Estimates for the stable dimension for holomorphic maps

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Abstract

We study the Hausdorff dimension of the intersection between stable manifolds and basic sets for an Axiom A holomorphic endomorphism on $\mathbb{P}^2$. For a map which is at least d'-to-1 on a basic set Λ, we are improving the upper estimate given in [11] by taking into consideration the number of preimages, and thus proving for non-invertible maps results parallel to those of Manning [10] and Verjovsky-Wu [16] from the case of diffeomorphisms.

Also a lower estimate for $HD(W^s_\varepsilon(x)\cap\Lambda)$ is given by using a concept of preimage entropy introduced by Hurley [6]. This preimage entropy plays the role of the entropy of f^{-1} from the case of homeomorphisms, but in general, if the map f is an endomorphism like in our situation, the preimage entropy does not coincide with $h(f|_\Lambda)$.

We also show that, given c with $|c|$ small, the perturbation $(z^2+c+\varepsilon w, w^2)$ of (z^2+c, w^2) is injective on its basic set Λ_ε close to $\Lambda := p_0(c)\times S^1$ (where $p_0(c)$ is the fixed attracting point for z^2+c) and also its stable dimension is strictly positive.

An interesting consequence of our results is the fact that $HD(W^s_\varepsilon(x)\cap\Lambda_g)$ does not depend continuously on g, (for $x\in\Lambda_g$), which is opposite to the situation of diffeomorphisms.

We study also the stable dimension for a large class of quadratic endomorphisms and show that it is either zero, or can be estimated easily using the derivative on the respective basic set.

Keywords: Hausdorff dimension, stable manifolds, preimage entropy, holomorphic endomorphisms, topological pressure.

§1. Introduction
§2. Upper estimate for $HD(W^s_\varepsilon(x)\cap\Lambda)$ using the number of preimages
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1 Introduction

We are considering Axiom A holomorphic functions $f\colon\ \mathbb{P}^2 \to \mathbb{P}^2$ on the 2-dimensional complex projective space. For definitions of hyperbolicity and Axiom A we refer to [5], [7], [15] .

Let Λ be a basic set of the saddle part S_1 of the nonwandering set Ω.

If $x \in \Lambda$ and one denotes by $\phi^s(x) := \log|Df|_{E^s_x}|$, where E^s_x is the stable tangent subspace at x, then we have the result of [11]:

Proposition 1. *Let f a holomorphic Axiom A endomorphism of $\mathbb{P}^2$ such that $C_f \cap S_1 = \emptyset$, where C_f = critical set of f. Then $HD(W^s_\varepsilon(x) \cap \Lambda) \leq t^s$, where t^s is the only zero of the function $t \to P(t \cdot \phi^s)$. The estimate does not depend on $x \in \Lambda$.* $\square$.

In the case of diffeomorphisms we have $HD(W^s_\varepsilon(x) \cap \Lambda) = t^s$ as was proved by Manning [10].

In this article, we denote the nonwandering set of f by Ω and we consider the partition of Ω as $S_0 \cup S_1 \cup S_2$, where the dimension of the unstable spaces over S_i is equal to i, $i \in \{0, 1, 2\}$. The function $P(\cdot)$ referred to in Proposition 1 is the topological pressure [17]. We will now define topological pressure and give some of its properties. Our purpose in doing this is two-fold: first as a convenience for the reader, second because later we shall define a related concept, that of preimage entropy. Parallels between these two notions shall prove very useful.

We will give now a brief description of selected topics from thermodynamic formalism ([17] for a more complete exposition).

The general setting is that of (X, d), a compact metric space, and $f\colon\ X \to X$ a continuous map. For n, a positive integer, $d_n(x, y) := \max\{d(f^i x, f^i y),\ i = 0, ..., n-1\}$ is a metric on X inducing the same topology as the metric d.

Definition 1. A subset $E \subset X$ is called (n, ε)-separated (for some $\varepsilon > 0$) if for all $x, y \in E$, $x \neq y$, we have $d_n(x, y) \geq \varepsilon$.

Definition 2. The *topological pressure* of f is the functional $P_f\colon\ \mathcal{C}(X, \mathbb{R}) \to \overline{\mathbb{R}}$ defined on the space of continuous functions by:

$$P_f(\varphi) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \exp\left(\sum_{i=0}^{n-1} \varphi(f^i x) \right), E \subset X, (n, \varepsilon)\text{-separated set.} \right\}.$$

Definition 3. When considering $\varphi \equiv 0$ in Definition 2, we obtain the notion of *topological entropy* of f.

There exists an interesting relationship between Borel invariant measures and P_f, contained in the following:

Theorem (Variational Principle). *In the above setting, $P_f(\varphi) = \sup_\mu\{h_\mu(f) + \int \varphi\ d\mu\}$, where the supremum is taken over all f-invariant Borel probability measures μ, and $h_\mu(f)$ = measure-theoretic entropy of μ.*

For the (long) definition of $h_\mu(f)$ and proofs of all these facts, as we mentioned, a good reference is [17].

Theorem (Properties of Pressure). *If $f\colon\ X \to X$ is a continuous transformation, and $\varphi, \psi \in \mathcal{C}(X, \mathbb{R})$, then:*

1) $\varphi \leq \psi \Rightarrow P_f(\varphi) \leq P_f(\psi)$

2) $P_f(\cdot)$ is either finitely valued or constantly ∞

3) P_f is convex

4) for a strictly negative function φ, the mapping $t \to P_f(t\varphi)$ is strictly decreasing if $P(0) < \infty$.

5) P_f is a topological conjugacy invariant.

In our situation $X = \Lambda$ (Λ a basic set of S_1); then, because $\phi^s < 0$, it follows that $t^s =$ zero of $P(t\phi^s)$ is uniquely defined, and $t^s \geq 0$ since $P(0) = h(f|_\Lambda) \geq 0$, $P(t\phi^s) < 0$ for large enough t, and $t \to P(t\phi^s)$ is strictly decreasing.

Define now the inverse limit of X, $\widehat{X} := \{(x_n)_{n\leq 0},\ f(x_{n-1}) = x_n,\ n \leq 0,\ x_n \in X\}$, and call an element $\hat{x} = (x_n)_{n\leq 0}$ of $\widehat{X}$ a prehistory of x_0. The metric on $\widehat{X}$ is the usual metric, see for example [12].

The map f extends to a homeomorphism $\hat{f}$ of $\widehat{X}$ given by $\hat{f}((x_n)_{n\leq 0}) = (x_{n+1})_{n\leq 0}$, where $x_1 = f(x_0)$. In this case, denoting by $p(\hat{x}) = x_0$ the canonical projection, $p\colon\ \widehat{X} \to X$, the following diagram commutes:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\hat{f}} & \widehat{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X \end{array}$$

It is well-known that, in this case, $h_f = h_{\hat{f}}$.

We now come back to $HD(W^s_\varepsilon(x) \cap \Lambda)$. In [11] the first author studied the maps $f(z, w) = (z^2 + c, w^2 + d)$, $c \neq 0$, $d \neq 0$ and showed that if c is chosen such that $|1 - \sqrt{1 - 4c}| = \frac{4}{5}$, then $P(2\phi^s) > 0$ implies $t^s > 2$.

So, the estimate $HD(W^s_\varepsilon(x) \cap \Lambda) \leq t^s$ is not sharp because $W^s_\varepsilon(x) \cap \Lambda$ consists of only one point. A different estimate for a certain class of maps will be proved in §2 to take this fact into consideration.

