Jarník and Julia;
a Diophantine analysis for parabolic rational maps

B.O. Stratmann, M. Urbański *

Abstract

In this paper we derive a Diophantine analysis for Julia sets of parabolic rational maps. We generalise two theorems of Dirichlet and Jarník in number theory to the theory of iterations of these maps. On the basis of these results, we then derive a 'weak multifractal analysis' of the conformal measure naturally associated with a parabolic rational map. The results in this paper contribute to a further development of Sullivan's famous dictionary translating between the theory of Kleinian groups and the theory of rational maps.

1 Statement of main results

In this paper we derive a Diophantine analysis for Julia sets $J(T)$ of parabolic rational maps $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We generalise two classical number theoretical theorems of Dirichlet and Jarník to the theory of iterations of rational maps. We then show that these results embed in the concept of conformal measures, where they admit a 'weak multifractal analysis' of the $\dim_H(J(T))$-conformal measure which is naturally associated with the dynamical system $(J(T), T)$. Also, a combination of the results in this paper with those for Kleinian groups which we obtained in [18], [21] and [23] adds another interesting chapter to Sullivan's famous 'Julia-Klein dictionary' [24] (see also [13], [22]).
Recall that for parabolic rational maps it is well-known that $J(T) = J_r(T) \cup J_p(T)$, i.e. the Julia set $J(T)$ admits a disjoint decomposition into the radial Julia set $J_r(T)$ and the countable set of pre-parabolic points $J_p(T) := \bigcup_{\omega\in\Omega}\bigcup_{n\in\mathbf{N}} T^{-n}(\omega)$, where Ω denotes the set of rationally indifferent periodic points ([26], [22]). For each $\omega \in \Omega$, we fix a standard neighbourhood $B(\omega, r_\omega)$ and consider, roughly speaking, all its holomorphic, inverse iterates $B(c(\omega), r_{c(\omega)})$. We call these balls *canonical balls* (see section 2, for the precise definition).
A major aim of this paper will be the fractal analysis of the *Jarník-Julia sets.* For $\omega \in \Omega$ and $\sigma > 0$, these sets are '$\limsup$ sets' which are defined by

$$\mathcal{J}_\sigma^\omega(T) := \bigcap_{n\in\mathbf{N}} \bigcup_{r_{c(\omega)}<1/n} B\left(c(\omega), r_{c(\omega)}^{1+\sigma}\right) \quad \text{and} \quad \mathcal{J}_\sigma(T) := \bigcup_{\omega\in\Omega} \mathcal{J}_\sigma^\omega(T).$$

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We call $\mathcal{J}_\sigma(T)$ the *σ-Jarník-Julia set* and $\mathcal{J}_\sigma^\omega(T)$ the *(σ,ω)-Jarník-Julia set.*
The following theorem is our first main result. The theorem is the natural generalisation to Julia sets of Jarník's Theorem in number theory ([12]) concerning the Hausdorff dimension of well-approximable irrational numbers (see section 5). (Note, analogous results for limit sets of geometrically finite Kleinian groups with parabolic elements are obtained in [18],[21],[10].)

Theorem 1
Let T be a parabolic rational map with Julia set of Hausdorff dimension h. For $\omega \in \Omega$ and $\sigma > 0$, the Hausdorff dimension $(\dim_H)$ of the σ-Jarník-Julia set and the (σ,ω)-Jarník-Julia set are determined by the following, where $p(\omega)$ denotes the number of attracting petals associated to ω, and $p_{min} := \min_{\eta\in\Omega} p(\eta)$.

- *If $h < 1$, then* $\dim_H(\mathcal{J}_\sigma(T)) = \frac{h}{1+\sigma}$.
- *If $h \geq 1$, then* $\dim_H(\mathcal{J}_\sigma^\omega(T)) = \begin{cases} \frac{h}{1+\sigma} & \textit{for} \quad \sigma \geq h-1 \\ \frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))} & \textit{for} \quad \sigma < h-1, \end{cases}$

  and hence, we have in particular that

$$\dim_H(\mathcal{J}_\sigma(T)) = \begin{cases} \frac{h}{1+\sigma} & \textit{for} \quad \sigma \geq h-1 \\ \frac{h+\sigma p_{min}}{1+\sigma(1+p_{min})} & \textit{for} \quad \sigma < h-1. \end{cases}$$

An essential ingredient in the proof of this theorem is to show that, much as for Kleinian groups ([23]), for parabolic rational maps there exists a generalisation of Dirichlet's Theorem in number theory (see section 3). Roughly speaking, this result shows that the Julia set admits economical, arbitrarily fine coverings and packings by finitely many canonical balls whose radii are diminished in a 'dynamically controlled' way. In fact, this generalisation implicitly reveals the 'hidden 3-dimensional dynamics' of the rational map. For the explicit statement of this result we refer to section 3, Theorem 3.
In our final result we apply Theorem 1 and derive some interesting insight into the multifractal nature of the associated h-conformal measures m. It is well-known that the scaling behaviour of m fluctuates between two extreme power laws, namely on the one hand the 'hyperbolic law' which is realised with the power h on a sequence of shrinking balls around elements in $J_r(T)$, and on the other hand the 'parabolic law' which for each $\omega \in \Omega$ is eventually realised uniformly with the power $h+p(\omega)(h-1)$ around the backward orbits of ω. Now, our weak multifractal analysis shows that these two extreme scaling behaviours of m are in fact partial aspects of certain continuous spectra of this measure. In order to state this application more precisely, we recall from [21] the following notion of the *weak singularity spectra* of a measure.

Definition
Let ν denote a Borel probability measure on $\mathbb{R}^n$. For $\theta > 0$, we define the following sets.

$$\mathcal{I}^\theta(\nu) := \{\xi \in supp(\nu) : \liminf_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \leq \theta\} \quad ;$$

$$\mathcal{I}_\theta(\nu) := \{\xi \in supp(\nu) : \liminf_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \geq \theta\} \quad ;$$

$$\mathcal{S}^\theta(\nu) := \{\xi \in supp(\nu) : \limsup_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \leq \theta\} \quad ;$$

$$\mathcal{S}_\theta(\nu) := \{\xi \in supp(\nu) : \limsup_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \geq \theta\} \quad .$$

The collections of Hausdorff dimensions of these sets, for $\theta > 0$ *, are referred to as the weak singularity spectra of* ν *.*

The following theorem will be the final result in this paper. The theorem gives a complete discription of the weak singularity spectra of the h-conformal measure associated with a parabolic rational map. (Note that for limit sets of geometrically finite Kleinian groups with parabolic elements the weak singularity spectra of the Patterson measure was derived in [21] (see also [19]).)

Theorem 2
The weak singularity spectra of the h *-conformal measure* m *of a parabolic rational map with Julia set of Hausdorff dimension* h *are determined by the following, where we have set* $p_{max} := \max_{\omega\in\Omega} p(\omega)$ *.*

- *If* $h = 1$ *, then the weak singularity spectra of* m *are trivial. Namely, in this case we have for all* $\xi \in J(T)$ *that*

$$\lim_{r\to 0} \frac{\log m(B(\xi,r))}{\log r} = h.$$

- *If* $h < 1$ *, then*

$$\dim_H(\mathcal{I}^\theta(m)) = \begin{cases} 0 & \text{for} \quad 0 < \theta \leq h + (h-1)p_{max} \\ \frac{h(\theta-(h+(h-1)p_{max}))}{(1-h)p_{max}} & \text{for} \quad h+(h-1)p_{max} < \theta < h \\ h & \text{for} \quad \theta \geq h \end{cases}$$

$$\dim_H(\mathcal{S}_\theta(m)) = \begin{cases} h & \text{for} \quad 0 < \theta \leq h \\ 0 & \text{for} \quad \theta > h. \end{cases}$$

- *If* $h > 1$ *, then*

$$\dim_H(\mathcal{S}_\theta(m)) =$$

$$\begin{cases} h & \text{for} \quad 0 < \theta \leq h \\ \frac{(h-1)(h+(h-1)p_{max})}{(\theta-1)p_{max}} - \frac{h-p_{max}}{p_{max}} & \text{for} \quad h < \theta < \frac{h(h+(h-1)p_{max})-(h-1)p_{max}}{h} \\ \frac{h(h+(h-1)p_{max}-\theta)}{(h-1)p_{max}} & \text{for} \quad \frac{h(h+(h-1)p_{max})-(h-1)p_{max}}{h} \leq \theta < h+(h-1)p_{max} \\ 0 & \text{for} \quad \theta \geq h+(h-1)p_{max}. \end{cases}$$

$$\dim_H(\mathcal{I}^\theta(m)) = \begin{cases} 0 & \text{for} \quad 0 < \theta < h \\ h & \text{for} \quad \theta \geq h. \end{cases}$$

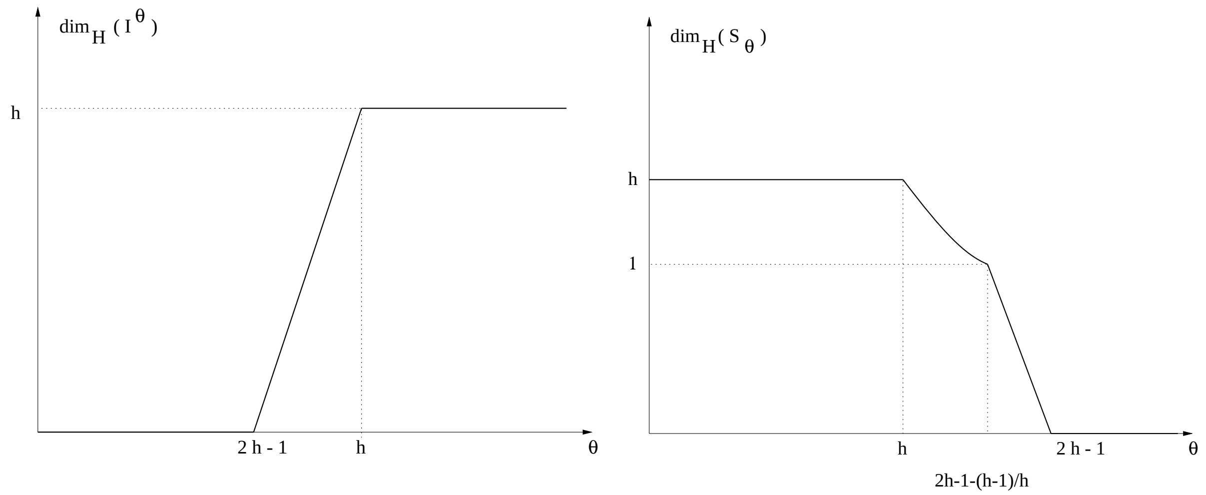

Figure 1: The most interesting spectra for $p_{max} = 1$

- *For* $h < 1$ *and* $h > 1$ *, we have that*

$$\dim_H(\mathcal{I}_\theta(m)) = \begin{cases} h & \textit{for} \quad 0 < \theta \leq h \\ 0 & \textit{for} \quad \theta > h. \end{cases}$$

$$\dim_H(\mathcal{S}^\theta(m)) = \begin{cases} 0 & \textit{for} \quad 0 < \theta < h \\ h & \textit{for} \quad \theta \geq h. \end{cases}$$

