f_w^{-k(w)}(B(x, \tau_\epsilon(\theta))) \neq \emptyset.$$

Let y be an element of the latter intersection. Without loss of generality we can assume that $k(w) \leq k(z)$. Using Koebe's distortion theorem, it follows that

$$|(f^{k(z)})'(y)| \leq K |(f^{k(z)})'(z)| \leq K M^{-1} 2^{l+1}$$

and

$$|(f^{k(w)})'(y)|^{-1} \leq K |(f^{k(w)})'(w)|^{-1} \leq K M 2^{-l}.$$

Consequently, applying Koebe's distortion theorem once again, we obtain

$$|(f^{k(z)-k(w)})'(f^{k(w)}(z))| \leq K |(f^{k(z)-k(w)})'(f^{k(w)}(y))| = K |(f^{k(z)})'(y)| \cdot |(f^{k(w)})'(y)|^{-1} \leq 2K^3.$$

Hence, it follows that $k(z) - k(w) \leq s$. Since for every $n \geq 0$ and distict $z, w \in f^{-n}(x) \cap K_\epsilon$ we have that $f_z^{-n}(B(x, \tau_\epsilon(\theta))) \cap f_w^{-n}(B(x, \tau_\epsilon(\theta))) = \emptyset$, it follows that the family $\Re_l$ has multiplicity at most s.

By [DU] (comp. [PU1]), there exists a $\mathrm{DD}(J(f))$-conformal measure m for $f : J(f) \to J(f)$. Hence, using Koebe's distortion theorem again, we get for every $l \geq 1$ that

$$\sum_{z \in Z_l} |(f^{k(z)})'(z)|^{-\mathrm{DD}(J(f))} \leq K^{\mathrm{DD}(J(f))} m(B(x, \tau_\epsilon(\theta)))^{-1} \sum_{z \in Z_l} m(f_z^{-k(z)}(B(x, \tau_\epsilon(\theta)))) \ll s K^{\mathrm{DD}(J(f))}.$$

(Here, $a \ll b$ (for $a, b > 0$) means that the quotient a/b is bounded from above by some constant.)

In the sequel, let $t > \mathrm{DD}(J(f))$. Using the latter estimate, we first make the following computation.

$$\begin{aligned}\Sigma_t &:= \sum_{n\geq 1}\sum_{z\in K_\epsilon\cap f^{-n}(x)} |(f^n)'(z)|^{-t} = \sum_{l\geq 0}\sum_{z\in Z_l} |(f^{k(z)})'(z)|^{-t} \\ &= \sum_{l\geq 0}\sum_{z\in Z_l} |(f^{k(z)})'(z)|^{-\mathrm{DD}(J(f))}\ |(f^{k(z)})'(z)|^{-(t-\mathrm{DD}(J(f)))} \\ &\leq \sum_{l\geq 0}\sum_{z\in Z_l} |(f^{k(z)})'(z)|^{-\mathrm{DD}(J(f))}\ M^{t-\mathrm{DD}(J(f))}\ 2^{-l(t-\mathrm{DD}(J(f)))} \\ &\ll sM^{t-\mathrm{DD}(J(f))}\ K^{\mathrm{DD}(J(f))} \sum_{l\geq 0} 2^{-l(t-\mathrm{DD}(J(f)))} < \infty.\end{aligned} \tag{2.2}$$

Now observe that for each $k \geq 1$, the family $\{B_z^\kappa : z \in K_\epsilon \cap f^{-n}(x), n \geq k\}$ represents a covering of $J_{x,\epsilon}^\kappa$, and that for k increasing the upper bound for the radii of this covering tends to zero (this follows since by (2.1) we have $\lim_{n\to\infty} \max\{|(f^n)'(z)| : z \in K_\epsilon \cap f^{-n}(x)\} = \infty$). Furthermore, by the computation in (2.2), for every $k \geq 1$ we have for the radii of the elements of this covering that

$$\sum_{n\geq k}\sum_{z\in K_\epsilon\cap f^{-n}(x)} \left((\rho_0|(f^n)'(z)|^{-1})^{1+\kappa}\right)^{\frac{t}{1+\kappa}} \ll \sum_{n\geq k}\sum_{z\in K_\epsilon\cap f^{-n}(x)} |(f^n)'(z)|^{-t} = \Sigma_t < \infty.$$

This immediately gives for the $\frac{t}{1+\kappa}$-dimensional Hausdorff measure $\mathcal{H}^{\frac{t}{1+\kappa}}$ that $\mathcal{H}^{\frac{t}{1+\kappa}}(J_{x,\epsilon}^\kappa) < \infty$, and hence that $\mathrm{HD}(J_{x,\epsilon}^\kappa) \leq \frac{t}{1+\kappa}$. By letting t tend to $\mathrm{DD}(J(f))$, we obtain

$$\mathrm{HD}(J_{x,\epsilon}^\kappa) \leq \frac{\mathrm{DD}(J(f))}{1+\kappa}. \tag{2.3}$$