Theorem 1. *Assume f is Axiom A, holomorphic endomorphism on $\mathbb{P}^2$ of degree d, and Λ is one of the basic sets with unstable index 1. Suppose that $C_f \cap \Lambda = \emptyset$ and that $f|_\Lambda\colon\ \Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least d' preimages in Λ, $d' \leq d$. Then $HD(W^s_\varepsilon(x) \cap \Lambda) \leq \frac{\log d' - h_{top}(f|_\Lambda)}{\log \alpha + \log \gamma}$, as long as $\alpha\gamma \leq 1$ where $\alpha := \frac{\sup\limits_{x\in\Lambda} |Df|_{E^s_x}|}{\inf\limits_{y\in\Lambda} |Df|_{E^s_y}|}$, and $\gamma := \sup\limits_{x\in\Lambda} |Df|_{E^s x}|$. The estimate is independent on $x \in \Lambda$.* □

This result complements the theorem from [12] stating that the interior of the union of all unstable manifolds corresponding to points in S_1, is empty.

In general, if $f|_\Lambda\colon\ \Lambda \to \Lambda$ is d'-to-1, then $h(f|_\Lambda) \geq \log d'$ (this is proved similarly to the Misiurewicz-Przytycki theorem, [7]). And, if $\alpha\gamma \leq 1$, $\log\alpha + \log\gamma \leq 0$, so $\frac{\log d' - h(f|_\Lambda)}{\log\alpha + \log\gamma} \geq 0$. For the example given above, $f(z,w) = (z^2+c, w^2+d)$, $h(f|_\Lambda) = \log 2, d' = 2$, hence the above estimate is sharp.

In the case of diffeomorphisms, $HD(W^s_\varepsilon(x) \cap \Lambda) = t^s$ ([16]). However in our case, that proof will not work mainly because the estimates using $h_\mu(f)$ break down due to the non-invertibility of f. The main problem is that the entropy was defined by considering the forward iterates of f, and, if f is not injective, $h(f)$ does not shed any light upon the growth of preimages. To compensate this, we will use in Theorem 2 the notion of *preimage (branch) entropy*, introduced by Hurley [6] and studied in [6] and [13].

Let us call a *branch* of length ℓ (or *prehistory* of length ℓ) in X, a sequence of preimages, $\beta = (z_0, z_{-1}, \ldots, z_{-\ell})$, with $z_i \in X$, $-\ell \leq i \leq 0$, such that $f(z_{i-1}) = z_i$, $-\ell+1 \leq i \leq 0$. Given another branch $\beta' = (z'_0, \ldots, z'_{-\ell})$ of same length, define their *branch distance* to be $d^b(\beta,\beta') = \max_{j=\overline{0,\ell}} d(z_{-j}, z'_{-j})$. The reader can notice the similarity between the branch distance and $d_n(\cdot,\cdot)$ introduced earlier. Like $d_n(\cdot,\cdot)$ for forward iterates, d^b measures the growth of inverse iterates. Using this, we now define a *branch metric* on X:

$$d^b_\ell(x,x') < \varepsilon,$$

if for every branch β of length ℓ with $z_0 = x$, there exists a branch β' of length ℓ with $z'_0 = x'$ such that $d^b(\beta,\beta') < \varepsilon$, and vice versa. Denote by $N_{\text{span}}(\varepsilon, d^b_\ell, X)$ the smallest cardinality of an ε-spanning set for X in the d^b_ℓ metric. Hence, if A is an ε-spanning set with $\#A = N_{\text{span}}(\varepsilon, d^b_\ell, X)$, then, $\forall x \in X$, $\exists y \in A$ with $d^b_\ell(x,y) < \varepsilon$.

Let also $N_{\text{sep}}(\varepsilon, d^b_\ell, X)$ be the largest cardinality of an ε-separated set for X. So, if A is ε-separated, then for all $x, y \in A$, $x \neq y$, $d^b_\ell(x,y) > \varepsilon$.

The following proposition gives the definition of the preimage entropy $h_i(f)$ and two ways to calculate it.

Proposition ([13]). *For $f\colon\ X \to X$ continuous, (X,d) compact metric space, we have*

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log N_{sep}(\varepsilon, d^b_n, X) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log N_{\text{span}}(\varepsilon, d^b_n, X)$$

and the common value is called the preimage (branch) entropy*, denoted by $h_i(f)$.* □

In general there is no relation between $h_i(f)$ and $h(f)$. Besides $h_i(f)$, there are several other notions of inverse entropy, a local one called $h_m(f)$ and a global one called preimage relation entropy, and denoted by $h_r(f)$, but we will not use them directly in our proofs.

Their definitions and properties can be found in the articles by Langevin and Walczak [9] and Hurley [6]. Hurley [6] proved the following.

Proposition. *For X compact metric space, $f\colon X \to X$ continuous,*

$$0 \leq h_m(f) \leq h(f) \leq h_m(f) + h_i(f) \leq h_m(f) + h_r(f).$$

If f is a homeomorphism, then $h_m(f) = 0$ and $h(f) = h_i(f) = h_r(f)$. □

Let us recall now two cases when $h_i(f) = 0$.

a. Forward-expansive coverings

If X is a metric space, then a continuous map $f\colon X \to X$ is called *forward expansive* if $\exists \varepsilon_0 > 0$ such that whenever $x, y \in X$, $x \neq y$, $\exists m \geq 0$ with

$$d(f^m x, f^m y) \geq \varepsilon_0 > 0.$$

For example, f is forward-expansive on any invariant subset of a Riemannian manifold on which Df is expanding by a constant factor $\lambda > 1$. Denker and Urbanski [4] (see also Coven and Reddy [3]) showed that, if X is a compact metric space, then a forward-expansive map is in fact *expanding*, i.e. for some metric d on X (equivalent to the original one), there exists $b > 0$, $\lambda > 1$ constants such that $d(x, y) < b$ implies $d(f(x), f(y)) \geq \lambda d(x, y)$. We are also saying that $f\colon X \to X$ is a *covering map* if $(\forall) x \in X, \exists$ a neighborhood U_x of x, such that $f^{-1}(U_x) = \bigcup_i V^i$ with $\{V^i\}$ open disjoint sets, and such that $f\colon V^i \to U_x$ is homeomorphism.

If f is forward-expansive on X compact metric space, then we can use Denker-Urbanski [4] characterization in connection to $h_i(f)$ because $h_i(f)$ is a topological concept which does not depend on the metric ([13]). This is the idea of the proof for the following:

Proposition ([13]). *If $f\colon X \to X$ is a* forward expansive covering *map, then $h_i(f) = 0$.* □

b. A second situation when $h_i(f) = 0$ is that of graph maps.

A *finite graph* is a compact metric space K with a distinguished finite set of points called *vertices*, whose complement has finitely many connected components, *edges*, homeomorphic to the open interval (0,1). We fix the metric on K by assigning length 1 to each edge and the distance between two points in K is the length of the shortest path connecting them.

Theorem (Nitecki-Przytycki, [13]). *Let K a finite graph and $f\colon K \to K$ continuous map. Then $h_i(f) = 0$.* □

Corollary. *For any continuous self-map f of a closed interval $[a, b]$, or of the circle S^1, we have $h_i(f) = 0$.* □

We are ready now to state Theorem 2 which gives a lower estimate for $HD(W^s_\varepsilon(x) \cap \Lambda)$.

Theorem 2. *If Λ is a basic set for an Axiom A holomorphic endomorphism $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ and $C_f \cap \Lambda = \emptyset$, then for any $x \in \Lambda$,*

$$HD(W^s_\varepsilon(x) \cap \Lambda) \geq \left| \log \inf_{y \in \Lambda} |Df|_{E^s_y}| \right|^{-1} \cdot h_i(f|_\Lambda) \geq 0$$ □

Remark. For $f(z,w) = (z^2 + c, w^2 + d), c, d$ small and non-zero, and $\Lambda = \{z_0\} \times J_{(w^2+d)}$, with $J_{(w^2+d)}$ the Julia set of $w \to w^2 + d$, and z_0 an attracting fixed point for $z^2 + c$, we already know that $W^s_\varepsilon(x) \cap \Lambda$ consists of a single point for $x \in \Lambda$, and therefore $HD(W^s_\varepsilon(x) \cap \Lambda) = 0$. This is in accordance with the fact from the Corollary, that $h_i(f|_\Lambda) = 0$ if Λ is a circle (or homeomorphic with a circle). □

We also show that, given c with $|c|$ small, the perturbation $(z^2 + c + \varepsilon w, w^2)$ of $(z^2 + c, w^2)$ is injective on its basic set Λ_ε close to $\Lambda := p_0(c) \times S^1$ (where $p_0(c)$ is the fixed attracting point for $z^2 + c$).