Remark: At the current state non of the existing general formalism in Fractal Geometry and Dynamical Systems allows to deduce the results which we obtain in this paper. For instance, if for $h \neq 1$ we combine our estimates of the weak singularity spectra and the fact that m has a flat Rényi dimension spectrum equal to h (cf. [22]), then we see that m can not be analysed by the currently existing multifractal formalism. Furthermore, for hyperbolic rational maps T one can define σ-Jarník-Julia sets $\mathcal{J}_\sigma^{hyp}(T)$ in a similar way as we did in this paper. Of course, in this expanding case the canonical balls are centred at elements of the uniformly-radial Julia set[1]. In this purely hyperbolic case we always have that $\dim_H(\mathcal{J}_\sigma^{hyp}(T)) = h/(1+\sigma)$, and in terms of the thermodynamical formalism this solution represents the (only) zero of the associated pressure function (cf. [9]). Now, one might suspect that the most natural extension of this thermodynamical interpretation to the parabolic case is that $\dim_H(\mathcal{J}_\sigma(T))$ is equal to the infimum of the set of all zeros of the pressure function. But, the results in this paper show that

[1]see section 6, and in particular the footnotes in there.

this certainly can not be the right extension. Namely, for $h > 1$ and $\sigma < h - 1$, Theorem 1 implies that if $\phi_\sigma := (1+\sigma)\log|T'|$ then $\dim_H(\mathcal{J}_\sigma(T))$ is strictly less than the least zero of the pressure function $P(\phi_\sigma)$.

2 Preliminaries

Julia sets revisited:-

As already mentioned in the introduction, throughout the paper $J(T)$ denotes the Julia set of a parabolic rational map T. For an introduction into the basic theory of iteration of rational maps we refer to [3],[4], [14]. Without loss of generality, we may assume that $J(T)$ is a compact subset of $\mathbb{C}$. Let $\Omega(T)$ denote the non-empty, finite set of rationally indifferent periodic points (parabolic points). If $\Omega_0(T) := \{\xi \in \Omega : T(\xi) = \xi, T'(\xi) = 1\}$, then (since $J(T^n) = J(T)$ for every $n \in \mathbb{N}$) we may assume without loss of generality that $\Omega_0(T) = \Omega(T)$.

Recall that for each $\omega \in \Omega$ we can find a ball $B(\omega, r_\omega)$ with centre ω and sufficiently small radius r_ω, such that on $B(\omega, r_\omega)$ there exists a unique holomorphic inverse branch T_ω^{-1} of T with the property that $T_\omega^{-1}(\omega) = \omega$. For the iterates of this branch on $B(\omega, r_\omega) \cap J(T) \setminus \{\omega\}$, the following two facts are obtained in [2], [7].

(LBP) *('local behaviour around parabolic fixed points')*
For $\xi \in B(\omega, r_\omega) \cap J(T) \setminus \{\omega\}$ and $n \in \mathbb{N}$ we have that

- $|\omega - T_\omega^{-n}(\xi)| \asymp 1/n^{1/p(\omega)}$;
- $|(T_\omega^{-n})'(\xi)| \asymp 1/n^{(1+p(\omega))/p(\omega)}$,

where the 'comparability constants' are dependent on the distance of the chosen point ξ from the parabolic point ω.

Recall that the set of *pre-parabolic points* $J_p(T)$ is given by $J_p(T) := \bigcup_{k=0}^{\infty} T^{-k}(\Omega(T))$, and that for parabolic rational maps the *radial Julia set* $J_r(T)$ is equal to $J(T) \setminus J_p(T)$ (cf. [26], [5], [22]). Also, here there exists a constant $\rho > 0$ such that to each $\xi \in J_r(T)$ we can associate a unique maximal sequence of integers $n_j(\xi)$ such that the inverse branches $T_\xi^{-n_j(\xi)}$ are well defined on $B(T^{n_j(\xi)}(\xi), \rho)$. Then, if we define $r_j(\xi) := |(T^{n_j(\xi)})'(\xi)|^{-1}$, the sequence of 'radii' $(r_j(\xi))_{j\in\mathbf{N}}$ is called the *hyperbolic zoom at* ξ. Similarly, to each $\xi \in J_p(T)$ we may associate its *terminating hyperbolic zoom* $(r_j(\xi))_{j=1,\dots,l(\xi)}$ (cf. [22]).

Furthermore, in the following, the concept 'canonical ball' will be crucial. For $\omega \in \Omega$, let $I(\omega) := T^{-1}(\{\omega\}) \setminus \{\omega\}$. Then, for each integer $n \geq 0$ and $\omega \in \Omega$, we define the canonical radius r_ξ at $\xi \in T^{-n}(I(\omega))$ by

$$r_\xi := |(T^n)'(\xi)|^{-1},$$

and call the ball $B(\xi, r_\xi)$ the *canonical ball* at ξ. Note that the canonical radius at ξ is comparable to the last element in the terminating hyperbolic zoom at ξ.

Conformal measures revisited:-

Recall from [2], [5] and [6] that for a parabolic rational map T there exists a unique h-conformal measure m supported on $J(T)$ (where h denotes the Hausdorff dimension of $J(T)$), i.e. a probability measure with the property that for each Borel set $F \subset J(T)$ on which T is injective, we have that

$$m(T(F)) = \int_F |T'(\xi)|^h \, dm(\xi).$$

In [22] we derived the following 'geometric formula' for the h-conformal measure, which describes the decay of the measure uniformly around arbitrary points in $J(T)$.

(GF) *('geometric formula for the h-conformal measure')*
With the notation above, there exists a function $\phi : J(T) \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\xi \in J(T)$ and for every positive $r < \operatorname{diam}(J(T))$ we have that

$$m(B(\xi, r)) \asymp r^h \cdot \phi(\xi, r) \quad .$$

The values of the conformal fluctuation function ϕ are determined, for positive $r < \operatorname{diam}(J(T))$, by the following.

- *If $\xi \in J_r(T)$, and r relates to the hyperbolic zoom at ξ such that $r_{j+1}(\xi) \leq r < r_j(\xi)$ and such that $T^k(\xi) \in B(\omega, r_\omega)$, for all $k \in (n_j(\xi), n_{j+1}(\xi)]$ and for some $\omega \in \Omega(T)$, then*
$$\phi(\xi, r) \asymp \begin{cases} \left(\frac{r}{r_j(\xi)}\right)^{(h-1)p(\omega)} & \text{for} \quad r > r_j(\xi) \left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{1/(1+p(\omega))} \\ \left(\frac{r_{j+1}(\xi)}{r}\right)^{(h-1)} & \text{for} \quad r \leq r_j(\xi) \left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{1/(1+p(\omega))} \end{cases} .$$
- *If $\xi \in J_p(T)$ and r exceeds the canonical radius r_ξ, then $\phi(\xi, r)$ is determined as above in the radial case by means of the terminating hyperbolic zoom at ξ. Otherwise, if $r \leq r_\xi$ and ξ is a pre-image of $\omega \in \Omega$, then*
$$\phi(\xi, r) \asymp \left(\frac{r}{r_\xi}\right)^{(h-1)p(\omega)} \quad .$$

3 The Julia set in the spirit of Dirichlet

In this section we give for parabolic rational maps a generalisation of a classical theorem in the theory of Diophantine approximation due to Dirichlet. This result will provide us with economical, finite coverings and packings of the Julia set which are closely connected to the 'hidden 3-dimensional dynamics' of the rational map. In order to motivate our generalisation, we first recall the classical Dirichlet theorem.

Dirichlet's Theorem: *There exists a universal constant $\kappa > 0$ such that for each sufficiently small $\alpha > 0$ the following holds. For every $x \in \mathbb{R}^+$ there exist $p, q \in \mathbb{N}$ co-prime with $1/q^2 > \alpha$, such that*

$$\left| x - \frac{p}{q} \right| < \kappa \sqrt{\alpha / q^2}.$$

We now generalise this theorem to the situation of a parabolic rational map T. The reader is asked to recall the notion of a canonical ball given in the previous section. For any small number $\alpha > 0$, we associate to each canonical ball $B(c(\omega), r_{c(\omega)})$, for which $r_{c(\omega)} > \alpha$ its α*-canonical Dirichlet ball* $B(c(\omega), r_{c(\omega),\alpha})$, which is determined by

$$r_{c(\omega),\alpha} := \alpha^{1/(1+p(\omega))}\ r_{c(\omega)}^{p(\omega)/(1+p(\omega))}.$$

Using this notation, we now state our generalisation of the Dirichlet theorem. (Note that this result has already been announced in [18], and also that for geometrically finite groups a similar generalisation of the Dirichlet Theorem was derived in [23].)

Theorem 3 *Let T be a parabolic rational map. There exist universal constants $\kappa_c, \kappa_p, \alpha_0 > 0$, depending only on T, such that for each $\omega \in \Omega$ and for each $0 < \alpha < \alpha_0$ the following holds.*

(i) The family $\left\{B(c(\omega), \kappa_p\, r_{c(\omega),\alpha}) : r_{c(\omega)} \geq \alpha\right\}$ provides a packing of $J(T)$.

(ii) The family $\left\{B(c(\omega), \kappa_c\, r_{c(\omega),\alpha}) : r_{c(\omega)} \geq \alpha\right\}$ provides a covering of $J(T)$.

