The Geometry of Conformal Measures for Parabolic Rational Maps

*B.O. Stratmann & M. Urbański**

Abstract

We study the h-conformal measure for parabolic rational maps, where h denotes the Hausdorff dimension of the associated Julia sets. We derive a formula which describes in a uniform way the scaling of this measure at arbitrary elements of the Julia set. Furthermore, we establish the Khintchine Limit Law for parabolic rational maps (the analogue of the 'logarithmic law for geodesics' in the theory of Kleinian groups), and show that this law provides some efficient control for the fluctuation of the h-conformal measure. We then show that these results lead to some refinements of the description of this measure in terms of Hausdorff and packing measures with respect to some gauge functions. Also, we derive a simple proof of the fact that the Julia set of a parabolic rational map is uniformly perfect. Finally, we obtain that the conformal measure is a regular doubling measure, we show that its Renyi dimension and its information dimension is equal to h, and we compute its logarithmic index.

1 Introduction and statement of results

Let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote a parabolic rational map, i.e. a rational map of degree at least 2 such that the associated Julia set $J(T)$ contains rationally indifferent periodic points but no critical points. In this paper we give a detailed discussion of the geometry of the h-conformal measure supported on $J(T)$, where h denotes the Hausdorff dimension of $J(T)$.

After we have recalled the notion of the *radial Julia set* and introduced the concept of *hyperbolic zoom* for elements of the Julia set, the first main result of the paper is the derivation of a *uniform estimate for the h-conformal measure* m of balls centred around arbitrary elements of $J(T)$. This formula gives precise information about the decay of the measure of such balls for uniformly

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shrinking radii. Namely, for Euclidean balls $B(\xi, r)$ centred at $\xi \in J(T)$ and of radius r less than the diameter of $J(T)$ we obtain that

$$m(B(\xi, r)) \asymp r^h \, \phi(\xi, r)$$

(where $\asymp$ denotes comparability of the two quantities, i.e. their quotient is bounded from above and below). The function ϕ will be called the *conformal fluctuation function*, and we give a description of this function purely in terms of the geometry of T. Roughly speaking, ϕ governs the individual fluctuation of the density of the measure m between the 'hyperbolic power law' r^h (which holds on the hyperbolic zooms) and the 'parabolic power law' $r^{h+(h-1)p(\omega)}$ (which holds around the parabolic points); where $p(\omega)$ denotes the number of petals of the parabolic point ω involved.

Our second main result is the derivation of the *Khintchine Limit Law for parabolic rational maps.* This law decribes the limit behaviour of neighbouring members of the hyperbolic zoom at points in a set of full h-conformal measure. Namely, if $r_j(\xi)$ and $r_{j+1}(\xi)$ denote such neighbours, then we obtain that for m-almost all $\xi \in J(T)$ one has that

$$\limsup_{j \to \infty} \frac{\log \left(r_j(\xi) \, / \, r_{j+1}(\xi) \right)}{\log \log \frac{1}{r_j(\xi)}} = \frac{1 + p_{\max}}{h + (h-1) p_{\max}}$$

(where, $p_{\max}$ denotes the maximal number of petals which can occur for the parabolic points of T). We then show that the Khintchine Law gives rise to good estimates on the decay (for $h > 1$) or growth (for $h < 1$) of the conformal fluctuation function, given that we restrict the discussion to a set of full measure. Roughly speaking, the outcome here is that generically the extreme values of the fluctuation function $\phi(\xi, r)$ behave like $\left(\log \frac{1}{r} \right)^{(1-h)p_{\max}/(h+(h-1)p_{\max})}$. This of course provides some deeper insight into the geometric nature of the h-conformal measure m, and for instance leads to a refined description of this measure in terms of Hausdorff and packing measures with respect to some gauge functions (see the table in Corollary 4.5).

Subsequently, we show that a combination of the above two main results, i.e. the uniform estimate for m and the Khintchine Limit Law, immediately gives some rather interesting statements concerning the fractal nature of the parabolic system $(J(T), T, m)$. Namely, we obtain an easy proof of the well-known fact that Julia sets are *uniformly perfect.* Furthermore, we conclude that the h-conformal measure is a *regular doubling measure*, and finally we compute its *Renyi dimension*, its *information dimension* and its *logarithmic index.*

Remark: We mention that the two main results in this paper, i.e. the uniform estimate for the h-conformal measure and the Khintchine limit law, are precise analogues of the 'global measure formula' and the 'Khintchine-type theorem' (which is also referred to as the 'logarithmic law for geodesics') which were derived in [?] for the Patterson measure on limit sets of geometrically finite Kleinian groups with parabolic elements (see also [?], [?]). The 'dictionary' for conceptual conversion between the dynamical systems arising from Kleinian groups on the one hand side and from rational maps on the hand, has been developed extremely well in the expanding hyperbolic case ([?], [?]). The results in this paper make a contribution to a completion of this dictionary also for the expansive parabolic case.

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2 Preliminaries

Throughout, let T denote a parabolic rational map and $J(T)$ the associated Julia set. Without loss of generality we shall assume that $J(T)$ is a compact subset of $\mathbb{C}$, i.e. $\{\infty\} \notin J(T)$. If $\Omega(T)$ denotes the set of rationally indifferent periodic points (parabolic points), then, by Fatou's theorem, $\Omega(T)$ is a finite subset of $J(T)$. It is well known that the ω-limit set of the critical points is contained in $F(T) \cup \Omega$, where $F(T) = \hat{\mathbb{C}} \setminus J(T)$ denotes the Fatou set of T. Also, if $\Omega_0(T) := \{\xi \in \Omega : T(\xi) = \xi, T'(\xi) = 1\}$, then without loss of generality we shall assume that $\Omega_0(T) = \Omega(T)$.

Recall that for each $\omega \in \Omega$ we can find a sufficiently small neighbourhood U_ω of ω such that there exists a unique holomorphic inverse branch T_ω^{-1} of T on U_ω with the property that $T_\omega^{-1}(\omega) = \omega$. For the iterates of this branch on $U_\omega \cap J(T) \setminus \{\omega\}$ the following facts were obtained in [?], [?]. In our discussion these geometric observations will be crucial and we shall make use of them without further notice. For $\xi \in U_\omega \cap J(T) \setminus \{\omega\}$ and $n \in \mathbb{N}$ we have that

- $|T_\omega^{-n}(\xi) - \omega| \asymp n^{-1/p(\omega)}$;
- $|(T_\omega^{-n})'(\xi)| \asymp n^{-(1+p(\omega))/p(\omega)}$.

