ABSOLUTELY CONTINUOUS MEASURES WITH L^q DENSITY ARISING FROM PARABOLIC IFS AND RANDOM CONTINUED FRACTIONS

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ABSTRACT. We continue to study parabolic iterated function systems (IFS) with overlaps on the real line and invariant measures associated with them. A shift-invariant Borel probability measure μ on the coding space projects into a measure ν on the limit set of the IFS. We consider families of IFS depending on parameter and satisfying a certain "transversality" condition. In [SSU2] sufficient conditions were found for the measure ν to be absolutely continuous for a.e. parameter value. Here we investigate when this measure has a density in L^q, for $q \in (1,2]$. We prove that if the q-dimension of the measure μ, with respect to a certain metric, is greater than one, then ν has a density in L^q for a.e. parameter value. On the other hand, if the q-dimension of μ is less than one, then the density, if it exists, cannot belong to L^q. These results are applied to a family of random continued fractions studied by R. Lyons. He proved that the distribution cannot have a density in L^2 above a certain threshold; we show that this threshold is sharp and establish the existence of L^2 density for a.e. parameter in some interval below the threshold.

1. INTRODUCTION

We continue to study parabolic iterated function systems with overlaps on the real line. In [SSU1] we investigated the Hausdorff dimension of the limit set. In [SSU2] we focused on the properties of invariant measures. An ergodic shift-invariant measure with positive entropy on the symbolic space induces an invariant (stationary) measure on the limit set of the iterated function system. The Hausdorff dimension of this measure equals the ratio of entropy over Lyapunov exponent if the iterated function system has no "overlaps". We investigated families of parabolic iterated function systems which do have overlaps but satisfy a "transversality condition". We show that for almost every member of such a family, if the entropy exceeds the Lyapunov exponent then the invariant measure is absolutely continuous, otherwise the above-mentioned formula for dimension still holds (comp. Theorem 2.7). In the present paper we explore the former case when the invariant measure is absolutely continuous.

The authors were supported in part by the NSF grants DMS 9800786 and DMS 9801583.

We would like to emphasize however that in contrast to the paper [SSU2] we do not assume here the reference measure on the shift space to be invariant. Only mixing properties play some role. We introduce two natural metrics on the shift space depending on the iterated function system considered. We recall the well-known fact that if the q-dimension of a Borel probability measure μ on the shift space is less that one, then the projection of this measure is not absolutely continuous with L^q density. If for a member of a family of iterated function systems described above the q-dimension, $q \in (1,2]$ with respect to the second measure is greater than 1 (the measure on the shift space is still assumed to be Borel probability only), then there exists a neighbourhood of this member such that the projection measure for almost every element from this neighbourhood is absolutely continuous with L^q density (see Theorem 3.3). If we assume that either the measure μ is an equilibrium (Gibbs) state for a Hölder continuous potential (for example if μ is a Bernoulli measure) or the family of iterated function systems consists of hyperbolic elements only, then the above result is true with the second metric replaced by the first oneTheorem 3.6. This statement remains also true if we consider an arbitrary Borel probability measure but we assume that its q-dimension with respect to the first measure is large enough (see Theorem 3.4). All the results described are applied for a class of random continued fractions considered by R. Lyons in [L].

2. Families of parabolic iterated function systems

Let $X \subset \mathbb{R}$ be a closed interval and $\theta \in (0,1]$. A $\mathcal{C}^{1+\theta}$ map $\phi : X \to X$ is hyperbolic if $0 < |\phi'(x)| < 1$ for all $x \in X$. We say that a $\mathcal{C}^{1+\theta}$ map $\phi : X \to X$ is **parabolic** if the following requirements are fulfilled:

- there is only one point $v \in X$ such that $\phi(v) = v$;
- $|\phi'(v)| = 1$ and $0 < |\phi'(x)| < 1$ for all $x \in X \setminus \{v\}$.
- There exists $C_1 \geq 1$ and $\beta = \beta(\phi) < \theta/(1-\theta)$ $(= \infty$ if $\theta = 1)$ such that

$$C_1^{-1} \leq \liminf_{x \to v} \frac{||\phi'(x)| - 1|}{|x-v|^\beta} \leq \limsup_{x \to v} \frac{||\phi'(x)| - 1|}{|x-v|^\beta} \leq C_1. \tag{2.1}$$

We will need the following basic lemma on iteration of a single parabolic function. For the proof see [U, L. 2.2,2.3] (see also [SSU1, L. 5.2,5.3]).

Lemma 2.1. *For every neighborhood V of v there exists a constant $L(V) \geq 1$ such that for all $x \in X \setminus V$ and all $n \geq 1$,*

$$L(V)^{-1} \leq |\phi^n(x) - v| \cdot (n+1)^{\frac{1}{\beta}} \leq L(V); \tag{2.2}$$

$$L(V)^{-1} \leq |(\phi^n)(x)| \cdot (n+1)^{\frac{\beta+1}{\beta}} \leq L(V). \tag{2.3}$$

Now, following [SSU1, SSU2], we define the class of parabolic iterated function systems (IFS) under investigation.

Definition 2.2. *Let $\Phi = \{\phi_1, \dots, \phi_k\}$ be a collection of $\mathcal{C}^{1+\theta}$ functions on a closed interval $X \subset \mathbb{R}$ such that ϕ_k is parabolic with the fixed point v and the other functions are hyperbolic. We shall write $\Phi \in \Gamma_X(\theta)$ if, in addition, $\phi_i(X) \subset \mathrm{Int}(X) \setminus \{v\}$ for all $i \leq k-1$.*