Proof:
Proof of (i):- For this it is sufficient to show that for all $\omega \in \Omega$ and for sufficiently small $\alpha, \kappa > 0$ the family

$$\mathcal{F}(\omega, \alpha, \kappa) \cup \{B(\omega, r_{\omega,\alpha})\}$$

provides a packing of $J(T)$. Here we have set

$$\mathcal{F}(\omega, \alpha, \kappa) := \{B(c(\omega), \kappa r_{c(\omega),\alpha}) : c(\omega) \in \bigcup_{n\geq 0} T^{-n}(I(\omega)), r_{c(\omega)} \geq \alpha\}.$$

For the following we shall assume that $\delta > 0$ is chosen sufficiently small such that $B(\omega, \delta) \cap B(\eta, \delta) = \emptyset$, for all distinct $\omega, \eta \in \Omega$. Also, recall that for each $y \in J(T) \setminus B(\Omega, \delta)$, $n \geq 0$ and $x \in T^{-n}(y)$ there exists a holomorphic inverse branch $T_x^{-n} : B(y, 2\theta) \to \hat{\mathbb{C}}$ of T^n such that $T_x^{-n}(y) = x$. Let us fix $\omega \in \Omega$ and $\alpha > 0$, where α will get adjusted throughout the construction. For convenience we write $p = p(\omega)$. Suppose that $\mathcal{F}(\omega, \alpha, \kappa)$ is not a packing. Then we have, for some positive $k \leq n$ and for some $x \in T^{-k}(I(\omega))$ and $y \in T^{-n}(I(\omega))$, that there exists

$$z \in B\left(x, \kappa\alpha^{1/(1+p)}|(T^k)'(x)|^{-p/(1+p)}\right) \cap B\left(y, \kappa\alpha^{1/(1+p)}|(T^n)'(y)|^{-p/(1+p)}\right) \qquad (1)$$

with the property that $|(T^k)'(x)|^{-1}$ and $|(T^n)'(y)|^{-1}$ both exceed α. Hence, our aim will be to show the coincidence of the two balls

$$B\left(x, \kappa\alpha^{1/(1+p)}|(T^k)'(x)|^{-p/(1+p)}\right) \text{ and } B\left(y, \kappa\alpha^{1/(1+p)}|(T^n)'(y)|^{-p/(1+p)}\right).$$

Using Koebe's $1/4$-distortion theorem (cf. [11]), we have that

$$\begin{aligned}
T_x^{-k}(B(T^k(x),\theta)) &\supset B\left(x, \frac{\theta}{4}\,|(T^k)'(x)|^{-1}\right) \\
&= B\left(x, \frac{\theta}{4}\,|(T^k)'(x)|^{-1/(1+p)}|(T^k)'(x)|^{-p/(1+p)}\right) \\
&\supset B\left(x, \frac{\theta}{4}\,\alpha^{1/(1+p)}|(T^k)'(x)|^{-p/(1+p)}\right) \\
&\supset B\left(x, \kappa\alpha^{1/(1+p)}|(T^k)'(x)|^{-p/(1+p)}\right), \qquad (2)
\end{aligned}$$

where in the last inclusion we assumed that $\kappa \leq \theta/4$. If $k = n$ then we have either that the two balls in (1) coincide (in the case when $x = y$) and we are done, or that they are disjoint (when $x \neq y$), which contradicts the fact that z belongs to both of these balls, and hence we are done as well. Thus, we may assume that $k < n$. Using (1) and applying Koebe's distortion theorem, we get, with K the positive constant originating from this theorem for the 'scale $1/2$' ([11]), that

$$\begin{aligned}
|T^k(z) - T^k(x)| &\leq K\kappa\alpha^{1/(1+p)}|(T^k)'(x)|^{-p/(1+p)}|(T^k)'(x)| \\
&= K\kappa\alpha^{1/(1+p)}|(T^k)'(x)|^{1/(1+p)}.
\end{aligned}$$

Hence, we have that

$$|T^{k+1}(z) - \omega| = |T(T^k(z)) - T(T^k(x))| \leq K\,||T'||\,\kappa\,(\alpha\,|(T^k)'(x)|)^{1/(1+p)}. \qquad (3)$$

Since (2) is obviously true with k replaced by n, an application of Koebe's distortion theorem gives that

$$\begin{aligned}
|(T^n)'(y)|^{-1} &\leq K\,|(T^n)'(z)|^{-1} = K\,|(T^{n-k})'(T^k(z))|^{-1}\cdot|(T^k)'(z)|^{-1} \\
&\leq K^2\,|(T^{n-k})'(T^k(z))|^{-1}\cdot|(T^k)'(x)|^{-1} \\
&\leq K^2\,||T'||\cdot|(T^{n-k-1})'(T^{k+1}(z))|^{-1}\,|(T^k)'(x)|^{-1}. \qquad (4)
\end{aligned}$$

It follows from (1) and (2) applied with k replaced by n that $T^{n-k-1}(T^{k+1}(z)) = T^n(z) \in B(T^n(y),\theta)$. Since $T^n(y) \in I(\omega)$, assuming that θ and δ are taken small enough, we may therefore conclude that $T^{n-k-1}(T^{k+1}(z)) \notin B(\Omega,\delta)$. Hence there exists a least l with $0 \leq l \leq n-k-1$ such that $T^l(T^{k+1}(z)) \notin B(\Omega,\delta)$. Since $T^l(T^{k+1}(z))$ and $T^{n-k-1-l}(T^{k+1+l}(z)) = T^n(z)$ are not in $B(\Omega,\delta)$, there exists an integer $t \geq 0$ such that $T^{n-k-1-l}(T^{k+1+l}(z)) = T^{*t}(T^{k+1+l}(z))$, where T^* denotes the jump transformation defined in [6] (also, cf. [2], [22] and [16]). By [6], the map T^* is expanding, which means that there exist constants $C > 0$ and $\gamma > 1$ such that $|(T^{*s})'(v)| \geq C\,\gamma^s$, for all $s \in \mathbb{N}$ and $v \in J_r(T)$. Hence, we have that

$$\begin{aligned}
|(T^{n-k-1})'(T^{k+1}(z))| &= |(T^l)'(T^{k+1}(z))|\cdot|(T^{n-k-1-l})'(T^{k+1+l}(z))| \\
&= |(T^l)'(T^{k+1}(z))|\cdot|(T^{*t})'(T^{k+1+l}(z))| \\
&\geq C\gamma^t|(T^l)'(T^{k+1}(z))| \geq C|(T^l)'(T^{k+1}(z))|.
\end{aligned}$$

Using (3) and (LBP), it now follows, for some universal constant $C_1 > 0$, that

$$\begin{aligned}|(T^{n-k-1})'(T^{k+1}(z))| &\geq C\, C_1\, |T^{k+1}(z) - \omega|^{-(1+p)} \\ &\geq C\, C_1\, (K\, ||T'||\, \kappa)^{-(1+p)}\, \alpha^{-1}\, |(T^k)'(x)|^{-1}.\end{aligned}$$

Combining this estimate and (4), we obtain with $D := K^{p+3}||T'||^{p+2}(CC_1)^{-1}$ that

$$\begin{aligned}\alpha &\leq |(T^n)'(y)|^{-1} \leq K^2||T'|| \cdot |(T^k)'(x)|^{-1}(CC_1)^{-1}(K||T'||\kappa)^{1+p}\alpha|(T^k)'(x)| \\ &= D\kappa^{1+p}\alpha < \alpha,\end{aligned}$$

where in the last inequality we assumed that $\kappa < D^{-1/(1+p)}$. This contradiction shows that the family $\mathcal{F}(\omega, \alpha, \kappa)$ is a packing. In order to complete the proof, assume that for some $q \geq 0$ and for some $x \in T^{-q}(I(\omega))$ such that $|(T^q)'(x)|^{-1} \geq \alpha$, we have that

$$B\left(\omega, \kappa\alpha^{1/(1+p)}\right) \cap B\left(x, \kappa\alpha^{1/(1+p)}|(T^q)'(x)|^{-p/(1+p)}\right) \neq \emptyset.$$

This assumption implies that for every $y \in I(\omega)$ it holds that

$$B\left(y, \kappa'\alpha'^{1/(1+p)}\right) \cap B\left(T_y^{-1}(x), \kappa'\alpha'^{1/(1+p)}|(T^{q+1})'(x)|^{-p/(1+p)}\right) \neq \emptyset,$$

as well as that $|(T^{q+1})'(x)|^{-1} \geq \alpha'$. Here we have put $\alpha' := \alpha||T'||^{-1}$ and κ' denotes some constant multiple of κ. For sufficiently small κ this non-empty intersection clearly contradicts the fact that the family $\mathcal{F}(\omega, \alpha', \kappa')$ is a packing. Hence, the statement (i) of the theorem follows.

Proof of (ii):- For this it is sufficient to show that there exist κ_c and $\alpha_0 > 0$ such that for any $\kappa \geq \kappa_c$ and $\alpha \leq \alpha_0$ the family $\mathcal{F}(\omega, \alpha, \kappa)$ provides a covering of $J(T)$, for each $\omega \in \Omega$. Hence, let us now fix $\omega \in \Omega$ and $\alpha > 0$, where α will get adjusted throughout the construction. Complementary to the previous discussion in (i), we now assume that δ is chosen sufficiently small such that $|T'(z)| \geq 1$ for every $z \in J(T) \cap B(\Omega, \delta)$. Furthermore, let δ and θ be so small that all inverse branches T_ω^{-n} are well-defined on θ-neighbourhoods of points in $J(T) \cap \bigcup_{\omega \in \Omega} B(\omega, ||T'||\delta + \theta)$. Now, since $T : J(T) \to J(T)$ is topologically exact, we have for sufficiently large $q \geq 0$ that the family $\{B(x, \theta) : 1 \leq n \leq q, x \in T^{-n}(T^{-1}(I(\omega)) \cap (J(T) \setminus B(\omega, \delta))\}$ forms a covering of $J(T) \setminus B(\omega, \delta)$. We define

$$u := \inf\{|T'(v)| : v \in J(T)\} \text{ and } C := (K||T'||^q)^{-1} \min\{1, u\}.$$

By the choice of $\delta > 0$, we have that after some number of forward iterates each point in $B(\omega, \delta) \setminus \{\omega\}$ eventually escapes from $B(\omega, \delta)$. For a fixed $z \in J(T) \setminus \{\omega\}$, we define

- $k(z) := \min\{n \geq 0 : |(T^n)'(z)| \geq C\alpha^{-1}\}$,
- $l(z) := \min\{n \geq 0 : T^n(z) \notin B(\omega, \delta)\}$,
- $j(z) := \min\{k(z) - 1, l(z)\}$.

