INVARIANT MEASURES FOR PARABOLIC IFS WITH OVERLAPS AND RANDOM CONTINUED FRACTIONS

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Abstract. We study parabolic iterated function systems (IFS) with overlaps on the real line. An ergodic shift-invariant measure with positive entropy on the symbolic space induces an invariant measure on the limit set of the IFS. The Hausdorff dimension of this measure equals the ratio of entropy over Lyapunov exponent if the IFS has no "overlaps". We focus on the overlapping case and consider parameterized families of IFS, satisfying a transversality condition. Our main result is that the invariant measure is absolutely continuous for a.e. parameter such that the entropy is greater than the Lyapunov exponent. If the entropy does not exceed the Lyapunov exponent, then their ratio gives the Hausdorff dimension of the invariant measure for a.e. parameter value, and moreover, the local dimension of the exceptional set of parameters can be estimated. These results are applied to a family of random continued fractions studied by R. Lyons. He proved singularity above a certain threshold; we show that this threshold is sharp and establish absolute continuity for a.e. parameter in some interval below the threshold.

1. Introduction

Let

$$[a_1, a_2, a_3, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Motivated by a problem in Ergodic Theory on Galton-Watson trees, R. Lyons [Ly] considered the distribution ν_α of the random continued fraction $[1, Y_1, 1, Y_2, 1, Y_3, \dots]$ where Y_i are i.i.d. and $Y_i = 0$ or α with probabilities $(\frac{1}{2}, \frac{1}{2})$. Here $\alpha > 0$ is a parameter. Equivalently, ν_α can be defined as the stationary measure of the random matrix product

$$\begin{pmatrix} 1 & Y_n \\ 1 & 1+Y_n \end{pmatrix} \cdots \begin{pmatrix} 1 & Y_1 \\ 1 & 1+Y_1 \end{pmatrix}, \tag{1.1}$$

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see [BL]. It is of interest to determine whether ν_α, for a given α, is singular or absolutely continuous (it is not hard to see that the distribution cannot be of mixed type). It turns out that ν_α is supported on a Cantor set of zero Lebesgue measure for $\alpha > 0.5$, hence ν_α is singular. However, for $\alpha \in (0, 0.5]$ the support of ν_a is the interval $X_\alpha := [0, \frac{1}{2}(-\alpha + \sqrt{a^2 + 4\alpha})]$, so the question becomes more delicate. Let $\widetilde{\chi}_\alpha$ be the top Lyapunov exponent of the random matrix product (1.1). Lyons [Ly] proved that ν_α is singular for all $\alpha \in (\alpha_c, 0.5]$ where $\alpha_c \in (0.2688, 0.2689)$ is the only positive number satisfying $\log 2 = 2\chi_{\alpha_c}$. Absolute continuity was not proved for any value of α, but Lyons conjectured that ν_α is absolutely continuous for all α sufficiently close to zero.

We make progress on this conjecture and show that the threshold α_c is sharp. In fact, we prove that ν_α is absolutely continuous for a.e. $\alpha \in (\alpha_0, \alpha_c)$, for some α_0. (The value of $\alpha_0 = 0.215$, that we obtain, has no special significance; we still don't know if the result holds for $\alpha_0 = 0$.)

This problem can be recast in the framework of iterated function systems (IFS). The measure ν_α is an invariant measure for the IFS $\Phi^\alpha = \{\phi_1^a, \phi_2\} := \{\frac{x+\alpha}{x+\alpha+1}, \frac{x}{x+1}\}$. The measure ν_α is supported on the **limit set of the IFS**, defined as the unique non-empty compact set satisfying $J_\alpha = \phi_1^a(J_\alpha) \cup \phi_2(J_\alpha)$. For $\alpha \in (0, 0.5]$ we have $J_\alpha = X_\alpha$, an interval, and for $\alpha \in (0, 0.5)$ the intersection $\phi_1^\alpha(J_\alpha) \cap \phi_2(J_\alpha)$ is itself a non-empty interval (see Figure 1 for the case $\alpha = \frac{1}{4}$). Thus, we say that this IFS has an **overlap**. Another complication is that this IFS is **parabolic** (therefore, not strictly contracting), because ϕ_2 has a neutral fixed point at $x = 0$.

Our approach is to consider Φ^α as a family of IFS depending on parameter and establish results for a.e. parameter value. Earlier work in this framework revealed the importance of a **transversality condition** in the parameter dependence, see [PoS, So1, PSo1, PSo2, So2, SSo, SSU]. We consider more general parabolic IFS; they are defined precisely in the next section. Projecting an ergodic shift-invariant measure μ from the symbolic space to the limit set, we obtain an invariant measure $\nu = \nu(\Phi, \mu)$ for the IFS. One can consider the entropy h_μ and the Lyapunov exponent χ_μ for the IFS. The Hausdorff dimension of ν_α is defined by $\dim_{\mathrm{H}} \nu_\alpha = \inf\{\dim_{\mathrm{H}}(Y) : \ \nu_\alpha(\mathbb{R} \setminus Y) = 0\}$. Our main result says, roughly speaking, that if a family of IFS satisfies the transversality condition, then the following holds for a.e. parameter value:

$$\dim_{\mathrm{H}} \nu = \min\{1, h_\mu/\chi_\mu\}, \textit{ and } \nu \textit{ is absolutely continuous if } h_\mu/\chi_\mu > 1.$$

We should note that the formula **"dimension = entropy/Lyapunov exponent"** has been established in many settings, but usually in the cases when there is no overlap, see, *e.g.*,

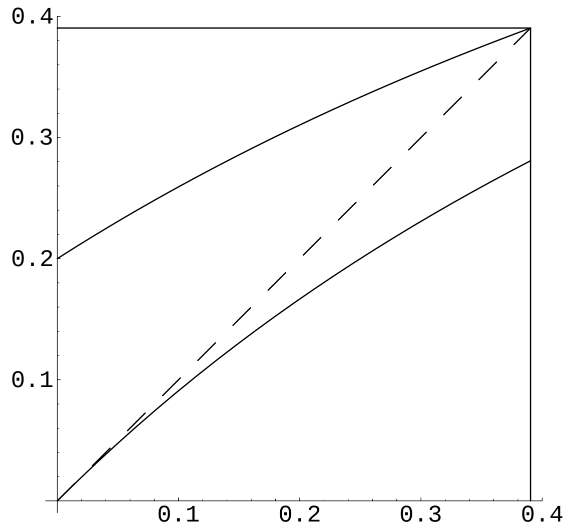

FIGURE 1. The iterated function system $\{\frac{x+1/4}{x+5/4}, \frac{x}{x+1}\}$

[Eg, KP, Mn, Y, Ma, MU]. Stationary measures for 2×2 random matrices (of which ν_α is an example), and their dimension properties, have been investigated by Ledrappier [L] (see also [BL]). Before [Ly], Pincus [Pi] studied Bernoulli random matrices and their stationary measures using the IFS approach; he found some sufficient conditions for singularity in the overlapping case.

The paper is organized as follows. The next section contains definitions and the statement of main result. Section 3 is devoted to preliminaries and proof of the upper estimate. In Section 4 the main theorem is proved. In Section 5 we estimate the local dimension of the exceptional set of parameters in our main theorem. In Section 6 we prove the results on random continued fractions; the main difficulty is checking transversality. Section 7 contains concluding remarks; in particular, we present the (much easier) hyperbolic analog of our main theorem.

A preliminary version of this paper was circulated as a preprint [SSU2].

2. Definitions and statement of main result

Let $X \subset \mathbb{R}$ be a closed interval and $\theta \in (0,1]$. A $\mathcal{C}^{1+\theta}$ map $\phi : X \to X$ is **hyperbolic** if $0 < |\phi'(x)| < 1$ for all $x \in X$. We say that a $\mathcal{C}^{1+\theta}$ map $\phi : X \to X$ is **parabolic** if the following requirements are fulfilled:

- there is only one point $v \in X$ such that $\phi(v) = v$;
- $|\phi'(v)| = 1$ and $0 < |\phi'(x)| < 1$ for all $x \in X \setminus \{v\}$.
- There exists $L_1 \geq 1$ and $\beta = \beta(\phi) < \theta/(1-\theta)$ ($= \infty$ if $\theta = 1$) such that
$$L_1^{-1} \leq \liminf_{x \to v} \frac{||\phi'(x)| - 1|}{|x-v|^\beta} \leq \limsup_{x \to v} \frac{||\phi'(x)| - 1|}{|x-v|^\beta} \leq L_1.$$

Now, following [SSU], we define the class of parabolic iterated function systems (IFS) under investigation. The interior of a set Y is denoted by $\mathrm{Int}(Y)$.

Definition 2.1. *Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be a collection of $\mathcal{C}^{1+\theta}$ functions on a closed interval $X \subset \mathbb{R}$ such that ϕ_k is parabolic with the fixed point v and the other functions are hyperbolic. We write $\Phi \in \Gamma_X(\theta)$ if, in addition,*

$$\phi_i(X) \subset \mathrm{Int}(X) \setminus \{v\} \quad \textit{for all} \ \ i \leq k-1. \tag{2.1}$$

Let $\mathcal{A} = \{1, \ldots, k\}$. We define the **natural projection map** $\pi_\Phi : \ \mathcal{A}^\infty \to \mathbb{R}$ by setting

$$\{\pi_\Phi(\omega)\} = \bigcap_{n \geq 1} \phi_{\omega|_n}(X)$$

where $\omega|_n = \omega_1 \ldots \omega_n$ and $\phi_{\omega|_n} = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$. If $\Phi \in \Gamma_X(\theta)$ then the map π_Φ is well-defined and continuous (see [U] and [SSU, Lemma 4.6]). We have

$$\pi_\Phi(\omega) = \phi_{\omega|_n}(\pi_\Phi(\sigma^n \omega)) \quad \text{for all } \omega \in \mathcal{A}^\infty \ \text{ and } \ n \geq 1 \tag{2.2}$$

where σ is the left shift on $\mathcal{A}^\infty$. The **limit set**, or attractor, of the IFS Φ is defined by

$$J_\Phi = \pi_\Phi(\mathcal{A}^\infty).$$

It is easy to see that J_Φ is the unique non-empty compact set such that $J_\Phi = \bigcup_{i \leq k} \phi_i(J_\Phi)$.