Then using Theorem 2 and the observation that for any homeomorphism g we have $h_i(g) = h(g)$, we will get also that its stable dimension is strictly positive.

Since the perturbation can be taken arbitrarily close to the map $(z,w) \to (z^2 + c, w^2)$, it will follow in Section 4 that $HD(W^s_\varepsilon(x) \cap \Lambda_g$ does not depend continuously on the map g, when $x \in \Lambda_g$, which is a contradiction to the caes of Henon maps, [16].

In Section 4 we study a large class of quadratic maps on $\mathbb{P}^2$ and prove that, if the basic set Λ is connected, then one can control the number of preimages and hence, by using Theorems 1 and 2, the stable dimension.

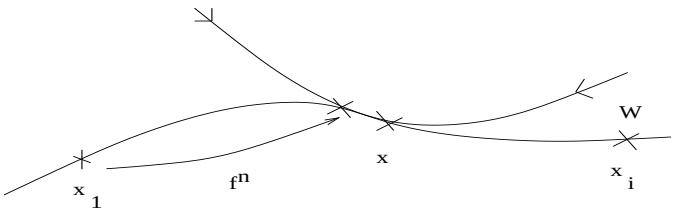
All this will be made precise in the sequel.

2 Upper Estimate for $HD(W^s_\varepsilon(x) \cap \Lambda)$

Theorem 1 *Assume f is an Axiom A, holomorphic map of degree $d \geq 2$ on $\mathbb{P}^2$, and Λ is one of the basic sets with unstable index equal to 1. Suppose $C_f \cap \Lambda = \emptyset$ (C_f= critical set of f) and also that $f|_\Lambda$: $\Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least $d' \leq d$ preimages in Λ. Then $HD(W^s_\varepsilon(x) \cap \Lambda) \leq \frac{\log d' - h_{top}(f|_\Lambda)}{\log \alpha + \log \gamma}$, where $\alpha := \frac{\sup_{x \in \Lambda} |Df|_{E^s_x}|}{\inf_{y \in \Lambda} |Df|_{E^s_y}|}$ and $\gamma := \sup_{x \in \Lambda} |Df|_{E^s_x}|$, as long as $\alpha \leq \gamma^{-1}$.*

Observation: In the case of diffeomorphisms, $HD(W^s_\varepsilon(x) \cap \Lambda) = t^s$, where t^s is the unique zero of the stable function $t \to P(t\phi^s)$, $\phi^s(x) := \log |Df|_{E^s_x}|$. Also, from the Variational Principle, it follows that there exists a probability measure μ_s such that $t^s = \frac{h\mu_s}{\lambda_s}$, λ_s being the Lyapunov exponent of μ_s. In the case of endomorphisms, it was showed in [11] that $HD(W^s_\varepsilon(x) \cap \Lambda) \leq t^s$, but in general, the inequality is strict. For example for the case of a map $f(z,w) = (P(z), Q(w))$, with P, Q hyperbolic on their Julia sets, $HD(W^s_\varepsilon(z,w) \cap \Lambda) = 0$, for $(z,w) \in \Lambda = \{(z_0, w_0)\} \times J_Q$, where Q_0 is an attracting periodic point for P. So, the formula in the Theorem explains that the gap between $HD(W^s_\varepsilon(x) \cap \Lambda)$ and t^s is due to the number of preimages.

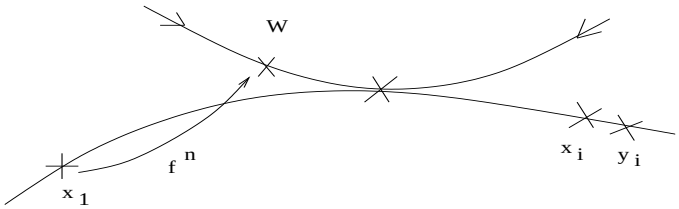
Proof. Denote $W := W^s_\varepsilon(x) \cap \Lambda$. Let E be an (n, δ)-separated set of maximal cardinality inside $f^{-n}(W) \cap \Lambda$ with $\delta << \varepsilon$ to be determined in the course of the proof.



We let now a point $x_1 \in E$, $f^n(x_1) \in W$, and $f^n(x_1)$ has at least d'^n different preimages belonging to Λ.

$$f^{-n}(f^n x_1) \cap \Lambda = \{x_1, x_2, \dots, x_{n(x_1)}\},$$

and denote by x_{i_1} the point of this set such that $\|Df^n|_{E^s_{x_{i_1}}}\| = \max\{\|Df^n|_{E^s_{x_i}}\|, i = 1, ..., n(x_1)\}$. Then there exists a universal constant A such that $A \cdot \|Df^n|_{E^s_{x_{i_1}}}\| \geq$ diam $f^n(W^s_\delta(x_i))$, $i = 1, ..., n(x_1)$, where this inequality follows from Lemma 1 of [11].



Since E is a maximal (n, δ)-separated set, there exist points $y_i \in E$ such that $y_1 = x_1$ and $x_i \in W^s_\delta(y_i)$, $i = 1, ..., n(x_1) \Leftrightarrow y_i \in W^s_\delta(x_i)$, $i = 1, ..., n(x_1)$ (in fact there could be more than one point from E inside $W^s_\delta(x_i)$, but we choose one).

We will show that $y_i \neq y_j$ for $i \neq j$. Suppose this is not true, then $y_i = y_j$ for some $i \neq j$, $\Rightarrow x_i, x_j \in W^s_\delta(y_i)$. But $f^n x_i = f^n x_j \Rightarrow \exists m$ such that $f^m(x_i) \neq f^m(x_j)$, $f^m(x_i), f^m(x_j) \in W^s_\delta(f^m(y_i))$, and $f^{m+1}(x_i) = f^{m+1}(x_j)$.

However $C_f \cap \Lambda = \emptyset$; hence $\exists \varepsilon_0 > 0$ such that $f\colon\ B(x', \varepsilon_0) \to \mathbb{P}^2$ injective, $(\forall) x' \in \Lambda$, and $\delta \ll \varepsilon_0$. We can choose $\delta << \varepsilon_0$, so one cannot have $f(f^m(x_i)) = f(f^m(x_j)), f^m(x_i) \neq f^m(x_j)$ and $d(f^m(x_i), f^m(x_j)) < 2\delta$. In conclusion $y_i \neq y_j$, $(\forall) i \neq j$.

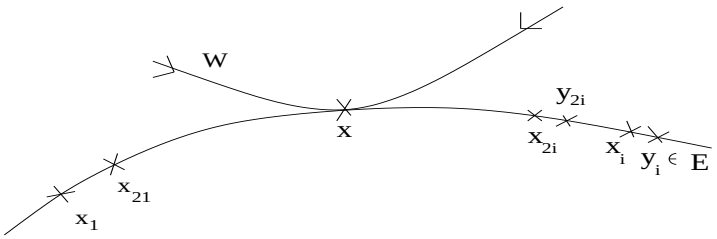
We assumed that $\|Df^n|_{E^s_{x_{i_1}}}\| = \max\{\|Df^n|_{E^s_{x_i}}\|, i = 1, ..., n(x_1)\}$ which implies that $f^n(W^s_\delta(x_i)) \subset f^n(W^s_{C\cdot\delta}(x_{i_1}))$, $(\forall) i = 1, ..., n(x_1)$ with C a universal constant which does not depend on x_1, x_i or n.