Since $l(z)$ is finite, we have in particular that $j(z)$ is finite. Now, let us assume first that $j(z) = l(z) = l$. In this case $l(z) \leq k(z) - 1$, which implies that $k(z) \geq 1$ (note that here we assume $\alpha < C||T'||^{-1}$). Hence, by our choice of q, there exist $0 \leq s \leq q$ and $y \in T^{-s}(I(\omega)) \setminus B(\omega, \delta)$ such that $T^l(z) \in B(y, \theta)$. If we let $x := T^s(y)$, then Koebe's distortion theorem implies that

$$\begin{aligned} z &\in B(T_z^{-l}(y), K\theta|(T_z^{-l})'(y)|) \\ &= B(T_z^{-(l+s)}(x), K\theta|(T_z^{-(l+s)})'(x)| \cdot |(T^s)'(y)|) \\ &\subset B(T_z^{-(l+s)}(x), \kappa|(T_z^{-(l+s)})'(x)|), \end{aligned}$$

where we have assumed that $\kappa > K\theta||T'||q \geq K\theta||T'||^s$, and where $T_z^{-l} : B(y, 2\theta) \to \hat{\mathbb{C}}$ and $T_z^{-(l+s)} : B(y, 2\theta) \to \hat{\mathbb{C}}$ denote the holomorphic inverse branches of T^l and T^{l+s} respectively, which respectively send $T^l(z)$ and $T^{l+s}(z)$ to z. By choice of the constant C and using Koebe's distortion theorem, we have that

$$\begin{aligned} |(T_z^{-(l+s)})'(x)| &\geq K^{-1}|(T_z^{-(l+s)})'(z)| = K^{-1}|(T^l)'(z)|^{-1}|(T^s)'(T^l(z))|^{-1} \\ &\geq K^{-1}C^{-1}\alpha||T'||^{-s} = (K||T'||^{s+1})^{-1}C^{-1}\alpha \geq \alpha. \end{aligned}$$

Hence, the proof for the case $j(z) = l(z)$ is complete.
We now consider the case $j(z) = k(z) - 1$. For simplicity, let us write k instead of $k(z)$ and l instead of $l(z)$. Here we have that $|(T^{k-1})'(z)| < C\alpha^{-1}$, that $|(T^k)'(z)| \geq C\alpha^{-1}$, that all points $z, T(z), \cdots, T^{k-1}(z), \cdots, T^{l-1}(z)$ are contained in $B(\omega, \delta)$, and that $T^l(z) \notin B(\omega, \delta)$. If we write as before $p = p(\omega)$, then, using (LBP), we have, for universal constants $C_1 \geq 1$ and $C_2 \geq 1$, that

$$C_1^{-1} l^{-1/p} \leq |z - \omega| \leq C_1 l^{-1/p}; \tag{5}$$

$$C_2^{-1}\, l^{-(1+p)/p} \leq |(T^l)'(z)|{-1} \leq C_2\, l^{-(1+p)/p}. \tag{6}$$

Hence, by our choice of k and l, since $|(T^l)'(z)| = |(T^{l-k})'(T^k(z))| \cdot |(T^k)'(z)| \geq |(T^k)'(z)|$ and assuming that $\kappa \geq 2C_1(C_2/C)^{1/(p+1)}$, it follows that

$$\begin{aligned} |z - \omega| &\leq C_1 l^{-1/p} \leq C_1 C_2^{1/(1+p)}\, |(T^l)'(z)|^{-1/(1+p)} && (7) \\ &\leq C_1 \left(C_2\, C^{-1}\, \alpha\right)^{1/(1+p)} \leq 2^{-1}\, \kappa\, \alpha^{1/(1+p)}. && (8) \end{aligned}$$

If we let $n \geq 0$ denote the largest integer such that

$$C_2 n^{(1+p)/p} \leq ||T'||^{-q}\alpha^{-1}, \tag{9}$$

then we have in particular that $n \geq 1$ (for $\alpha < C_2^{-1}||T'||^{-q}$), and that

$$n^{(p+1)/p} \geq 2^{-(p+1)/p}(n+1)^{(p+1)/p} \geq 2^{-(p+1)/p} C_2^{-1}||T'||^{-q}\alpha^{-1}. \tag{10}$$

Our choice of q implies the existence of s with $0 \leq s \leq q$ and $v \in T^{-s}(T^{-1}(B(\omega, \delta||T'||+\theta) \setminus \{\omega\}) \setminus B(\omega, \delta))$, such that for $x = T_\omega^{-n}(v) \in T^{-(n+s)}(I(\omega))$ (using (9) and (LBP), and recalling that $x = T^s(y)$) we have

$$|(T^{s+n})'(x)| = |(T^n)'(x)| \cdot |(T^s)'(v)| \leq C_2\, n^{(p+1)/p}||T'||^q \leq \alpha^{-1}. \tag{11}$$

On the other hand, if we combine (10) and (LBP), we have that

$$|x-\omega| \le C_1\, n^{-1/p} \le C_1\, 2^{1/p} C_2^{1/p}\, ||T'||^{q/(p+1)} \alpha^{1/(p+1)} < 2^{-1}\, \kappa\, \alpha^{1/(1+p)}; \qquad (12)$$

where we assumed that $\kappa > C_1 C_2^{1/p}\, 2^{(1+p)/p}\, ||T'||^{q/(p+1)}$. Combining this inequality and (8), we get that $|z-x| < \kappa\alpha^{1/(1+p)}$, which of course, as follows from (12), is true in particular for $z=\omega$. This completes the proof of the statement *(ii)* in the theorem.

□

4 Counting canonical balls

In this section we derive an estimate for the number of equally sized canonical balls contained in a small neighbourhood around a pre-parabolic point. More precisely, for fixed $\omega, \eta \in \Omega$ and for $\sigma > 0$ we estimate the cardinality of the set of roughly equally sized canonical balls of the kind $B(c(\eta), r_{c(\eta)})$ which are contained in a σ-reduced canonical ball $B(c(\omega), r_{c(\omega)}^{1+\sigma})$. We show that this cardinality is governed by the quotient of the conformal measure of these two balls. This estimate will be crucial in the following section.
We introduce the following notation. For $0 < \rho < 1$, $n \in \mathbb{N}$ and $\omega, \eta \in \Omega$, we define

$$\Pi_{\omega,n}(\rho) := \{c(\omega) \in J_p(T) : \rho^{n+1} \le r_{c(\omega)} < \rho^n\},$$

$$\Sigma_{\eta,n}(c(\omega), \sigma, \rho) := \{c(\eta) \in \Pi_{\eta,n}(\rho) : B(c(\eta), r_{c(\eta)}) \subset B(c(\omega), r_{c(\omega)}^{1+\sigma})\}.$$

Proposition 1
There exist $\lambda, c_0, c_1, c_2 > 0$ and an increasing function $\iota : \mathbb{N} \to \mathbb{R}^+$ with the following property. For any $\omega, \eta \in \Omega$ and $c(\omega) \in \Pi_{\omega,n}(\lambda)$ for some $n \ge c_0$, we have for $m > \iota(n)$ that

$$c_1\, \lambda^{h(n-m)+\sigma n(h+(h-1)p(\omega))} \le \mathrm{card}(\Sigma_{\eta,m}(c(\omega), \sigma, \lambda)) \le c_2\, \lambda^{h(n-m)+\sigma n(h+(h-1)p(\omega))}.$$

Note: This estimate of $\mathrm{card}(\Sigma_{\eta,m}(c(\omega), \sigma, \lambda))$ does not depend on $\eta \in \Omega$.
Proof:
Since our proof follows closely the proof of the corresponding result for geometrically finite groups, we here give only the crucial estimates. For further details we refer to [18] (Proposition 3).
Let $c(\omega) \in J_p(T)$ be fixed such that $r_{c(\omega)}$ is sufficiently small (i.e. more precisely, such that $r_{c(\omega)} < \min\{\alpha_0, (4\kappa_p)^{-1/\sigma}\}$). For $\eta \in \Omega$, we define $\Sigma_\eta := \{c(\eta) \in J_p(T) : B(c(\eta), r_{c(\omega)}) \subset B(c(\omega), r_{c(\omega)}^{1+\sigma})\}$. Now, using Theorem 3 and after performing some elementary calculations (cf. [18]), we obtain for sufficiently small $\alpha > 0$ (i.e. more precisely, for $\alpha < r_{c(\omega)}^{1+\sigma(1+p(\omega))}/(4\kappa_c)$, where κ_c is the 'covering-constant' of Theorem 3) that

$$m(B(c(\omega), r_{c(\omega)}^{1+\sigma})) \asymp m(B(c(\omega), r_{c(\omega),\alpha})) + \sum_{\substack{c(\eta)\in\Sigma_\eta \\ r_{c(\eta)} \ge \alpha}} m(B(c(\eta), r_{c(\eta),\alpha})). \qquad (13)$$

Using (GF), we see that for $r_{c(\eta)} \geq \alpha$ we have

$$\begin{aligned} m(B(c(\eta), r_{c(\eta),\alpha})) &\asymp \left(r_{c(\eta)} \left(\frac{\alpha}{r_{c(\eta)}} \right)^{1/(1+p(\eta))} \right)^h \left(\left(\frac{\alpha}{r_{c(\eta)}} \right)^{1/(1+p(\eta))} \right)^{(h-1)p(\eta)} \\ &= \alpha^h \left(\frac{r_{c(\eta)}}{\alpha} \right)^{p(\eta)/(1+p(\eta))}. \end{aligned}$$

Using this estimate, we derive from (12) that

$$\alpha^{-h}\ m(B(c(\omega), r_{c(\omega)}^{1+\sigma})) \asymp \left(\frac{r_{c(\omega)}}{\alpha} \right)^{p(\omega)/(1+p(\omega))} + \sum_{\substack{c(\eta) \in \Sigma_\eta \\ r_{c(\eta)} \geq \alpha}} \left(\frac{r_{c(\eta)}}{\alpha} \right)^{p(\eta)/(1+p(\eta))}. \quad (14)$$