(Note that here the 'comparability constants' are dependent on the distance of the chosen point ξ to the parabolic point ω.)

In order to specify certain partial aspects of $J(T)$, we introduce the set of *pre-parabolic points* $J_p(T)$. This set comprises the pre-images of the parabolic points, i.e.

$$J_p(T) := \bigcup_{k=0}^{\infty} T^{-k}(\Omega(T)).$$

Now, the following definition of a radial point was originally stated in [**?**] (see also [**?**], [**?**]).

Definition 2.1 *Let T denote some arbitrary rational map. For $\kappa > 0$ an element $\xi \in J(T)$ is called a* κ-radial point *if and only if there exists an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers $n_j = n_j(\xi)$, such that for each $j \in \mathbb{N}$ the inverse branch $T_\xi^{-n_j}$ which sends $T^{n_j}(\xi)$ back to ξ is well defined on $B(T^{n_j}(\xi), \kappa)$.*

If we return with our discussion to the class of parabolic rational maps, then the following result was obtained in [**?**].

Lemma 2.2 *If T is a parabolic rational map, then there exists a positive constant $c = c(T)$ such that ξ is a c-radial point for each $\xi \in J(T) \setminus J_p(T)$.*

This lemma now enables us to give the following definition.

Definition 2.3 *For a parabolic rational map T let $J_r(T)$ denote the set of all c-radial points, where the constant $c = c(T)$ is chosen according to the previous lemma. In this situation we shall simply refer to a c-radial point as a* radial point*, and call $J_r(T)$ the* radial Julia set*.*

We may now state the following immediate result concerning Julia sets of parabolic rational maps, which is analogous to the theorem of Beardon and Maskit [**?**] in the theory of Kleinian groups. Note that as for Kleinian groups, the property that the limit set 'splits' into the radial Julia set and the set of pre-parabolic points may also be used as an alternative definition for 'parabolic rational'.

Corollary 2.4 *For a parabolic rational map T every element of its Julia set is either pre-parabolic or radial, i.e. $J(T) = J_r(T) \cup J_p(T)$.*

Finally, we now introduce the notions of 'optimal sequence' and 'hyperbolic zoom'. These notions will be crucial for our analysis in the following sections. Note that these concepts provide some 'rough coding' for the elements in the Julia set.

Definition 2.5 *For T a parabolic rational map, let $c = c(T)$ denote the constant in the above definition of the radial Julia set. Then, to each $\xi \in J_r(T)$ we can associate a unique maximal sequence $(n_j(\xi))_{j\in\mathbb{N}}$ such that $T^{n_j(\xi)}(\xi) \notin \bigcup_{\omega\in\Omega} U_\omega$ for all $j \in \mathbb{N}$ and hence in particular, such that the inverse branches $T_\xi^{-n_j(\xi)}$ are well defined on $B(T^{n_j(\xi)}(\xi), c)$. This sequence will be called the* optimal *sequence at* ξ *. Also, we let* $r_j(\xi) := |(T^{n_j(\xi)})'(\xi)|^{-1}$ *, and call* $(r_j(\xi))_{j\in\mathbb{N}}$ *the hyperbolic zoom at* ξ *. Furthermore, to each pre-parabolic point* $\xi \in J_p(T)$ *we can associate in a similar way a unique maximal terminating optimal sequence* $(n_j(\xi))_{j=1,\cdots,l}$ *, such that* $T^{n_j(\xi)}(\xi) \notin \bigcup_{\omega\in\Omega} U_\omega$ *for all* $j = 1, \cdots, l$ *. In this situation* $T^{n_l(\xi)}(\xi) = \omega$ *for some* $\omega \in \Omega$ *. We define* $r_j(\xi) := |(T^{n_j(\xi)})'(\xi)|^{-1}$ *, and call* $(r_j(\xi))_{j=1,\cdots,l}$ *the terminating hyperbolic zoom at* ξ *.*

3 The 'uniform formula' for the conformal measure

Let T denote a parabolic rational map with Julia set $J(T)$ of Hausdorff dimension h. In this section we shall derive a new formula for the associated h-conformal measure.

Recall from [?], [?] and [?] that for a parabolic rational map there exists a unique h-conformal measure m supported on $J(T)$, i.e. a probability measure with the property that for each Borel set $E \subset J(T)$ on which T is injective, we have that

$$m(T(E)) = \int_E |T'(\xi)|^h \, dm(\xi).$$

It was shown in [?] that h is the infimum of all real numbers s for which there exists an s-conformal measure. It is known [?] that m is a non-atomic measure and that the Haudorff measure H_h on $J(T)$ is absolutely continuous with respect to m ([?]). Furthermore, by constructing a suitable Markov partition for the 'jump transformation' T^*, it has been shown that T^* is ergodic with respect to m^*, where m^* denotes the T^*-invariant measure in the measure class of m.

Now, an easy consequence of the h-conformality is that for each radial point ξ and for any $j \in \mathbb{N}$ the h-conformal measure of the ball of radius $r_j(\xi)$ centred at ξ is comparable to $r_j(\xi)^h$, i.e. on a sequence the measure follows a 'hyperbolic power law'. Also, using the mapping structure of the inverse branches of T which stabilise the parabolic points, it is easy to see that the h-conformal measure of balls centred around pre-parabolic points eventually follows a 'parabolic power law', i.e. the measure scales like the radius of the ball raised to the power $h+p(\omega)(h-1)$, where $p(\omega)$ denotes the number of petals of the parabolic point which is associated to that particular pre-parabolic point. Already in [?] a more general formula for the h-conformal measure, unifying these two different aspects of the measure, was obtained. This formula arose from considerations of the interplay between the local mapping structure of the rational map and the conformal measure, and concentrated only to a lesser extent on the geometry of the map. Now, the following new formula reflects in a more satisfying way the interplay between the h-conformal measure and the geometry of the Julia set. For balls centred around arbitrary elements of the Julia set the formula gives precise geometric estimates on the fluctuation of the measure between the hyperbolic and the parabolic power law. We mention that this formula is the precise analogue for parabolic rational maps of the 'global measure formula' for the Patterson measure on limit sets of geometrically finite Kleinian groups [**?**].