We have $\bigcup_{i=1}^n (x_1) f^n(W^s_\delta(y_i)) \subset f^n(W^s_{2C\delta}(x_1))$, hence we choose only the point $x_1 = y_1$ out of the set $\{y_1, .., y_{n(x_1)}\}$.

If for all points $z \in E$, we have $f^n(z) \in f^n(W^s_{2C\delta}(x_{i_1}))$, then $W \subset f^n(W^s_{(2C+1)\delta}(x_{i_1}))$ for n large enough and the procedure stops.

So, suppose there exists a point $x_{2,1} \in E$ such that $f^n(x_{2,1}) \in W \backslash f^n(W^s_{2\delta C}(x_{i_1}))$.

$f^n(x_{2,1})$ has $n(x_{2,1})$ preimages $x_{2,1}, x_{2,2}, ..., x_{2,n(x_{2,1})}$ in $f^{-n}(f^n(x_{2,1})) \cap \Lambda$, where $n(x_{2,1}) \geq d'^n$.



Now for each $x_{2,i}$, a preimage of $f^n(x_{2,1})$, $\exists y_{2,i} \in E$ such that $x_{2,i} \in W^s_\delta(y_{2,i})$, $(\forall)i$ (equivalently $y_{2,i} \in W^s_\delta(x_{2,i})$, $(\forall)i$). By the same argument as before, $y_{2,i} \neq y_{2,j}$, $i \neq j$.

Is it possible to have $y_i = y_{2,j}$ for some i, j?

If $y_i = y_{2,j} \Rightarrow x_i, x_{2,j} \in W^s_\delta(y_i)$.

$$x_{2,j} \in W^s_{2\delta}(x_i) \Rightarrow f^n(x_{2,j}) \in f^n(W^s_{2\delta}(x_i)).$$

But $f^n(x_{2,j}) = f^n(x_{2,1}) \notin f^n(W^s_{2C\delta}(x_1))$ and $f^n(W^s_{2\delta}(x_i)) \subset f^n(W^s_{2C\delta}(x_1))$. So $y_i \neq y_{2,j}, \forall i, j$.

Therefore $\{y_1, \dots, y_{n(x_1)}\} \cap \{y_{2,1}, \dots, y_{2,n(x_{2,1})}\} = \emptyset$. Now, assuming for instance that y_{2,i_2} satisfies

$$\|Df^n|_{E^s_{y_{2,i_2}}}\| = \max\{\|Df^n|_{E^s_{y_{2,i}}}\|, i = 1, ..., n(x_{2,1})\},$$

we can cover all $f^n(W^s_\delta(y_{2,i}))$ with $f^n(W^s_{2C\delta}(y_{2,i_2}))$. So, from the set $\{y_{2,1}, \dots, y_{2,n(x_{2,1})}\} \subset E$ we can keep only y_{2,i_2}, and throw away the other points. Also, from $\{y_1, \dots, y_{n(x_1)}\}$ we kept only y_{i_1}, as long as $\|Df^n|_{E^s_{y_{i_1}}}\| = \max\{\|Df^n|_{E^s_{y_i}}\|, i = 1, ..., n(x_1)\}$. This procedure is repeated until we exhaust the entire finite set E. But, since E is (n, δ)-spanning $\Rightarrow$

$$\Rightarrow W \subset \bigcup_{y \in E} f^n W^s_\delta(y) \subset \bigcup_{z_k \in \widetilde{E}} f^n(W^s_{(2C+1)\delta}(z_k)),$$

where $\widetilde{E}$ is the subset obtained by the above procedure: $y_{i_1} \in \widetilde{E}$, $y_{2,i_2} \in \widetilde{E}$, etc. (points with maximum $\|Df|_{E^s}\|$ among preimages).

Now $\sum_{z_k \in \widetilde{E} \subset E} [\text{diam } f^n(W^s_{(2C+1)\delta}(z_k))]^t \leq A \cdot \sum_{z_k \in \widetilde{E}} \|Df^n|_{E^s_{z_k}}\|^t \leq A \cdot \frac{\alpha^t}{d'^n} \cdot \sum_{y \in E} \|Df^n|_{E^s_y}\|^t$.

From the Variational Principle, $P(t\phi^s) \leq h_{\text{top}}(f|_\Lambda) + t \log \gamma$, $\gamma := \sup_{x \in \Lambda} \|Df|_{E^s_x}\|$.

Assume that $t > \frac{\log d' - h_{\text{top}}(f|_\Lambda)}{\log \alpha + \log \gamma}$ and let $t > t' > \frac{\log d' - h_{\text{top}}(f|_\Lambda)}{\log \alpha + \log \gamma}$. Then $P(t\phi^s) < P(t' \cdot \phi^s) \leq h_{\text{top}}(f|_\Lambda) + t' \cdot \log \gamma$, so $\sum_{y \in E} \|Df^n|_{E^s_y}\|^t < e^{n(h_{\text{top}} f|_\Lambda) + t' \log \gamma)}$, for all n large enough .

This implies that $\sum_{z_k \in \widetilde{E}} [\text{diam } f^n(W^s_{(2C+1)\delta}(z_k))]^t \leq e^{n(h_{\text{top}}(f|_\Lambda) + t' \log \gamma + t \log \alpha - \log d')}$.

But $t > \frac{\log d' - h_{\text{top}}(f|_\Lambda)}{\log \alpha + \log \gamma}$ and we can choose t' as close to t as we want. Hence, using also here that $\alpha\gamma \leq 1$, we can make $h_{\text{top}}(f|_\Lambda) + t' \log \gamma + t \log \alpha - \log d' < 0 \Rightarrow \sum_{z_k \in \widetilde{E}} [\text{diam}(f^n(W^s_{(2C+1)\delta}(z_k))]^t \underset{n \to \infty}{\longrightarrow} 0 \Rightarrow$ $HD(W) \leq t$.

But t has been chosen arbitrarily larger than $\frac{\log d' - h_{top}(f|_\Lambda)}{\log \alpha + \log \gamma}$, therefore we obtain the desired inequality.

□

3 Lower Estimate for $HD(W^s_\varepsilon(x) \cap \Lambda)$

We will start by defining another concept of preimage entropy, which does not require X to be compact, or even a metric space. Suppose $\mathcal{A}$ is an open cover of X. For a subset $E \subset X$, define $n^-_{\mathcal{A}}(E)$ as the largest integer such that for all $0 \leq k < n^-_{\mathcal{A}}(E)$, and all $x, y \in E$, and every prehistory of x of length k, $\hat{x} = (x, x_{-1}, \dots, x_{-k})$, $\exists \hat{y} = (y, y_{-1}, \dots, y_{-k})$ prehistory of y of length k, and $\exists k$ sets from $\mathcal{A}, E^0, \dots, E^k$, such that $(x, y) \in E^0 \times E^0$, $(x_{-1}, y_{-1}) \in E^1 \times E^1, \dots, (x_{-k}, y_{-k}) \in E^k \times E^k$. If $\mathcal{B} = \{B_1, B_2, \dots\}$ has union covering X, set

$$\mathcal{D}_{\mathcal{A}}(\mathcal{B}, \lambda) := \sum_{i=1}^{\infty} \exp(-\lambda n^-_{\mathcal{A}}(B_i)).$$

Like in the definition of Hausdorff measures, let

$$m^-_{\mathcal{A},\lambda}(X) := \lim_{\varepsilon \to 0} \inf\{\mathcal{D}_{\mathcal{A}}(\mathcal{B}, \lambda); \mathcal{B} = \{B_1, B_2, \dots\}, X \subset \bigcup_{i=1}^{\infty} B_i, \text{ and } \exp(-n^-_{\mathcal{A}}(B_i)) < \varepsilon, (\forall) i\}$$

Then, let $h^-_{\mathcal{A}}(f) := \inf\{\lambda, m^-_{\mathcal{A},\lambda}(X) = 0\}$, and $h^-(f) := \sup_{\mathcal{A}} h^-_{\mathcal{A}}(f)$. It is easy to check that $h^-_{\mathcal{A}} \geq h^-_{\mathcal{A}'}$, when $\mathcal{A}, \mathcal{A}'$ are two open covers and $\mathcal{A}$ refines $\mathcal{A}'$, so $h^-(f)$ is the $\limsup h^-_{\mathcal{A}}(f)$ when $\text{mesh}(A) \to 0$.