Let $\mathcal{A} = \{1, \dots, k\}$. We define the natural projection map $\pi_\Phi : \mathcal{A}^\infty \to \mathbb{R}$ by setting

$$\{\pi_\Phi(\omega)\} = \bigcap_{n \geq 1} \phi_{\omega|_n}(X)$$

where $\omega|_n = \omega_1 \dots \omega_n$ and $\phi_{\omega|_n} = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$. If $\Phi \in \Gamma_X(\theta)$ then the map π_Φ is well-defined and continuous (see [SSU1, Lemma 5.6]). We have

$$\pi_\Phi(\omega) = \phi_{\omega|_n}(\pi_\Phi(\sigma^n \omega)) \quad \text{for all } \omega \in \mathcal{A}^\infty \text{ and } n \geq 1,$$

where σ is the left shift on $\mathcal{A}^\infty$. The **limit set**, or attractor, of the IFS Φ is defined by

$$J_\Phi = \pi_\Phi(\mathcal{A}^\infty).$$

It is easy to see that J_Φ is the unique non-empty compact set such that $J_\Phi = \bigcup_{i \leq k} \phi_i(J_\Phi)$. Let $U \subset \mathbb{R}^d$ be an open set. Consider a family of parabolic IFS

$$\Phi^{\mathbf{t}} = \{\phi_1^{\mathbf{t}}, \dots, \phi_{k-1}^{\mathbf{t}}, \phi_k\} \in \Gamma_X(\theta), \ \mathbf{t} \in \overline{U}. \tag{2.4}$$

Although the parabolic function *does not* depend on the parameter, it is sometimes convenient to write $\phi_k^{\mathbf{t}} \equiv \phi_k$ for $\mathbf{t} \in U$. Let $\pi_{\mathbf{t}} : \mathcal{A}^\infty \to \mathbb{R}$ be the natural projection associated with $\Phi^{\mathbf{t}}$ and denote $J_{\mathbf{t}} = J_{\Phi^{\mathbf{t}}}$. Two conditions which control the dependence on $\mathbf{t}$ will be needed.

Continuity: the maps

$$\mathbf{t} \mapsto \phi_i^{\mathbf{t}} \text{ are continuous from } \overline{U} \text{ to } \mathcal{C}^{1+\theta}(X) \text{ for } i \leq k-1. \tag{2.5}$$

Transversality Condition: there exists a constant C_2 such that for all ω and τ in $\mathcal{A}^\infty$ with $\omega_1 \neq \tau_1$,

$$\mathcal{L}_d\{\mathbf{t} \in U : \ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\} \le C_2 r \quad \text{for all} \ \ r > 0. \tag{2.6}$$

A mild additional condition on the parabolic map ϕ_k will be needed in our theorems.

Definition 2.3. *Let us say that a parabolic function ϕ on X with the fixed point v is* **well-behaved** *on a connected open neighborhood V of v if $x = v$ is the only local extremum for $|\phi'(x)|$ on $V \cap X$.*

A function ϕ is well-behaved on some neighborhood if for some $\delta_1 > 0$ the function $|\phi'(x)|$ is concave down on $(v, v+\delta_1] \cap X$ and concave up on $[v-\delta_1, v) \cap X$ (one of the intersections is empty if v is an endpoint). Clearly, any real-analytic parabolic function is well-behaved on a neghborhood of the parabolic point.

We write $B_\delta(\mathbf{t}_0)$ for the open ball of radius δ centered at $\mathbf{t}_0$ and $\mathcal{L}_d$ for the Lebesgue measure in $\mathbb{R}^d$. If μ is a measure we often write μF without parentheses. Recall that $\mathcal{A} = \{1, \ldots, k\}$. For a finite word $w \in \mathcal{A}^n$ the corresponding cylinder set in $\mathcal{A}^\infty$ is denoted by $[w]$. For ω and τ in $\mathcal{A}^\infty$ we denote by $\omega \wedge \tau$ their common initial segment, so that $\omega, \tau \in [\omega \wedge \tau]$ and $\omega_{n+1} \neq \tau_{n+1}$ for $n = |\omega \wedge \tau|$. For $\phi \in \mathcal{C}^{1+\theta}(X)$ we shall write

$$||\phi'||_\theta = \sup\{|\phi'(x) - \phi'(y)| \cdot |x-y|^{-\theta} : \ x, y \in X\},$$

and $||\Phi'||_\theta := \max\{||\phi_i'||_\theta : \ i \in \mathcal{A}\}$ if $\Phi = \{\phi_i\}_{i \in \mathcal{A}}$ is an IFS. We denote by $||\cdot||$ the supremum norm on X. If Φ and Ψ are two IFS we write

$$||\Phi - \Psi|| = \max_{i \le k-1} ||\phi_i - \psi_i|| \quad \text{and} \quad ||\Phi' - \Psi'|| = \max_{i \le k-1} ||\phi_i' - \psi_i'||.$$

Following [SSU1, SSU2] we introduce additional notation useful for families of IFS. We write

$$\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$$

for $\Phi \in \Gamma_X(\theta)$ if V is a connected open neighborhood of the parabolic point v such that

$$V \cap \bigcup_{i=1}^{k-1} \phi_i(X) = \emptyset, \tag{2.7}$$

$$\max\{||(\phi_i)'|| : \ i \le k-1\} \le \gamma \in (0,1), \tag{2.8}$$

$$\min\{|\phi_i'(x)| : \ x \in X, \ i \le k\} \ge u \in (0,1),$$

and $||\Phi'||_\theta \le M$. By Definition 2.2, every $\Phi \in \Gamma_X(\theta)$ belongs to $\Gamma_X(\theta, V, \gamma, u, M)$ for some V, γ, u and M. Given a finite word $\omega \in \mathcal{A}^*$, let $h(\omega)$ denote the number of all hyperbolic letters (i.e. $\neq k$) appearing in ω. By (2.8),

$$\| \phi_w' \| \le \gamma^{h(w)} \qquad \text{for all} \;\; w \in \mathcal{A}^*. \tag{2.9}$$

We will need three technical lemmas from [SSU1, SSU2]. The first of them is a basic result on distortion for parabolic IFS; the second one says that if ϕ_k is well-behaved, then the derivative of a map ϕ_w cannot be too small near the parabolic point, and the third one compares the derivatives at a single point but for two distinct IFS.