Given an ergodic shift-invariant measure μ on $\mathcal{A}^\infty$ with positive entropy h_μ, we consider the "push-down" measure on the limit set:

$$\nu(\Phi, \mu) = \mu \circ \pi_\Phi^{-1}.$$

It is called the **invariant measure** for the IFS (corresponding to μ). We are going to study the following questions:

(a) When is the measure $\nu(\Phi, \mu)$ singular? absolutely continuous?

(b) Estimate or compute the Hausdorff dimension of the measure $\nu(\Phi,\mu)$, defined for an arbitrary positive measure ν on the real line by

$$\dim_{\mathrm{H}} \nu = \inf\{\dim_{\mathrm{H}}(Y) : \ \nu(\mathbb{R}\setminus Y)=0\}. \tag{2.3}$$

Clearly, the questions are related since an absolutely continuous measure has Hausdorff dimension equal to one. A standard argument shows that $\nu(\Phi,\mu)$ has pure type since μ is ergodic. Positivity of h_μ and ergodicity of μ imply that $\nu(\Phi,\mu)$ is non-atomic.

The **Lyapunov exponent** of the IFS Φ corresponding to the measure μ is

$$\chi_\mu(\Phi) = -\int_{\mathcal{A}^\infty} \log|\phi'_{\omega_1}(\pi_\Phi(\sigma\omega))|\, d\mu(\omega). \tag{2.4}$$

In the important special case when μ is Bernoulli, the Lyapunov exponent can be rewritten as follows:

$$\mu=(p_1,\dots,p_k)^{\mathbb{N}} \ \Rightarrow\ \chi_\mu(\Phi) = -\sum_{i=1}^{k} p_i \int \log|\phi_i'|\, d\nu(\Phi,\mu). \tag{2.5}$$

It is known [MU] that

$$\dim_{\mathrm{H}} \nu(\Phi,\mu) \le \frac{h_\mu}{\chi_\mu(\Phi)}\,, \tag{2.6}$$

and this inequality becomes equality if $\nu(\Phi,\mu)(\phi_i(J_\Phi)\cap\phi_j(J_\Phi))=0$ for all $i\ne j$. In the next section we provide a short proof of the estimate (2.6) for the reader's convenience.

In this paper we are interested in the "overlapping" case. What do we mean by that? Strictly speaking, an IFS has an overlap if the Open Set Condition (OSC) is not satisfied (the OSC is said to hold if there exists an open set U such that $\phi_i(U)\subset U$ for all $i\le m$ and $\phi_i(U)\cap\phi_j(U)=\emptyset$ for all $i\ne j$). Since this is not always easy to check, sometimes the word "overlap" is used more loosely to include cases when $\phi_i(X)\cap\phi_j(X)$ is a non-empty interval for some $i\ne j$. This property certainly depends on X and does not guarantee that the OSC fails. However, if we know, in addition, that $X=J_\Phi$, then there is an overlap in the strict sense.

To deal with the overlapping case, we consider families of IFS and obtain results for a typical member of the family. The following set-up is taken from [SSU].

Let $U\subset\mathbb{R}^d$ be an open set. Consider a family of parabolic IFS

$$\Phi^{\mathbf{t}} = \{\phi_1^{\mathbf{t}},\dots,\phi_{k-1}^{\mathbf{t}},\phi_k\}\in\Gamma_X(\theta),\ \ \mathbf{t}\in\overline{U}. \tag{2.7}$$

Although the parabolic function *does not* depend on the parameter, it is sometimes convenient to write $\phi_k^{\mathbf{t}}\equiv\phi_k$ for $\mathbf{t}\in U$. We let $\pi_{\mathbf{t}}:\ \mathcal{A}^\infty\to\mathbb{R}$ be the natural projection associated with $\Phi^{\mathbf{t}}$ and denote $J_{\mathbf{t}}=J_{\Phi^{\mathbf{t}}}$. Two conditions which control the dependence on $\mathbf{t}$ will be needed.

Continuity: the maps

$$\mathbf{t} \mapsto \phi_i^{\mathbf{t}} \ \text{ are continuous from } \ \overline{U} \ \text{ to } \ \mathcal{C}^{1+\theta}(X) \ \text{ for } \ i \le k-1. \tag{2.8}$$

Transversality Condition: there exists a constant C_1 such that for all ω and τ in $\mathcal{A}^\infty$ with $\omega_1 \neq \tau_1$,

$$\mathcal{L}_d\{\mathbf{t} \in U : \ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\} \le C_1 r \quad \text{for all } \ r > 0. \tag{2.9}$$

A mild additional condition on the parabolic map ϕ_k will be required in our main theorem.

Definition 2.2. *Say that a parabolic function ϕ on X with the fixed point v is* **well-behaved** *on a connected open neighborhood V of v if $x = v$ is the only local extremum for $|\phi'(x)|$ on $V \cap X$.*

Clearly, any real-analytic parabolic function is well-behaved on a neighborhood of the parabolic point. Now we can state the main result of the paper.

Theorem 2.3. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of parabolic IFS (2.7) satisfying (2.8) and (2.9), such that ϕ_k is well-behaved on some neighborhood of v. Let μ be a shift-invariant ergodic Borel probability measure on $\mathcal{A}^\infty$ with positive entropy and let $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$. Then*

(i) *for Lebesgue-a.e. $\mathbf{t} \in U$,*

$$\dim_{\mathrm{H}} \nu_{\mathbf{t}} = \min\left\{\frac{h_\mu}{\chi_\mu(\Phi^{\mathbf{t}})}\, , 1\right\}; \tag{2.10}$$

(ii) *the measure $\nu_{\mathbf{t}}$ is absolutely continuous for a.e. t in $\{t \in U : \ \frac{h_\mu}{\chi_\mu(\Phi^{\mathbf{t}})} > 1\}$.*

Remark. The statement of the theorem cannot be true for *all* (rather than almost all) IFS in every family considered in the theorem. Indeed, the formula may break down at $\mathbf{t}_0$ for a trivial reason, when $\phi_i^{\mathbf{t}_0}$ and $\phi_j^{\mathbf{t}_0}$ are identically equal for some $i \neq j$. More generally, exceptions may occur if two maps corresponding to distinct words over $\mathcal{A}$ are identical. However, such trivial exceptions are rare (there are none in many families), and we do not know of other exceptional cases. The local dimension of the exceptional set of parameters in Theorem 2.3 (i) is estimated from above in Section 5, assuming a slightly stronger transversality condition. Analogous estimates in part (ii) are much harder, but one might be able to obtain them using the methods of [PSc].

3. Preliminaries and estimate above

Notation. We denote by $B_\delta(\mathbf{t}_0)$ the open ball of radius δ centered at $\mathbf{t}_0$ and $\mathcal{L}_d$ denotes the Lebesgue measure on $\mathbb{R}^d$. Recall that $\mathcal{A} = \{1, \dots, k\}$. For a finite word $w \in \mathcal{A}^n$ the corresponding cylinder set in $\mathcal{A}^\infty$ is denoted by $[w]$. For ω and τ in $\mathcal{A}^\infty$ we denote by $\omega \wedge \tau$ their common initial segment, so that $\omega, \tau \in [\omega \wedge \tau]$ and $\omega_{n+1} \neq \tau_{n+1}$ for $n = |\omega \wedge \tau|$. For $\phi \in \mathcal{C}^{1+\theta}(X)$ we write

$$\| \phi' \|_\theta = \sup\{|\phi'(x) - \phi'(y)| \cdot |x-y|^{-\theta} : \ x, y \in X\},$$

and $\| \Phi' \|_\theta := \max\{\| \phi_i' \|_\theta : \ i \in \mathcal{A}\}$ for an IFS $\Phi = \{\phi_i\}_{i \in \mathcal{A}}$. We denote by $\| \cdot \|$ the supremum norm on X. Given two IFS $\Phi = \{\phi_1, \dots, \phi_k\}$ and $\Psi = \{\psi_1, \dots, \psi_k\}$,we write

$$\| \Phi - \Psi \| = \max_{i \le k} \| \phi_i - \psi_i \| \qquad \text{and} \qquad \| \Phi' - \Psi' \| = \max_{i \le k} \| \phi_i' - \psi_i' \| .$$

Next we recall several classical results. By the Shannon-McMillan-Breiman Theorem,

$$-\frac{1}{n} \log \mu[\omega|_n] \to h_\mu \quad \text{for } \ \mu\text{-a.e. } \omega \tag{3.1}$$

if μ is a shift-invariant ergodic probability measure on $\mathcal{A}^\infty$.

Let $\Phi \in \Gamma_X(\theta)$ be an IFS and let π_Φ the natural projection map. Let

$$f(\omega) = \log |\phi'_{\omega_1}(\pi_\Phi(\sigma\omega))|.$$

Then

$$\sum_{i=0}^{n-1} f(\sigma^i \omega) = \log |\phi'_{\omega|_n}(\pi_\Phi(\sigma^n \omega))|,$$

and in view of (2.4), Birkhoff's Ergodic Theorem implies

$$-\frac{1}{n} \log |\phi'_{\omega|_n}(\pi_\Phi(\sigma^n \omega))| \to \chi_\mu(\Phi) \quad \text{for } \ \mu\text{-a.e. } \omega\,. \tag{3.2}$$

Recall Billingsley's Theorem (see [F3, p.171]): for any Borel measure ν on $\mathbb{R}$,

$$\dim_{\mathrm{H}} \nu = \nu\text{-ess}\sup \left\{ \liminf_{r \searrow 0} \frac{\log \nu[x-r, x+r]}{\log(2r)} \right\}. \tag{3.3}$$

By Frostman's Theorem, see [F1, Theorem 4.13],

$$\dim_{\mathrm{H}} \nu \ge \sup \left\{ \alpha > 0 : \ \iint_{\mathbb{R}^2} \frac{d\nu(x)\, d\nu(y)}{|x-y|^\alpha} < \infty \right\} \tag{3.4}$$

(the expression in the right-hand side is the correlation dimension of the measure ν).