If we let $\alpha := \lambda^m$, for some sufficiently small $\lambda > 0$, then a simple calculation (cf. [18], p. 394) shows that (13) implies

$$\sum_{\substack{c(\eta) \in \Sigma_\eta \\ \lambda^{m+1} \leq r_{c(\eta)} < \lambda^m}} \mathbf{1} \asymp \lambda^{-(m+1)\, h}\ \ m(B(c(\omega), r_{c(\omega)}^{1+\sigma})).$$

Now, if we choose $n \in \mathbb{N}$ such that $c(\omega) \in \Pi_{\omega,n}(\lambda)$, and apply once again (GF), then it follows that

$$\begin{aligned} m(B(c(\omega), r_{c(\omega)}^{1+\sigma})) &\asymp r_{c(\omega)}^{h(1+\sigma)}\ r_{c(\omega)}^{\sigma(h-1)p(\omega)} \\ &\asymp \lambda^{nh+n\sigma(h+(h-1)p(\omega))} \end{aligned}$$

Hence, by combining the two latter estimates, it follows that

$$\sum_{\substack{c(\eta) \in \Sigma_\eta \\ \lambda^{m+1} \leq r_{c(\eta)} < \lambda^m}} \mathbf{1} \asymp \lambda^{h(n-m)+n\sigma(h+(h-1)p(\omega))},$$

which gives the statement in the proposition. □

5 The Julia set in the spirit of Jarník

In this section we give the proof of Theorem 1, i.e. we give for parabolic rational maps a generalisation of a classical theorem in the theory of Diophantine approximation due to Jarník [12] (which was obtained slightly later independently also by Besicovitch [1]). In order to motivate this generalisation, we first recall this classical result of Jarník.

> **Jarník's Theorem:** *The Hausdorff dimension of the set of* well-approximable irrational numbers *is determined by the following. For* $\sigma > 0$, *we have that*
>
> $$\dim_H \left(\left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \left(q^{-2} \right)^{1+\sigma} \text{ for infinitely many reduced } \frac{p}{q} \right\} \right) = \frac{1}{1+\sigma}.$$

We now generalise this theorem to the situation of a parabolic rational map. The proof follows closely the construction in [18] and [21], where we derived a similar generalisation of Jarník's theorem to the theory of Kleinian groups.
Throughout, we assume that $\sigma > 0$ and $\omega \in \Omega$ are given, and that $\lambda > 0$ is chosen according to Proposition 1. The key for getting the lower bound of $\dim_H(\mathcal{J}_\sigma^\omega(T))$ is first of all the explicit construction of a set $\mathcal{C}^\sigma(\omega) \subset \mathcal{J}_\sigma^\omega(T)$. Similar to a 2-dimensional Cantor set, this set is the $\limsup$ set of infinitely many approximations (or generations) of the set with an increasing resolution. Here it is important that each of these generations consists of roughly equally sized, σ-reduced canonical balls, and that the ratio of the diameters of members of 'successive generations' decreases to 0, whereas the number of elements of a generation which are contained in exactly one member of the previous generation increases exponentially fast. The task will then be to give a sufficiently good quantitative description of this set.
We start with the construction of the set $\mathcal{C}^\sigma(\omega)$. For this let $\{s_k\}_{k\in\mathbf{N}}$ denote a strictly increasing sequence of positive integers such that s_0 is sufficiently large, $s_k > \iota(s_{k-1})$ for all k, and further that $s_n^{-1}\sum_{j=0}^{n-1} s_j \to 0$ for $n \to \infty$. Now, fix an element $z_0 \in \Pi_{\omega,s_0}(\lambda)$ and let $\mathcal{C}_0 := B(z_0, r_{z_0}^{1+\sigma})$. Then define inductively the generation $\mathcal{C}_k$ for $k \in \mathbb{N}$ by:-

$$\text{if} \quad \mathcal{C}_{k-1} \quad \text{is defined, then} \quad \mathcal{C}_k := \{B(c(\omega), r_{c(\omega)}^{1+\sigma}) : c(\omega) \in \Sigma_{\omega,s_k}(z,\sigma,\lambda)$$
$$\text{for some} \quad z \in \Pi_{\omega,s_{k-1}}(\lambda) \quad \text{such that} \quad B(z, r_z^{1+\sigma}) \in \mathcal{C}_{k-1}\}.$$

Without loss of generality, we may assume that each element in $\mathcal{C}_{k-1}$ contains exactly N_k elements of $\mathcal{C}_k$, where we have set $N_k := \min_{z\in\Pi_{\omega,s_{k-1}}(\lambda)} \operatorname{card} \Sigma_{\omega,s_k}(z,\sigma,\lambda)$. Hence, we can now define $\mathcal{C}^\sigma(\omega) := \bigcap_{k\geq 0}\bigcup_{C\in\mathcal{C}_k} C$, and instead of $\mathcal{C}^\sigma(\omega)$ we shall usually just write $\mathcal{C}^\sigma$, where it is clear which parabolic point ω is involved.
Next, we construct a probability measure on $\mathcal{C}^\sigma$ by renormalising the h-conformal measure m on each $\mathcal{C}_k$, i.e. for all $k \in \mathbb{N}$ define a probability measure $m_{\sigma,k}$ on $\mathcal{C}_k$ such that for Borel sets $F \subset \hat{\mathbb{C}}$ we have

$$m_{\sigma,k}(F) = \sum_{I\in\mathcal{C}_k} (N_1 \cdot \ldots \cdot N_k)^{-1} m(F \cap I)/m(I).$$

(Note that we could have defined $m_{\sigma,k}$ also just as a 'counting measure', i.e. for the purposes in this paper it is not relevant that $m_{\sigma,k}$ depends on m.) Using *Helly's Theorem*, we obtain a probability measure m_σ on $\mathcal{C}^\sigma$ as the weak limit of the sequence of measures $\{m_{\sigma,k}\}$. Note that $m_{\sigma,k}(I) = m_\sigma(I)$, for each $k \in \mathbb{N}$ and $I \in \mathcal{C}_k$.

Lemma 1
For each $\xi \in \mathcal{C}^\sigma$ and r such that $\lambda^{s_k+2} \leq r < \lambda^{s_{k-1}+2}$ for some $k \in \mathbb{N}$, the ball $B(\xi, r)$ intersects exactly one element in $\mathcal{C}_{k-1}$ and

$$\operatorname{card}\{C \in \mathcal{C}_k : C \cap B(\xi, r) \neq \emptyset\} \ll \lambda^{-hs_k}\, m(B(\xi, r)) \quad .$$

Proof.
Let ξ and r be given as stated in the lemma. Now, first note that, by Theorem 3 (i), we may assume without loss of generality that the canonical balls $B(z, 2r_z)$, which have the property that $B(z, r_z^{1+\sigma}) \in \mathcal{C}_{k-1}$, are pairwise disjoint. In order to see that $B(\xi, r)$ intersects exactly one element of $\mathcal{C}_{k-1}$, note first that since $\xi \in \mathcal{C}^\sigma$, there exists a unique $B(c(\omega), r_{c(\omega)}^{1+\sigma}) \in \mathcal{C}_{k-1}$ containing ξ. Now, if $B(\xi, r)$ would not be fully contained in $B(c(\omega), r_{c(\omega)})$, then it would follow that

$$r > r_{c(\omega)} - r_{c(\omega)}^{1+\sigma} \geq \lambda^{s_{k-1}+1}(1 - \lambda^{\sigma s_{k-1}}) > \lambda^{s_{k-1}+2},$$

which contradicts our assumption concerning the size of r.
For the second assertion in the lemma note that if $B(c(\omega), r_{c(\omega)}^{1+\sigma}) \in \mathcal{C}_k$ intersects $B(\xi, r)$, then we have that $B(c(\omega), r_{c(\omega)}) \subset B(\xi, r + r_{c(\omega)} + r_{c(\omega)}^{1+\sigma})$. Using this observation and the pairwise disjointness of the canonical balls which we mentioned at the beginning of the proof, it follows that

$$\begin{aligned} &\operatorname{card}\{C \in \mathcal{C}_k : C \cap B(\xi, r) \neq \emptyset\} \min_{B(z, r_z^{1+\sigma}) \in \mathcal{C}_k} m(B(z, r_z)) \\ &\leq \max_{B(z, r_z^{1+\sigma}) \in \mathcal{C}_k} m(B(\xi, r + r_z + r_z^{1+\sigma})) \\ &\ll m(B(\xi, r)), \end{aligned}$$

where in the last inequality we made use of the fact that m is a doubling measure, which is an immediate consequence of (GF). Now, since for $B(z, r_z^{1+\sigma}) \in \mathcal{C}_k$ we have that $m(B(z, r_z)) \asymp \lambda^{h s_k}$, the lemma follows. $\square$

Lemma 2
For each $\epsilon > 0$ there exists $r_o(\epsilon) > 0$ with the following property. For all $\xi \in \mathcal{C}^\sigma$ and $0 < r < r_o(\epsilon)$ such that $\lambda^{s_k} \leq r < \lambda^{s_{k-1}}$ for some $k \in \mathbb{N}$,

$$m_\sigma(B(\xi, r)) \ll m(B(\xi, r)) \, \lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} .$$

Proof.
Let ξ and r be given as stated in the lemma. By construction of the measure m_σ and using Lemma 1, it follows that

$$m_\sigma(B(\xi, r)) \leq \prod_{j=0}^{k} N_j^{-1} \operatorname{card}\{C \in \mathcal{C}_k : C \cap B(\xi, r) \neq \emptyset\} \ll \lambda^{-h s_k} m(B(\xi, r)) \prod_{j=0}^{k} N_j^{-1} .$$

Hence, using Proposition 1, it follows that

$$\begin{aligned} m_\sigma(B(\xi, r)) &\ll \lambda^{-s_k h} m(B(\xi, r)) c_1^{-(k-1)} \lambda^{h(s_k - s_{k-1})} \lambda^{-\sigma s_{k-1}(h+(h-1)p(\omega))} \\ &\qquad \times \lambda^{h(s_{k-1} - s_o)} \lambda^{-\sigma(h+(h-1)p(\omega)) \sum_{j=0}^{k-2} s_j} \\ &= m(B(\xi, r)) \\ &\times \quad \lambda^{s_{k-1}(-\sigma(h+(h-1)p(\omega)) - s_{k-1}^{-1}(s_o + \sigma(h+(h-1)p(\omega)) \sum_{j=0}^{k-2} s_j + (k-1)(\log c_1)(\log \lambda)^{-1}))} . \end{aligned}$$

