# Estimates for the stable dimension for holomorphic maps

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**Abstract**

We study the Hausdorff dimension of the intersection between stable manifolds and basic sets for an Axiom $A$ holomorphic endomorphism on the complex projective space $\mathbb{P}^2$. For a map which is at least $d'$-to-1 on a basic set $\Lambda$, we are improving the upper estimate given in [11] by taking into consideration the number of preimages, and thus proving for non-invertible maps results parallel to those of Verjovsky-Wu [17] from the case of Henon diffeomorphisms. Also a lower estimate for $HD(W^s_\delta(x) \cap \Lambda)$ is given by using a concept of preimage entropy modeled after Bowen. This preimage entropy plays the role of the entropy of $f^{-1}$ from the case of homeomorphisms, but in general, if the map $f$ is an endomorphism like in our situation, the preimage entropy does not coincide with the usual forward entropy $h(f|_\Lambda)$. We also show that, given $c$ with $|c|$ small, the perturbation $(z^2 + c + \varepsilon w, w^2)$ of $(z^2 + c, w^2)$ is injective on its basic set $\Lambda_\varepsilon$ close to $\Lambda := p_0(c) \times S^1$ (where $p_0(c)$ is the fixed attracting point for $z^2 + c$) and also its stable dimension is strictly positive. An interesting consequence of our results is the fact that $HD(W^s_\delta(x) \cap \Lambda_g)$ does not depend continuously on $g$, (for $x \in \Lambda_g$), which is opposite to the situation of diffeomorphisms. We study also the stable dimension for a large class of quadratic endomorphisms and show that under a mild technical condition it is either zero, or it can be estimated easily using the derivative on the respective basic set.

**Keywords:** Hausdorff dimension, stable manifolds, preimage entropy, holomorphic endomorphisms, topological pressure.



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# 1 Introduction

We consider Axiom $A$ holomorphic functions $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ on the 2-dimensional complex projective space. For definitions of hyperbolicity and Axiom $A$ we refer to [5], [16]. We will use the notations from [5], where the nonwandering set $\Omega$ of $f$ is partitioned into three invariant subsets $S_0, S_1, S_2$, according to the dimension of the unstable tangent space at points from these sets.

Let $\Lambda$ be a basic set of the saddle part $S_1$ of the nonwandering set $\Omega$. This means that at every point of $\Lambda$ there exist both stable and unstable directions. According to [5], at a point $x \in \Lambda$ there exist a local stable manifold of size $\delta$, denoted by $W^s_\delta(x)$ and, for every prehistory $\hat{x}$ of $x$, a local unstable manifold of size $\delta$, $W^u_\delta(\hat{x})$. These local manifolds are analytic disks in our case.

**Definition 1.** Given a point $x$ of the basic set $\Lambda$ for an Axiom A map as above, we will call **stable dimension** at $x$, the Hausdorff dimension $HD(W^s_\delta(x) \cap \Lambda)$, for some positive, small $\delta$. The stable dimension will be denoted by $\delta_s(x)$ or by $\delta_s$ when no confusion about the point $x$ is possible.

If $x \in \Lambda$ and one denotes by $\phi^s(x) := \log |Df|_{E^s_x}|$, where $E^s_x$ is the stable tangent subspace at $x$, then we have the result of [11]:

**Proposition 1.** *Let $f$ a holomorphic Axiom A endomorphism of $\mathbb{P}^2$ such that $C_f \cap S_1 = \emptyset$, where $C_f$ = critical set of $f$. Then $HD(W^s_\delta(x) \cap \Lambda) \leq t^s$, where $t^s$ is the only zero of the function $t \to P(t \cdot \phi^s)$. Hence the estimate does not depend on $x \in \Lambda$.* □

In the case of diffeomorphisms we have $HD(W^s_\delta(x) \cap \Lambda) = t^s$ as was proved in [10].