This definition is inspired by Bowen's definition of topological entropy for non-compact spaces in [1].

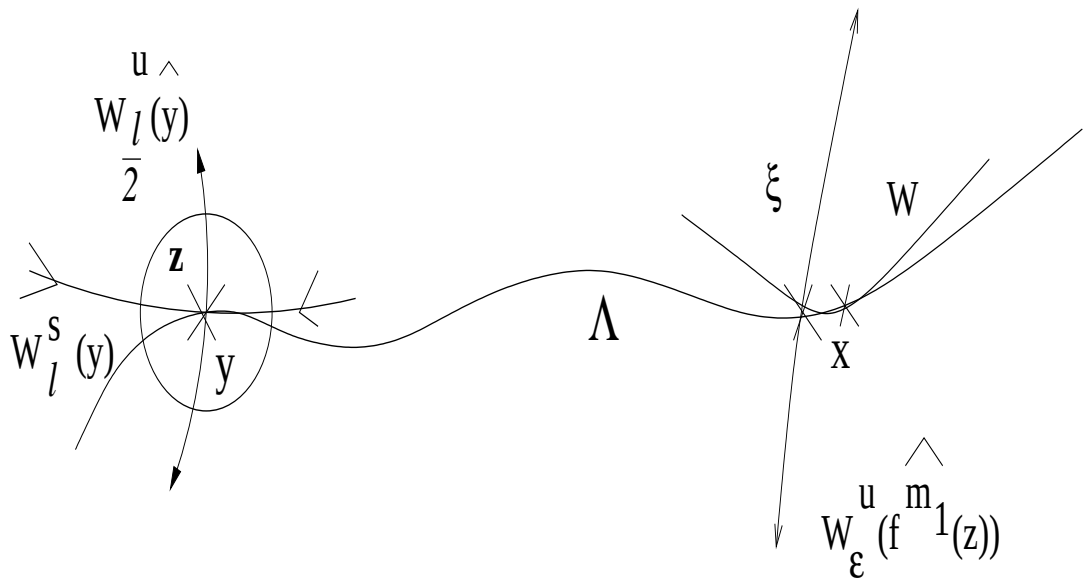
Bowen proved that his definition coincides with the usual one for compact spaces. Noticing that the same idea of taking prehistories of points and defining "closeness" with regard to them appears in both $h^-(f)$ and $h_i(f)$, one can use the methods of Bowen ([1], pg. 126) to show that:

Proposition 2. *If $f\colon X \to X$ continuous map on a compact metric space X, then $h^-(f) = h_i(f)$.* □

The use of Bowen's definition to show estimates on HD was employed successfully in [10]. We are now ready to prove:

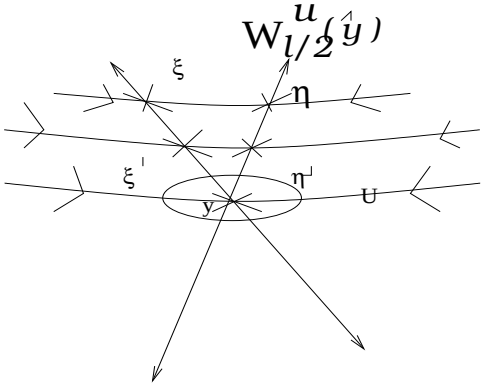
Theorem 2. *If Λ is a basic set of unstable index 1 for the Axiom A map $f\colon \mathbb{P}^2 \to \mathbb{P}^2$, and $C_f \cap \Lambda = \emptyset$, then $HD(W^s_\varepsilon(x) \cap \Lambda) \geq \left| \log \inf_{y \in \Lambda} |Df|_{E^s_y}| \right|^{-1} \cdot h_i(f|_\Lambda)$, for all $x \in \Lambda$.*

Proof. In view of Proposition 2, and since Λ is compact, $h_i(f|_\Lambda)$ can in fact be calculated as $h^-(f|_\Lambda)$. Let $W := W^s_\varepsilon(x) \cap \Lambda$ for a point x in Λ, and consider $y \in \Lambda$. Let also $\mathcal{A}$ be an open cover of Λ with $mesh(\mathcal{A}) =: \ell$. We will prove that there exists an integer m such that $f^{-m}(W) \cap \Lambda$ intersects any unstable local manifold $W^u_{\ell/2}(\hat{y})$, $(\forall)\hat{y} \in \hat{\Lambda}$. Denote by $B(y, \ell/4)$ the $\ell/4$-ball around y.



Since Λ is a basic set, hence $f|_\Lambda$ is transitive, it follows that $\exists\ m_1$ integer and $z \in B(y, \ell/4) \cap \Lambda$ such that $f^{m_1}(z) \in B\left(x, \frac{\varepsilon}{2}\right)$. Now take the unstable manifold $W^u_\varepsilon(\hat{f}^{m_1}(z))$, where $\widehat{f^{m_1}(z)}$ is any prehistory of $f^{m_1}(z)$ such that $(f^{m_1}(z))_{-m_1} = z$. From the local product structure, $W^u_\varepsilon(\widehat{f^{m_1}(z)}) \cap W = \xi$. The point ξ has a prehistory $\hat{\xi}$ with $d(\xi_{-m_1}, z) < \ell/4$, $\xi_{-m_1} \in \Lambda$, if m_1 is big enough. So, $W^s_{M\cdot\ell}(\xi_{-m_1})$ intersects $W^u_{\ell/2}(\hat{y})$ transversally, for any prehistory $\hat{y}$ of y, where M is some constant which does not depend on y, giving the maximum inclination of the unstable spaces w.r.t. stable ones.

Also, $f^{m_1}(\xi_{-m_1}) \in W$ and $f^{m_1}(W^s_{M\cdot\ell}(\xi_{-m_1}) \cap \Lambda) \subset W$, so $W^s_{M\cdot\ell}(\xi_{-m_1}) \cap \Lambda \subset f^{-m_1}(W) \cap \Lambda$. We cover now the set $f^{-m}(W) \cap \Lambda$ with open sets, $\mathcal{U} = \{U\}$, such that for some $\varepsilon' > 0$, $\sum_{U\in\mathcal{U}} (\text{diam } U)^{\delta+\varepsilon'} < \frac{1}{2}$, with $\delta := HD(W^s_\varepsilon(x) \cap \Lambda)$. For each $U \in \mathcal{U}$, let $U^* := \left(\bigcup_{y\in U} W^u_{\ell/2}(\hat{y})\right) \cap \Lambda$, obtaining in this way a cover of Λ.



Let us take a point $\xi \in U^*$; then $W^s_{\ell'}(\xi) \cap W^u_{\ell/2}(\hat{y}) = \{\eta\}, \ell' := M \cdot \ell$.

Assume that in the prehistory $\hat{y}$, we have $\hat{y}_{-k} = y^i_k$, where $\{y^i_k\}$ is a way to index the preimages of y under f^k.