In order to obtain the second upper estimate of the lemma, note that Fatou's flower theorem implies that for each $n \geq 1$ and $z \in K_\epsilon \cap f^{-n}(x)$ the ball $B(\omega, (\rho_0|(f^n)'(z)|^{-1})^\kappa)$ admits a covering by balls of radii $(\rho_0|(f^n)'(z)|^{-1})^{\kappa(p(\omega)+1)}$ such that the cardinality of this covering is comparable to $|(f^n)'(z)|^{\kappa p(\omega)}$. Consequently, by Lemma 2.1 and Koebe's distortion theorem, each ball B_z^κ with $z \in K_\epsilon \cap f^{-n}(x)$ can be covered by at most a number comparable to $|(f^n)'(z)|^{\kappa p(\omega)}$ of balls with radii comparable to $(\rho_0|(f^n)'(z)|^{-1})^{1+\kappa(1+p(\omega))}$. Hence, using this observation and (2.2), we obtain for each $\lambda > 0$ that

$$\begin{aligned}\mathcal{H}^{\frac{\mathrm{DD}(J(f))+\kappa p(\omega)}{1+\kappa(1+p(\omega))}+\lambda}(J_{x,\epsilon}^\kappa) &\ll \lim_{k\to\infty}\sum_{n\geq k}\sum_{z\in K_\epsilon\cap f^{-n}(x)} (\rho_0|(f^n)'(z)|^{-1})^{(1+\kappa(1+p(\omega)))\left(\frac{\mathrm{DD}(J(f))+\kappa p(\omega)}{1+\kappa(1+p(\omega))}+\lambda\right)}|(f^n)'(z)|^{\kappa p(\omega)} \\ &\ll \lim_{k\to\infty}\sum_{n\geq k}\sum_{z\in K_\epsilon\cap f^{-n}(x)} |(f^n)'(z)|^{-(\mathrm{DD}(J(f))+\lambda(1+\kappa(1+p(\omega))))} = 0.\end{aligned}$$

This implies that $\mathrm{HD}(J_{x,\epsilon}^\kappa) \leq \frac{\mathrm{DD}(J(f))+\kappa p(\omega)}{1+\kappa(1+p(\omega))} + \lambda$. Hence, by letting λ tend to 0, we conclude that

$$\mathrm{HD}(J_{x,\epsilon}^\kappa) \leq \frac{\mathrm{DD}(J(f)) + \kappa p(\omega)}{1+\kappa(1+p(\omega))}.$$

■

As an immediate consequence of this lemma we have the following corollary.

Corollary 2.3. *For each $\omega \in \Omega$ and for every $\kappa > 0$ we have that*

$$\mathrm{HD}(J_\omega^\kappa) \le \min\left\{ \frac{\mathrm{DD}(J(f))}{1+\kappa}, \frac{\mathrm{DD}(J(f)) + \kappa p(\omega)}{1+\kappa(1+p(\omega))} \right\}$$

and

$$\mathrm{HD}(J_\kappa) \le \min\left\{ \frac{\mathrm{DD}(J(f))}{1+\kappa}, \frac{\mathrm{DD}(J(f)) + \kappa p_{\min}}{1+\kappa(1+p_{\min})} \right\}.$$

For the remaining part of the proof of Theorem 1.1, we assume additionally that $\theta > 0$ is chosen so small that for every $z \in (J(f)\setminus\Omega)\cap B(\Omega,\theta)$ there exists $n \ge 1$ with $f^n(z) \notin B(\Omega,\theta)$. For fixed $\epsilon > 0$ and for $z \in K_\epsilon \setminus \bigcup_{n\ge 0} f^{-n}(\Omega)$ let $\{n_j\}_{j=1}^\infty$ denote the increasing sequence of natural numbers which is given by $f^{n_j}(z) \notin B(\Omega,\theta)$ for all j, and $f^k(z) \in B(\Omega,\theta)$ for all $k \notin \{n_1, n_2, \cdots\}$. Clearly, such a sequence is infinite. Also, for every $j \ge 1$ we let $r_j(z) := |(f^{n_j})'(z)|^{-1}$, and for every $r > 0$ sufficiently small we define

$$r_{min}(z) := \min\{r_k(z) : r_k(z) > r\} \quad \text{and} \quad r_{max}(z) := \max\{r_k(z) : r_k(z) \le r\}.$$

Furthermore with $x = x(\omega)$ chosen as before for $\omega \in \Omega$, if $z \in K_\epsilon \cap f^{-n}(x)$ for some $n \ge 0$, then we define

$$r_\omega(z) := |(f^n)'(z)|^{-1}.$$

For each $\epsilon > 0$ sufficiently small such that $K_\epsilon \ne \emptyset$, we obtained in [DU] that there exists a Borel probability measure m_ϵ supported on K_ϵ and a number $h_\epsilon \le \mathrm{DD}(J(f))$ such that the following properties are satisfied.