In this article, we denote the nonwandering set of $f$ by $\Omega$ and we consider the partition of $\Omega$ as $S_0 \cup S_1 \cup S_2$, where the dimension of the unstable spaces over $S_i$ is equal to $i$, $i \in \{0, 1, 2\}$. The function $P(\cdot)$ referred to in Proposition 1 is the topological pressure. We will now define the topological pressure and give some of its properties, following [18]. Our purpose in doing this is two-fold: first as a convenience for the reader, second because later we shall define a related concept, that of preimage entropy. Parallels between these two notions shall prove very useful.

The general setting is that of $(X, d)$, a compact metric space, and $f\colon X \to X$ a continuous map. For $n$, a positive integer, $d_n(x, y) := \max\{d(f^i x, f^i y),\ i = 0, ..., n-1\}$ is a metric on $X$ inducing the same topology as the metric $d$.

**Definition 2.** A subset $E \subset X$ is called $(n, \varepsilon)$-separated (for some $\varepsilon > 0$) if for all $x, y \in E$, $x \neq y$, we have $d_n(x, y) \geq \varepsilon$.

**Definition 3.** The *topological pressure* of $f$ is the functional $P_f\colon \mathcal{C}(X, \mathbb{R}) \to \overline{\mathbb{R}}$ defined on the space of continuous functions by:

$$P_f(\varphi) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \exp \left( \sum_{i=0}^{n-1} \varphi(f^i x) \right), E \subset X, (n, \varepsilon)\text{-separated set.} \right\}.$$

**Definition 4.** When considering $\varphi \equiv 0$ in Definition 3, we obtain the notion of *topological entropy* of $f$, denoted by $h_{top}(f)$ or $h(f)$.

There exists an interesting relationship between Borel invariant measures and $P_f$, contained in the following:

**Theorem (Variational Principle).** *In the above setting,* $P_f(\varphi) = \sup\limits_{\mu}\{h_\mu(f) + \int \varphi \, d\mu\}$, *where the supremum is taken over all* $f$*-invariant Borel probability measures* $\mu$, *and* $h_\mu(f)$ = *measure-theoretic entropy of* $\mu$.

For the (long) definition of $h_\mu(f)$ and proofs of all these facts, as we mentioned, a good reference is [18].

**Theorem (Properties of Pressure).** *If* $f\colon\ X \to X$ *is a continuous transformation, and* $\varphi, \psi \in \mathcal{C}(X, \mathbb{R})$, *then:*

*1)* $\varphi \leq \psi \Rightarrow P_f(\varphi) \leq P_f(\psi)$

*2)* $P_f(\cdot)$ *is either finitely valued or constantly* $\infty$

*3)* $P_f$ *is convex*

*4) for a strictly negative function* $\varphi$, *the mapping* $t \to P_f(t\varphi)$ *is strictly decreasing if* $P(0) < \infty$.

*5)* $P_f$ *is a topological conjugacy invariant.*

In our situation $X = \Lambda$ (with $\Lambda$ a basic set of $S_1$); then, because $\phi^s < 0$, it follows that there exists a unique zero of the map $t \to P(t\phi^s)$, since $t \to P(t\phi^s)$ is strictly decreasing. We shall denote this unique zero by $t^s$. It is true that $t^s \geq 0$ since $P(0) = h(f|_\Lambda) \geq 0$, and $P(t\phi^s) < 0$ for large $t$.