Lemma 2.4. (see [SSU1, Lemma 5.8]). *There exists a constant $C_3 = C_3(X, \theta, V, \gamma, u, M) \ge 1$ such that for every $\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$, all $\omega \in \mathcal{A}^\infty$ and all $n \ge 1$,*

$$C_3^{-1} \le \frac{|\phi'_{\omega|_n}(y)|}{|\phi'_{\omega|_n}(x)|} \le C_3 \qquad \textit{for all} \;\; x, y \in X \setminus V. \tag{2.10}$$

Lemma 2.5. (see [SSU2, Lemma 4.3]). *There exists a constant $C_4 = C_4(X, \theta, V, \gamma, u, M) \ge 1$ such that for any parabolic IFS $\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$ which has the additional property that ϕ_k is well-behaved on V, the following property holds: For all $\omega \in \mathcal{A}^\infty$ and all $n \in \mathbb{N}$,*

$$\frac{|\phi'_{\omega|_n}(y)|}{|\phi'_{\omega|_n}(x)|} \le C_4 \qquad \textit{for all} \;\; x \in X \;\; \textit{and} \;\; y \in X \setminus V.$$

Lemma 2.6. (see [SSU1, Cor. 6.3]). *There exists a constant $C_5 = C_5(X, \theta, V, \gamma, u, M) > 0$ such that for any $\Phi = \{\phi_1, \dots, \phi_k\}$ and $\Psi = \{\psi_1, \dots, \psi_k\}$, two IFS in $\Gamma_X(\theta, V, \gamma, u, M)$ with $\phi_k = \psi_k$, for all $\omega \in \mathcal{A}^*$ and all $x \in X$,*

$$\frac{|\phi'_\omega(x)|}{|\psi'_\omega(x)|} \le \exp\left(C_5 h(\omega) \left(||\Phi - \Psi||^\theta + ||\Phi' - \Psi'||\right)\right) . \tag{2.11}$$

The following was the main result of [SSU2].

Theorem 2.7. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t} \in \overline{U}}$ is a family of parabolic IFS (2.4) satisfying (2.5) and (2.6), and such that ϕ_k is well-behaved on some neighborhood of v. Let μ be a shift-invariant ergodic Borel probability measure with positive entropy on $\mathcal{A}^\infty$ and $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$. Let h_μ denote its entropy with respect to the shift map $\sigma : \mathcal{A}^\infty \to \mathcal{A}^\infty$ and let*

$$\chi_\mu(\Phi) = -\int_{\mathcal{A}^\infty} \log |\phi'_{\omega_1}(\pi_\Phi(\sigma\omega))| \, d\mu(\omega)$$

be the Lyapunov exponent of the IFS Φ *with respect to the measure* μ. *Then for Lebesgue-a.e.* $\mathbf{t} \in U$,

$$\dim_{\mathrm{H}} \nu_{\mathbf{t}} = \min\left\{\frac{h_\mu}{\chi_\mu(\Phi_{\mathbf{t}})}, 1\right\}$$

and $\nu_{\mathbf{t}}$ *is absolutely continuous for a.e.* $\mathbf{t} \in \{\mathbf{t} : \ h_\mu/\chi_\mu(\Phi_{\mathbf{t}}) > 1\}$.

It is therefore tempting to ask about the properties of the Radon-Nikodym derivative of the measure $\nu_{\mathbf{t}}$ with respect to the Lebesgue measure in the case when the former measure is absolutely continuous. The next section is devoted to this problem.

3. Results and proofs

In this section we explore the problem of when the projection measures $\nu_{\mathbf{t}}$ are absolutely continuous with L^q densities. We begin by recalling the notion of the q-dimension of a probability measure on a metric space. So, if (X, ϱ) is a metric space, $q > 1$ and μ is a Borel probability measure on X, then

$$D_q(\mu) = \sup\{t : I_{t,q}(\mu) < \infty\}$$

is the (lower) q-dimension dimension of the measure μ, where $I_{t,q}(\mu)$, the (t,q)-energy of the measure μ, is given by the formula

$$I_{t,q}(\mu) = \int \left(\int \frac{d\mu(y)}{\varrho(x,y)^t}\right)^{q-1} d\mu(x).$$

In the case when $q = 2$, the q-dimension of measure μ is also called its correlation dimension.

Now consider a parabolic iterated function system $\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$ and equip the coding space $\mathcal{A}^\infty$ with two metrics d_1 and d_2 given by the formulas (for $\omega \neq \tau$):

$$d_1(\omega, \tau) = |I_{\omega\wedge\tau}|_1 := |\phi_{\omega\wedge\tau}(X)|$$

and

$$d_2(\omega, \tau) = |I_{\omega\wedge\tau}|_2 := |\phi_{\omega\wedge\tau}(X \setminus V)|,$$

where $|\cdot|$ denotes the Lebesgue measure. Notice that Lemma 2.4 implies the existence of a constant $C_6 > 0$ such that if $\kappa \in \{\omega, \tau\}$ is such that $(\sigma^{|\omega\wedge\tau|}\kappa)_1 \neq k$, then

$$C_6^{-1}|\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\kappa))| \leq |I_{\omega\wedge\tau}|_2 \leq C_6|\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\kappa))|. \tag{3.1}$$

Therefore, the metric d_2 is equivalent to the metric given by the formula

$$\widetilde{d}_2(\omega, \tau) = \max\left\{|\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\omega))|, |\phi'_{\omega\wedge\tau}(\pi(\sigma^{|\omega\wedge\tau|}\tau))|\right\}.$$

Clearly, the q-dimension depends only on the equivalence class of the metric; thus, the q-dimension corresponding to d_2 is independent of the set V. Let us denote by $D_q^{(i)}(\mu)$, $i = 1, 2$, the q-dimension of μ computed with respect to the metric d_i. It is immediate from the definition that

$$D_q^{(1)}(\mu) \geq D_q^{(2)}(\mu).$$

Of course, the metrics d_i, $i = 1, 2$, and dimensions $D_q^{(i)}(\mu)$, $i = 1, 2$, depend on the iterated function system under consideration and sometimes dealing with families of IFS, in order to avoid confusion, we will write $d_i^{(\mathbf{t})}$ and $D_q^{(i,\mathbf{t})}(\mu)$ to indicate the parameter. We begin with the proof of the following.