By our choice of the sequence $\{s_k\}$, we have for each $\epsilon > 0$ that for sufficiently large k it holds that

$$\frac{1}{s_{k-1}} \left(s_o + \sigma(h + (h-1)p(\omega)) \sum_{j=0}^{k-2} s_j + (k-1)(\log c_1)(\log \lambda)^{-1}) \right) < \epsilon \quad .$$

Using this inequality in the latter estimate, the lemma follows. $\square$

Proposition 2

• *If $h \geq 1$, then for each $\omega \in \Omega$, $\xi \in \mathcal{C}^\sigma(\omega)$ and $\epsilon > 0$, there exists $r_1 = r_1(\xi, \epsilon) > 0$ such that for all $0 < r < r_1$ we have that*

$$m_\sigma(B(\xi, r)) \ll \begin{cases} r^{h/(1+\sigma)-\epsilon} & \text{for} \quad \sigma \geq h-1 \\ r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega)))-\epsilon} & \text{for} \quad \sigma \leq h-1. \end{cases}$$

• *If $h < 1$ and $\omega \in \Omega$ such that $p(\omega) = p_{max}$, then it holds that for each $\xi \in \mathcal{C}^\sigma(\omega)$ and $\epsilon > 0$, there exists $r_1 = r_1(\xi, \epsilon) > 0$ such that for all $0 < r < r_1$ we have that*

$$m_\sigma(B(\xi, r)) \ll r^{h/(1+\sigma)-\epsilon}.$$

Proof.
Let $\xi \in \mathcal{C}^\sigma(\omega)$ be given. Without loss of generality assume that $\lambda^{s_k} \leq r < \lambda^{s_{k-1}}$ and that $\xi \in B(c(\omega), r_{c(\omega)}^{1+\sigma}) \in \mathcal{C}_{k-1}$. We then have the following.

Case 1: *For each $\epsilon > 0$, there exist $k_1 = k_1(\xi, \epsilon)$ such that for all $k \geq k_1$ the following holds. If $(\lambda^{s_{k-1}})^{1+\sigma} \leq r < \lambda^{s_{k-1}}$, then*

$$m_\sigma(B(\xi, r)) \ll r^{h/(1+\sigma)-\epsilon/(1+\sigma)}.$$

Proof: By construction of m_σ, we have that $m_\sigma(B(c(\omega), r_{c(\omega)})) \asymp m_\sigma(B(c(\omega), r_{c(\omega)}^{1+\sigma}))$. Using this observation, Lemma 2 implies that

$$\begin{aligned} m_\sigma(B(\xi, r)) &\ll m_\sigma(B(\xi, r_{c(\omega)}^{1+\sigma})) \\ &\ll m(B(\xi, r_{c(\omega)}^{1+\sigma}))\ \lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} \\ &\asymp r_{c(\omega)}^h\ r_{c(\omega)}^{\sigma(h+(h-1)p(\omega))}\ \lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} \\ &\asymp r_{c(\omega)}^h\ r_{c(\omega)}^{-\epsilon} \\ &\asymp \left(r_{c(\omega)}^{1+\sigma}\right)^{(h-\epsilon)/(1+\sigma)} \\ &\ll r^{h/(1+\sigma)}\ r^{-\epsilon/(1+\sigma)}. \end{aligned}$$

Case 2: *For each $\epsilon > 0$, there exist $k_2 = k_2(\xi, \epsilon)$ such that for all $k \geq k_2$ the following holds. If $(\lambda^{s_{k-1}})^{1+\sigma(1+p(\omega))} \leq r < (\lambda^{s_{k-1}})^{1+\sigma}$, then*

$$m_\sigma(B(\xi, r)) \ll \begin{cases} r^{h/(1+\sigma)}\ r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma \geq h-1 \\ r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega))}\ r^{-\epsilon'} & \text{for} \quad \sigma < h-1, \end{cases}$$

where ϵ' denotes some constant multiple of ϵ.

Proof: Let $r = \lambda^{s_{k-1}(1+\sigma+\tau)}$ for some $0 < \tau \leq \sigma p(\omega)$. Also, without loss of generality we may assume that $\xi \in \mathcal{C}^\sigma \setminus \mathcal{J}^\omega_{\sigma+\delta}(T)$, for some sufficiently small $\delta > 0$. For the hyperbolic zoom at ξ we have that $r_j(\xi) = r_{c(\omega)}$, for some $j \in \mathbb{N}$. An elementary calculation, using (LBP), gives that

$$r_{c(\omega)}^{1+(\sigma+\delta)(1+p(\omega))} < r_{j+1}(\xi) \leq r_{c(\omega)}^{1+\sigma(1+p(\omega))}.$$

We first consider the case '$h > 1$'. Applying (GF), it follows that (note that, by choosing δ sufficiently small, we can guarantee that we are in the situation of 'the second part in (GF) concerning $\xi \in J_r(T)$')

$$m(B(\xi,r)) \ll r^h \left(\frac{r_{j+1}(\xi)}{r}\right)^{h-1} \ll r^h \left(\frac{r_{c(\omega)}^{1+\sigma(1+p(\omega))}}{r}\right)^{h-1}.$$

Thus, using Lemma 2, it follows that

$$\begin{aligned}
m_\sigma(B(\xi,r)) &\ll m(B(\xi,r))\,\lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} \\
&\ll r^h \left(\frac{r_{c(\omega)}^{1+\sigma(1+p(\omega))}}{r}\right)^{h-1} \lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} \\
&\ll \lambda^{s_{k-1}h(1+\sigma+\tau)} \left(\frac{\lambda^{s_{k-1}(1+\sigma(1+p(\omega)))}}{\lambda^{s_{k-1}(1+\sigma+\tau)}}\right)^{h-1} \lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} \\
&\ll \lambda^{s_{k-1}(h+\tau-\epsilon)} \\
&= \left(\lambda^{s_{k-1}(1+\sigma+\tau)}\right)^{(h+\tau-\epsilon)/(1+\sigma+\tau)} \\
&= r^{(h+\tau-\epsilon)/(1+\sigma+\tau)} \\
&\ll \begin{cases} r^{h/(1+\sigma)}\; r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma \geq h-1 \\ r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega))}\; r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma < h-1. \end{cases}
\end{aligned}$$

Hence, the statement follows in the case that $h > 1$. Now, for $h \leq 1$, we similarly see that

$$\begin{aligned}
m(B(\xi,r)) &\ll r^h \left(\frac{r_{j+1}(\xi)}{r}\right)^{h-1} \\
&\ll r^h \left(\frac{r_{c(\omega)}^{1+(\sigma+\delta)(1+p(\omega))}}{r}\right)^{h-1} \\
&= r \cdot r_{c(\omega)}^{(h-1)(1+(\sigma+\delta)(1+p(\omega)))},
\end{aligned}$$

and hence, that

$$m_\sigma(B(\xi,r)) \ll m(B(\xi,r))\,\lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)}$$

$$
\begin{aligned}
&\ll \dots \\
&\ll \lambda^{s_{k-1}(h+\tau)}\ \lambda^{s_{k-1}\delta(h-1)(1+p(\omega))}\ \lambda^{-s_{k-1}\epsilon} \\
&= r^{(h+\tau)/(1+\sigma+\tau)}\ r^{-\epsilon'} \\
&\ll r^{h/(1+\sigma)}\ r^{-\epsilon'},
\end{aligned}
$$

where we have set $\epsilon' := \delta(1-h)(1+p(\omega))+\epsilon$, and where the last inequality follows since here, we trivially have that $h < 1+\sigma$.

Case 3: *For each $\epsilon > 0$, there exist $k_3 = k_3(\xi, \epsilon)$ such that for all $k \geq k_3$ and $\lambda^{s_k} \leq r < (\lambda^{s_{k-1}})^{1+\sigma(1+p(\omega))}$ the following holds.*

- *If $h \geq 1$, then*

$$
m_\sigma(B(\xi,r)) \ll \begin{cases} r^{h/(1+\sigma)}\ r^{-\epsilon/(1+\sigma)} & \textit{for} \quad \sigma \geq h-1 \\ r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega))}\ r^{-\epsilon'} & \textit{for} \quad \sigma < h-1, \end{cases}
$$

where ϵ' denotes some constant multiple of ϵ.

- *If $h < 1$ and $\xi \in \mathcal{C}^\sigma(\omega)$ for some $\omega \in \Omega$ such that $p(\omega) = p_{max}$, then*

$$
m_\sigma(B(\xi,r)) \ll r^{h/(1+\sigma)}\ r^{-\epsilon/(1+\sigma)}.
$$

Proof: We first consider the case $h \geq 1$. Here, we have that

$$
m(B(\xi,r)) \ll r^h,
$$

and hence, using Lemma 2, that

$$
\begin{aligned}
m_\sigma(B(\xi,r)) &\ll r^h\ \lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} \\
&\ll r^h r^{-(\sigma(h+(h-1)p(\omega))+\epsilon)/(1+\sigma(1+p(\omega)))} \\
&\ll r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega)))}\ r^{-\epsilon/(1+\sigma(1+p(\omega)))} \\
&\ll \begin{cases} r^{h/(1+\sigma)}\ r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma \geq h-1 \\ r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega))}\ r^{-\epsilon'} & \text{for} \quad \sigma < h-1, \end{cases}
\end{aligned}
$$

which gives the proof in the case $h \geq 1$.
For $h < 1$ and $\omega \in \Omega$ such that $p(\omega) = p_{max}$, we assume without loss of generality that $\xi \in \mathcal{C}^\sigma(\omega) \setminus \mathcal{J}_{\sigma+\delta}(T)$, for sufficiently small $\delta > 0$. Also, let r be related to the hyperbolic zoom at ξ such that, for some $l \in \mathbb{N}$, we have that $r_{l+1}(\xi) \leq r < r_l(\xi)$, and that $\eta \in \Omega$ is associated to this particular part of the hyperbolic zoom . Using once more Lemma 2 and (GF), and the fact that $\lambda^{-s_{k-1}} < r^{-1/(1+\sigma(1+p(\omega))}$, we obtain that

$$
\begin{aligned}
m_\sigma(B(\xi,r)) &\ll r^h\ \phi(\xi,r)\ \lambda^{-s_{k-1}(\sigma(h+(h-1)p(\omega))+\epsilon)} \\
&\ll \dots \\
&\ll r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega)))}\ \phi(\xi,r)\ r^{-\epsilon/(1+\sigma(1+p(\omega)))}.
\end{aligned}
$$