Define now the inverse limit of $X$, $\widehat{X} := \{(x_n)_{n\leq 0},\ f(x_{n-1}) = x_n,\ n \leq 0,\ x_n \in X\}$, and call an element $\hat{x} = (x_n)_{n\leq 0}$ of $\widehat{X}$ a prehistory of $x_0$. The metric on $\widehat{X}$ is the usual metric, i.e

$$d(\hat{x}, \hat{y}) := \sum_{i\leq 0} \frac{d(x_i, y_i)}{2^{|i|}}.$$

The map $f$ induces a homeomorphism $\hat{f}$ of $\widehat{X}$ given by $\hat{f}((x_n)_{n\leq 0}) = (x_{n+1})_{n\leq 0}$, where $x_1 = f(x_0)$. In this case, denoting by $p(\hat{x}) = x_0$ the canonical projection, $p\colon\ \widehat{X} \to X$, the following diagram commutes:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\hat{f}} & \widehat{X} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X \end{array}$$

It is well-known that, in this case, $h(f) = h(\hat{f})$. We now come back to $HD(W^s_\delta(x) \cap \Lambda)$. In [11] the first author studied the Axiom A maps $f(z, w) = (z^2 + c, w^2 + d)$, $c \neq 0$, $d \neq 0$ and showed that if $c$ is chosen such that $|1 - \sqrt{1 - 4c}| = \frac{4}{5}$, then $P(2\phi^s) > 0$ implies $t^s > 2$. So, the estimate $HD(W^s_\delta(x) \cap \Lambda) \leq t^s$ is not sharp because $W^s_\delta(x) \cap \Lambda$ consists of only one point. A different estimate for a certain class of maps will be proved in §2 to take this fact into consideration.

**Theorem 1.** *Assume $f$ is Axiom A, holomorphic endomorphism on $\mathbb{P}^2$ of degree $d$, and $\Lambda$ is one of the basic sets with unstable index 1. Suppose that $C_f \cap \Lambda = \emptyset$ and that $f|_\Lambda \colon \Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least $d'$ preimages in $\Lambda$, $d' \leq d$. Then $HD(W^s_\delta(x) \cap \Lambda) \leq t^s_0$, where $t^s_0$ is the unique zero of the function $t \to P(t \log |Df|_{E^s_y}| - \log d')$.*

*The estimate is independent on $x \in \Lambda$.* □

This result complements the theorem from [12] stating that the union of all unstable manifolds corresponding to points in $S_1$, has empty interior. In general, if $f|_\Lambda \colon \Lambda \to \Lambda$ is $d'$-to-1, then $h(f|_\Lambda) \geq \log d'$ (this is proved similarly to the Misiurewicz-Przytycki theorem, [8]). For the example given above, $f(z,w) = (z^2 + c, w^2 + d)$, $h(f|_\Lambda) = \log 2, d' = 2$, hence the above estimate is sharp. In the case of complex hyperbolic Hénon diffeomorphisms, $HD(W^s_\delta(x) \cap \Lambda) = t^s$ ([17]). However in our case, that proof will not work mainly because the estimates using $h_\mu(f)$ break down due to the non-invertibility of $f$. The main problem is that the entropy was defined by considering the forward iterates of $f$, and, if $f$ is not injective, $h(f)$ does not shed any light upon the growth of preimages. To compensate this, we will use in Theorem 2 a notion of *preimage (branch) entropy*. Let us call a *branch* of length $\ell$ (or *prehistory* of length $\ell$) in $X$, a sequence of preimages, $\beta = (z_0, z_{-1}, \ldots, z_{-\ell})$, with $z_i \in X$, $-\ell \leq i \leq 0$, such that $f(z_{i-1}) = z_i$, $-\ell + 1 \leq i \leq 0$. Given another branch $\beta' = (z'_0, \ldots, z'_{-\ell})$ of same length, define their *branch distance* to be $d^b(\beta, \beta') = \max_{j=\overline{0,\ell}} d(z_{-j}, z'_{-j})$. The reader can notice the similarity between the branch distance and $d_n(\cdot,\cdot)$ introduced earlier. Like $d_n(\cdot,\cdot)$ for forward iterates, $d^b$ measures the growth of inverse iterates. Using this, we now define a *branch metric* on $X$:

$$d^b_\ell(x, x') < \varepsilon,$$

if for every branch $\beta$ of length $\ell$ with $z_0 = x$, there exists a branch $\beta'$ of length $\ell$ with $z'_0 = x'$ such that $d^b(\beta, \beta') < \varepsilon$, and vice versa. Denote by $N_{\text{span}}(\varepsilon, d^b_\ell, X)$ the smallest cardinality of an $\varepsilon$-spanning set for $X$ in the $d^b_\ell$ metric. Hence, if $A$ is an $\varepsilon$-spanning set with $\#A = N_{\text{span}}(\varepsilon, d^b_\ell, X)$, then, $\forall x \in X$, $\exists y \in A$ with $d^b_\ell(x,y) < \varepsilon$. Let also $N_{\text{sep}}(\varepsilon, d^b_\ell, X)$ be the largest cardinality of an $\varepsilon$-separated set for $X$ in the $d^\beta_\ell$ metric. So, if $A$ is $\varepsilon$-separated, then for all $x, y \in A$, $x \neq y$, $d^b_\ell(x,y) > \varepsilon$.

The following proposition gives the definition of the preimage entropy $h_i(f)$ and two ways to calculate it.

**Proposition ([13]).** *For $f\colon X \to X$ continuous, $(X,d)$ compact metric space, we have*

$$\lim_{\varepsilon \to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log N_{sep}(\varepsilon, d^b_n, X) = \lim_{\varepsilon \to 0} \underline{\lim_{n\to\infty}} \frac{1}{n} \log N_{\text{span}}(\varepsilon, d^b_n, X)$$

*and the common value is called the* preimage (branch) entropy, *denoted by $h_i(f)$.* □

In general there is no relation between $h_i(f)$ and $h(f)$. Let us recall now two cases when $h_i(f) = 0$.

**a. Forward-expansive coverings**

If $X$ is a metric space, then a continuous map $f\colon\ X \to X$ is called *forward expansive* if $\exists \varepsilon_0 > 0$ such that whenever $x, y \in X$, $x \neq y$, $\exists m \geq 0$ with

$$d(f^m x, f^m y) \geq \varepsilon_0 > 0.$$

For example, $f$ is forward-expansive on any invariant subset of a Riemannian manifold on which $Df$ is expanding by a constant factor $\lambda > 1$. Denker and Urbański [4] (see also Coven and Reddy [3]) showed that, if $X$ is a compact metric space, then a forward-expansive map is in fact *expanding*, i.e. for some metric $d$ on $X$ (equivalent to the original one), there exists $b > 0$, $\lambda > 1$ constants such that $d(x, y) < b$ implies $d(f(x), f(y)) \geq \lambda d(x, y)$. We also say that $f\colon\ X \to X$ is a *covering map* if $(\forall) x \in X, \exists$ a neighborhood $U_x$ of $x$, such that $f^{-1}(U_x) = \bigcup\limits_i V^i$ with $\{V^i\}$ open disjoint sets, and such that $f : V^i \to U_x$ is a homeomorphism. If $f$ is forward-expansive on $X$ compact metric space, then we can use Denker-Urbański [4] characterization in connection to $h_i(f)$ because $h_i(f)$ is a topological concept which does not depend on the metric ([13]). This is the idea of the proof for the following:

**Proposition ([13]).** *If $f\colon\ X \to X$ is a* forward expansive covering *map, then $h_i(f) = 0$.* □

**b. A second situation when $h_i(f) = 0$ is that of graph maps.**

A *finite graph* is a compact metric space $K$ with a distinguished finite set of points called *vertices*, whose complement has finitely many connected components, *edges*, homeomorphic to the open interval (0,1). We fix the metric on $K$ by assigning length 1 to each edge and the distance between two points in $K$ is the length of the shortest path connecting them.

**Theorem (Nitecki-Przytycki, [13]).** *Let $K$ a finite graph and $f\colon\ K \to K$ continuous map. Then $h_i(f) = 0$.* □

**Corollary.** *For any continuous self-map $f$ of a closed interval $[a, b]$, or of the circle $S^1$, we have $h_i(f) = 0$.* □

We will start by defining another concept of preimage entropy, modeled after Bowen's covering type entropy ([1]).