Theorem 3.1. *If μ is a Borel probability measure on $\mathcal{A}^\infty$ and $D_q^{(1)}(\mu) < 1$, then $\mu \circ \pi^{-1}$ is not absolutely continuous with L^q density.*

Proof. Indeed, since the projection $\pi : (\mathcal{A}^\infty, d_1) \to \mathbb{R}$ is Lipshitz continuous with Lipshitz constant 1, we have $D_q(\mu \circ \pi^{-1}) \leq D_q^{(1)}(\mu) < 1$. Since it is well-known that for any Borel measure ν with compact support on $\mathbb{R}$, inequality $D_q(\nu) < 1$ implies that ν is not absolutely continuous with L^q density, we are done. □

We shall now prove a straightforward but useful lemma.

Lemma 3.2. *Suppose that μ is a Borel probability measure on $\mathcal{A}^\infty$ and $s < D_q^{(i)}(\mu)$, $i = 1, 2$. Suppose that for μ-almost every $\omega \in \mathcal{A}^\infty$ there exist two increasing sequences $\{k_n(\omega)\}_{n \geq 0}$ and $\{l_n(\omega)\}_{n \geq 0}$ such that $k_n(\omega) \leq l_n(\omega) \leq k_{n+1}(\omega)$ for every $n \geq 0$ and the functions $\omega \mapsto k_n(\omega), l_n(\omega)$ are Borel measurable, then*

$$\int_{\mathcal{A}^\infty} \left(\sum_{n \geq 0} \frac{\mu\big([\omega|_{k_n(\omega)}] \setminus [\omega|_{l_n(\omega)}]\big)}{|I_{\omega|_{k_n(\omega)}}|_i^s} \right)^{q-1} d\mu(\omega) \leq I_{s,q}^{(i)}(\mu) < \infty.$$

Note that the sequences $\{k_n(\omega)\}$ and $\{l_n(om)\}$ may be finite or infinite.

Proof. Since $s < D_q^{(i)}(\mu)$, we have $I_{s,q}^{(i)}(\mu) < \infty$. Therefore, since all the sets $\{[\omega|_{k_n(\omega)}] \setminus [\omega|_{l_n(\omega)}]\}_{n\geq 0}$ are mutually disjoint, we can estimate as follows

$$\begin{aligned}\infty > I_{s,q}^{(i)}(\mu) &= \int_{\mathcal{A}^\infty} \left(\int_{\mathcal{A}^\infty} \frac{d\mu(\tau)}{|I_{\omega\wedge\tau}|_i^s} \right)^{q-1} d\mu(\omega) \\ &\geq \int_{\mathcal{A}^\infty} \left(\sum_{n\geq 0} \int_{[\omega|_{k_n(\omega)}]\setminus[\omega|_{l_n(\omega)}]} \frac{d\mu(\tau)}{|I_{\omega\wedge\tau}|_i^s} \right)^{q-1} d\mu(\omega) \\ &\geq \int_{\mathcal{A}^\infty} \left(\sum_{n\geq 0} \frac{\mu\big([\omega|_{k_n(\omega)}] \setminus [\omega|_{l_n(\omega)}]\big)}{|I_{\omega|_{k_n(\omega)}}|_i^s} \right)^{q-1} d\mu(\omega).\end{aligned}$$

The proof is complete. □

We are now in a position to prove the first new result of this paper. Its proof combines the methods of [PSo2, Th.4.1] and [SSU2].

Theorem 3.3. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS (2.4) satisfying (2.5) and (2.6), and such that ϕ_k is well-behaved on some neighborhood of v. Suppose that μ is a Borel atomless probability measure on $\mathcal{A}^\infty$ and that $D_q^{(2,\mathbf{t}_0)}(\mu) > 1$ for some $q \in (1,2]$ and $\mathbf{t}_0 \in U$, where $D_q^{(2,\mathbf{t}_0)}(\mu)$ is the q-dimension of μ computed with respect to the metric d_2 associated with the IFS $\Phi^{\mathbf{t}_0}$. Then there exists $\delta > 0$ such that for $\mathcal{L}_d$-a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$, the measure $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$ is absolutely continuous with L^q density.*

Proof. Write $\gamma = e^{-b}$, with $b > 0$, and $\Phi = \Phi^{\mathbf{t}_0} = \{\phi_i : i \in \mathcal{A}\}$. Fix $1 < s < D_q^{(2,\mathbf{t}_0)}(\mu)$ and $\varepsilon = b(s-1)$. In view of Lemma 2.6, there exists $\delta > 0$ so small that for all $w \in \mathcal{A}^*$ and all $x \in X$,

$$|\mathbf{t} - \mathbf{t}_0| \leq \delta \implies \frac{|\phi_w'(x)|}{|(\phi_w^{\mathbf{t}})'(x)|} \leq e^{\varepsilon h(w)}. \tag{3.2}$$

We are going to show that

$$\mathcal{I} = \int_{B_\delta(\mathbf{t}_0)} \int_{\mathbb{R}} \underline{D}(\nu_{\mathbf{t}}, x)^{q-1} \, d\nu_{\mathbf{t}}(x) \, d\mathbf{t} < \infty, \tag{3.3}$$

where

$$\underline{D}(\nu_{\mathbf{t}}, x) = \liminf_{r\searrow 0} \frac{\nu_{\mathbf{t}}[x-r, x+r]}{2r}$$

is the lower density of the measure $\nu_{\mathbf{t}}$ at the point x. Then by [Mat, 2.12], the measure $\nu_{\mathbf{t}}$ is absolutely continuous for a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$. For such $\mathbf{t}$ we will have $\underline{D}(\nu_{\mathbf{t}}, x) = \frac{d\nu_{\mathbf{t}}}{dx}$, so (3.3) will imply that $\frac{d\nu_{\mathbf{t}}}{dx} \in L^q(\mathbb{R})$ for a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$.