Following [SSU], we introduce notation useful for families of IFS: we write

$$\Phi \in \Gamma_X(\theta, V, \gamma, u, M) \tag{3.5}$$

for $\Phi \in \Gamma_X(\theta)$ if V is a connected open neighborhood of the parabolic point v such that

$$V \cap \bigcup_{i=1}^{k-1} \phi_i(X) = \emptyset, \tag{3.6}$$

$$\max\{\| (\phi_i)' \|: \ i \le k-1\} \le \gamma \in (0,1), \tag{3.7}$$

$$\min\{|\phi_i'(x)| : \ x \in X, \ i \le k\} \ge u \in (0,1), \tag{3.8}$$

and

$$\| \Phi' \|_\theta \le M. \tag{3.9}$$

By Definition 2.1, every $\Phi \in \Gamma_X(\theta)$ belongs to $\Gamma_X(\theta, V, \gamma, u, M)$ for some V, γ, u and M. Next we state a basic distortion result for parabolic IFS.

Lemma 3.1. (see [U] and [SSU, Lemma 4.8]). *There exists a constant* $L_2 = L_2(X, \theta, V, \gamma, u, M) > 1$ *such that for every* $\Phi \in \Gamma_X(\theta, V, \gamma, u, M)$*, all* $\omega \in \mathcal{A}^\infty$*, and all* $n \ge 1$*,*

$$L_2^{-1} \le \frac{|\phi'_{\omega|_n}(y)|}{|\phi'_{\omega|_n}(x)|} \le L_2 \qquad \textit{for all} \ \ x, y \in X \setminus V. \tag{3.10}$$

Now we give a proof of the upper estimate (2.6) for the reader's convenience. A more general result is contained in [MU].

Proof of (2.6). Let $\pi = \pi_\Phi$ and $\nu = \nu(\Phi, \mu)$. If x is in the support of ν_Φ then $x = \pi(\omega)$ for some $\omega \in \mathcal{A}^\infty$. We are going to use (3.3) so we can take x from an arbitrary set of full ν-measure. Since $\nu = \mu \circ \pi^{-1}$, we can assume that ω lies in any given set of full μ-measure. Thus, we can assume that (3.1) and (3.2) hold for ω, and also

$$\#\{n : \ \omega_n \ne k\} = \infty, \tag{3.11}$$

since μ has positive entropy. By (2.2) and the Mean Value Theorem,

$$\operatorname{diam}(\pi([\omega|_n]) \le \| \phi'_{\omega|_n} \| \operatorname{diam}(X).$$

Let $r_n = \| \phi'_{\omega|_n} \| \operatorname{diam}(X)$; observe that $\lim_{n\to\infty} r_n = 0$. We have

$$\nu[x - r_n, x + r_n] = \mu\{\tau : \ |\pi(\tau) - \pi(\omega)| \le r_n\} \ge \mu[\omega|_n]. \tag{3.12}$$

Fix a neighborhood V of v and $u > 0$ so that (3.6) and (3.8) hold. Then we have for every n such that $\omega_n \ne k$ and all $y \in X$ by Lemma 3.1,

$$\frac{|\phi'_{\omega|_n}(y)|}{|\phi'_{\omega|_n}(\sigma^n\omega)|} = \frac{|\phi'_{\omega|_{n-1}}(\phi_{\omega_n}(y))|}{|\phi'_{\omega|_{n-1}}(\phi_{\omega_n}(\pi(\sigma^n\omega))|} \cdot \frac{|\phi'_{\omega_n}(y)|}{|\phi'_{\omega_n}(\pi(\sigma^n\omega))|} \le L_2 \cdot \frac{1}{u}\,.$$

Therefore,

$$\| \phi'_{\omega|_n} \| \leq L_2 u^{-1} |\phi'_{\omega|_n}(\sigma^n \omega)|,$$

and we can estimate by (3.12) for all n such that $\omega_n \neq k$ and $2r_n < 1$:

$$\frac{\log \nu[x - r_n, x + r_n]}{\log(2r_n)} \leq \frac{\log \mu[\omega|_n]}{\log |\phi'_{\omega|_n}(\sigma^n \omega)| + \log(2\mathrm{diam}(X)) + \log(L_2 u^{-1})}.$$

Recall that (3.1) and (3.2) hold for ω so, in view of (3.11),

$$\liminf_{r \searrow 0} \frac{\log \nu[x - r, x + r]}{\log(2r)} \leq \liminf_{n \to \infty} \frac{\log \nu[x - r_n, x + r_n]}{\log(2r_n)} = \frac{h_\mu}{\chi_\mu(\Phi)},$$

and the proof is finished applying (3.3). □

4. Proof of Theorem 2.3

We begin with several lemmas which may be of independent interest. Denote $\chi_{\mathbf{t}} := \chi_\mu(\Phi^{\mathbf{t}})$.

Lemma 4.1. *Suppose that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t} \in \overline{U}}$ is a family of parabolic IFS (2.7) satisfying (2.8). Then the function $\mathbf{t} \mapsto \chi_{\mathbf{t}}$ is continuous on U.*

Proof. Recall that $\chi_{\mathbf{t}} = -\int_{\mathcal{A}^\infty} \log |(\phi^{\mathbf{t}}_{\omega_1})'(\pi_{\mathbf{t}}(\sigma\omega))| \, d\mu(\omega)$ by (2.4). By the continuity condition (2.8), we can choose a connected neighborhood V of v and $\gamma, u \in (0,1)$, $M > 0$ so that $\Phi^{\mathbf{t}} \in \Gamma_X(\theta, V, \gamma, u, M)$ for all $\mathbf{t} \in U$. The desired statement will immediately follow from (2.8) once we prove the following

Sublemma. *Let $\Phi = \{\phi_1, \ldots, \phi_{k-1}, \phi_k\}$ and $\Psi = \{\psi_1, \ldots, \psi_{k-1}, \psi_k\}$ be two IFS in $\Gamma_X(\theta, V, \gamma, u, M)$, with $\phi_k = \psi_k$. Then for all $\omega \in \mathcal{A}^\infty$,*

$$\left| \log \left| \frac{\phi'_{\omega_1}(\pi_\Phi(\omega))}{\psi'_{\omega_1}(\pi_\Psi(\omega))} \right| \right| \leq \frac{1}{u} \left(M \| \Phi - \Psi \|^\theta (1 - \gamma)^{-\theta} + \| \Phi' - \Psi' \| \right).$$

Proof of Sublemma. First we show that

$$|\pi_\Phi(\omega) - \pi_\Psi(\omega)| \leq \| \Phi - \Psi \| (1 - \gamma)^{-1}. \tag{4.1}$$

Indeed, let $\omega = k^n i\tau$ where $n \geq 0, i \neq k$ and $\tau \in \mathcal{A}^\infty$ (in the exceptional case when $\omega = k^\infty$ we have $\pi_\Phi(\omega) = \pi_\Psi(\omega) = v$ since $\phi_k = \psi_k$). Then by (2.2) and the Mean Value Theorem,

$$\begin{aligned}|\pi_\Phi(\omega) - \pi_\Psi(\omega)| &= |\phi_k^n \phi_i(\pi_\Phi(\tau)) - \phi_k^n \psi_i(\pi_\Psi(\tau))| \\ &= |(\phi_k^n)'(c)|\, |\phi_i(\pi_\Phi(\tau)) - \psi_i(\pi_\Psi(\tau))| \\ &\leq |\phi_i(\pi_\Phi(\tau)) - \psi_i(\pi_\Psi(\tau))| \\ &\leq |\phi_i(\pi_\Phi(\tau)) - \psi_i(\pi_\Phi(\tau))| + |\psi_i(\pi_\Phi(\tau)) - \psi_i(\pi_\Psi(\tau))| \\ &\leq \ \| \Phi - \Psi \| + \gamma |\pi_\Phi(\tau) - \pi_\Psi(\tau)|,\end{aligned}$$

and repeating this inductively we obtain (4.1). Now, by the elementary inequality $\left|\log |\frac{x}{y}|\right| \leq \frac{|x-y|}{\min\{|x|,|y|\}}$, in view of (3.8) and (4.1),

$$\begin{aligned}\left|\log \left| \frac{\phi'_{\omega_1}(\pi_\Phi(\omega))}{\psi'_{\omega_1}(\pi_\Psi(\omega))} \right|\right| &\leq \frac{|\phi'_{\omega_1}(\pi_\Phi(\omega)) - \psi'_{\omega_1}(\pi_\Psi(\omega))|}{u} \\ &\leq \frac{1}{u}\left(\left|\phi'_{\omega_1}(\pi_\Phi(\omega)) - \phi'_{\omega_1}(\pi_\Psi(\omega))\right| + \left|\phi'_{\omega_1}(\pi_\Psi(\omega)) - \psi'_{\omega_1}(\pi_\Psi(\omega))\right| \right) \\ &\leq \frac{1}{u}\left(\| \Phi' \|_\theta \, |\pi_\Phi(\omega) - \pi_\Psi(\omega)|^\theta + \| \Phi' - \Psi' \| \right) \\ &\leq \frac{1}{u}\left(M \, \| \Phi - \Psi \|^\theta \, (1-\gamma)^{-\theta} + \| \Phi' - \Psi' \| \right).\end{aligned}$$