Theorem 3.1 *Let T be a parabolic rational map with Julia set $J(T)$ of Hausdorff dimension h. Let m denote the associated h-conformal measure supported on $J(T)$. Then there exists a function $\phi : J(T) \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\xi \in J(T)$ and for every positive $r < \mathrm{diam}(J(T))$ we have that*

$$m(B(\xi, r)) \asymp r^h \cdot \phi(\xi, r) \quad .$$

The function ϕ is called the conformal fluctuation function. *For $r < \mathrm{diam}(J(T))$, the values of ϕ are determined as follows.*

- *For $\xi \in J_r(T)$, assume that r relates to the optimal sequence and the hyperbolic zoom at ξ such that $r_{j+1}(\xi) \leq r < r_j(\xi)$ and such that $T^k(\xi) \in U_\omega$, for all $k \in (n_j(\xi), n_{j+1}(\xi)]$ and for some $\omega \in \Omega(T)$. In this situation we have that*
$$\phi(\xi, r) \asymp \begin{cases} \left(\frac{r}{r_j(\xi)}\right)^{(h-1)p(\omega)} & \text{for} \quad r > r_j(\xi) \left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{1/(1+p(\omega))} \\ \left(\frac{r_{j+1}(\xi)}{r}\right)^{(h-1)} & \text{for} \quad r \leq r_j(\xi) \left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{1/(1+p(\omega))} \end{cases} .$$

- *For $\xi \in J_p(T)$ consider its terminating optimal sequence $(n_j(\xi))_{j=1,\cdots,l}$ and its terminating hyperbolic zoom $(r_j(\xi))_{j=1,\cdots,l}$. Let $T^{n_l(\xi)}(\xi) = \omega \in \Omega$. For $r > r_l(\xi)$ the value $\phi(\xi, r)$ is determined as above in the radial case. Otherwise, if $r \leq r_l(\xi)$, we have that*
$$\phi(\xi, r) \asymp \left(\frac{r}{r_l(\xi)}\right)^{(h-1)p(\omega)} \quad .$$

Proof:

We consider first the case in which $\xi \in J_r(T)$. Using [**?**] (Lemma 4.2 and Lemma 4.8), it follows that there exists a constant $\sigma > 0$ such that if we let $1 \geq R \geq \sigma \, |T^{n_j(\xi)+1}(\xi) - \omega|$ and put $r = R \ \left| \left(T^{n_j(\xi)+1}\right)' (\xi)\right|^{-1}$, then on $B(T^{n_j(\xi)+1}(\xi), R)$ we obtain the existence of the inverse branches $T_\xi^{-(n_j(\xi)+1)}$

sending $T^{n_j(\xi)+1}(\xi)$ back to ξ. Using the Köbe Distortion Theorem [?] and the h-conformality of m, it follows that

$$\begin{aligned} m(B(\xi,r)) &\asymp |\left(T^{n_j(\xi)+1}\right)'(\xi)|^{-h}\cdot m(B(T^{n_j(\xi)+1}(\xi),R)) \\ &\asymp |\left(T^{n_j(\xi)}\right)'(\xi)|^{-h}\cdot R^{h+(h-1)p(\omega)} \\ &\asymp |\left(T^{n_j(\xi)}\right)'(\xi)|^{-h}\cdot\left(r\,|(T^{n_j(\xi)})'(\xi)|\right)^{h+(h-1)p(\omega)} \\ &\asymp r^h\cdot\left(\frac{r}{r_j(\xi)}\right)^{(h-1)p(\omega)} \end{aligned}$$

In order to specify the range of values of r in which this estimate holds to be true, note that by the chain rule we have that

$$\begin{aligned} 1 &\asymp r_{j+1}(\xi)\,|\left(T^{n_j(\xi)}\right)'(\xi)|\,|\left(T^{n_{j+1}(\xi)-n_j(\xi)}\right)'(T^{n_j(\xi)}(\xi))| \\ &\asymp \frac{r_{j+1}(\xi)}{r_j(\xi)}\,(n_{j+1}(\xi)-n_j(\xi))^{(1+p(\omega))/p(\omega)}, \end{aligned}$$

which implies that

$$(n_{j+1}(\xi)-n_j(\xi))^{1/p(\omega)}\asymp\left(\frac{r_j(\xi)}{r_{j+1}(\xi)}\right)^{1/(1+p(\omega))}.$$

Using this estimate, it follows that

$$\begin{aligned} r &\asymp R\,r_j(\xi) \\ &\geq r_j(\xi)\,\sigma\,|T^{n_j(\xi)+1}(\xi)-\omega| \\ &\asymp r_j(\xi)\,(n_{j+1}(\xi)-(n_j(\xi)+1))^{-1/p(\omega)} \\ &\asymp r_j(\xi)\,\left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{1/(1+p(\omega)).} \end{aligned}$$

This gives the first part of the case in which $\xi\in J_r(T)$. For the second part choose $k\in[n_j(\xi)+1,n_{j+1}(\xi)$ such that

$$r\asymp|(T^k)'(\xi)|^{-1}\,\sigma\,|T^k(\xi)-\omega|.$$

Again, using [?] (Lemma 4.8) and the h-conformality of m, we obtain that

$$\begin{aligned} m(B(\xi,r)) &\asymp |(T^k)'(\xi)|^{-h}\cdot m(B(T^k(\xi),\sigma\,|T^k(\xi)-\omega|)) \\ &\asymp |(T^k)'(\xi)|^{-h}\cdot|T^k(\xi)-\omega|^{h+(h-1)p(\omega)}. \end{aligned}$$

Now, observe that by [?] (Theorem 8.4), [?] (Lemma 1) and the chain rule, we have that

$$\begin{aligned} 1 &\asymp r_{j+1}(\xi)\,|(T^k)'(\xi)|\,|(T^{n_{j+1}(\xi)-k})'(T^k(\xi))| \\ &\asymp r_{j+1}(\xi)\,|(T^k)'(\xi)|\,(n_{j+1}(\xi)-k)^{(1+p(\omega))/p(\omega)}, \end{aligned}$$

and also that

$$|T^k(\xi) - \omega|^{-(p(\omega)+1)} \asymp (n_{j+1}(\xi) - k)^{(1+p(\omega))/p(\omega)}.$$