First we apply Fatou's Lemma and then make a change of variable to get

$$\begin{aligned}\mathcal{I} &\leq \liminf_{r\searrow 0}\int_{B_\delta(\mathbf{t}_0)}\int_{\mathbb{R}}\frac{\big(\nu_{\mathbf{t}}[x-r,x+r]\big)^{q-1}}{(2r)^{q-1}}\,d\nu_{\mathbf{t}}(x)\,d\mathbf{t}\\ &= \liminf_{r\searrow 0}\int_{B_\delta(\mathbf{t}_0)}\int_{\mathcal{A}^\infty}\frac{\big(\nu_{\mathbf{t}}B_r(\pi_{\mathbf{t}}(\omega))\big)^{q-1}}{(2r)^{q-1}}\,d\mu(\omega)\,d\mathbf{t}.\end{aligned}\tag{3.4}$$

Next we reverse the order of integration and use Hölder's inequality (recall that $q\in(1,2]$) to obtain

$$\mathcal{I}\preceq \liminf_{r\searrow 0}(2r)^{1-q}\int_{\mathcal{A}^\infty}\left(\int_{B_\delta(\mathbf{t}_0)}\nu_{\mathbf{t}}B_r(\pi_{\mathbf{t}}(\omega))\,d\mathbf{t}\right)^{q-1}\,d\mu(\omega).\tag{3.5}$$

We have, denoting by $\mathbf{1}_E$ the characteristic function of a set E,

$$\begin{aligned}\int_{B_\delta(\mathbf{t}_0)}\nu_{\mathbf{t}}B_r(\pi_{\mathbf{t}}(\omega))\,d\mathbf{t} &= \int_{B_\delta(\mathbf{t}_0)}\int_{\mathbb{R}}\mathbf{1}_{B_r(\pi_{\mathbf{t}}(\omega))}\,d\nu_{\mathbf{t}}\,d\mathbf{t}\\ &= \int_{B_\delta(\mathbf{t}_0)}\int_{\mathcal{A}^\infty}\mathbf{1}_{\{\tau\in\mathcal{A}^\infty:|\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)|\leq r\}}\,d\mu(\tau)\,d\mathbf{t}\\ &= \int_{\mathcal{A}^\infty}\mathcal{L}_d\{\mathbf{t}\in B_\delta(\mathbf{t}_0):\ |\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)|\leq r\}\,d\mu(\tau)\\ &= \int_{\mathcal{A}^\infty\setminus\{\omega\}}\mathcal{L}_d\{\mathbf{t}\in B_\delta(\mathbf{t}_0):\ |\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)|\leq r\}\,d\mu(\tau).\end{aligned}\tag{3.6}$$

We could write the last equality since μ is atomless. Now take an arbitrary $\tau\in\mathcal{A}^\infty\setminus\{\omega\}$ and denote $\rho=\omega\wedge\tau$. Then we have for some $c\in[\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega),\pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)]$ using (3.2):

$$\begin{aligned}|\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)| &= |(\phi_\rho^{\mathbf{t}})'(c)|\cdot|\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega)-\pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)|\\ &\geq |\phi_\rho'(c)|e^{-h(\rho)\varepsilon}\cdot|\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega)-\pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)|\,.\end{aligned}$$

Since $\omega_{|\rho|+1}\neq\tau_{|\rho|+1}$, at least for one element $\kappa\in\{\omega,\tau\}$ we have $\kappa_{|\rho|+1}\neq k$. Since $(\sigma^{|\rho|}\kappa)_1\neq k$, we have $\pi(\sigma^{|\rho|}\kappa)\notin V$ by (2.7). Therefore, by Lemma 2.5 and (3.1),

$$\begin{aligned}|\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)| &\geq C_4^{-1}|\phi_\rho'(\pi(\sigma^{|\rho|}\kappa))|\cdot e^{-h(\rho)\varepsilon}|\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega)-\pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)|\\ &\geq (C_4C_6)^{-1}|I_\rho|_2\cdot e^{-h(\rho)\varepsilon}|\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega)-\pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)|.\end{aligned}\tag{3.7}$$

It follows that

$$
\begin{aligned}
&\mathcal{L}_\delta\{\mathbf{t}\in B_d(\mathbf{t}_0):\ |\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)|\le r\}\\
\le&\mathcal{L}_d\left\{\mathbf{t}\in B_\delta(\mathbf{t}_0):\ |\pi_{\mathbf{t}}(\sigma^{|\rho|}\omega)-\pi_{\mathbf{t}}(\sigma^{|\rho|}\tau)|\le \frac{C_4C_6e^{h(\rho)\varepsilon}r}{|I_\rho|_2}\right\}\\
\le&C_2C_4C_6e^{h(\rho)\varepsilon}r|I_\rho|_2^{-1},
\end{aligned}
$$

by the transversality condition (2.6). Combining this with (3.5) and (3.6) yields

$$
\begin{aligned}
\mathcal{I}&\preceq\int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}}e^{h(\rho)\varepsilon}|I_\rho|_2^{-1}\,d\mu(\tau)\right)^{q-1}d\mu(\omega)\\
&=\int_{\mathcal{A}^\infty}\left(\sum_{n=0}^\infty e^{h(\omega|_n)\varepsilon}|I_{\omega|_n}|_2^{-1}\mu\big([\omega|_n]\setminus[\omega|_{n+1}]\big)\right)^{q-1}d\mu(\omega).
\end{aligned}
\tag{3.8}
$$