The sublemma is proved and Lemma 4.1 follows. $\square$

Lemma 4.2. (see [SSU, Corollary 6.3]) *There exists a positive constant* $L_3 = L_3(X, \theta, V, \gamma, u, M)$ *such that for any* $\Phi = \{\phi_1, \ldots, \phi_k\}$ *and* $\Psi = \{\psi_1, \ldots, \psi_k\}$*, two parabolic IFS in* $\Gamma_X(\theta, V, \gamma, u, M)$ *with* $\phi_k = \psi_k$*, for all* $\omega \in \mathcal{A}^\infty$*, all* $n \geq 0$*, and all* $x \in X$*,*

$$\frac{|\phi'_{\omega|_n}(x)|}{|\psi'_{\omega|_n}(x)|} \leq \exp\left(L_3 n \left(\| \Phi - \Psi \|^\theta + \| \Phi' - \Psi' \| \right) \right).$$

Lemma 4.3. *There exists a positive constant* $L_4 = L_4(X, \theta, V, \gamma, u, M)$ *such that for any parabolic IFS* $\Phi = \{\phi_1, \ldots, \phi_k\} \in \Gamma_X(\theta, V, \gamma, u, M)$*, with* ϕ_k *well-behaved on* V*, the following property holds:*

For all $\omega \in \mathcal{A}^\infty$ *and all* $n \in \mathbb{N}$*,*

$$\frac{|\phi'_{\omega|_n}(y)|}{|\phi'_{\omega|_n}(x)|} \leq L_4 \qquad \textit{for all} \ \ x \in X \ \ \textit{and} \ \ y \in X \setminus V.$$

Note that here, in contrast with Lemma 3.1, there is no symmetry between x and y.

Proof of Lemma 4.3. Let $\rho = \omega|_n$. Suppose $\rho = wik^l$ where $l \ge 0$ and $i \ne k$. Then $\phi_{ik^l}(X) \subset X \setminus V$ and hence, by Lemma 3.1,

$$\frac{|\phi'_\rho(y)|}{|\phi'_\rho(x)|} = \frac{|\phi'_w(\phi_{ik^l}(y))|}{|\phi'_w(\phi_{ik^l}(x))|} \cdot \frac{|\phi'_i(\phi^l_k(y))|}{|\phi'_i(\phi^l_k(x))|} \cdot \frac{|(\phi^l_k)'(y)|}{|(\phi^l_k)'(x)|} \le L_2 \cdot \frac{1}{u} \cdot \frac{|(\phi^l_k)'(y)|}{|(\phi^l_k)'(x)|}.$$

If $\rho = k^l$ then

$$\frac{|\phi'_\rho(y)|}{|\phi'_\rho(x)|} = \frac{|(\phi^l_k)'(y)|}{|(\phi^l_k)'(x)|} =: Q,$$

so in both cases it remains to estimate Q from above.

If $x = v$, then we are done, since $|(\phi^l_k)'(v)| = 1$ and $|(\phi^l_k)'(y)| \le 1$. Suppose that ϕ_k is increasing and $x > v$. Let $V \cap [v, +\infty) = [v, v + \delta_1)$; we can assume that $v + \delta_1 \in X$. If $x \in (v, v + \delta_1)$ then

$$(\phi^l_k)'(x) = \prod_{i=0}^{l-1} \phi'_k(\phi^i_k(x)) \ge \prod_{i=0}^{l-1} \phi'_k(\phi^i_k(v + \delta_1)) = (\phi^l_k)'(v + \delta_1)$$

where we used that ϕ_k is increasing and ϕ'_k is decreasing on $[v, v + \delta_1]$ by Definition 2.2, and also that $\phi_k([v, v + \delta_1]) \subset [v, v + \delta_1]$. Thus we may assume that $x \ge v + \delta_1$. But then the desired estimate follows from Lemma 3.1 (see also [U, Lemma 2.3]).

The case when ϕ_k is increasing and $x < v$ is considered similarly. If ϕ_k is decreasing, which is only possible when $v \in \mathrm{Int}(X)$, the proof follows by passing to the second iterate ϕ^2_k, which is an increasing well-behaved parabolic function in a neighborhood of v. □

Lemma 4.4. *Suppose that the family $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ satisfies (2.9). Then for every $0 < \alpha < 1$ there exists $C_2 = C_2(\alpha) > 0$ such that for all $\omega, \tau \in \mathcal{A}^\infty$ with $\omega_1 \ne \tau_1$,*

$$\int_U \frac{d\mathbf{t}}{|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)|^\alpha} \le C_2.$$

Proof is elementary, writing the integral in terms of the cumulative distribution function; see [SSU, Lemma 3.3] for details. □

Now we are ready for the proof of the main theorem. It combines the methods of [PoS, PSo2, SSo, SSU] but also uses a new idea which is related to Lemma 4.3.

Proof of Theorem 2.3. **(i)** Since $\dim_{\mathrm{H}} \nu_{\mathbf{t}} \le h_\mu/\chi_{\mathbf{t}}$ for all $\mathbf{t} \in U$ by (2.6), we only need to establish the estimate from below (for a.e. $\mathbf{t} \in U$). We are going to prove that

$$\forall \mathbf{t}_0 \in U,\ \forall \epsilon > 0,\ \exists \eta > 0:\ \dim_{\mathrm{H}} \nu_{\mathbf{t}} \ge \min\{h_\mu/\chi_{\mathbf{t}_0}, 1\} - \epsilon \quad \text{for } \mathcal{L}_d\text{-a.e. } \mathbf{t} \in B_\eta(\mathbf{t}_0) \tag{4.2}$$

This will imply the desired statement: indeed, if $\dim_H \nu_{\mathbf{t}} < h_\mu/\chi_{\mathbf{t}}$ on a set of positive measure, taking $\mathbf{t}_0$ to be a Lebesgue density point leads to a contradiction with (4.2) by Lemma 4.1. (A similar argument is worked out in detail at the end of [SSU, Section 3].)

Thus, we begin the proof of (4.2) by fixing $\mathbf{t}_0 \in U$ and $\epsilon > 0$. It is convenient to let $\Phi = \Phi^{\mathbf{t}_0}$, $\pi = \pi_{\mathbf{t}_0}$, and $\chi = \chi_{\mathbf{t}_0}$. By Lemma 4.2 and (2.8), there exists $\eta > 0$ such that for all $\omega \in \mathcal{A}^\infty$, $n \geq 1$, and $x \in X$,

$$|\mathbf{t} - \mathbf{t}_0| \leq \eta \implies \frac{|\phi'_{\omega|_n}(x)|}{|(\phi^{\mathbf{t}}_{\omega|_n})'(x)|} \leq e^{\frac{1}{4}n\epsilon\chi}. \tag{4.3}$$

By Egorov's Theorem, choose a set $\Omega \subset \mathcal{A}^\infty$ such that $\mu(\Omega) > 0$ and convergence in (3.1) and (3.2) is uniform on Ω. Let

$$\widetilde{\mu} = \mu|_\Omega \qquad \text{and} \qquad \widetilde{\nu}_{\mathbf{t}} = \widetilde{\mu} \circ \pi_{\mathbf{t}}^{-1}.$$

It is clear from (2.3) that $\dim_H \widetilde{\nu}_{\mathbf{t}} \leq \dim_H \nu_{\mathbf{t}}$ so it suffices to estimate $\dim_H \widetilde{\nu}_{\mathbf{t}}$ from below. Let $s := \min\{1, h_\mu/\chi\}$. The inequality (4.2) will follow if we prove

$$\mathcal{S} := \int_{B_\eta(\mathbf{t}_0)} \iint_{\mathbb{R}^2} \frac{d\widetilde{\nu}_{\mathbf{t}}(x)\, d\widetilde{\nu}_{\mathbf{t}}(y)}{|x-y|^{s-\epsilon}}\, d\mathbf{t} < \infty, \tag{4.4}$$

see (3.4). For a finite (possibly empty) word ρ over the alphabet $\mathcal{A}$ we denote

$$A_\rho = \{(\omega,\tau) \in \Omega^2 : \ \omega \wedge \tau = \rho\}.$$

We also let $\mu_2 = \mu \times \mu$. Changing variables, reversing the order of integration, and decomposing $\Omega^2 = \bigcup_{n\geq 0} \bigcup_{\rho \in \mathcal{A}^n} A_\rho$ we obtain

$$\begin{aligned} \mathcal{S} &= \iint_{\Omega^2} \int_{B_\eta(\mathbf{t}_0)} \frac{d\mathbf{t}}{|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)|^{s-\epsilon}}\, d\mu_2(\omega,\tau) \\ &= \sum_{n\geq 0} \sum_{\rho\in\mathcal{A}^n} \iint_{A_\rho} \left(\int_{B_\eta(\mathbf{t}_0)} \frac{d\mathbf{t}}{|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)|^{s-\epsilon}} \right) d\mu_2(\omega,\tau). \end{aligned} \tag{4.5}$$

Let $|\rho| = n$ and $(\omega,\tau) \in A_\rho$. Then we have for some $c \in [\pi_{\mathbf{t}}(\sigma^n\omega), \pi_{\mathbf{t}}(\sigma^n\tau)]$ using (4.3):

$$|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| = |(\phi^{\mathbf{t}}_{\omega|_n})'(c)| \cdot |\pi_{\mathbf{t}}(\sigma^n\omega) - \pi_{\mathbf{t}}(\sigma^n\tau)| \geq |\phi'_{\omega|_n}(c)| e^{-\frac{1}{4}n\epsilon\chi} |\pi_{\mathbf{t}}(\sigma^n\omega) - \pi_{\mathbf{t}}(\sigma^n\tau)|.$$