Let $X$ a compact metric space, $Y \subset X$, and $f : X \to X$ a continuous surjective map.

Denote by $\mathcal{C}_m(\varepsilon)$ the set of collections of length $m$ of balls of radius $\varepsilon$ centered at points of a certain prehistory of $x$, $C = \{U_0 = B(x_0, \varepsilon), ..., U_{m-1} = B(x_{-m+1}, \varepsilon)\}$, where $f(x_{-1}) = x_0, ..., f(x_{-m+1}) = x_{-m+2}$, with $x = x_0$ an arbitrary point of $X$. We denote by $n(C)$ the number of elements of $C$. We shall call $C$ a **branch modeled after** the prehistory $(x_0, ..., x_{-m+1})$.

Now let $C = \{U_0, ..., U_{m-1}\} \in \mathcal{C}_m(\varepsilon)$, and define

$$X(C) := \{y \in U_0, \exists y_{-1} \in f^{-1}(y) \cap U_1, \exists y_{-2} \in f^{-1}(y_{-1}) \cap U_2, ...\}.$$

Let $\mathcal{C}(\varepsilon) := \bigcup_{m=1}^{\infty} \mathcal{C}_m(\varepsilon)$.

**Definition 5.** We shall call $(n, \varepsilon)$-**inverse ball** centered at $x$, the set $\bigcup\limits_C X(C)$, where $C$ ranges over all branches modeled after the $n$-prehistories of $x$. It will be denoted by $B_n^-(x, \varepsilon)$.

For an arbitrary $\lambda$ real, $\varepsilon > 0$, integer $N$ and subset $Y \subset X$ we introduce

$$H_-(\lambda, Y, N, \varepsilon) := \inf\{\sum_F \exp(-\lambda n_x), \text{where } Y \subset \bigcup_{x\in F} B^-_{n_x}(x,\varepsilon), \text{and } n_x \geq N, \forall x \in F\},$$

When $N$ increases, the pool of possible candidate spanning sets $F$ appearing in the definition of $H_-(\lambda, Y, N, \varepsilon)$ decreases. Hence, there exists the limit $\lim_{N\to\infty} H_-(\lambda, Y, N, \varepsilon) =: h_-(\lambda, Y, \varepsilon)$. The notation $h_-(\lambda, Y, \varepsilon)$ emphasizes the nature of the construction in the spirit of Hausdorff outer measure. Now let $h_-(Y, \varepsilon) := \inf\{\lambda, h_-(\lambda, Y, \varepsilon) = 0\}$.

Let us note now that, similar to the case of usual (forward) entropy ([18]), $\lim\limits_{\varepsilon\to 0} h_-(Y, \varepsilon)$ exists. This limit will be denoted by $h_-(Y)$ and will be called **the inverse entropy of $f$ on $Y$**. When we will want to emphasize the dependence of $h_-$ on $f$, we will write $h_-(f, Y)$. In the case $Y = X$, we will often write just $h_-(f)$ or $h_-(X)$, when no confusion about the map $f$ can appear.

We can notice now that the balls used in the definition of $h_i$ are smaller than the ones used for $h_-$. Indeed if $y$ is a point in $X$, then the $\varepsilon$-ball around $y$ in the $d^b_n$ metric is given by the intersection $\cap X(C)$, when $C$ ranges over all branches modeled after the $n$-prehistories of $y$. On the other hand, the inverse ball $B^-_n(y, \varepsilon)$ is equal to the union $\cup X(C)$ when $C$ ranges over all the $n$-prehistories of $y$. Therefore we will need more balls in the $d^b_n$ metric than balls of the type $B^-_n$, in order to cover $X$. This fact is used to prove the following Proposition:

**Proposition 2.** *If $f\colon X \to X$ continuous map on a compact metric space $X$, then $0 \leq h_-(f) \leq h_i(f)$. Also, if $f$ is a homeomorphism on $X$, then $h_-(f) = h(f^{-1}) = h(f) = h_i(f)$.* $\square$

**Remark:**

In general we do not have the equality $h_-(f) = h_i(f)$. Let us consider as a counterexample the map $f : S^2 \to S^2$, constructed in [9] (where $S^2$ represents the two dimensional real sphere). $f$ is a smooth map which sends the square $K_1 := [0,1] \times [0,1]$ on a horseshoe $H_1$ touching the line $y = 4$ in such a way that the unstable directions are vertical.

$f$ also sends the square $K_2 := [0,1] \times [2,3]$ onto a horizontal horseshoe $H_2$ whose unstable directions are horizontal as well. The square $[0,1] \times [1,2]$ is sent to a connecting image between the two horseshoes $H_1$ and $H_2$; this image lies outside the rectangle $[0,1] \times [0,4]$. Denote by $\mathcal{B}^u_1$ the unstable lamination of the basic set of the restriction of $f$ to $K_1$ and by $\mathcal{B}^u_2$ the unstable lamination of the basic set of $f|_{K_2}$. Let $K$ denote the intersection $K_2 \cap \mathcal{B}^u_1 \cap \mathcal{B}^u_2$.

If two points $\xi$ and $\eta$ from $K$ belong to distinct components of $(f|_{K_1})^n(K_1) \cap K_2$, for some $n$, then their inverse iterates $(f|_{K_1})^{-j}(\xi)$ and $(f|_{K_1})^{-j}(\eta)$ are in different components of $(f|_{K_1})^{-1}(K_1) \cap K_1$ for some $j$, $1 \leq j \leq n$. It is the same if $\xi$ and $\eta$ belong to different components of $K_2 \cap (f|_{K_2})^n(K_2)$.

Now, there are $2^n$ different components of $(f|_{K_1})^n(K_1) \cap K_2$, (called **components of first type of order $n$**), and $2^n$ different components of $K_2 \cap (f|_{K_2})^n(K_2)$, called **components of second type of order $n$**. The components of first type are transversal to the ones of second type.

A given point in $K$ has prehistories in both the components of first type of order $n$, as well as in the components of second type of order $n$. Therefore when calculating $h_i$, one must take into consideration at least $4^n$ balls in the $d^b_n$ metric, obtained from the intersections of components of

first type of order $n$ with the components of second type of order $n$; these $4^n$ balls are necessary to cover $K$. Hence $h_i(K) \geq \log 4$.
On the other hand, in order to cover $K$ we need only $2 \cdot 2^n - 1$ inverse balls $B_n^-(x, \varepsilon)$, which are given by the components of first type of order $n$ union with those of second type of order $n$ (if $\varepsilon$ is chosen apropiately); hence we conclude that $h_-(K) \leq \log 2$. This implies that in this case $h_i$ and $h_-$ do not coincide. □

From the definition of inverse entropy we obtain the following Proposition :

**Proposition 3.** *In the above setting, given a continuous map $f : X \to X$, we have that:*

*a) If $Y = \bigcup_i Y_i$, then $h_-(Y) = \sup_i h_-(Y_i)$.*

*b) If $Y_1 \subset Y_2 \subset X$, then $h_-(Y_1) \leq h_-(Y_2)$.*

*c) In case $f$ is a homeomorphism of $X$, the inverse entropy coincides with the usual entropy of $f$, i.e $h^-(X) = h(X)$.*

**Remark:** Let us note that Proposition 2 and the discussion above about $h_i$, imply that $h^-(f) = 0$ for forward expansive maps and graph maps.

We are ready now to state Theorem 2 which gives a lower estimate for $HD(W^s_\delta(x) \cap \Lambda)$.