Combining these two estimates, we obtain that

$$|(T^k)'(\xi)|^{-1} \asymp r_{j+1}(\xi)\ |T^k(\xi) - \omega|^{-(1+p(\omega))}.$$

Using this latter comparability and the above representation of r, it follows that

$$\begin{aligned} r &\asymp |(T^k)'(\xi)|^{-1}\ |T^k(\xi) - \omega| \\ &\asymp r_{j+1}(\xi)\ |T^k(\xi) - \omega|^{-(p(\omega)+1)}\ |T^k(\xi) - \omega| \\ &\asymp r_{j+1}(\xi)\ |T^k(\xi) - \omega|^{-p(\omega)}. \end{aligned}$$

Hence,

$$|T^k(\xi) - \omega| \asymp \left(\frac{r_{j+1}(\xi)}{r}\right)^{1/p(\omega)},$$

which implies that

$$\begin{aligned} |(T^k)'(\xi)| &\asymp r^{-1}\ |T^k(\xi) - \omega| \\ &\asymp r^{-1}\ \left(\frac{r_{j+1}(\xi)}{r}\right)^{1/p(\omega)}. \end{aligned}$$

Using the two latter observations, we can now continue our estimate for the measure as follows:

$$\begin{aligned} m(B(\xi, r)) &\asymp r^h\ \left(\frac{r_{j+1}(\xi)}{r}\right)^{-h/p(\omega)}\ \left(\frac{r_{j+1}(\xi)}{r}\right)^{(h+(h-1)p(\omega))/p(\omega)} \\ &\asymp r^h\ \left(\frac{r_{j+1}(\xi)}{r}\right)^{h-1}, \end{aligned}$$

which proves the second part of the case in which ξ is a radial point.

The case in which $\xi \in J_p(T)$ and $r > r_l(\xi)$ (where $r_l(\xi)$ denotes the last element of the terminating hyperbolic zoom at ξ) can be dealt with analogously to the above radial case. For the remaining case $r \leq r_l(\xi)$, the assertion is an immediate consequence of [**?**] (Lemma 2). □

The preceding theorem has the following immediate corollary.

Corollary 3.2 *Let T be a parabolic rational map with Julia set of Hausdorff dimension h. Let m denote the associated h-conformal measure. Then we have the following.*

- *If* $\xi \in J_r(T)$ *, then for the hyperbolic zoom* $(r_j(\xi))_j$ *it holds that*

$$m(B(\xi, r_j(\xi)) \asymp r_j(\xi)^h.$$

- *If* $\xi \in J_p(T)$ *is given such that* $\xi = T^{-k}(\omega)$ *for some* $k \in \mathbb{N}$ *and* $\omega \in \Omega$ *, then there exists a constant* $\rho = \rho(\xi) > 0$ *such that for all positive* $r < \rho$ *it holds that*

$$m(B(\xi, r)) \asymp r^{h+(h-1)p(\omega)}.$$

4 The Khintchine limit law

The main result in this section is the derivation of the Khintchine limit law for parabolic rational maps T. In terms of the geometry of the map this limit law provides useful approximations of the essential support of the h-conformal measure. During the preparations for its proof, we obtain in particular the result that the associated jump transformation T^* acts rationally ergodically. Furthermore, we derive four separate limit laws concerning various relevant functions.

We assume that the reader is familiar with the notion of the associated *jump transformation* T^* and some of its properties (cf. [?], [?], [?]).

In the following, fix a *Markov partition* for T^* and consider a sufficiently small neighbourhood U_ω around each of the parabolic points $\omega \in \Omega$. Assume that the elements of the Markov partition which intersects U_ω are labelled by $\mathbb{N}$ in 'increasing order' towards ω. With these preliminaries, the function $Q_\omega : U_\omega \to \mathbb{N}$ is defined by: $Q_\omega(\xi) = n \in \mathbb{N} \setminus \{0, 1\}$ if and only if ξ is an element of the atom of the Markov partition labelled by n (and hence, if and only if $T^*(\xi) = T^n(\xi)$); (for $\xi \notin U_\omega$ we let $Q_\omega(\xi) = 1$).

For $\epsilon \in \mathbb{R}, \omega \in \Omega$ and $n \in \mathbb{N}$ let

$$A_{\omega,n}(\epsilon) := \{\xi \in J(T) : Q_\omega(\xi) \geq n^{p(\omega)/(h+(h-1)p(\omega))-\epsilon}\},$$

and define the following 'limsup-set' concerning the forward orbit of T^*:

$$A_{\omega,\infty}(\epsilon) := \{\xi \in J(T) : (T^*)^n(\xi) \in A_{\omega,n}(\epsilon) \text{ for infinitely many } n\}.$$

We shall now see that the jump transformation T^* acts *rationally ergodically* in the neighbourhoods of parabolic points.

Lemma 4.1 *With the notation above, we have for all* $\omega \in \Omega$ *that* $m(A_{\omega,\infty}(\epsilon)) > 0$ *if and only if* $\epsilon \geq 0$.

Proof: Let us fix some parabolic point $\omega \in \Omega$. Using the h-conformality of m, we obtain for $\epsilon \in \mathbb{R}$ that

$$\begin{aligned}\sum_{n\in\mathbb{N}} m(A_{\omega,n}(\epsilon)) &\asymp \sum_{n\in\mathbb{N}} \sum_{k\geq n^{p(\omega)/(h+(h-1)p(\omega))-\epsilon}} k^{-\frac{1+p(\omega)}{p(\omega)}h} \\ &\asymp \sum_{n\in\mathbb{N}} n^{-(1-\epsilon\,\frac{(h+(h-1)p(\omega)}{p(\omega)})}.\end{aligned}$$

Hence, using this measure estimate and the fact that $\frac{h+(h-1)p(\omega)}{p(\omega)}$ is positive for all $\omega \in \Omega$ (cf. [**?**]), the 'only-if-part' of the statement of the lemma is an immediate consequence of the Borel-Cantelli lemma and the existence of the T^*-invariant measure m^* in the measure class of m (with bounded Radon-Nikodym derivative).