Before continueing this estimate, we now first give an upper estimate for the conformal fluctuation $\phi(\xi, r)$. It is sufficient to consider the extreme case where the fluctuation is largest. Here, we have for sufficiently small r that

$$(r_l(\xi))^{1+(\sigma+\delta)(1+p(\eta))} \leq r_{l+1}(\xi) < (r_l(\xi))^{1+\sigma(1+p(\eta))}. \tag{15}$$

Also, (GF) immediately gives that, for $r_{l+1}(\xi) \leq r < r_l(\xi)$, the fluctuation $\phi(\xi, \cdot)$ attains its maximal value for

$$r = r_l(\xi) \left(\frac{r_{l+1}(\xi)}{r_l(\xi)} \right)^{1/(1+p(\eta))}.$$

In the following we fix r to be equal to this value. Also, note that, with this 'maximal choice' of r, (15) and (16) imply that

$$(r_l(\xi))^{1+\sigma+\delta} \leq r < (r_l(\xi))^{1+\sigma}.$$

We can now estimate the maximal conformal fluctuation as follows.

$$\begin{aligned}
\phi(\xi, r) &\asymp \left(\frac{r}{r_l(\xi)} \right)^{(h-1)p(\eta)} \asymp \left(\frac{r_{l+1}(\xi)}{r_l(\xi)} \right)^{(h-1)p(\eta)/(1+p(\eta))} \\
&\ll (r_l(\xi))^{(\sigma+\delta)(h-1)p(\eta)} \\
&\ll \left(r^{1/(1+\sigma)} \right)^{(\sigma+\delta)p(\eta)(h-1)}.
\end{aligned}$$

Using the latter inequality, we now continue the above estimate for $m_\sigma(B(\xi, r))$. Let $\epsilon' := \frac{\epsilon}{1+\sigma(1+p(\omega))} + \delta \frac{(1-h)\sigma p(\eta)}{1+\sigma}$, then, with an elementary argument, we see that

$$\begin{aligned}
m_\sigma(B(\xi, r)) &\ll r^{(h+\sigma p(\omega))/(1+\sigma(1+p(\omega)))} \, r^{(h-1)\sigma p(\eta)/(1+\sigma)} \, r^{-\epsilon'} \\
&\ll r^{h/(1+\sigma)} \, r^{-\epsilon'}.
\end{aligned}$$

Here, the latter inequality follows since

$$\begin{aligned}
p(\omega) \geq p(\eta) \quad &\Rightarrow \quad \frac{p(\omega)}{1+p(\omega)} \geq \frac{p(\eta)(1+p(\omega)) - p(\omega)}{p(\eta)(1+p(\omega))} \\
&\Rightarrow \quad h > \frac{p(\eta)(1+p(\omega)) - p(\omega)}{p(\eta)(1+p(\omega))} \quad \left(\text{since } h > \frac{p(\omega)}{1+p(\omega)} \text{(cf. [2]} \right) \\
&\Rightarrow \quad p(\omega) + (h-1)(1 + p(\omega)p(\eta)) > 0 \\
&\Rightarrow \quad \sigma(p(\omega) + (h-1)(1 + p(\omega)p(\eta))) > (h-1)(p(\omega) - p(\eta)) \\
&\Rightarrow \quad \frac{h + \sigma p(\omega)}{1 + \sigma(1+p(\omega))} + \frac{\sigma p(\eta)(h-1)}{1+\sigma} > \frac{h}{1+\sigma}.
\end{aligned}$$

This completes the proof in the third case.

The statement of the proposition now follows by summing up the above three cases. $\square$

Proof of Theorem 1:
We apply the *mass distribution principle* (cf. e.g. [8]). For $h \geq 1$, $\omega \in \Omega$ and $\sigma > 0$, Proposition 2 implies that

$$\dim_H(\mathcal{J}_\sigma^\omega(T)) \geq \begin{cases} \frac{h}{1+\sigma} & \text{for} \quad \sigma \geq h-1 \\ \frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))} & \text{for} \quad \sigma < h-1. \end{cases}$$

For $h < 1$, note that $\dim_H(\mathcal{J}_\sigma(T)) \geq \dim_H(\mathcal{J}_\sigma^\omega(T))$, for any $\omega \in \Omega$. Hence, in this case, Proposition 2 implies that

$$\dim_H(\mathcal{J}_\sigma(T)) \geq \frac{h}{1+\sigma}.$$

For the upper bounds of $\dim_H(\mathcal{J}_\sigma^\omega(T))$ and $\dim_H(\mathcal{J}_\sigma(T))$, note that $\{B(z, r_z^{1+\sigma}) : z \in J_p^\omega(T)\}$, where $J_p^\omega(T) := \bigcup_{n\geq 0} T^{-n}(\omega)$, provides a 'natural cover' of $\mathcal{J}_\sigma^\omega(T)$. Using this and the convergence of $\sum_{z\in J_p^\omega(T)} \left(r_z^{1+\sigma}\right)^{h/(1+\sigma)}$ and $\sum_{z\in J_p(T)} \left(r_z^{1+\sigma}\right)^{h/(1+\sigma)}$, it follows that

$$\dim_H(\mathcal{J}_\sigma^\omega(T)) \leq \frac{h}{1+\sigma} \quad \text{and} \quad \dim_H(\mathcal{J}_\sigma(T)) \leq \frac{h}{1+\sigma}.$$

Also, note that the intersection of $J(T)$ with some arbitrary $B(c(\omega), r_{c(\omega)}^{1+\sigma})$ is contained in $\Pi(c(\omega))$, the pull-back to $c(\omega)$ of the *Fatou flower* at ω. Using (LBP), we see that for each petal $\pi_i \subset \Pi(c(\omega))$ we have that the largest canonical ball which is contained in $B(c(\omega), r_{c(\omega)}^{1+\sigma}) \cap \pi_i$, lies at the rim of $B(c(\omega), r_{c(\omega)}^{1+\sigma})$ and is of size comparable to $r_{c(\omega)}^{1+\sigma(1+p(\omega))}$. Now, a simple Euclidean argument gives that we may cover $B(c(\omega), r_{c(\omega)}^{1+\sigma}) \cap J(T)$ with Euclidean balls of the size $r_{c(\omega)}^{1+\sigma(1+p(\omega))}$, such that the number of these balls is comparable to $r_{c(\omega)}^{-\sigma p(\omega)}$. We call this particular cover the 'associated cover' (note that the balls in the associated cover are of course not necessarily canonical balls). If in the above 'natural cover' of $\mathcal{J}_\sigma^\omega(T)$ we replace each of the σ-reduced canonical balls by its associated cover, then this gives an alternative way of covering $\mathcal{J}_\sigma^\omega(T)$. For this cover we have that

$$\sum_{z\in J_p^\omega(T)} r_z^{-\sigma\, p(\omega)}\; r_z^{s(1+\sigma(1+p(\omega)))} \begin{cases} \text{converges} & \text{for} \quad s \geq \frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))} \\ \text{diverges} & \text{for} \quad s < \frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))}. \end{cases}$$

Hence, it follows that

$$\dim_H(\mathcal{J}_\sigma^\omega(T)) \leq \frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))}.$$

For $h \geq 1$, a combination of the above two upper bounds for $\dim_H(\mathcal{J}_\sigma^\omega(T))$, together with an elementary calculation, now gives that

$$\dim_H(\mathcal{J}_\sigma^\omega(T)) \leq \begin{cases} \frac{h}{1+\sigma} & \text{for} \quad \sigma \geq h-1 \\ \frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))} & \text{for} \quad \sigma < h-1. \end{cases}$$

Hence, this completes the calculation of the Hausdorff dimension for $\mathcal{J}_\sigma^\omega(T)$. In order to derive $\dim_H(\mathcal{J}_\sigma^\omega(T))$, note that for $\omega, \eta \in \Omega$ with $p(\eta) \leq p(\omega)$, we have for $\sigma \leq h-1$ that

$$\frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))} \leq \frac{h+\sigma p(\eta)}{1+\sigma(1+p(\eta))}.$$

Hence, it follows that

$$\dim_H(\mathcal{J}_\sigma(T)) = \begin{cases} \frac{h}{1+\sigma} & \text{for} \quad \sigma \geq h-1 \\ \frac{h+\sigma p_{min}}{1+\sigma(1+p_{min})} & \text{for} \quad \sigma < h-1. \end{cases}$$

Finally, for $h<1$, we immediately derive from the above that

$$\dim_H(\mathcal{J}_\sigma(T)) = \frac{h}{1+\sigma}.$$

Thus, the proof of Theorem 1 is complete.

□

6 Weak singularity spectra of the h-conformal measure

In this section we give the proof of Theorem 2. We apply Theorem 1 in order to derive the weak singularity spectra of the h-conformal measure m.

Proof of Theorem 2:
We consider the cases '$h=1$', '$h<1$' and '$h>1$' separately.