In view of (3.1) and (2.9), and using that $\gamma=e^{-b}$, $\varepsilon=b(s-1)$, we can continue our estimate of $\mathcal{I}$ as follows

$$
\begin{aligned}
\mathcal{I}&\preceq\int_{\mathcal{A}^\infty}\left(\sum_{n=0}^\infty e^{h(\omega|_n)\varepsilon}|I_{\omega|_n}|_2^{s-1}|I_{\omega|_n}|_2^{-s}\mu\big([\omega|_n]\setminus[\omega|_{n+1}]\big)\right)^{q-1}d\mu(\omega)\\
&\preceq\int_{\mathcal{A}^\infty}\left(\sum_{n=0}^\infty e^{(\varepsilon-(s-1)b)h(\omega|_n)}|I_{\omega|_n}|_2^{-s}\mu\big([\omega|_n]\setminus[\omega|_{n+1}]\big)\right)^{q-1}d\mu(\omega)\\
&\preceq\int_{\mathcal{A}^\infty}\left(\sum_{n=0}^\infty |I_{\omega|_n}|_2^{-s}\mu\big([\omega|_n]\setminus[\omega|_{n+1}]\big)\right)^{q-1}d\mu(\omega)<\infty.
\end{aligned}
$$

In the last line we used Lemma 3.2, with $k_n(\omega)=n$ and $l_n(\omega)=n+1$. The proof is complete. $\square$

The last result of this section concerning general (not necessarily "mixing") measures is the following. Recall that $\beta>0$ appears in the definition of parabolic map (2.1).

Theorem 3.4. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS (2.4) satisfying (2.5) and (2.6), and such that ϕ_k is well-behaved on some neighborhood of v. Suppose that μ is a Borel atomless probability measure on $\mathcal{A}^\infty$ and that $D_q^{(1,\mathbf{t}_0)}(\mu)>1+\beta$ for some $\mathbf{t}_0\in U$, where $D_q^{(1,\mathbf{t}_0)}(\mu)$ is the q-dimension of μ calculated with respect to the metric d_1 associated with the IFS $\Phi^{\mathbf{t}_0}$. Then there exists $\delta>0$ such that for $\mathcal{L}_d$-a.e. $\mathbf{t}\in B_\delta(\mathbf{t}_0)$, the measure $\nu_{\mathbf{t}}=\mu\circ\pi_{\mathbf{t}}^{-1}$ is absolutely continuous with L^q density.*

Proof. We repeat the proof of Theorem 3.3 until (3.8) almost word by word. We assume that $\beta+1<s<D_q^{(1)}(\mu)$ but still take $\varepsilon=b(s-1)$. We use only the first line of (3.7), keeping $|\phi'_\rho(\pi(\sigma^{|\rho|}\kappa))|$ instead of $|I_\rho|_2$ and obtain

$$\mathcal{I}\preceq\int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}}e^{h(\rho)\varepsilon}|\phi'_\rho(\pi(\sigma^{|\rho|}\kappa))|^{-1}d\mu(\tau)\right)^{q-1}d\mu(\omega)|\phi'_\rho(\pi(\sigma^{|\rho|}\kappa))|.$$

Write $\rho=\alpha k^{p(\rho)}$, where the last letter of α is different from k. Since $(\sigma^{|\rho|}\kappa)_1\neq k$, we have $\pi(\sigma^{|\rho|}\kappa)\notin V$. Therefore, applying Lemma 2.1 and Lemma 2.4, we get

$$|I_\rho|_1\asymp||\phi'_\alpha||(p(\rho)+1)^{-\frac{1}{\beta}}\quad\text{ and }\quad|\phi'_\rho(\pi(\sigma^{|\rho|}\kappa))|\asymp||\phi'_\alpha||(p(\rho)+1)^{-\frac{\beta+1}{\beta}}.$$

Hence

$$\begin{aligned}\mathcal{I}&\preceq\int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}}e^{h(\rho)\varepsilon}||\phi'_\alpha||^{-1}(p(\rho)+1)^{\frac{\beta+1}{\beta}}d\mu(\tau)\right)^{q-1}d\mu(\omega)\\&=\int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}}e^{h(\rho)\varepsilon}\left(||\phi'_\alpha||(p(\rho)+1)^{-\frac{1}{\beta}}\right)^{-s}||\phi'_\alpha||^{s-1}(p(\rho)+1)^{\frac{\beta+1}{\beta}-\frac{s}{\beta}}d\mu(\tau)\right)^{q-1}d\mu(\omega)\\&\preceq\int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}}e^{h(\rho)\varepsilon-bh(\rho)(s-1)}(p(\rho)+1)^{\frac{\beta+1-s}{\beta}}|I_\rho|_1^{-s}d\mu(\tau)\right)^{q-1}d\mu(\omega)\\&\leq\int_{\mathcal{A}^\infty}\left(\int_{\mathcal{A}^\infty\setminus\{\omega\}}|I_\rho|_1^{-s}d\mu(\tau)\right)^{q-1}d\mu(\omega)\leq I_{s,q}(\mu)<\infty.\end{aligned}$$

Above, in the third line we used that $\|\phi'_\alpha\|\leq\gamma^{h(\alpha)}=\gamma^{h(\rho)}=e^{-bh(\rho)}$. The proof is complete. □

We would like to conclude this section by showing that for certain shift-invariant measures (including Gibbs states and, in particular, Bernoulli measures), Theorem 3.3 is in a sense optimal. We call a Borel probability measure μ on $\mathcal{A}^\infty$ well-mixing if $\mu[i]<1$ for every $i\in\mathcal{A}$ and if there exist constants $C_7>0$ and $\ell\geq1$ such that if $\alpha,\eta,\zeta\in\mathcal{A}^*$, and $|\eta|\geq\ell$, then

$$\mu[\alpha\eta\zeta]\leq C_7\mu[\alpha]\mu[\zeta].$$

It can be demonstrated by an easy induction that if μ is well-mixing then there exist constants $C_8>0$ and $0<\lambda<1$ such that for every $n\geq0$, every $\alpha\in\mathcal{A}^*$ and every $\eta\in\mathcal{A}^n$

$$\mu[\alpha\eta]\leq C_8\lambda^n\mu[\alpha].\tag{3.9}$$

Lemma 3.5. *If μ is a Borel probability well-mixing measure on $\mathcal{A}^\infty$, then $D_q^{(1)}(\mu)=D_q^{(2)}(\mu)$ for every $q>1$.*