**Theorem 2.** *Suppose that $\Lambda$ is a basic set of saddle type for an Axiom A holomorphic endomorphism $f\colon\ \mathbb{P}^2 \to \mathbb{P}^2$, $C_f \cap \Lambda = \emptyset$. Then for any $x \in \Lambda$,*

$$HD(W^s_\delta(x) \cap \Lambda) \geq \left| \log \inf_{y \in \Lambda} |Df|_{E^s_y}| \right|^{-1} \cdot h_-(f|_\Lambda) \geq 0 \qquad \square$$

*Remark.* For $f(z, w) = (z^2 + c, w^2 + d), c, d$ small and non-zero, and $\Lambda = \{z_0\} \times J_{(w^2+d)}$, with $J_{(w^2+d)}$ the Julia set of $w \to w^2 + d$, and $z_0$ an attracting fixed point for $z^2 + c$, we already know that $W^s_\delta(x) \cap \Lambda$ consists of a single point for $x \in \Lambda$, and therefore $HD(W^s_\delta(x) \cap \Lambda) = 0$. This is in accordance with the fact from the Corollary, that $h_i(f|_\Lambda) = 0$ if $\Lambda$ is a circle (or homeomorphic with a circle), hence also $h_-(f|_\Lambda) = 0$. □

We also show that, given $c$ with $|c|$ small, the perturbation $g(z, w) = (z^2 + c + \varepsilon w, w^2)$ of $(z^2 + c, w^2)$ is injective on its basic set $\Lambda_g$ close to $\Lambda := p_0(c) \times S^1$ (where $p_0(c)$ is the fixed attracting point for $z^2+c$). Then using Theorem 2 and the observation that for any homeomorphism $g$ we have $h_-(g) = h(g) = \log 2$, we will get also that its stable dimension is greater than a positive number independent of $g$. Since the perturbation can be taken arbitrarily close to the map $(z, w) \to (z^2+c, w^2)$, it will follow in Section 4 that $HD(W^s_\delta(x) \cap \Lambda_g)$ does not depend continuously on the map $g$, when $x \in \Lambda_g$, which is in contrast with the case of Hénon maps, [17]. In Section 4 we study a large class of quadratic maps on $\mathbb{P}^2$ and prove that, if the basic set $\Lambda$ is connected and we have an additional technical assumption, then one can control the number of preimages and hence, by using Theorems 1 and 2, the stable dimension. All this will be made precise in the sequel.

## 2 Upper Estimate for $HD(W^s_\delta(x) \cap \Lambda)$

**Theorem 1** *Assume $f$ is an Axiom A, holomorphic map of degree $d \geq 2$ on $\mathbb{P}^2$, and $\Lambda$ is one of the basic sets with unstable index equal to 1. Suppose $C_f \cap \Lambda = \emptyset$ ($C_f$ denotes the critical set of $f$) and also that $f|_\Lambda\colon\ \Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least $d' \leq d$ preimages in $\Lambda$. Then $HD(W^s_\delta(x) \cap \Lambda) \leq t^s_0$, where $t^s_0$ is the unique zero of the function $t \to P(t \log |Df|_{E^s_y}| - \log d')$. As a consequence, $HD(W^s_\delta(x) \cap \Lambda) \leq \frac{h(f|_\Lambda) - \log d'}{|\log \sup_{y\in\Lambda} |Df|_{E^s_y}||}$.*