Now η has a preimage η^i_{-k} by f^k such that y^i_{-k} and η^i_{-k} are very close to each other, where $f^k(y^i_{-k}) = y$. From the definition of $n^-_{\mathcal{A}}(U^*), \xi$ has a prehistory $(\xi^i_{-1}, \dots, \xi^i_{-k})$ such that ξ^i_{-k} and η^i_{-k} are in the same set A^i_{-k} from $\mathcal{A}$, for all $0 < k < n_{\mathcal{A}}(U^*)$. Since mesh $(\mathcal{A}) \leq \ell$ for example, we get

$$\xi^i_{-k} \in W^s_\ell(\eta^i_{-k}) \Rightarrow d(\xi, \eta) = d(f^k\xi^i_{-k}, f^k\eta^i_{-k})$$
$$\leq d(\xi^i_{-k}, \eta^i_{-k}) \cdot \|Df^k|_{E^s_\omega}\| \text{ for some } \omega \in W^s_\ell(\eta^i_{-k}).$$

Similarly, if ξ, ξ' are arbitrary points in U^*, and if they have prehistories such that $\xi^i_{-k}, \xi^{'i}_{-k}$ are in the same set A^i_{-k} of $\mathcal{A}$, then

$$d(\xi^i_{-k}, \xi^{'i}_{-k}) \leq d(\xi^i_{-k}, \eta^i_{-k}) + d(\eta^i_{-k}, \eta^{'i}_{-k}) + d(\xi^{'i}_{-k}, \eta^{'i}_{-k}).$$

Since $\eta^i_{-k}, \eta^{'i}_{-k}$ follow the unstable manifold $W^u_{\ell/2}(\hat{y})$, one gets $d(\eta^i_{-k}, \eta^{'i}_{-k}) < \frac{\ell}{2}$ and for $d(\xi^i_{-k}, \eta^i_{-k})$ and $d(\xi^{'i}_{-k}, \eta^{'i}_{-k})$ we use the estimate (3).

Hence,

$$d(\xi^i_{-k}, \xi^{'i}_{-k}) \leq \frac{\ell}{2} + d(\xi, \eta) \cdot \|Df^k|_{E^s_\omega}\|^{-1} + d(\xi', \eta') \cdot \|Df^k|_{E^s_{\omega'}}\|^{-1},$$

and diam $W^s_\ell(\eta) \cap U^* \leq M \cdot$ diam U, $k < \eta_{\mathcal{A}}(U^*)$. Again by using the maximality of $n = n^-_{\mathcal{A}}(U^*)$, we will obtain for some points ξ, ξ' and some of their prehistories that

$$\ell < d(\xi^i_{-n}, \xi^{'i}_{-n}) \leq \frac{\ell}{2} + 2M \cdot \text{diam } U \cdot \sup_\omega \|Df^n|_{E^s_\omega}\|^{-1}.$$

So, $\frac{\ell}{2} \leq 2M \cdot \text{diam } U \cdot \left(\inf_{y \in W^s_\ell(\Lambda)} \|Df|_{E^s_y}\| \right)^{-n^-_{\mathcal{A}}(U^*)}$. Denote $\inf_{y \in W^s_\ell(\Lambda)} \|Df\|_{E^s_y}\|$ by λ_s; since $C_f \cap \Lambda = \emptyset$ it follows $\lambda_s > 0$ and then,

$$\sum_{U^*} \left(\lambda_s^{n^-_{\mathcal{A}}(U^*)} \cdot \frac{\ell}{4M} \right)^{\delta+\varepsilon'} \leq \sum_{U \in \mathcal{U}} (\text{diam } U)^{\delta+\varepsilon'} < \frac{1}{2}.$$

Therefore, $h^-_{\mathcal{A}} \leq |\log \lambda_s| \cdot (\delta + \varepsilon')$.

And now, since $h^- = \overline{\lim}_{\text{mesh}(\mathcal{A}) \to 0} h^-_{\mathcal{A}}$, and ε' was chosen arbitrarily,

$$h^-(f|_\Lambda) \leq |\log \lambda_s| \cdot \delta$$
$$\Rightarrow HD(W^s_\varepsilon(x) \cap \Lambda) \geq h^-(f|_\Lambda) \cdot |\log \lambda_s|^{-1} =$$
$$= h_i(f|_\Lambda) \cdot |\log \lambda_s|^{-1}.$$

□

We conclude this section with the remark that $h_i(f|_\Lambda) \neq h_i(\hat{f}|_{\hat{\Lambda}})$ and that the value of h_i is not stable under perturbations.

Remark: Indeed, let us take Λ, a basic set for an Axiom A endomorphism f without cycles among the basic sets of its saddle part S_1. Then, if f_ε is sufficiently close to f, f_ε will have a basic

set Λ_ε close to Λ in the Hausdorff metric. And we know from the Stability Theorem that there exists a homeomorphism $\hat{g}\colon\ \hat{\Lambda} \to \hat{\Lambda}_\varepsilon$ which commutes with a surjection $g\colon\ \hat{\Lambda} \to \Lambda_\varepsilon$

$$\begin{array}{ccc} \hat{\Lambda} & \xrightarrow{\ \hat{g}\ } & \hat{\Lambda}_\varepsilon \\ p\big\downarrow & \searrow & \big\downarrow p \\ \Lambda & g & \Lambda_\varepsilon \end{array} \qquad p \text{ is the canonical projection.}$$

We know that $h(f|_\Lambda) = h(\hat{f}|_{\hat{\Lambda}}) = h(\hat{f}_\varepsilon|_{\hat{\Lambda}_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon})$, so the value of the entropy remains constant w.r.t. small perturbations. However $h_i(f|_\Lambda) \neq h_i(\hat{f}|_{\hat{\Lambda}})$ in general. For example when $f(z,w) = (z^2+c, w^2)$, ($|c|$ small, $\Lambda = \{z_0\} \times J_{(w^2)}$) we have $h_i(f|_\Lambda) = 0$ as we showed in the Remark from §1, but at the same time $h_i(\hat{f}|_{\hat{\Lambda}}) = h(\hat{f}|_{\hat{\Lambda}})$ since $\hat{f}\colon\ \hat{\Lambda} \to \hat{\Lambda}$ is a homeomorphism. Combining this with $h(\hat{f}|_{\hat{\Lambda}}) = h(f|_\Lambda) = \log 2$ shows $h_i(f|_\Lambda) \neq h_i(\hat{f}|_{\hat{\Lambda}})$. If we consider the example discussed in Section 4 of a perturbation f_ε such that $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism, we see that $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$. However $h_i(f) = 0$ since Λ is a quasicircle, hence the value of h_i is not stable under perturbation.

4 Examples and applications

First we study a large class of maps obtained as perturbations of (z^2+c, w^2) ($0 \neq |c|$ small) and identify those elements of the class which are injective on their respective basic sets.

Theorem 3. *Given the map $f_\varepsilon(z,w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, there exist small positive constants $c(a,b,d,e)$ and $\varepsilon(a,b,c,d,e)$ such that, for $0 \neq |c| < c(a,b,d,e)$ and $0 < \varepsilon < \varepsilon(a,b,c,d,e)$ we have that f_ε is injective on its basic set Λ_ε close to $p_0(c) \times S^1$ (where $p_0(c)$ is the attracting fixed point for z^2+c), if and only if $b \neq 0$ or $d \neq 0$.*

Proof. Assume that $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ for two points $(z,w), (z',w') \in \Lambda_\varepsilon$

$$\begin{aligned} &\Rightarrow z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 = z'^2 + a\varepsilon z' + b\varepsilon w' + c + d\varepsilon z'w' + e\varepsilon w'^2 \\ &\Rightarrow (z^2 - z'^2) + \varepsilon[a(z-z') + b(w-w') + d(zw - z'w') + e(w^2 - w'^2)] = 0. \end{aligned}$$

Assume $w \neq w'$. Then, $w' = -w$ since $w^2 = w'^2$

$$\Rightarrow (z-z')(z+z'+\varepsilon a) + \varepsilon[2bw + dw(z+z')] = 0.$$

$$\Rightarrow (z-z')(z+z'+\varepsilon a) = -\varepsilon[2bw + dw(z+z')]. \tag{1}$$