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$$\lim_{\epsilon \searrow 0} h_\epsilon = \mathrm{DD}(J(f)); \tag{2.4}$$
- for every Borel set $E \subset J(f)$ such that $f|_E$ is injective, we have that
$$m_\epsilon(f(E)) \ge \int_E |f'(z)|^{h_\epsilon} dm_\epsilon(z); \tag{2.5}$$
- for every Borel set $F \subset K_\epsilon \setminus \partial B(\mathrm{Crit}(f),\epsilon)$ such that $f|_F$ is injective, we have that
$$m_\epsilon(f(F)) = \int_F |f'(z)|^{h_\epsilon} dm_\epsilon(z).$$

Finally, the conformal fluctuation function ζ_ϵ of m_ϵ is defined for $z \in K_\epsilon$ and $r > 0$ by

$$\zeta_\epsilon(z,r) := \frac{m_\epsilon(B(z,r))}{r^{h_\epsilon}}.$$

Now, the remaining part of the proof of Theorem 1.1 is completely analogous to the constructions in [SU3], and we refer to this paper for the details. More precisely, in order to obtain uniform estimates for the conformal fluctuation function, we can now proceed as in section 3 of [SU3]. Note that the proof of these estimates uses (2.5), Lemma 2.1, Corollary 2.3 and the local properties of f around parabolic points.

Lemma 2.4. • *If $z \in K_\epsilon \setminus \bigcup_{n\geq 0} f^{-n}(\Omega)$ and $r > 0$ such that $r_{min}(z) = r_j(z)$ and $f^{n_j+1}(z) \in B(\omega,\theta)$ for some $j \in I\!N$ and $\omega \in \Omega$, then we have that*

$$\zeta_\epsilon(z,r) \ll \begin{cases} \left(\frac{r}{r_{min}(z)}\right)^{(h_\epsilon-1)p(\omega)} & \text{for } \ r_{min}(z) \geq r \geq r_{min}(z)\left(\frac{r_{max}(z)}{r_{min}(z)}\right)^{\frac{1}{p(\omega)+1}} \\ \left(\frac{r_{max}(z)}{r}\right)^{h_\epsilon-1} & \text{for } \ r_{max}(z) \leq r \leq r_{min}(z)\left(\frac{r_{max}(z)}{r_{min}(z)}\right)^{\frac{1}{p(\omega)+1}}. \end{cases}$$

• *If $z \in K_\epsilon \cap f^{-n}(x)$ for some $n \geq 0$ and $\omega \in \Omega$, then we have for all $0 < r \leq r_\omega(z)$ that*

$$\zeta_\epsilon(z,r) \asymp r^{(h_\epsilon-1)p(\omega)}.$$

Furthermore, on the basis of Lemma 2.1, Lemma 2.4 and the local properties of f around parabolic points, we can then proceed as in section 4.1 of [SU3] to obtain the following result. Note that in [SU3], in order to derive these lower estimates for the Hausdorff dimension, we employed precisely the type of estimates for the conformal fluctuation function, which we have just derived for ζ_ϵ in Lemma 2.4 above.

Lemma 2.5. *For each $\omega \in \Omega$ and every $\epsilon > 0$ we have that*

$$\mathrm{HD}(J^\kappa_\omega) \geq \begin{cases} \frac{h_\epsilon}{1+\kappa} & \text{if } \ \kappa \geq h_\epsilon - 1 \\ \frac{h_\epsilon+\kappa p(\omega)}{1+\kappa(1+p(\omega))} & \text{if } \ \kappa \leq h_\epsilon - 1. \end{cases}$$

Consequently, it follows that

$$\mathrm{HD}(J^\kappa) \geq \begin{cases} \frac{h_\epsilon}{1+\kappa} & \text{if } \kappa \geq h_\epsilon - 1 \\ \frac{h_\epsilon+\kappa p_{\min}}{1+\kappa(1+p_{\min})} & \text{if } \kappa \leq h_\epsilon - 1. \end{cases}$$

Finally, the proof of Theorem 1.1 now follows from Lemma 2.2 and Lemma 2.5, where we let ϵ tend to 0 and use the limit behaviour as stated in (2.4). ■

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Bernd O. Stratmann; Mathematical Institute, University of St Andrews, St Andrews KY16 9SS, Scotland. http://www.maths.st-and.ac.uk/~bos
bos@maths.st-and.ac.uk

Mariusz Urbański; Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA. http://www.math.unt.edu/~urbanski
urbanski@unt.edu

Michel Zinsmeister; MAPMO, Mathématiques, Universite d'Orleans, BP 6759 45067 Orléans Cedex, France. http:// www.labomath.univ-orleans.fr/descriptions/zins
Michel.Zinsmeister@labomath.univ-orleans.fr