Now observe that $\omega_{n+1} \neq \tau_{n+1}$, and therefore, both cannot be equal to k. Suppose that $\omega_{n+1} \neq k$ (the other case is completely similar). Then $\pi(\sigma^n\omega) \in \phi_{\omega_{n+1}}(X) \subset X \setminus V$, and we have by Lemma 4.3:

$$|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \geq L_4^{-1} |\phi'_{\omega|_n}(\pi(\sigma^n\omega))| e^{-\frac{1}{4}n\epsilon\chi} |\pi_{\mathbf{t}}(\sigma^n\omega) - \pi_{\mathbf{t}}(\sigma^n\tau)|. \tag{4.6}$$

Since $\omega_{n+1} \neq \tau_{n+1}$ and $s-\epsilon < 1$, we can apply Lemma 4.4 to obtain a constant $C_2 = C_2(s-\epsilon)$ such that

$$\int_{B_\eta(\mathbf{t}_0)} \frac{d\mathbf{t}}{|\pi_{\mathbf{t}}(\sigma^n\omega) - \pi_{\mathbf{t}}(\sigma^n\tau)|^{s-\epsilon}} \leq C_2. \tag{4.7}$$

Now we estimate the expression in (4.5) with the help of (4.6) and (4.7) to get

$$\mathcal{S} \leq \sum_{n\geq 0}\sum_{\rho\in\mathcal{A}^n} C_2 L_4^{s-\epsilon} e^{\frac{1}{4}n\epsilon\chi(s-\epsilon)} |\phi'_{\omega|_n}(\pi(\sigma^n\omega))|^{-(s-\epsilon)}\mu_2(A_\rho). \tag{4.8}$$

Recall that convergence in (3.1) and (3.2) is uniform on Ω, so we can find $N \in \mathbb{N}$ such that for all $\omega \in \Omega$ and $n \geq N$,

$$\mu[\omega|_n] \leq e^{-n(h_\mu - \frac{1}{4}\epsilon\chi)} \tag{4.9}$$

and

$$|\phi'_{\omega|_n}(\pi(\sigma^n\omega))| \geq e^{-n(\chi+\frac{1}{4}\epsilon\chi)}. \tag{4.10}$$

We have $\mu_2(A_\rho) \leq \mu[\rho]^2$. If ρ is a restriction of some $\omega \in \Omega$ we have $\mu[\rho] = \mu[\omega|_n] \leq e^{-n(h_\mu - \frac{1}{4}\epsilon\chi)}$ by (4.9). Otherwise, $A_\rho = \emptyset$, so in any case,

$$\mu_2(A_\rho) \leq \mu[\rho]e^{-n(h_\mu-\frac{1}{4}\epsilon\chi)}.$$

To prove that the series in (4.8) converges we estimate the truncated series for $n \geq N$ as follows:

$$\sum_{n\geq N} \leq C_2 L_4^{s-\epsilon}\exp[\tfrac{1}{4}n\epsilon\chi(s-\epsilon) + n(\chi+\tfrac{1}{4}\epsilon\chi)(s-\epsilon) - n(h_\mu - \tfrac{1}{4}\epsilon\chi)]$$

taking into account (4.10) and that $\sum_{\rho\in\mathcal{A}^n}\mu[\rho] = 1$. But $s = \min\{1, \mathrm{h}_\mu/\chi\}$, and a simple computation shows that the exponential term above is less than $e^{-\frac{1}{4}n\epsilon\chi}$. So $\mathcal{S} < \infty$, (4.4) is proved, and the proof of Theorem 2.3(i) is concluded.

Proof of **(ii).** Let $U' = \{\mathbf{t} \in U : \ h_\mu/\chi_{\mathbf{t}} > 1\}$. This is an open set by Lemma 4.1; assume it is non-empty, otherwise there is nothing to prove. Fix an arbitrary $\mathbf{t}_0 \in U'$. It is enough to show that $\nu_{\mathbf{t}}$ is absolutely continuous with respect to the Lebesgue measure for $\mathcal{L}_d$-a.e. $\mathbf{t}$ in some neighborhood of $\mathbf{t}_0$. Let $\Phi = \Phi^{\mathbf{t}_0}$, $\pi = \pi_{\mathbf{t}_0}$, and $\chi = \chi_{\mathbf{t}_0}$. Since $\mathbf{t}_0 \in U'$ we can fix $\epsilon > 0$ such that

$$\chi < h_\mu - 3\epsilon.$$

Then by Lemma 4.2 and (2.8), there exists $\eta > 0$ such that for all $\omega \in \mathcal{A}^\infty$, $n \geq 1$, and $x \in X$,

$$|\mathbf{t} - \mathbf{t}_0| \leq \eta \implies \frac{|\phi'_{\omega|_n}(x)|}{|(\phi^{\mathbf{t}}_{\omega|_n})'(x)|} \leq e^{n\epsilon}. \tag{4.11}$$

By Egorov's Theorem, for any $\epsilon' > 0$ there exists $\Omega \subset \mathcal{A}^\infty$ such that $\mu(\Omega) > 1 - \epsilon'$ and convergence in (3.1) and (3.2) is uniform on Ω. Let

$$\widetilde{\mu} = \mu|_\Omega \qquad \text{and} \qquad \widetilde{\nu}_{\mathbf{t}} = \widetilde{\mu} \circ \pi_{\mathbf{t}}^{-1}.$$

If we show that $\widetilde{\nu}_{\mathbf{t}}$ is absolutely continuous with respect to the Lebesgue measure for a.e. $\mathbf{t} \in B_\eta(\mathbf{t}_0)$, then letting $\epsilon' \to 0$ along a countable set we will be able to conclude that $\nu_{\mathbf{t}}$ is absolutely continuous for a.e. $\mathbf{t} \in B_\eta(\mathbf{t}_0)$. To this end we are going to show that

$$\mathcal{I} = \int_{B_\eta(\mathbf{t}_0)} \int_{\mathbb{R}} \underline{D}(\widetilde{\nu}_{\mathbf{t}}, x)\, d\widetilde{\nu}_{\mathbf{t}}\, d\mathbf{t} < \infty$$

where

$$\underline{D}(\widetilde{\nu}_{\mathbf{t}}, x) = \liminf_{r \searrow 0} \frac{\widetilde{\nu}_{\mathbf{t}}[x-r, x+r]}{2r}$$

is the lower density of the measure $\widetilde{\nu}_{\mathbf{t}}$ at the point x. This will be sufficient since then for a.e. $\mathbf{t} \in B_\eta(\mathbf{t}_0)$ we will have $\underline{D}(\widetilde{\nu}_{\mathbf{t}}, x) < \infty$ for $\widetilde{\nu}_{\mathbf{t}}$-a.e. x and [Mat, Th. 2.12] will imply that $\widetilde{\nu}_{\mathbf{t}}$ is absolutely continuous. (Actually, this will imply that $\widetilde{\nu}_{\mathbf{t}}$ has a density in L^2 for a.e. $\mathbf{t} \in B_\eta(\mathbf{t}_0)$; this property, however, may disappear as we let $\epsilon' \to 0$.)

The argument below follows the scheme of [PSo2]. First we apply Fatou's Lemma to get

$$\mathcal{I} \le \liminf_{r \searrow 0} \int_{B_\eta(\mathbf{t}_0)} \int_{\mathbb{R}} \frac{\widetilde{\nu}_{\mathbf{t}}[x-r, x+r]}{2r}\, d\widetilde{\nu}_{\mathbf{t}}\, d\mathbf{t}. \tag{4.12}$$

Next we use the definition of $\widetilde{\nu}_{\mathbf{t}}$ to change the variable, write $\widetilde{\nu}_{\mathbf{t}}[x-r, x+r]$ as an integral of the indicator function, and change the variable once again to obtain

$$\int_{\mathbb{R}} \widetilde{\nu}_{\mathbf{t}}[x-r, x+r]\, d\nu_{\mathbf{t}} = \iint_{\Omega^2} \mathbf{1}_{\{\omega \in \Omega:\ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\}}\, d\mu_2(\omega, \tau).$$

Substituting this into (4.12) and exchanging the order of integration leads to

$$\begin{aligned} \mathcal{I} &\le \liminf_{r \searrow 0} (2r)^{-1} \iint_{\Omega^2} \mathcal{L}_d\{\mathbf{t} \in B_\eta(\mathbf{t}_0) :\ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\}\, d\mu_2(\omega, \tau) \\ &= \liminf_{r \searrow 0} (2r)^{-1} \sum_{n \ge 0} \sum_{\rho \in \mathcal{A}^n} \iint_{A_\rho} \mathcal{L}_d\{\mathbf{t} \in B_\eta(\mathbf{t}_0) :\ |\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \le r\}\, d\mu_2(\omega, \tau)\,. \end{aligned} \tag{4.13}$$

Here we again denote $A_\rho = \{(\omega, \tau) \in \Omega^2 :\ \omega \wedge \tau = \rho\}$ for a finite (possibly empty) word ρ over the alphabet $\mathcal{A}$.