In order to prove the remaining part of the lemma let us recall the following general result [**?**]. We remark that the 'mixing condition' in this statement is often refered to as *rational ergodicity.*

- If for a sequence $(E_n)_{n\in\mathbb{N}}$ of events in a probability space (X, P) we have that $\sum_{n\in\mathbb{N}} P(E_n) = \infty$ and that $P(E_n \cap E_k) \ll P(E_n)\,P(E_k)$ for arbitrary distinct $n, k \in \mathbb{N}$, then $P(\limsup_{n\to\infty} E_n) \gg 1$.

Using once again the above measure estimate, the 'if-part' of the lemma follows from this general result once we have shown that for all $n, k \in \mathbb{N}$ with $n > k$ we have that

$$m\left((T^*)^{-k}(A_{\omega,k}(\epsilon)) \cap (T^*)^{-n}(A_{\omega,n}(\epsilon))\right) \ll m\left((T^*)^{-k}(A_{\omega,k}(\epsilon))\right)\, m\left((T^*)^{-n}(A_{\omega,n}(\epsilon))\right).$$

In order to obtain this inequality, recall that there exists a unique T^*-invariant probability measure in the measure class of m. Hence, it is sufficient to show that

$$m\left(A_{\omega,k}(\epsilon) \cap (T^*)^{-(n-k)}(A_{\omega,n}(\epsilon))\right) \ll m(A_{\omega,k}(\epsilon))\, m(A_{\omega,n}(\epsilon)).$$

Since the set $A_{\omega,k}(\epsilon)$ can be written as a union of 'jump cylinders' of length 1, it may also be written as a union of cylinders of length $(n-k)$. If $A_{\omega,k}(\epsilon) = \bigcup B_k(\epsilon)$ denotes such a representation by cylinders of length $(n-k)$, then the 'bounded distortion property' (Renyi's property for jump cylinders [**?**]) gives that

$$m\left((T^*)^{-(n-k)}(A_{\omega,n}(\epsilon)) \cap B_k(\epsilon)\right) \asymp |((T_\xi^*)^{-(n-k)})'(y)|^h\, m(A_{\omega,n}(\epsilon)) \cap (T^*)^{n-k}(B_k(\epsilon)),$$

which implies that

$$\frac{m((T^*)^{-(n-k)}(A_{\omega,n}(\epsilon)) \cap B_k(\epsilon))}{m(B_k(\epsilon))} \asymp \frac{|((T^*)^{-(n-k)})'(y)|^h\, m(A_{\omega,n}(\epsilon) \cap (T^*)^{n-k}(B_k(\epsilon)))}{|((T^*)^{-(n-k)})'(y)|^h\, m((T^*)^{n-k}(B_k(\epsilon)))}.$$

Since the $B_k(\epsilon)$ are cylinders of length $(n-k)$, there are only finitely many sets of the form $(T^*)^{n-k}(B_k(\epsilon))$, and hence the infimum of the measure of these sets is positive. This implies that

$$m((T^*)^{-(n-k)}(A_{\omega,n}(\epsilon)) \cap B_k(\epsilon)) \ll m(A_{\omega,n}(\epsilon))\ m(B_k(\epsilon)).$$

If we now sum up these inequalities over all sets $B_k(\epsilon)$, then we obtain that

$$m((T^*)^{-(n-k)}(A_{\omega,n}(\epsilon)) \cap A_{\omega,k}(\epsilon)) \ll m(A_{\omega,n}(\epsilon))\ m(A_{\omega,k}(\epsilon)),$$

which is the desired inequality.

□

Lemma 4.2 *With the notation above, for all $\omega \in \Omega$ and for all $\epsilon \geq 0$ we have that $m(A_{\omega,\infty}(\epsilon)) = 1$.*

Proof:
For each $\omega \in \Omega$ and for all $\epsilon > 0$, we clearly have that $T^*(A_{\omega,\infty}(\epsilon)) \subseteq A_{\omega,\infty}(\epsilon)$. Hence, using the previous lemma and the ergodicity of the jump transformation, the statement of the lemma follows.

□

We are now in the position to state our first limit law.

First Limit Law:
For m-almost every $\xi \in J(T)$ and for all $\omega \in \Omega$ we have that

$$\limsup_{n\to\infty} \frac{\log Q_\omega((T^*)^n(\xi))}{\log n} = \frac{p(\omega)}{h + (h-1)p(\omega)}.$$

Proof:
In order to obtain the lower bound for the '$\limsup$' in the lemma, fix some $\omega \in \Omega$ and note that by Lemma 4.1 we have that $m(A_{\omega,\infty}(0)) = 1$. If $\xi \in A_{\omega,\infty}(0)$, then by definition, there exists a sequence $(k_j)_{j\in\mathbb{N}}$ of natural numbers k_j, such that $(T^*)^{k_j}(\xi) \in A_{\omega,k_j}(0)$ for all $j \in \mathbb{N}$. This implies for all j that

$$Q_\omega((T^*)^{k_j}(\xi)) \geq k_j^{p(\omega)/(h+(h-1)p(\omega))},$$

and hence that

$$\limsup_{n\to\infty} \frac{\log Q_\omega((T^*)^n(\xi))}{\log n} \geq \frac{p(\omega)}{h + (h-1)p(\omega)}.$$

In order to obtain the upper bound for the '$\limsup$' in the lemma, let $\epsilon < 0$ and $\omega \in \Omega$. By the previous lemma, there exists a set $B_\omega(\epsilon)$ such that $m(B_\omega(\epsilon)) = 1$, and such that if $\xi \in B_\omega(\epsilon)$ then there exists a number $n_0 = n_0(\xi) \in \mathbb{N}$ with the property that $(T^*)^n(\xi) \notin A_{\omega,n}(\epsilon)$ for all $n \geq n_0$.

Hence, for $\xi \in B_\omega(\epsilon)$ we have for all $n \geq n_0$ that

$$\limsup_{n\to\infty} \frac{\log Q_\omega((T^*)^n(\xi))}{\log n} \leq \frac{p(\omega)}{h + (h-1)p(\omega)} - \epsilon.$$

If we define the set $B_\omega := \bigcap_{n\in\mathbb{N}} B_\omega(-\frac{1}{n})$, then we have that $m(B_\omega) = 1$, and furthermore that each $\xi \in B_\omega$ has the property that

$$\limsup_{n\to\infty} \frac{\log Q_\omega((T^*)^n(\xi))}{\log n} \leq \frac{p(\omega)}{h + (h-1)p(\omega)}.$$

Hence, for $\xi \in A_{\omega,\infty}(0) \cap B_\omega$ we obtain the equality stated in the First Limit Law.