Proof. Since $D_q^{(1)}(\mu) \geq D_q^{(2)}(\mu)$, we are only left to show that $D_q^{(1)}(\mu) \leq D_q^{(2)}(\mu)$. So, fix $s < D_q^{(1)}(\mu)$. We need to demonstrate that $s < D_q^{(2)}(\mu)$. And indeed, fix $\omega, \tau \in \mathcal{A}^\infty$, $\omega \neq \tau$, and write $\rho := \omega \wedge \tau = \alpha k^{p(\rho)}$, where the last letter of $\alpha = \alpha(\rho)$ is different from k. As we have already seen in the proof of Theorem 3.4, $|\phi_\rho'(\pi(\sigma^{|\rho|}\kappa))| \asymp ||\phi_\alpha'||(p(\rho)+1)^{-\frac{\beta+1}{\beta}}$, where $\kappa \in \{\omega, \tau\}$ is that element for which $(\sigma^{|\rho|}\kappa)_1 \neq k$. Therefore, using (3.1), we can estimate as follows

$$\begin{aligned} I_{s,q}(\mu) &= \int_{\mathcal{A}^\infty} \left(\frac{d\mu(\tau)}{d_2(\omega,\tau)^s}\right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\frac{d\mu(\tau)}{|\phi_\rho'(\pi(\sigma^{|\rho|}\kappa))|^s}\right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\int_{\mathcal{A}^\infty} ||\phi_\alpha'||^{-s}(p(\rho)+1)^{\frac{\beta+1}{\beta}s}\, d\mu(\tau)\right)^{q-1} d\mu(\omega) \\ &\preceq \int_{\mathcal{A}^\infty} \left(\int_{\mathcal{A}^\infty} |I_\alpha|_1^{-s}(p(\rho)+1)^{\frac{\beta+1}{\beta}s}\, d\mu(\tau)\right)^{q-1} d\mu(\omega) \end{aligned} \tag{3.10}$$

We shall now make preparations for applying Lemma 3.2. First, given $\omega \in \mathcal{A}^\infty$ we define the sequence $\{k_n(\omega)\}_{n\geq 0}$ to be the increasing enumeration of all elements $i \geq 1$ such that ω_i is hyperbolic (we let $k_0(\omega) = 0$). Then we fix an arbitrary $\ell \geq 1$ such that $C_8\lambda^\ell < 1$ and define

$$l_n(\omega) = \min\{k_n(\omega) + \ell, k_{n+1}(\omega)\} \tag{3.11}$$

and

$$C_9 = C_8/(1 - C_8\lambda^\ell). \tag{3.12}$$

Note that both $\omega \mapsto k_n(\omega)$ and $\omega \mapsto l_n(\omega)$ are Borel measurable. Note that

$$p(\omega|_{k_n(\omega)+i}) = p(\omega|_{k_n(\omega)}k^i) = i \qquad \text{for } \ 0 \leq i \leq k_{n+1}(\omega) - k_n(\omega) - 1,$$

so we can now continue (3.10) as follows

$$I_{s,q}(\mu) \leq \int_{\mathcal{A}^\infty} \left(\sum_{n\geq 0} |I_{\omega|_{k_n(\omega)}}|_1^{-s} \sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} (i+1)^{\frac{\beta+1}{\beta}s} \mu\big([\omega_{k_n(\omega)+i}] \setminus [\omega_{k_n(\omega)+i+1}]\big)\right)^{q-1} d\mu(\omega). \tag{3.13}$$

In order to apply Lemma 3.2 we will need the estimate

$$\mu[\omega|_{k+j\ell}] - \mu[\omega|_{k+(j+1)\ell}] \leq C_9\lambda^j \ (\mu[\omega|_k] - \mu[\omega|_{k+\ell}]) \tag{3.14}$$

for every $k \geq 1$ and every $j \geq 0$. In order to prove (3.14), it suffices to demonstrate that

$$\mu[\omega|_{k+j\ell}] + C_9\lambda^j\mu[\omega|_{k+\ell}] \leq C_9\lambda^j\mu[\omega|_k].$$

And indeed, applying (3.9) and (3.12), we get

$$\begin{aligned}
\mu[\omega|_{k+j\ell}] + C_9\lambda^j\mu[\omega|_{k+\ell}] &\leq \mu[\omega|_k]C_8\lambda^{j\ell} + C_9\lambda^j\mu[\omega|_k]C_8\lambda^\ell \\
&= \mu[\omega|_k]C_8\lambda^j(\lambda^{j(\ell-1} + C_9\lambda^\ell) \\
&\leq \mu[\omega|_k]C_8\lambda^j(1 + C_9\lambda^\ell) \\
&= C_9\lambda^j\mu[\omega|_k].
\end{aligned}$$

Now fix $n \geq 0$. There are two possibilities for $l_n(\omega)$ in (3.11). First suppose that $l_n(\omega) = k_n(\omega) + \ell$. We are going to estimate the inner sum in (3.13) by writing $\sum_i \leq \sum_{i=0}^\infty = \sum_{j=0}^\infty \sum_{i=j\ell}^{(j+1)\ell}$. Using this and later (3.14) we obtain

$$\begin{aligned}
&\sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} (i+1)^{\frac{b+1}{\beta}s}\mu\big([\omega_{k_n(\omega)+i}] \setminus [\omega_{k_n(\omega)+i+1}]\big) \\
&\leq \sum_{i=0}^{\infty}(i+1)^{\frac{b+1}{\beta}s}\big(\mu[\omega_{k_n(\omega)+i}] - \mu[\omega_{k_n(\omega)+i+1}]\big) \\
&\leq \sum_{j=0}^{\infty}((j+1)\ell+1)^{\frac{\beta+1}{\beta}s}\big(\mu[\omega_{k_n(\omega)+j\ell}] - \mu[\omega_{k_n(\omega)+(j+1)\ell}]\big) \\
&\preceq \left(\sum_{j=0}^{\infty}(j+1)^{\frac{\beta+1}{\beta}s}\lambda^j\right)\big(\mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{k_n(\omega)+\ell}]\big) \\
&\preceq \mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{l_n(\omega)}].
\end{aligned} \tag{3.15}$$