The next step is almost the same as in the proof of part (i). Let $(\omega, \tau) \in A_\rho$. Then we have for some $c \in [\pi_{\mathbf{t}}(\sigma^n\omega), \pi_{\mathbf{t}}(\sigma^n\tau)]$ using (4.11):

$$|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| = |(\phi^{\mathbf{t}}_{\omega|_n})'(c)| \cdot |\pi_{\mathbf{t}}(\sigma^n\omega) - \pi_{\mathbf{t}}(\sigma^n\tau)| \ge |\phi'_{\omega|_n}(c)| e^{-n\epsilon} |\pi_{\mathbf{t}}(\sigma^n\omega) - \pi_{\mathbf{t}}(\sigma^n\tau)|\,.$$

Since $\omega_{n+1} \ne \tau_{n+1}$, we can assume, without loss of generality, that $\omega_{n+1} \ne k$. Then we have by Lemma 4.3:

$$|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)| \ge L_4^{-1} |\phi'_{\omega|_n}(\pi(\sigma^n\omega))| e^{-n\epsilon} |\pi_{\mathbf{t}}(\sigma^n\omega) - \pi_{\mathbf{t}}(\sigma^n\tau)|\,.$$

It follows that

$$\begin{aligned}&\mathcal{L}_d\{\mathbf{t}\in B_\eta(\mathbf{t}_0):\ |\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)|\le r\}\\ \le&\mathcal{L}_d\left\{\mathbf{t}\in B_\eta(\mathbf{t}_0):\ |\pi_{\mathbf{t}}(\sigma^n\omega)-\pi_{\mathbf{t}}(\sigma^n\tau)|\le \frac{L_4e^{n\epsilon}r}{|\phi'_{\omega|_n}(\pi(\sigma^n\omega))|}\right\}\le C_1L_4e^{n\epsilon}r|\phi'_{\omega|_n}(\pi(\sigma^n\omega))|^{-1},\end{aligned}$$

by the transversality condition (2.9). Substituting this into (4.13) gives

$$\mathcal{I}\le \tfrac{1}{2}C_1L_4\sum_{n\ge 0}e^{n\epsilon}\sum_{\rho\in\mathcal{A}^n}\iint_{A_\rho}|\phi'_{\omega|_n}(\pi(\sigma^n\omega)|^{-1}\,d\mu_2(\omega,\tau)\,, \tag{4.14}$$

where $\omega\in A_\rho$. Recall that convergence in (3.1) and (3.2) is uniform on Ω, so we can find $N\in\mathbb{N}$ such that for all $\omega\in\Omega$ and $n\ge N$,

$$\mu[\omega|_n]\le e^{-n(h_\mu-\epsilon)} \tag{4.15}$$

and

$$|\phi'_{\omega|_n}(\pi(\sigma^n\omega))|\ge e^{-n(\chi+\epsilon)}\,. \tag{4.16}$$

Similarly to the proof of part (i) we have

$$\mu_2(A_\rho)\le\mu[\rho]e^{-n(h_\mu-\epsilon)},$$

hence the truncated series in (4.14) is estimated as follows:

$$\sum_{n\ge N}\le\sum_{n\ge N}\exp[n\epsilon+n(\chi+\epsilon)-n(h_\mu-\epsilon)]\,,$$

which is finite since $\chi-h_\mu<-3\epsilon$. This concludes the proof of Theorem 2.3. □

5. Exceptional parameters

In this section, following the scheme of Kaufman [Ka], we obtain an estimate from above for the local Hausdorff dimension of the set of exceptional parameters in Theorem 2.3(i). As before, we assume that $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ is a family of IFS in $\Gamma_X(\theta)$ satisfying (2.8), but we will need the following stronger transversality condition. Denote by $N_r(F)$ the minimal number of balls of radius r needed to cover the set $F\subset\mathbb{R}^d$.

Strong Transversality Condition: there exists a constant $C_3>0$ such that for all ω and τ in $\mathcal{A}^\infty$ with $\omega_1\ne\tau_1$,

$$N_r\left(\{\mathbf{t}\in U:\ |\pi_{\mathbf{t}}(\omega)-\pi_{\mathbf{t}}(\tau)|\le r\}\right)\le C_3r^{1-d}. \tag{5.1}$$

Of course, the strong transversality condition implies the transversality condition (2.9).

The following lemma is analogous to Lemma 4.4 and [SSU, Lemma 5.1]; its proof is elementary.

Lemma 5.1. *Suppose that the family $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ satisfies the strong transversality condition. Let m be a Borel probability measure in $\mathbb{R}^d$ such that $m(B_r(x)) \le Cr^u$ for some $C,\ u > 0$ and all $x \in \mathbb{R}^d,\ r > 0$. Then for every $\alpha < u - d + 1$ and for all $\omega, \tau \in \mathcal{A}^\infty$ with $\omega_1 \ne \tau_1$, there exists $C_4 = C_4(\alpha) > 0$ such that*

$$\int_U \frac{dm(\mathbf{t})}{|\pi_{\mathbf{t}}(\omega) - \pi_{\mathbf{t}}(\tau)|^\alpha} \le C_4.$$

In the sequel any measure with the properties required in Lemma 5.1 will be called a Frostman measure with exponent u. Next we prove the analog of (4.2). We fix an ergodic shift-invariant probability measure μ on $\mathcal{A}^\infty$ and let $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$.

Lemma 5.2. *Suppose that the family $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in U}$ satisfies (2.8) and the strong transversality condition (5.1). Then for any $\mathbf{t}_0 \in U$ and any $\epsilon > 0$ there exists $\eta = \eta(\mathbf{t}_0, \epsilon) > 0$ such that if m is a Frostman measure on $B_\eta(\mathbf{t}_0)$ with exponent u, then*

$$\dim_{\mathrm{H}}(\nu_{\mathbf{t}}) \ge \min\left\{\frac{h_\mu}{\chi_{\mathbf{t}_0}}, u - d + 1\right\} - \epsilon$$

for m-a.e. $\mathbf{t} \in B_\eta(\mathbf{t}_0)$.

Proof. We let $s = \min\{u - d + 1, h_\mu/\chi_{\mathbf{t}_0}\}$ and then repeat the proof of (4.2) almost word by word. We define

$$\mathcal{S}' := \int_{B_\eta(\mathbf{t}_0)} \iint_{\mathbb{R}^2} \frac{d\widetilde{\nu}_{\mathbf{t}}(x)\, d\widetilde{\nu}_{\mathbf{t}}(y)}{|x-y|^{s-\epsilon}}\, dm(\mathbf{t})$$

and prove that $\mathcal{S}' < \infty$ using Lemma 5.1 in the place where Lemma 4.4 was used. This finishes the proof since $\dim_{\mathrm{H}}(\nu_{\mathbf{t}}) \ge \dim_{\mathrm{H}}(\widetilde{\nu}_{\mathbf{t}})$. □

Next we prove the main result of this section.

Theorem 5.3. *Suppose that the d-parameter family of IFS $\{\Phi^{\mathbf{t}}\}_{\mathbf{t}\in\overline{U}}$ satisfies (2.8) and the strong transversality condition (5.1). If G is an arbitrary subset of U, then for every $\xi > 0$ we have*

$$\dim_{\mathrm{H}}\left(\left\{\mathbf{t} \in G :\ \dim_{\mathrm{H}}(\nu_t) < \min\{\xi, h_\mu/\chi_{\mathbf{t}}\}\right\}\right) \le \min\{\xi, \sup_G h_\mu/\chi_{\mathbf{t}}\} + d - 1.$$

Proof. Denote $\kappa := \min\{\xi, \sup_G h_\mu/\chi_{\mathbf{t}}\} + d - 1$. By the countable stability of the Hausdorff dimension, it is enough to prove that for all $n \in \mathbb{N}$,

$$\dim_{\mathrm{H}}\left(\left\{\mathbf{t} \in G :\ \dim_{\mathrm{H}}(J_t) < \min\{\xi, h_\mu/\chi_{\mathbf{t}}\} - n^{-1}\right\}\right) \le \kappa.$$

Fix n and observe that it suffices to show that for all $\mathbf{t}_0 \in G$ there exists $\eta = \eta(\mathbf{t}_0)$ such that

$$\dim_{\mathrm{H}}\left(\left\{\mathbf{t} \in B_\eta(\mathbf{t}_0) :\ \dim_{\mathrm{H}}(J_t) < \min\{\xi, h_\mu/\chi_{\mathbf{t}}\} - n^{-1}\right\}\right) \le \kappa$$

(just use that any cover of G contains a countable subcover and again the countable stability of the Hausdorff dimension). To establish our claim, suppose that it is false. Then there exists $\mathbf{t}_0 \in G$ such that for all $\eta > 0$

$$\dim_{\mathrm{H}}\Big(\Big\{\mathbf{t} \in B_\eta(\mathbf{t}_0) :\ \dim_{\mathrm{H}}(J_t) < \min\{\xi, h_\mu/\chi_{\mathbf{t}}\} - n^{-1}\Big\}\Big) > \kappa.$$

Choose $\eta > 0$ so small that the statement of Lemma 5.2 holds with $\epsilon = \frac{1}{2n}$ and $|\frac{h_\mu}{\chi_{\mathbf{t}}} - \frac{h_\mu}{\chi_{\mathbf{t}_0}}| < \frac{1}{2n}$ for all $\mathbf{t} \in B_\eta(\mathbf{t}_0)$ (by the continuity of $\chi_{\mathbf{t}}$). Then

$$\begin{aligned}&\Big\{\mathbf{t} \in B_\eta(\mathbf{t}_0) :\ \dim_{\mathrm{H}}(\nu_t) < \min\{\xi, h_\mu/\chi_{\mathbf{t}}\} - n^{-1}\Big\}\\ \subset&\Big\{\mathbf{t} \in B_\eta(\mathbf{t}_0) :\ \dim_{\mathrm{H}}(\nu_t) < \min\{\xi, h_\mu/\chi_{\mathbf{t}_0}\} - (2n)^{-1}\Big\} =: E,\end{aligned}$$

hence $\dim_{\mathrm{H}}(E) > \kappa$. By Frostman's Lemma (see [Mat, Th.8.8]), there is a Frostman measure m on the set E with exponent $u = \kappa$. By Lemma 5.2, for m-a.e. $\mathbf{t}$ we have

$$\dim_{\mathrm{H}}(\nu_{\mathbf{t}}) \geq \min\{h_\mu/\chi_{\mathbf{t}_0}, \kappa - d + 1\} - (2n)^{-1} = \min\Big\{h_\mu/\chi_{\mathbf{t}_0}, \min\{\xi, \sup_G h_\mu/\chi_{\mathbf{t}}\}\Big\} - (2n)^{-1}.$$

This is a contradiction since for all $\mathbf{t} \in E$ we have $\dim_{\mathrm{H}}(\nu_t) < \min\{\xi, h_\mu/\chi_{\mathbf{t}_0}\} - (2n)^{-1}$ and

$$\min\{\xi, h_\mu/\chi_{\mathbf{t}_0}\} \leq \min\Big\{h_\mu/\chi_{\mathbf{t}_0}, \min\{\xi, \sup_G h_\mu/\chi_{\mathbf{t}}\}\Big\}.$$

The proof is complete. □

Since the function $\mathbf{t} \mapsto \chi_{\mathbf{t}}$, $\mathbf{t} \in U$, is continuous, as an immediate consequence of Theorem 5.3 we get the following estimate for the local dimension of the exceptional set.