□

Note that if $Q_\omega(\xi) = n$, then it follows that $|\omega - \xi| \asymp n^{-1/p(\omega)}$. This now leads to our second limit law.

Second Limit Law:
For m-almost every $\xi \in J(T)$ we have for all $\omega \in \Omega$ that

$$\limsup_{n\to\infty} \frac{-\log|(T^*)^n(\xi) - \omega|}{\log n} = \frac{1}{h + (h-1)p(\omega)}.$$

Proof:
Fix $\xi \in J_r(T)$ and $\omega \in \Omega$. By definition of Q_ω, we have for $n \in \mathbb{N}$ that

$$|(T^*)^n(\xi) - \omega| \asymp Q_\omega((T^*)^n(\xi))^{-1/p(\omega)},$$

and hence

$$\lim_{n\to\infty} \left| \frac{-\log|(T^*)^n(\xi) - \omega|}{\log n} - \frac{\log Q_\omega((T^*)^n(\xi))}{p(\omega)\ \log n} \right| = 0.$$

Using the First Limit Law, this then implies for m-almost all $\xi \in J(T)$ that

$$\begin{aligned} \limsup_{n\to\infty} \frac{-\log|(T^*)^n(\xi) - \omega|}{\log n} &= \frac{1}{p(\omega)} \limsup_{n\to\infty} \frac{\log Q_\omega((T^*)^n(\xi))}{\log n} \\ &= \frac{1}{h + (h-1)p(\omega)}. \end{aligned}$$

□

Obviously, the forward orbit of m-almost every $\xi \in J(T)$ is visiting arbitrarily small neighbourhoods of each parabolic point infinitely often. Taking this fact into account, we now modify on a set of full measure the definition of the hyperbolic zoom $(r_j(\xi))_j$ and optimal sequence $(n_j(\xi))_j$. Namely, for a given $\omega \in \Omega$ we abstract only those elements in these sequences which result exclusively from visits of the forward orbit of ξ to the neighbourhood U_ω. With other words, we consider subsequences $(r_{i_k}(\xi))_k$ and $(n_{i_k}(\xi))_k$, such that $T^{n_{i_k}(\xi)+1}(\xi) \in U_\omega$ for all $k \in \mathbb{N}$. These subsequences will be referred to as *ω-restricted hyperbolic zoom*, and *ω-restricted optimal sequence* respectively.

Third Limit Law:
For each $\omega \in \Omega$ the ω-restricted optimal sequence at m-almost every $\xi \in J(T)$ has the property that

$$\limsup_{k\to\infty} \frac{\log(n_{i_k+1}(\xi) - n_{i_k}(\xi))}{\log i_k} = \frac{p(\omega)}{h + (h-1)p(\omega)}.$$

Proof:
Let $\omega \in \Omega$ and $\xi \in J_r(T)$. For $n \in \mathbb{N}$ and $\omega \in \Omega$ the function N_n is defined by $(T^*)^n(\xi) = T^{N_n(\xi)}(\xi)$. Then we have by induction that $N_j = n_j$, for all $j \in \mathbb{N}$ (this follows, since $n_1 = N_1$ and, assuming that $n_i = N_i$, since $n_{i+1}(\xi) = n_i(\xi) + N_1(T^{n_i(\xi)}(\xi)) = N_{i+1}(\xi)$).

Using the fact that $|T^{N_{i_k}(\xi)}(\xi) - \omega| \asymp (N_{i_k+1}(\xi) - N_{i_k}(\xi))^{-1/p(\omega)}$ as well as the above Second Limit Law, it follows that for m-almost all ξ we have that

$$\begin{aligned}
\limsup_{k\to\infty} \frac{\log(n_{i_k+1}(\xi) - n_{i_k}(\xi))}{\log i_k} &= \limsup_{k\to\infty} \frac{\log(N_{i_k+1}(\xi) - N_{i_k}(\xi))}{\log i_k} \\
&= \limsup_{k\to\infty} \frac{-p(\omega)\ \log|T^{N_{i_k}(\xi)}(\xi) - \omega|}{\log i_k} \\
&= \limsup_{k\to\infty} \frac{-p(\omega)\ \log|(T^*)^{i_k}(\xi) - \omega|}{\log i_k} \\
&= \frac{p(\omega)}{h + (h-1)p(\omega)}.
\end{aligned}$$

□

Forth Limit Law:
For each $\omega \in \Omega$ the ω-restricted hyperbolic zoom at m-almost every $\xi \in J(T)$ has the property that

$$\limsup_{k\to\infty} \frac{\log\left(r_{i_k}(\xi) \,/\, r_{i_k+1}(\xi)\right)}{\log i_k} = \frac{1 + p(\omega)}{h + (h-1)p(\omega)}.$$

Proof:
For $\omega \in \Omega$ and $\xi \in J_r(T)$, we already saw during the proof of Theorem 3.1 that for $k \in \mathbb{N}$ we have that

$$\frac{r_{i_k}(\xi)}{r_{i_k+1}(\xi)} \asymp (n_{i_k+1}(\xi) - n_{i_k}(\xi))^{(1+p(\omega))/p(\omega)}.$$

Combining this estimate and the Third Limit Law, it follows for m-almost all $\xi \in J(T)$ that

$$\begin{aligned} \limsup_{k\to\infty} \frac{\log\left(r_{i_k}(\xi)\,/\,r_{i_k}(\xi)\right)}{\log i_k} &= \limsup_{k\to\infty} \frac{1+p(\omega)}{p(\omega)} \; \frac{\log(n_{i_k+1}(\xi) - n_{i_k}(\xi))}{\log i_k} \\ &= \frac{1+p(\omega)}{h+(h-1)p(\omega)}. \end{aligned}$$

□

Theorem 4.3 *(The Khintchine Limit Law for parabolic rational maps)*
The hyperbolic zoom at m-almost every $\xi \in J(T)$ has the property that

$$\limsup_{j\to\infty} \frac{\log\left(r_j(\xi)\,/\,r_{j+1}(\xi)\right)}{\log\log \frac{1}{r_j(\xi)}} = \frac{1+p_{\max}}{h+(h-1)p_{\max}}.$$