Let $p_0(c)$ be the fixed attracting point for $z \to z^2+c$, $0 \neq |c|$ small. And consider $\alpha := \sup_{(z,w)\in\Lambda_\varepsilon} |z - p_0(c)|$. Let (z_0, w_0) a point in Λ_ε, where the supremum in Λ_ε is attained. Hence $\alpha = |z_0 - p_0(c)|$, and $p_0^2(c) + c = p_0(c)$. Now, we can find a point $(z,w) \in \Lambda_\varepsilon$ such that

$$\begin{aligned} f_\varepsilon(z,w) = (z_0,w_0) &\Rightarrow z_0 = z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 \\ &\Rightarrow z_0 - p_0(c) = z^2 - p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \\ &= (z - p_0(c))^2 + 2zp_0(c) - 2p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \Rightarrow \\ &\alpha \leq \alpha^2 + 2|p_0(c)|\alpha + K\varepsilon, \end{aligned}$$

with K, a positive constant depending on the parameters of the map. Now $\alpha^2+\alpha(2|p_0(c)|-1)+K\varepsilon \geq 0$, and since $\alpha \ll 1$ (since Λ_ε is very close to $\{p_0(c)\} \times S^1$), we obtain (for some constant K'):

$$0 \leq \alpha \leq \frac{2K\varepsilon}{1-2|p_0(c)|+\sqrt{(1-2|p_0(c)|)^2-4K\varepsilon}} \leq K' \cdot \varepsilon.$$

From (2) one gets $2\alpha(2|p_0(c)|+\varepsilon|a|) > |2bw+dw(z+z')|$ for $\varepsilon < \varepsilon(a,b,c,d,e)$, since z, z' are both ε-close to $p_0(c)$.

But for $(z,w) \in \Lambda_\varepsilon$, one has $|w|=1$, and $|z+z'|$ is ε-close to $2|p_0(c)|$. Therefore, assuming first that $b \neq 0$, if $|c|$ is small enough in comparison to $|b|$, then $|p_0(c)|$ becomes so small that $|2bw+dw(z+z')| > |b| > 0$. Hence $2K'\varepsilon(2|p_0(c)|+\varepsilon|a|) \geq |b| > 0$, which is a contradiction if $|c|$ is small enough (since $|c|$ small will imply that $|p_0(c)|$ is also small, and we can always reduce ε accordingly).

Hence, we proved that $w' = w$. Then, from $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ it follows that $z^2 - z'^2 = -\varepsilon[a(z-z')+dw(z-z')]$. If $z \neq z'$, we would then get $z+z' = -\varepsilon(a+dw)$.

But z, z' are both ε -close to $p_0(c) \neq 0$, so if we choose $\varepsilon < \varepsilon(a,b,c,d,e)$ appropiately, then $|z+z'| > |p_0(c)| > 0$. On the other hand, if $|a+dw| \neq 0$, we can take ε small enough such that $\varepsilon|a+dw| < |p_0(c)|$, which gives a contradiction. If $|a+dw| = 0$, then we get a contradiction again since $p_0(c) \neq 0$.

In conclusion $z = z', w = w'$ and in consequence $f_\varepsilon|_{\Lambda_\varepsilon}\colon\ \Lambda_\varepsilon \to \Lambda_\varepsilon$ is an injective map. The case when $b = 0, d \neq 0$ is treated similarly.

We now only have to show that if $b = 0, d = 0$, then f_ε is non-injective. Actually we prove that each point in Λ_ε has at least two preimages in Λ_ε.

Let us fix a point $(z_0, w_0) \in \Lambda_\varepsilon$ and show that $(z_0, -w_0) \in \Lambda_\varepsilon$ too. Consider a neighbourhood of (z_0, w_0), $U = D(z_0, \delta) \times D(w_0, \delta)$. Then from the fact that (z_0, w_0) is a nonwandering point, we obtain that there exists an increasing sequence of integers n_k such that $f_\varepsilon^{n_k}(U) \cap U \neq \emptyset$. If $p_2(z,w) = w$ is the projection on the second factor, then for n_k large enough, $p_2 \circ f_\varepsilon^{n_k}(U) = S^1$. So if we take $V := D(z_0, \delta) \times D(-w_0, \delta)$ as a neighbourhood of $(z_0, -w_0)$, then since w does not appear in the first term of f_ε, we get $f_\varepsilon^{n_k}(V) \cap V \neq \emptyset$, and as $\delta > 0$ was arbitrary, it follows that $(z_0, -w_0) \in \Lambda_\varepsilon$.

Since obviously $f_\varepsilon(z_0, w_0) = f_\varepsilon(z_0, -w_0)$, f_ε is at least 2-to-1 on Λ_ε.

In conclusion, the only instances when f_ε is injective are when $b \neq 0$ or $d \neq 0$. □

Corollary 1. *In the setting of Theorem 3, if $f_\varepsilon(z,w)$ is injective on Λ_ε and $|c|$ is small enough, then $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$ and $HD(W^s_{\varepsilon'}(x) \cap \Lambda_\varepsilon) > \frac{\log 2}{\log|\frac{1+\sqrt{1-4c}}{2c}|}$.*

Moreover, in this case Λ_ε is not a Jordan curve.

Proof. If $f_\varepsilon|_{\Lambda_\varepsilon}$ is injective, then it is a homeomorphism of Λ_ε and hence $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon})$. But

$$h(f_\varepsilon|_{\Lambda_\varepsilon}) = h(\hat{f}_\varepsilon) = h(\hat{f}) = h(f) = \log 2$$

In the above, $\hat{f}$ represents the lifting of f to a homeomorphism of $\hat{\Lambda}$ and we can use the conjugacy between $\hat{f}$ and $\hat{f}_\varepsilon$ ([14]); also the last equality is due to the fact that $f|_\Lambda$ is just the map $w \to w^2$ whose entropy is $\log 2$.

The estimate for the stable dimension in our claim is proved by using Theorem 2 and the fact that $\left| \inf_{y \in \Lambda} |Df|_{E^s_y}| \right| > |p_0(c)|$, where $p_0(c) = \frac{2c}{1+\sqrt{1-4c}}$ is the unique fixed attracting point of the map $z^2 + c$, for $|c|$ small.

The last consequence to show is that the basic set Λ_ε is not a Jordan curve.

Indeed, if Λ_ε were a Jordan curve, the Nitecki-Przytycki Theorem in Section 1 would imply that $h_i(f_\varepsilon|_{\Lambda_\varepsilon})$ is zero, but as we saw, $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$.

□

In the case of diffeomorphisms the stable dimension depends continuously on the map, even real analytically in the case of Henon maps ([16]). However this property is not present anymore for holomorphic endomorphisms.

Indeed, from Corollary 5, if f_ε is a perturbation of $f(z, w) = (z^2 + c, w^2)$, such that f_ε injective on its basic set Λ_ε, then $HD(W^s_\delta(x_\varepsilon) \cap \Lambda_\varepsilon)$ is bounded below by a constant which does not depend on ε, no matter how small ε is.

On the other hand $HD(W^s_\delta(x) \cap \Lambda) = 0$, if Λ is the basic set of f. Therefore we get a contradiction, since in any neighbourhood of f we were able to find a map having stable dimension bounded below by a constant not depending on ε. This will be formulated in the following:

Corollary 2. *If f is a holomorphic map on $\mathbb{P}^2$ which has a basic set Λ, f_ε is an ε- perturbation of f and Λ_ε is the basic set of f_ε close to Λ, then $HD(W^s_\delta(x_\varepsilon) \cap \Lambda_\varepsilon)$ does not converge to $HD(W^s_\delta(x) \cap \Lambda)$, when $\varepsilon \to 0$ and $x_\varepsilon \in \Lambda_\varepsilon, x \in \Lambda, x_\varepsilon \to x$. Hence the dependence of the stable dimension on the map is not continuous.*

Now we ask what can be said about the number of preimages that a point in the basic set Λ can have in Λ, in order to shed more light on the upper estimate from Theorem 1.