Corollary 5.4. *For every $\mathbf{t}_0 \in U$ we have*

$$\lim_{r\to 0} \dim_{\mathrm{H}}\Big(\{t \in B_r(\mathbf{t}_0) :\ \dim_{\mathrm{H}}(\nu_{\mathbf{t}}) < \min\{\xi, h_\mu/\chi_{\mathbf{t}}\}\}\Big) \leq \min\{\xi, h_\mu/\chi_{\mathbf{t}_0}\} + d - 1.$$

6. Example: random continued fractions

Here we apply our results to the family of parabolic IFS $\{\frac{x}{x+1}, \frac{x+\alpha}{x+\alpha+1}\}$, with parameter α, associated with the class of continued fractions considered by Lyons [Ly]. First we prove a sufficient condition for strong transversality which extends [SSU, Proposition 7.2].

Suppose that $Y \subset \mathbb{R}$ is a closed interval and $\phi \in \mathcal{C}^{1+\theta}(Y)$ is increasing and satisfies

$$|\phi'(x)| \geq u > 0 \qquad \text{for all } \ x \in Y.$$

Let $\phi_i(x) = \phi(x + a_i)$, for $i = 1, \ldots, k$, and $a_i \in \mathbb{R}$ are such that $\Phi = \{\phi_1, \ldots, \phi_k\} \in \Gamma_X(\theta)$ for some closed interval X. For $w \in \mathcal{A}^n$ denote

$$M_w = \sup\{|\phi'_{w_1}(x)| :\ x \in \phi_{\sigma w}(X)\}$$

(if $w = w_1$ then $M_w = \| \phi'_{w_1} \|$). Let

$$\Phi^{\mathbf{t}} = \{\phi_1(x+t_1), \ldots, \phi_{k-1}(x+t_{k-1}), \phi_k(x)\} \quad \text{for} \quad \mathbf{t} = (t_1, \ldots, t_{k-1}) \in \mathbb{R}^{k-1}.$$

Denote by $\pi_{\mathbf{t}}$ the projection map corresponding to $\Phi^{\mathbf{t}}$. Below $\delta_{i,j}$ is the Kronecker's symbol.

Lemma 6.1. (i) *If there exists $\epsilon > 0$ such that*

$$\frac{\partial}{\partial t_i} \pi_{\mathbf{t}}(\omega)|_{\mathbf{t}=0} \le 1 - \epsilon \tag{6.1}$$

for all $\omega \in \mathcal{A}^\infty$ and $1 \le i \le k-1$, then there exists $\eta > 0$ such that the family $\{\Phi^{\mathbf{t}} : \mathbf{t} \in B_\eta(\mathbf{0})\}$ satisfies the strong transversality condition (5.1).

(ii) *Suppose that $\mathcal{Z} \subset \mathcal{A}^\infty$, $W \subset \bigcup_{n\ge 1}^{\infty} \mathcal{A}^n$, and $\epsilon > 0$ are such that*

$$[i] = \mathcal{Z} \cup \bigcup_{w \in W} [w], \tag{6.2}$$

the inequality (6.1) holds for all $\omega \in \mathcal{Z}$, and

$$M_w \left(\delta_{w_1,i} + M_{\sigma w} \left(\delta_{w_2,i} + \cdots + M_{\sigma^{n-1} w}(\delta_{w_n,i} + 1)\right)\right) \le 1 - \epsilon \tag{6.3}$$

for all $w \in W$. Then (6.1) holds for all $\omega \in \mathcal{A}^\infty$.

Remark. The simplest special case of (6.2) is $[i] = [i]$ and $\mathcal{Z} = \emptyset$. Then (6.3) becomes $M_i < 1/2$ which exactly corresponds to [SSU, Proposition 7.2(ii)].

Proof. Part (i) is a special case of [SSU, Proposition 7.2(i)] so we only need to prove part (ii). We have for $\omega \in [w]$, with $w = w_1 \ldots w_n$:

$$\pi_{\mathbf{t}}(\omega) = \phi_{w_1}\left(t_{w_1} + \phi_{w_2}\left(t_{w_2} + \cdots + \phi_{w_n}(t_{w_n} + \pi_{\mathbf{t}}(\sigma^n \omega))\right)\right),$$

with the convention that $t_k \equiv 0$. Thus,

$$\frac{\partial}{\partial t_i} \pi_{\mathbf{t}}(\omega)|_{\mathbf{t}=0} \le M_w \left(\delta_{w_1,i} + M_{\sigma w}\left(\delta_{w_2,i} + \cdots + M_{\sigma^{n-1}w}\left(\delta_{w_n,i} + \frac{\partial}{\partial t_i}\pi_{\mathbf{t}}(\sigma^n \omega)|_{\mathbf{t}=0}\right)\right)\right).$$

We are going to estimate

$$A_i = \sup\left\{\frac{\partial}{\partial t_i}\pi_{\mathbf{t}}(\omega)|_{\mathbf{t}=0} : \ \omega \in \mathcal{A}^\infty\right\}.$$

Since $\| \phi'_j \| \le 1$ for all $j \le k$, we can restrict ourselves to the case $w_1 = i$ when estimating $A_i = \sup\left\{\frac{\partial}{\partial t_i}\pi_{\mathbf{t}}(\omega)|_{\mathbf{t}=0} : \ \omega \in \mathcal{A}^\infty\right\}$. Taking into account (6.2), we obtain that A_i satisfies

$$A_i \le \max\left\{1 - \epsilon, \ \sup_{w \in W}\{M_w\left(\delta_{w_1,i} + M_{\sigma w}\left(\delta_{w_2,i} + \cdots + M_{\sigma^{n-1}w}\left(\delta_{w_n,i} + A_i\right)\right)\right)\}\right\}. \tag{6.4}$$

The right-hand side is the supremum of a set of linear functions in A_i, each having a slope in $[0,1)$. By (6.3), the value of each of these functions at 1 does not exceed $1-\epsilon$. Therefore,

$A_i \le G(A_i)$ where G is an increasing function on $[0,+\infty)$ such that $G(y)-G(x) \le y-x$ for all $x<y$ and $G(1) \le 1-\epsilon$. The desired inequality $A_i \le 1-\epsilon$ is now straightforward. $\square$

Now we are going to apply Lemma 6.1 to the family

$$\Phi^\alpha = \{\phi_1^\alpha(x), \phi_2^\alpha(x)\} \quad \text{where } \phi_2^\alpha(x) = \phi(x) = \frac{x}{x+1} \text{ and } \phi_1^\alpha(x) = \phi(x+\alpha).$$

It is easy to see that $\Phi^\alpha \in \Gamma_{[0,1]}(1)$ for all $\alpha > 0$.

Lemma 6.2. *The family $\{\Phi^\alpha\}$ satisfies the strong transversality condition (5.1) for $\alpha \in (0.215, 0.5)$.*

Remarks. 1. We are only interested in $\alpha \in (0, 0.5)$ since for $\alpha \ge 0.5$ the IFS satisfies the Open Set Condition.

2. The proof of this lemma uses Lemma 6.1(ii). The relevant result in our previous paper, [SSU, Proposition 7.2(ii)], would only give transversality of the family $\{\Phi^\alpha\}$ on $(\sqrt{2}-1, 0.5)$.

3. One can enlarge the interval where the strong transversality condition (5.1) holds by increasing the amount of numerical computations. However, a computation of $\frac{\partial}{\partial\alpha}\pi_\alpha(110^\infty)$, which can be made exact, implies that this method cannot give more than the interval $(0.17, 0.5)$. Moreover, we believe that the transversality condition fails for small α.

Proof. We are going to apply Lemma 6.1 with $k=2$. Denote by π_α the projection map associated with Φ^α. In view of Lemma 6.1, it is enough to prove that there exists $\epsilon > 0$ such that

$$\frac{\partial}{\partial\alpha}\pi_\alpha(\omega) \le 1-\epsilon \tag{6.5}$$

for all $\omega \in [1] \subset \mathcal{A}^\infty$ and $\alpha \in (0.215, 0.5)$. We have

$$[1] = \{12^\infty, 112^\infty, 1212^\infty\} \cup \bigcup_{k=0}^{\infty}[112^k1] \cup \bigcup_{k=0}^{\infty}[1212^k1] \cup \bigcup_{k=2}^{\infty}[12^k1]\,.$$

Let us begin by checking the three single points listed above. We have

$$\pi_\alpha(12^\infty) = \phi_1(0) = \phi(\alpha),$$

hence $\frac{\partial}{\partial\alpha}\pi_\alpha(12^\infty) = (\alpha+1)^{-2} \le 0.8$ for for all $\alpha \ge 0.215$. Next,

$$\pi_\alpha(112^\infty) = \phi_1(\phi_1(0)) = \phi(\alpha + \phi(\alpha)).$$

Since ϕ is increasing and concave down, the derivative $\frac{\partial}{\partial\alpha}\pi_\alpha(112^\infty)$ is decreasing in α. We used *Mathematica* for the simple numerical computations needed in this lemma and found that

$$\frac{\partial}{\partial\alpha}\pi_\alpha(112^\infty) < 1 \quad \text{for } \alpha \ge 0.171.$$