Proof:
Observe that for m-almost all $\xi \in J(T)$ we have that

$$\begin{aligned} \lim_{j\to\infty} \frac{-\log r_j(\xi)}{j} &= \lim_{j\to\infty} \frac{\log|(T^{n_j(\xi)})'(\xi)|}{j} \\ &= \lim_{j\to\infty} \frac{\log|(T^{N_j(\xi)})'(\xi)|}{j} \\ &= \lim_{j\to\infty} \frac{\log|((T^*)^j)'(\xi)|}{j} \\ &= \chi; \end{aligned}$$

where the latter equality follows from the Birkhoff Ergodic Theorem, using the fact that $(J(T), T^*, m^*)$ is an ergodic system and that

$$\chi = \int_{J(T)} \log|(T^*)'(\xi)| \; dm^*(\xi) < \infty.$$

Hence, we have that

$$\lim_{j\to\infty} \frac{\log\log \frac{1}{r_j(\xi)}}{\log j} = 1.$$

Now the theorem follows by combining this equality and the Forth Limit Law, and noting that

$$\max_{\omega\in\Omega} \frac{1+p(\omega)}{h+(h-1)p(\omega)} = \frac{1+p_{\max}}{h+(h-1)p_{\max}}.$$

□

Corollary 4.4 *For the conformal fluctuation function* ϕ *of the* h*-conformal measure* m *associated to a parabolic rational map* T *the following holds.*

(i) For $h=1$*, we have for all* $\xi \in J_r(T)$ *and positive* $r < \operatorname{diam}(J(T))$ *that*

$$\phi(\xi, r) \asymp 1.$$

(ii) For $h<1$*, we have for* m*-almost every* $\xi \in J(T)$ *that*

$$\limsup_{r\to 0} \frac{\log \phi(\xi,r)}{\log\log\frac{1}{r}} = \frac{(1-h)p_{\max}}{h+(h-1)p_{\max}}.$$

(iii) For $h>1$*, we have for* m*-almost every* $\xi \in J(T)$ *that*

$$\liminf_{r\to 0} \frac{\log \phi(\xi,r)}{\log\log\frac{1}{r}} = \frac{(1-h)p_{\max}}{h+(h-1)p_{\max}}.$$

Proof:
The statement (i) of the corollary is an immediate consequence of Theorem 3.1. In order to prove the statement (ii), let $\xi \in J_r(T)$ and $r>0$ sufficiently small be given. Without loss of generality we may assume that $r_{j+1}(\xi) \le r < r_j(\xi)$ and that $T^{n_j(\xi)+1}(\xi) \in U_\omega$, for some $\omega \in \Omega$. For r in this range, an elementary calculation gives that the maximal value of $\phi(\xi, r)$ is achieved for r comparable to $r_{j,\max}(\xi) := r_j(\xi)\left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{1/(1+p(\omega))}$. For this value of r we have that

$$\phi(\xi, r_{j,\max}(\xi)) \asymp \left(\frac{r_j(\xi)}{r_{j+1}(\xi)}\right)^{(1-h)p(\omega)/(1+p(\omega))}.$$

We have seen above in the proof of the Khintchine law that without loss of generality, for m-almost all $\xi \in J(T)$ it is sufficient to concentrate on the visits of the forward orbit of ξ to those $\omega \in \Omega$ for which $p(\omega) = p_{\max}$. ¿From this it follows that for all $\epsilon > 0$ and for m-almost all $\xi \in J(T)$ we eventually have that

$$\frac{(1-\epsilon)\,(1+p(\omega))}{h+(h-1)p(\omega)}\,\log\log\frac{1}{r_j(\xi)} \le_{i.o.} \log\frac{r_j(\xi)}{r_{j+1}(\xi)} \le \frac{(1+\epsilon)\,(1+p(\omega))}{h+(h-1)p(\omega)}\,\log\log\frac{1}{r_j(\xi)};$$

(where ' $\le_{i.o.}$ ' indicates that the inequality holds 'infinitely often', i.e. for some infinite subsequence $(r_{j_i}/r_{j_i+1})_i$). Hence, the above estimate implies that

$$\left(\log \frac{1}{r_j(\xi)}\right)^{(1-\epsilon)(1-h)p_{\max}/(h+(h-1)p_{\max})}$$
$$\ll_{i.o.} \phi(\xi, r_{j,\max}(\xi)) \ll \left(\log \frac{1}{r_j(\xi)}\right)^{(1+\epsilon)(1-h)p_{\max}/(h+(h-1)p_{\max})}.$$

This proves the statement (ii) of the corollary. The statement (iii) of the corollary follows from a similar argument, and we omit its proof.

□

Recall that for a parabolic rational maps with Julia set of Hausdorff dimension $h \ge 1$, it is known that the associated h-conformal measure is a multiple of the h-dimensional Hausdorff measure (cf. [?]). We are now in the position to derive a refinement of this description of the geometric nature of the h-conformal measure. Namely, using the latter corollary, we have the following statements concerning its relationship to the *packing measure* $\mathcal{P}_{\psi_\rho}$ and *Hausdorff measure* $\mathcal{H}_{\psi_\rho}$ with respect to the dimension function ψ_ρ. Where ψ_ρ is given for $\rho \in \mathbb{R}$ and positive r by

$$\psi_\rho(r) := r^h \left(\log \frac{1}{r}\right)^{(1+\rho)(1-h)p_{\max}/(h+(h-1)p_{\max})}.$$

We remark that for $h < 1$ similar results were obtained in [?].

Corollary 4.5 *For a parabolic rational map T with Julia set $J(T)$ of Hausdorff dimension h and with h-conformal measure m, we have the following table. Where we have used '$<<$' to denote absolute continuity between two measures.*

ρ vs. h	$h<1$	$h>1$
$\rho > 0$	$m << \mathcal{H}_{\psi_\rho}$ *and* $\mathcal{H}_{\psi_\rho}(J(T)) = \infty$	$\exists E_\rho, m(E_\rho) = 1$ *s.t.* $\mathcal{P}_{\psi_\rho}(E_\rho) = 0$
$\rho \le 0$	$\exists F_\rho, m(F_\rho) = 1$ *s.t.* $\mathcal{H}_{\psi_\rho}(F_\rho) = 0$	$m << \mathcal{P}_{\psi_\rho}$ *and* $\mathcal{P}_{\psi_\rho}(J(T)) = \infty$

5 Some 'fractal' consequences

In this section we shall see that a combination of the uniform estimate for the conformal measure (Theorem 3.1) and the Khintchine Limit Law (Theorem 4.3) immediately gives rise to a comprehensive understanding of the coarse geometry arising from parabolic rational maps. In particular, we derive various interesting facts concerning the fractal nature of their Julia sets and their associated conformal measures. Note that similar results were obtained in the context of limit sets of geometrically finite Kleinian groups (see [?]).