The second possibility is that $l_n(\omega) = k_{n+1}(\omega)$. Then $k_{n+1}(\omega) - k_n(\omega) \leq \ell$ and we get

$$\begin{aligned}
&\sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} (i+1)^{\frac{\beta+1}{\beta}s}\mu\big([\omega_{k_n(\omega)+i}] \setminus [\omega_{k_n(\omega)+i+1}]\big) \\
&\leq \big(k_{n+1}(\omega) - k_n(\omega)\big)^{1+\frac{\beta+1}{\beta}s} \sum_{i=0}^{k_{n+1}(\omega)-k_n(\omega)-1} \big(\mu[\omega_{k_n(\omega)+i}] - \mu[\omega|_{k_n(\omega)+i+1}]\big) \\
&\leq \ell^{1+\frac{\beta+1}{\beta}s}\big(\mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{k_{n+1}(\omega)}]\big) \\
&\preceq \mu[\omega|_{k_n(\omega)}] - \mu[\omega|_{l_n(\omega)}].
\end{aligned}$$

Using this and (3.15) and applying then Lemma 3.2, we can continue (3.13) as follows

$$I_{s,q}(\mu) \le \int_{\mathcal{A}^\infty} \left(\sum_{n\ge 0} |I_{\omega|_{k_n(\omega)}}|_1^{-s} \big(\mu([\omega|_{k_n(\omega)}]) - \mu([\omega|_{l_n(\omega)}])\big) \right)^{q-1} d\mu(\omega) < \infty.$$

Thus, the proof of Lemma 3.5 is complete. □

Theorem 3.6. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS (2.4) satisfying (2.5) and (2.6), and such that ϕ_k is well-behaved on some neighborhood of v. Suppose that μ is a well-mixing measure on $\mathcal{A}^\infty$ and that $D_q^{(1,\mathbf{t}_0)}(\mu) > 1$ for some $q \in (1, 2]$ and $\mathbf{t}_0 \in U$, where $D_q^{(1,\mathbf{t}_0)}(\mu)$ is the q-dimension of μ computed with respect to the metric d_1 associated with the IFS $\Phi^{\mathbf{t}_0}$. Then there exists $\delta > 0$ such that for $\mathcal{L}_d$-a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$, the measure $\nu_{\mathbf{t}} = \mu \circ \pi_t^{-1}$ is absolutely continuous with L^q density.*

Proof. This theorem is an immediate consequence of Theorem 3.3 and Lemma 3.5. □

Since each Bernoulli measure is obviously well-mixing and since all the Gibbs (equilibrium) states of Hölder continuous potential (by the way, each Bernoulli measure, say, generated by a probability vector $(p_1, p_2, \dots, p_k)$ is the Gibbs state of the Hölder continuous potential $\omega \mapsto -\log p_{\omega_1}$, $\omega \in \mathcal{A}^\infty$) are also well-mixing (see [Bo]) we get the following.

Corollary 3.7. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS (2.4) satisfying (2.5) and (2.6), and such that ϕ_k is well-behaved on some neighborhood of v. Suppose that μ is a a Gibbs (equilibrium) state of Hölder continuous potential (in particular a Bernoulli measure) on $\mathcal{A}^\infty$ and that $D_q^{(1,\mathbf{t}_0)}(\mu) > 1$ for some $\mathbf{t}_0 \in U$, where $D_q^{(1,\mathbf{t}_0)}(\mu)$ is the correlation dimension of μ computed with respect to the metric d_2 associated with the IFS $\Phi^{\mathbf{t}_0}$. Then there exists $\delta > 0$ such that for $\mathcal{L}_d$-a.e $\mathbf{t} \in B_\delta(\mathbf{t}_0)$, the measure $\mu \circ \pi_t^{-1}$ is absolutely continuous with L^2 density.*

The following corollary is an immediate consequence of Corollary 3.7 and the fact that $\frac{d}{dx}\left(\frac{x+\alpha}{x+\alpha+1}\right) = (\frac{1}{x+\alpha+1})^2$ decreases with α.

Corollary 3.8. *There exists $\alpha_q \le \alpha_0$ such that for Lebesgue almost every $\alpha \in (0.215, \alpha_1)$, ν_a is absolutely continuous, and for every $\alpha > \alpha_q$, ν_α cannot have a density in L^q.*

In terminology of Corollary 3.8 the main Lyons' from Section 3 of [L] says that $\alpha_1 \le \sqrt{6}/2 - 1 = 0.2247\dots$.

Since for hyperbolic systems the two metrics d_1 and d_2 introduced in Section 5 are equal, the proof of Theorem 5.4 (simplified by the observation that in hyperbolic case $h(\rho) = |\rho|$) demonstrates also the following.

Theorem 3.9. *Let $U \subset \mathbb{R}^d$ be an open set. Suppose that $\Phi^{\mathbf{t}} = \{\phi_1^{\mathbf{t}}, \dots, \phi_k^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}} \in \Xi_X(\theta)$ is a family of hyperbolic IFS such that the mappings $\mathbf{t} \mapsto \phi_i^{\mathbf{t}}$ are continuous from $\overline{U}$ to $\mathcal{C}^{1+\theta}(X)$ for all $i \le k$ and the transversality condition holds. Suppose that μ is a Borel atomless probability measure on $\mathcal{A}^\infty$ and that $D_q^{(1,\mathbf{t}_0)}(\mu) > 1$ for some $q \in (1, 2]$ and $\mathbf{t}_0 \in U$, where $D_q^{(1,\mathbf{t}_0)}(\mu)$ is the q-dimension of μ computed with respect to the metric d_1 associated with the IFS $\Phi^{\mathbf{t}_0}$. Then there exists $\delta > 0$ such that for $\mathcal{L}_d$-a.e. $\mathbf{t} \in B_\delta(\mathbf{t}_0)$, the measure $\mu \circ \pi_t^{-1}$ is absolutely continuous with L^q density.*

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