Lemma 1. *Let f be a holomorphic map on $\mathbb{P}^2$, and Λ one of its basic sets of infinite cardinality, such that $C_f \cap \Lambda = \emptyset$, where C_f is the critical set of f. Then the set $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ is open with respect to the induced topology on Λ.*

Proof. If $C_f \cap \Lambda = \emptyset$, then there exists $\delta > 0$ such that ,if $f(z_1) = f(z_2), z_1, z_2 \in \Lambda, z_1 \neq z_2$, then $d(z_1, z_2) > \delta$. First let us notice that Λ cannot have isolated points. This follows from the fact that $f|_\Lambda$ is transitive and since Λ is assumed to be infinite. Now assume that the statement of our lemma is not true. Then there would exist $x \in \Lambda$ such that $f^{-1}(x) \cap \Lambda = \{z\}$ and, since we saw that Λ does not have isolated points, there would exist a sequence $y_n \in \Lambda$, $y_n \to x$, with $\{z^1_n, z^2_n\} \subset f^{-1}(y_n) \cap \Lambda, z^1_n \neq z^2_n, \forall n$. From the discussion above, it follows that $d(z^1_n, z^2_n) > \delta$.

But since Λ is compact, there exists a subsequence of $(z^1_n)_n$, denoted also by $(z^1_n)_n$ such that $z^1_n \to \xi^1 \in \Lambda$.

By taking yet another subsequence, one can assume also that $z^2_n \to \xi^2 \in \Lambda$.

Since $f(z_n^1) = y_n$ and $y_n \to x$, we have $f(\xi^1) = x$ and similarly $f(\xi^2) = x$. But $d(z_n^1, z_n^2) > \delta$, hence $d(\xi^1, \xi^2) \geq \delta$. Therefore we get a contradiction with the fact that the set $f^{-1}(x) \cap \Lambda$ has only one element. So the set of points $x \in \Lambda$ having only one preimage in Λ, is open in the induced topology. □

Lemma 2. *In the same setting as in Lemma 1, the set* $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ *is closed.*

Proof. First we will show that $f|_\Lambda : \Lambda \to \Lambda$ is open.

Indeed, $f|_\Lambda$ can be lifted to a homeomorphism $\hat{f}$ of $\hat{\Lambda}$ on the inverse limit set, such that $p \circ \hat{f} = f \circ p$, where $p : \hat{\Lambda} \to \Lambda$ is the canonical projection $p(x_n)_{n \leq 0} = x_0$.

Let U an open set in the topology induced on Λ from the usual topology on $\mathbb{P}^2$.

Then, since p is continuous, $p^{-1}(U)$ is open in $\hat{\Lambda}$. But $\hat{f}$ is a homeomorphism, hence $\hat{f}(p^{-1}(U))$ is open in $\hat{\Lambda}$.

Now let us prove that p is open. For this we need to show that for any $\delta > 0$, there is some $\delta' > 0$ such that for any $\hat{x} \in \hat{\Lambda}$, $p(B(\hat{x}, \delta)) \supset B(x, \delta')$.

Indeed, if M denotes the diameter of $\mathbb{P}^2$, then take an integer N large enough such that $\frac{M}{2^N} < \frac{\delta}{2}$ and $\delta' > 0$ such that for any $x, y \in \mathbb{P}^2$ with $d(x, y) < \delta'$, and any prehistory of x, $\hat{x}$, there is a prehistory $\hat{y}$ of y with

$$d(x, y) + \frac{d(x_{-1}, y_{-1})}{2} + ... + \frac{d(x_{-n}, y_{-n})}{2^N} < \frac{\delta}{2}$$

Therefore $d(\hat{x}, \hat{y}) < \delta$ and $p(B(\hat{x}, \delta)) \supset B(x, \delta')$. This shows that p is open.

Coming back to the set $\hat{f}(p^{-1}(U))$ which we proved to be open, it follows immediately that $f(U) = p(\hat{f}(p^{-1}(U)))$ is open also in Λ since p is an open map.

In conclusion $f|_\Lambda : \Lambda \to \Lambda$ is open.

Now assume that the set of points with only one preimage in Λ is not closed.

Then, since Λ does not have isolated points, there would exist $x \in \Lambda$ having two preimages y_1, y_2 in Λ and a sequence $x_n \in \Lambda, x_n \to x$ with $f^{-1}(x_n) \cap \Lambda = \{z_n\}$.

By taking eventually a subsequence, one can suppose that $d(y_1, z_n) > \delta, \forall n$, for some positive constant δ.

Let now U an open ball around y_1 of radius $\frac{\delta}{2}$. Since $f|_\Lambda$ is open and Λ has no isolated points, it follows that $x_n \in f(U)$ for n large enough. So, there exists $\xi_n \in U$ such that $f(\xi_n) = x_n$. Also, since we assumed $d(y_1, z_n) > \delta$, it is clear that $z_n \neq \xi_n$.

So, this implies that $\{\xi_n, z_n\} \subset f^{-1}(x_n) \cap \Lambda$, which is a contradiction with our assumption.

Therefore the set $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ is closed in Λ.

□

Lemmas 1, 2 imply the following:

Corollary 3. *In the same setting as in Lemma 1, and assuming also that the basic set* Λ *is connected, one has that, if there exists a point* $x \in \Lambda$ *with only one preimage in* Λ*, then every point in* Λ *has exactly one preimage in* Λ*.*

Similarly one can show a more general result:

Corollary 4. *In the same setting as in Lemma 1, and assuming also that Λ is connected, the number of preimages that a point from Λ has in Λ is constant.*

As can be seen from the proofs of Lemmas 1, 2, this statement is true in a more general setting: f does not have to be necessarily holomorphic for example.

A natural question is how we can check which basic sets are connected.

In fact it turns out that it is enough to perturb a map with a basic set which we know to be connected in order to obtain other such examples.

Lemma 3. *In the same setting as in Lemma 1, if Λ is connected and g is sufficiently close to f, with its corresponding basic set Λ_g, then Λ_g is connected as well.*

Proof. We know from a result of Przytycki ([14]) that there exists a conjugacy at the level of inverse limits, between $\hat{\Lambda}$ and $\widehat{\Lambda_g}$ if g is close enough to f.

Also an easy result from topology shows that Λ is connected if and only if $\hat{\Lambda}$ is connected. Hence if we assumed Λ to be connected, it follows also that $\hat{\Lambda}$, $\widehat{\Lambda_g}$, and Λ_g are all connected.

□

Corollary 3 and Lemma 3, together with Theorems 1 and 2 give now a lot of information now about the stable dimension of perturbations of the map $(z^2 + c, w^2)$.

Corollary 5. *For a perturbation g of the map $(z^2 + c, w^2)$, where $0 \neq |c|$ small, and the basic set Λ_g close to $p_0(c) \times S^1$, we have that $HD(W^s_\varepsilon(x) \cap \Lambda_g)$ is either equal to zero, and in this case $g|_{\Lambda_g}$ not injective, or $\frac{\log 2}{|\log \inf\limits_{y \in \Lambda_g} |Df|_{E^s_y}||} \leq HD(W^s_\varepsilon(x) \cap \Lambda_g) \leq \frac{log2}{|\log \frac{(\sup\limits_{\xi \in \Lambda_g} |Df|_{E^s_\xi}|)^2}{\inf\limits_{z \in \Lambda_g} |Df|_{E^s_z}|}|}$, and in this case $g|_{\Lambda_g}$ is injective.*

Obviously the same reasoning can be applied to other perturbations of maps f on $\mathbb{P}^2$ for which $C_f \cap \Lambda = \emptyset$, (see for example [5] for more examples, like product maps, solenoids, etc.)

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