We start our list of immediate applications with the following result. We remark that this result is not new. In fact, it is known that the Julia set of an arbitrary rational map is uniformly perfect (see e.g. [?]).

Lemma 5.1 *The Julia set of a parabolic rational map is uniformly perfect.*

Proof:
This is an immediate consequence of Theorem 3.1. Namely, recall that there exist constants $c_1, c_2 > 0$ such that for $\xi \in J(T)$ and $r < \operatorname{diam}(J(T))$ we have that $c_1 \, r^h \, \phi(\xi, r) \leq m(B(\xi, r)) \leq c_2 \, r^h \, \phi(\xi, r)$.

Then, choose a positive constant c_0 such that for all $\xi \in J(T)$ and $r < \operatorname{diam}(J(T))$ it holds that

$$c_0 < \min \left\{ 1; \left(\frac{c_1 \, \phi(\xi, r)}{c_2 \, \phi(\xi, c_0 r)} \right)^{1/h} \right\}.$$

Now, c_0 depends only on T, and we have that

$$\begin{aligned} m(B(\xi, c_0 r)) &\leq r^h \, c_2 \, c_0^h \;\; \phi(\xi, c_0 r) \\ &< r^h \, c_1 \, \phi(\xi, r) \\ &\leq m(B(\xi, r)). \end{aligned}$$

In particular this gives that the annulus $B(\eta, r) \setminus B(\eta, c_0 r)$ and $J(T)$ are of non-empty intersection, and hence the lemma follows.

□

Recall that a measure ν on $\mathbb{R}^N$ is called a *doubling measure* if and only if for some constant $c > 1$ we have that $\nu(B(\xi, cr)) \ll \nu(B(\xi, r))$, for all $\xi \in$ *supp*(ν) and uniformly for all sufficiently small $r > 0$. Now, the following is an immediate consequence of Theorem 3.1.

Lemma 5.2 *The h-conformal measure of a parabolic rational map whose Julia set has Hausdorff dimension h is a doubling measure.*

Also, recall that in fractal geometry a measure ν on $\mathbb{R}^N$ is called *regular* if and only if $\liminf_{r \to 0} \frac{\log \nu(B(\xi, r))}{\log r} = \limsup_{r \to 0} \frac{\log \nu(B(\xi, r))}{\log r}$ for ν-almost all $\xi \in$ *supp*(ν) (cf. [?]). Applying Thorem 3.1 and the Corollary 4.4, we obtain the following immediate result.

Lemma 5.3 *The h-conformal measure of a parabolic rational map whose Julia set has Hausdorff dimension h is a regular measure.*

It has been shown in [?], [?] that if h denotes the Hausdorff dimension ($\dim_H$) of the Julia set $J(T)$ of a parabolic rational map T, then we have that the packing dimension ($\dim_P$) and the box-counting dimension ($\dim_B$) of $J(T)$ coincide and are equal to h, i.e.

$$\dim_H(J(T)) = \dim_P(J(T)) = \dim_B(J(T)) = h \quad .$$

We now add to these results the following characterisation of the fractal nature of the h-conformal measure supported on $J(T)$.

Proposition 5.4 *Let m denote the h-conformal measure of a parabolic rational map T with Julia set of Hausdorff dimension h. For $q \neq 0$, the generalised q-th Renyi dimension ($\mathcal{R}_{m,q}$), the q-th logarithmic index ($\mathcal{L}_{m,q}$) and the information dimension ($\mathcal{I}_m$) of $J(T)$ are given by the following.*

- $\mathcal{I}_m(J(T)) = \mathcal{R}_{m,q}(J(T)) = h;$
- $\mathcal{L}_{m,q+1}(J(T)) = qh.$

Proof.
Using Theorem 3.1 and Corollary 4.4, we obtain for $q \neq 0$ that

$$\lim_{r\to 0} \frac{\log \int_{J(T)} r^{hq} \left(\log \frac{1}{r}\right)^{q(1+\epsilon)(1-h)p_{\max}/(h+(h-1)p_{\max})} dm(\xi)}{q \log r}$$
$$= h + \lim_{r\to 0} \frac{\log \int_{J(T)} \left(\log \frac{1}{r}\right)^{q(1+\epsilon)(1-h)p_{\max}/(h+(h-1)p_{\max})} dm(\xi)}{q \log r} = h;$$

This implies that

$$h = \mathcal{R}_{m,q}(J(T)) \left(:= \lim_{r\to 0} \frac{\log \int_{J(T)} m(B(\xi,r))^q dm(\xi)}{q \log r} \right).$$

Similarly, we compute

$$h = \mathcal{I}_m(J(T)) \left(:= \lim_{r\to 0} \frac{\int_{J(T)} \log m(B(\xi,r)) dm(\xi)}{\log r} \right).$$

For the definition of the q-th logarithmic index $\mathcal{L}_{m,q}$ we refer to [?]. Since m is a doubling measure, we may apply [?] (Theorem 2.24), from which we deduce that $\mathcal{L}_{m,q+1}(J(T)) = qh$.

□

Finally, we can draw the following conclusion concerning the 'fractality' of the conformal measure. For this, first recall that by definition (cf.[?], [?]), a measure ν on $\mathbf{R}^N$ is called *fractal* if and only if $\mathcal{I}_\nu < \dim_H(supp\ \nu)$.

Corollary 5.5 *For a parabolic rational map whose Julia set has Hausdorff dimension h, the associated h-conformal measure is not a fractal measure.*

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B.O. Stratmann,
Mathematical Institute,
University of St Andrews,
St Andrews KY16 9SS,
Scotland.
e-mail: bos@maths.st-and.ac.uk

M. Urbański,
Dept. of Mathematics,
University of North Texas,
Denton, TX 76203-5118,
USA.
e-mail: urbanski@unt.edu