Similarly, we have verified that

$$\frac{\partial}{\partial\alpha}\pi_\alpha(1212^\infty) < 1 \quad \text{for} \quad \alpha \geq 0.15.$$

Next we check the cylinders. For example, if $\omega \in [1221]$ the inequality (6.3) becomes

$$M_{1221}(1 + M_{221}M_{21}M_1 \cdot 2) \leq 1 - \epsilon$$

which is equivalent to

$$\phi'(\alpha + \phi^3(\alpha)) \cdot \Big(1 + 2(\phi^3)'(\alpha)\Big) \leq 1 - \epsilon.$$

A numerical computation shows that this is true for $\alpha \geq 0.21$ (for some $\epsilon > 0$). In a similar way we made the following table where for each cylinder set it is shown where (6.3) is satisfied (for some $\epsilon > 0$).

cylinder set	$\alpha \geq$
[111]	0.213
[1121]	0.201
[11221]	0.192
[112221]	0.186
[1122221]	0.181
[1211]	0.178
[12121]	0.170
[1221]	0.204
[12221]	0.184

The remaining cylinders are $[112^k1]$, $k \geq 5$, $[1212^k1]$, $k \geq 2$, and $[12^k1]$, $k \geq 4$. They are checked using less accurate estimates. For example, for $w = 12^k1$ the inequality (6.3) becomes

$$\phi'(\alpha + \phi^{k+1}(\alpha)) \cdot (1 + (\phi^{k+1})'(\alpha)) \leq 1 - \epsilon.$$

Clearly, $(\alpha+1)^{-2} = \phi'(\alpha) \geq \phi'(\alpha + \phi^{k+1}(\alpha))$, and $\phi^{k+1}(x) = \frac{x}{(k+1)x+1}$, so $(\phi^{k+1})'(\alpha) = ((k+1)\alpha+1)^{-2}$. Thus, if

$$(\alpha+1)^{-2}(1 + ((k+1)\alpha+1)^{-2}) < 1, \tag{6.6}$$

then the desired condition (6.3) holds for some $\epsilon > 0$. A numerical check yields that (6.6) is true for $\alpha \geq 0.213$ and $k = 4$, and since the left-hand side of (6.6) is monotone decreasing in k, this implies (6.3) for all the cylinders $[12^k1]$, $k \geq 4$. Similarly, by estimating $M_w \leq \phi'(\alpha)$ in (6.3) which makes the expression decreasing in k, we have verified that (6.3) holds for $[112^k1]$, $k \geq 5$, if $\alpha \geq 0.215$, and for $[1212^k1]$, $k \geq 2$, if $\alpha \geq 0.214$ (for some $\epsilon > 0$). Combining all these estimates yields the desired result. □

Now we can apply our results to the family $\Phi^\alpha = \{\frac{x+\alpha}{x+\alpha+1}, \frac{x}{x+1}\}$. The interesting interval of parameters is $0 < \alpha < 0.5$ when the limit set of Φ^α is the interval $[0, \frac{1}{2}(-\alpha + \sqrt{\alpha^2 + 4\alpha})]$ and the IFS has an overlap in the strict sense. Let ν_α be the invariant measure corresponding to Bernoulli $\mu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$. We have $h_\nu = \log 2$. Let χ_α be the corresponding Lyapunov exponent. By (2.5),

$$\chi_\alpha = \int \log[(1+x)(1+x+\alpha)]\, d\nu_\alpha(x). \tag{6.7}$$

The connection with continued fractions is as follows: denote

$$[a_1, a_2, a_3, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Then ν_α is the distribution of the random continued fraction $[1, Y_1, 1, Y_2, 1, Y_3, \dots]$ where $Y_i \in \{0, \alpha\}$ are i.i.d. and have the distribution of Y which equals 0 or α with probabilities $(\frac{1}{2}, \frac{1}{2})$. Alternatively, ν_α can be viewed as the stationary measure for the random matrix $\begin{pmatrix} 1 & Y \\ 1 & 1+Y \end{pmatrix}$. Lyons [Ly] proved that $\dim_{\mathrm{H}} \nu_\alpha \le \log 2/(2\widetilde{\chi}_\alpha)$ where $\widetilde{\chi}_\alpha$ is the top Lyapunov exponent of the random matrix. This estimate is equivalent to (2.6) since the Lyapunov exponent for the IFS satisfies $\chi_\alpha = 2\widetilde{\chi}_\alpha$; this can be verified directly or seen from [Ly, (2.6)] which agrees with (6.7). Lyons estimated numerically the critical value α_c for which $\log 2/\chi_{\alpha_c} = 1$ and found that $\alpha_c \in (0.2688, 0.2689)$. He conjectured that ν_α is absolutely continuous for sufficiently small positive α. In the next corollary we make some progress on this conjecture; in particular, we show that the threshold α_c in Lyons' result is sharp.

Corollary 6.3. *The measure ν_α is absolutely continuous for Lebesgue-a.e. $\alpha \in (0.215, \alpha_c)$ and has Hausdorff dimension equal to $\log 2/\chi_\alpha$ for Lebesgue-a.e. $\alpha \in (\alpha_c, 0.5)$. Moreover, for any set $G \subset (\alpha_c, 0.5)$ we have*

$$\dim_{\mathrm{H}}\{\alpha \in G :\ \dim_{\mathrm{H}}(\nu_\alpha) < \log 2/\chi_\alpha\} \le \sup_G \log 2/\chi_\alpha.$$

Proof is immediate from Lemma 6.2, Theorem 2.3 and Theorem 5.3. □

Remarks. 1. The following questions remain open:

(a) Is there $\alpha \in (0, \alpha_c)$ such that ν_α is singular?

(b) Is it true that ν_α is absolutely continuous for a.e. $\alpha \in (0, 0.215)$?

2. The family ν_α has some resemblance with the well-known family of infinite Bernoulli convolutions which arise from the family of **linear** IFS $\{\lambda x - 1, \lambda x + 1\}$. For Bernoulli convolutions, a countable number of exceptions (of number-theoretic nature) is known [E]. Also, for Bernoulli convolutions the absolute continuity was recently proven for a.e. λ in the

"overlap region" $(0.5, 1)$ [So1, PSo1]. Observe that in our non-linear case ν_α is not an infinite convolution.

3. Lyons [Ly] proved also that ν_α cannot have a density in L^2 for $\alpha > \sqrt{6}/2 - 1 = 0.2247\ldots$ We have recently shown that this threshold is sharp as well. This and other results on L^q densities of invariant measures for IFS will be presented in another paper.

7. Concluding remarks

Here we state the hyperbolic analog of Theorem 2.3.

Definition 7.1. *Let $X \subset \mathbb{R}$ be a closed interval and $\theta \in (0, 1]$. We write $\Phi = \{\phi_1, \ldots, \phi_k\} \in \Xi_X(\theta)$ if ϕ_i are hyperbolic $\mathcal{C}^{1+\theta}$-maps from X into $\mathrm{Int}(X)$ for all $i \le k$.*

Theorem 7.2. *Let $U \subset \mathbb{R}^d$ be an open set. Suppose that $\Phi^{\mathbf{t}} = \{\phi_1^{\mathbf{t}}, \ldots, \phi_k^{\mathbf{t}}\}_{\mathbf{t} \in \overline{U}} \in \Xi_X(\theta)$ is a family of hyperbolic IFS such that the mappings $\mathbf{t} \mapsto \phi_i^{\mathbf{t}}$ are continuous from $\overline{U}$ to $\mathcal{C}^{1+\theta}(X)$ for all $i \le k$ and the transversality condition (2.9) holds. Let μ be a shift-invariant ergodic Borel probability measure with positive entropy on $\mathcal{A}^\infty$ and let $\nu_{\mathbf{t}} = \mu \circ \pi_{\mathbf{t}}^{-1}$. Then*

(i) *for Lebesgue-a.e. $\mathbf{t} \in U$, $\dim_{\mathrm{H}} \nu_{\mathbf{t}} = \min\left\{\frac{h_\mu}{\chi_\mu(\Phi^{\mathbf{t}})}, 1\right\}$;*

(ii) *the measure $\nu_{\mathbf{t}}$ is absolutely continuous for a.e. t in $\{t \in U : \frac{h_\mu}{\chi_\mu(\Phi^{\mathbf{t}})} > 1\}$.*

The proof is analogous to that of Theorem 2.3 but it is much easier, in view of the classical bounded distortion principle for hyperbolic IFS. □

Corollary 7.3. *Suppose that $\Phi \in \Xi_X(\theta)$ is such that $\| \phi_i' \| < 1/2$ for all $i \le k$. Consider the family $\Phi^{\mathbf{t}} = \{\phi_1(x) + t_1, \ldots, \phi_k(x) + t_d\}$. Then the conclusion of Theorem 7.2 holds in some neighborhood of $\mathbf{0} \in \mathbb{R}^k$.*

Proof. Transversality of this family follows by [SSo, Lemma 3.3] so Theorem 7.2 applies. □

Remarks. 1. Hunt [H] considered $\mathcal{C}^{1+\theta}$ families of hyperbolic IFS in any dimension. In the one-dimensional case his result is similar to our Corollary 7.3, however, he does not address the question of absolute continuity and confines himself to the case of Bernoulli measures on the symbolic space. In higher dimensions Hunt [H] assumed that the largest Lyapunov exponent is less than $(1 + \theta)$ times the smallest Lyapunov exponent and showed that the dimension of the measure (actually, the pointwise dimension almost everywhere) equals the Lyapunov dimension.

2. It should be possible to extend the results of this paper to the case when the maps ϕ_i are **conformal**, in any dimension $\mathbb{R}^m$. Also, one can handle parabolic IFS with more than

one parabolic functions if their parabolic points are distinct. The appropriate setting for such IFS is developed in [MU].

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