- For $h=1$, the weak singularity spectra are trivial. This follows from (GF), since in this case we have for all $\xi \in J(T)$ and $0<r<\operatorname{diam}(J(T))$ that $\phi(\xi,r) \asymp 1$, which implies that $m(B(\xi,r)) \asymp r^h$, and hence,

$$\lim_{r\to 0} \frac{\log m(B(\xi,r))}{\log r} = h.$$

- For $h<1$, we define

$$\mathcal{M}_\sigma := \left\{\xi \in J_r(T) : m(B(\xi,r)) \gg_{i.o.} r^h \, r^{\sigma(h-1)p_{max}/(1+\sigma)}\right\},$$

where '$\gg_{i.o.}$' indicates that the inequality holds 'infinitely often', i.e. for some decreasing sequence of radii tending to zero. An elementary calculation shows that if, for $\omega \in \Omega$, we let

$$\sigma(\omega) := \frac{\sigma p_{max}}{p(\omega)+\sigma(p(\omega)-p_{max})},$$

then we may write

$$\mathcal{M}_\sigma = \bigcup_{\omega\in\Omega} \mathcal{J}_{\sigma(\omega)}^\omega(T).$$

Now, an application of Theorem 1 gives that

$$\dim_H(\mathcal{J}_{\sigma(\omega)}^\omega(T)) \leq \dim_H(\mathcal{J}_{\sigma(\omega)}(T)) = \frac{h\,(1-\sigma(p_{max}-p(\omega))/p(\omega)))}{1+\sigma} \leq \frac{h}{1+\sigma}.$$

Hence, since in particular, for $\eta \in \Omega$ such that $p(\eta) = p_{max}$, we have that $\mathcal{J}^{\eta}_{\sigma(\eta)}(T) \subset \mathcal{M}_\sigma$ and $\dim_H(\mathcal{J}^{\eta}_{\sigma(\eta)}(T)) = \dim_H(\mathcal{J}^{\eta}_{\sigma}(T)) = h/(1+\sigma)$, it follows that

$$\dim_H(\mathcal{M}_\sigma) = \frac{h}{1+\sigma}.$$

If we let $\theta := h + \sigma(h-1)p_{max}/(1+\sigma)$, or what is equivalent $\sigma = (\theta - h)/(h - \theta + (h-1)p_{max})$, then it follows for $h + (h-1)p_{max} < \theta < h$ that

$$\dim_H(\mathcal{I}^\theta(m)) = \frac{h}{(1-h)p_{max}} \left(\theta - (h + (h-1)p_{max})\right).$$

Furthermore, for $\theta \geq h$ we have that $\mathcal{I}^\theta(m) = J_r(T)$, and hence that $\dim_H(\mathcal{I}^\theta(m)) = h$. Finally, if $\theta = h + (h-1)p_{max}$ then $\mathcal{I}^\theta(m) = \bigcup_{\omega \in \Omega : p(\omega) = p_{max}} J^\omega_p(T)$, and if $\theta < h + (h-1)p_{max}$ then $\mathcal{I}^\theta(m) = \emptyset$. Hence, for $\theta \leq h + (h-1)p_{max}$ we have that $\dim_H(\mathcal{I}^\theta(m)) = 0$.
For the remaining spectra in this case, note that (GF) implies that for all $\xi \in J(T)$ and all positive $r < \operatorname{diam}(J(T))$ we have that $m(B(\xi, r)) \gg r^h$. Now, note that for $\theta \leq h$ the inequality $m(B(\xi, r)) \ll r^\theta$ holds r*-eventually* (i.e. uniformly for arbitrary small values of r) at least for all ξ in the *uniformly-radial Julia set* $J_{ur}(T)$, where [2]

$$J_{ur}(T) := \{\xi \in J(T) : \exists\, c = c(\xi) > 0 \quad \text{such that} \quad \frac{r_i(\xi)}{r_{i+1}(\xi)} \leq c \;\; \forall\, i \in \mathbb{N}\}.$$

For $\theta > h$ this inequality is r-eventually never satisfied, for any $\xi \in J(T)$. Using the fact[3] that $\dim_H(J_{ur}(T)) = h$, it follows that

$$\dim_H(\mathcal{I}_\theta(m)) = \begin{cases} h & \text{for} \quad 0 < \theta \leq h \\ 0 & \text{for} \quad \theta > h. \end{cases}$$

Also, for $\theta \leq h$ the inequality $m(B(\xi, r)) \ll r^\theta$ holds for each $\xi \in J_r(T)$ at least for values of r in the hyperbolic zoom $(r_j(\xi))_j$ (i.e. on a decreasing sequence of radii). For $\theta > h$ there exists no such sequence which satisfies this inequality, for any $\xi \in J(T)$. Hence, we have that

$$\dim_H(\mathcal{S}_\theta(m)) = \begin{cases} h & \text{for} \quad 0 \leq \theta \leq h \\ 0 & \text{for} \quad \theta > h. \end{cases}$$

Furthermore, we see that for $\theta \geq h$ the inequality $m(B(\xi, r)) \gg r^\theta$ holds r-eventually for any $\xi \in J(T)$. For $\theta < h$ this inequality is r-eventually never satisfied, for any $\xi \in J_r(T)$. Hence, it follows that

$$\dim_H(\mathcal{S}^\theta(m)) = \begin{cases} 0 & \text{for} \quad 0 < \theta < h \\ h & \text{for} \quad \theta \geq h. \end{cases}$$

[2] Note that $J_{ur}(T) = \{\xi \in J(T) : \operatorname{dist}(T^n(\xi), \Omega) > 0 \;\; \forall n \in \mathbb{N}\}$. Also, note that $J_{ur}(T)$ is equal to the so called *hyperbolic part* of $J(T)$ (cf. e.g. [17]).

[3] which is an immediate consequence of the fact that $\dim_H(J_{ur}(T)) = \inf\{s : \exists\; s-\text{conformal measure}\}$ for all rational maps T (cf. [15]), combined with the fact that $\dim_H(J(T)) = \inf\{s : \exists\; s-\text{conformal measure}\}$ for parabolic rational maps T (cf. [5]). Alternatively, this can also be obtained, using (GF) and Theorem 3, by the methods in [20].

This gives the weak singularity spectra of m for $h<1$.

- For $h>1$, we consider the set

$$\mathcal{M}^{\sigma} := \left\{ \xi \in J_r(T) : m(B(\xi,r)) \ll_{i.o.} r^h \, r^{\sigma(h-1)p_{max}/(1+\sigma)} \right\}.$$

For $\sigma \geq h-1$, a similar argumentation as in the case '$h<1$' above gives that

$$\dim_H(\mathcal{M}^{\sigma}) = \frac{h}{1+\sigma}.$$

Then, if we let as before $\theta := h+\sigma(h-1)p_{max}/(1+\sigma)$, it follows for $h+(h-1)p_{max} - \frac{(h-1)p_{max}}{h} \leq \theta < h+(h-1)p_{max}$ that

$$\dim_H(\mathcal{S}_{\theta}(m)) = \frac{h}{(h-1)p_{max}} \left(h+(h-1)p_{max}-\theta\right).$$

Also, similar as before, we see for $\theta \geq h+(h-1)p_{max}$ that $\dim_H(\mathcal{S}_{\theta}(m))=0$, and that for $0<\theta \leq h$ we have that $\dim_H(\mathcal{S}_{\theta}(m)) = h$.
For $h<\theta<h+(h-1)p_{max}-\frac{(h-1)p_{max}}{h}$, or what is equivalent for $0<\sigma<h-1$, we see that

$$\mathcal{M}^{\sigma} = \bigcup_{\omega\in\Omega} \mathcal{J}^{\omega}_{\sigma(\omega)}(T).$$

Now, using Theorem 1, it follows, for $\omega,\eta \in \Omega$ such that $p(\eta)=p_{max}$, that

$$\begin{aligned}
\dim_H(\mathcal{J}^{\omega}_{\sigma(\omega)}(T)) &= \frac{h+\sigma(\omega)p(\omega)}{1+\sigma(\omega)(1+p(\omega))} \\
&= \ldots \\
&= \frac{h+\sigma p_{max}+h\sigma(1-p_{max}/p(\omega))}{1+\sigma(1+p_{max})} \\
&\leq \frac{h+\sigma p_{max}}{1+\sigma(1+p_{max})} \\
&= \dim_H(\mathcal{J}^{\eta}_{\sigma}(T)).
\end{aligned}$$

Hence, we have that

$$\dim_H(\mathcal{M}^{\sigma}) = \frac{h+\sigma p_{max}}{1+\sigma(1+p_{max})}.$$

Expressing this equality in terms of θ, we deduce for $h<\theta<h+(h-1)p_{max}-\frac{(h-1)p_{max}}{h}$ that

$$\dim_H(\mathcal{S}_{\theta}(m)) = \frac{(h-1)(h+(h-1)p_{max})}{(\theta-1)p_{max}} - \frac{h-p_{max}}{p_{max}}.$$

For the remaining spectra in this case, note that (GF) implies that we have, for all $\xi \in J(T)$ and positive $r<\mathrm{diam}(J(T))$, that $m(B(\xi,r)) \ll r^h$. Also, note that for $h<\theta\leq h+(h-1)p_{max}$ the inequality $m(B(\xi,r)) \ll r^{\theta}$ holds r-eventually exclusively only for certain $\xi \in J_p(T)$. For $\theta > h+(h-1)p_{max}$ we even have that for small values of r this inequality never holds. Using these observations, we derive that

$$\dim_H(\mathcal{I}_\theta(m)) = \begin{cases} h & \text{for} \quad 0 < \theta \le h \\ 0 & \text{for} \quad \theta > h. \end{cases}$$

Also, for $\theta \ge h$ the inequality $m(B(\xi, r)) \gg r^\theta$ holds for each $\xi \in J_r(T)$ at least for $r \in \{r_1(\xi), r_2(\xi), \ldots\}$ (i.e. for a decreasing sequence of radii). For $\theta < h$ there exists no such sequence which satisfies this inequality, for any $\xi \in J(T)$. Hence, we have that

$$\dim_H(\mathcal{I}^\theta(m)) = \begin{cases} 0 & \text{for} \quad 0 < \theta < h \\ h & \text{for} \quad \theta \ge h. \end{cases}$$

Finally, we see that for $\theta \ge h$ the inequality $m(B(\xi, r)) \gg r^\theta$ holds r-eventually for any $\xi \in J_{ur}(T)$. For $\theta < h$ this inequality is r-eventually never satisfied, for any $\xi \in J(T)$. Hence, using once again the fact that $\dim_H(J_{ur}(T)) = h$, it follows that

$$\dim_H(\mathcal{S}^\theta(m)) = \begin{cases} 0 & \text{for} \quad 0 < \theta < h \\ h & \text{for} \quad \theta \ge h. \end{cases}$$

This gives the weak singularity spectra of m for $h > 1$, which then completes the proof of Theorem 2.

$\square$

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B.O. Stratmann,
Mathematical Institute,
University of St Andrews,
St Andrews KY16 9SS,
Scotland.
e-mail: bos@maths.st-and.ac.uk

M. Urbański,
Department of Mathematics,
University of North Texas,
Denton, TX 76203-5118,
USA.
e-mail: urbanski@unt.edu