**Observation:** In the case of diffeomorphisms, $HD(W^s_\delta(x) \cap \Lambda) = t^s$, where $t^s$ is the unique zero of the pressure of the stable function $t \to P(t\phi^s)$, $\phi^s(y) := \log |Df|_{E^s_y}|$. Also, it follows from the Variational Principle, that there exists a probability measure $\mu_s$ such that $t^s = \frac{h_{\mu_s}}{\lambda_s}$, $\lambda_s$ being the Lyapunov exponent of $\mu_s$. In the case of endomorphisms, it was shown in [11] that $HD(W^s_\delta(x) \cap \Lambda) \leq t^s$, but in general, the inequality is strict. For example for the case of a map $f(z, w) = (P(z), Q(w))$, with $Q$ hyperbolic on its Julia set $J_Q$ and $\Lambda := \{z_0\} \times J_Q$, where $z_0$ is an attracting periodic point for $P$, we obtain $HD(W^s_\delta(z, w) \cap \Lambda) = 0$, for all $(z, w) \in \Lambda$. So, the formula in the Theorem explains that the gap between $HD(W^s_\delta(x) \cap \Lambda)$ and $t^s$ is due to the number of preimages.

*Proof.* First of all, consider the function $t \to P(t \log |Df|_{E^s_y}| - \log d')$ which is well defined since $C_f \cap \Lambda = \emptyset$. Notice also that $P(t \log |Df|_{E^s_y}| - \log d') = P(t \log |Df|_{E^s_y}|) - \log d'$. It is strictly decreasing and at $t = 0$ takes the value $h(f|_\Lambda) - \log d' \geq 0$ and for $t$ very large, it takes negative values. Henceforth it has exactly one zero denoted by $t^s_0$, and $t^s_0 \geq 0$. Denote now $W := W^s_\delta(x) \cap \Lambda$ and $\hat{W}$ its lift inside $\hat{\Lambda}$, i.e $\hat{W} := \pi^{-1}(W)$, where $\pi(\hat{x}) = x_0$ is the canonical projection from $\hat{\Lambda}$ to $\Lambda$. In this proof we will use the map $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$. Also it is well known that $P(\phi) = P(\phi \circ \pi)$, for any continuous real function $\phi$ on $\Lambda$, so the topological pressure does not change by lifting the function to $\hat{\Lambda}$. Let $\hat{E}$ be an $(n + 1, \delta)$-separated set of maximal cardinality inside $\hat{f}^{-n}(\hat{W})$ with $\delta << \varepsilon$ to be determined in the course of the proof.

Since $t^s_0$ is the unique zero of the pressure function, it follows that if we consider an arbitrary $t > t^s_0$, then there exists $\beta < 0$ such that

$$P(t \log |Df|_{E^s_y}| - \log d') < \beta < 0.$$

Therefore, from the definition of pressure, if $n$ is large enough,

$$\frac{1}{n+1} \log \sum_{\hat{z}\in\hat{E}} e^{S_{n+1}\Phi(\hat{z})} < \beta < 0,$$

where $\Phi(\hat{y}) := t \log |Df|_{E^s_y}| - \log d'$. Hence,

$$\sum_{\hat{z}\in\hat{E}} |Df^n|_{E^s_z}|^t < e^{(n+1)\beta} \cdot (d')^n.$$

But $\hat{E}$ has been taken as a separated set of maximal cardinality, hence it is also $(n + 1, \delta)$ - spanning for the compact set $\hat{f}^{-n}(\hat{W})$ in the metric $d(\hat{\cdot}, \hat{\cdot})$ from $\hat{\Lambda}$. This means that the balls

$B_{n+1}(\hat{y}, \delta) := \{\hat{z} \in \hat{\Lambda}, d(\hat{f}^k \hat{z}, \hat{f}^k \hat{y}) < \delta, k = 0, .., n\}, \hat{y} \in \hat{E}$, cover the entire set $\hat{f}^{-n}(\hat{W})$. From above, it follows that $\{\hat{f}^n(B_n(\hat{y}, \delta))\}_{\hat{y} \in \hat{E}}$ cover the set $\hat{W}$, and for brevity, we will denote this collection of sets by $\{\hat{B}_j\}_{j \in J}$, where $\hat{E} = (\hat{y}_j)_{j \in J}$, $J$ finite. Let us consider now a point $y$ from $W$ and $\hat{y}, \hat{y}'$ two prehistories of $y$ which are different as $n$-prehistories, i.e there exists $0 < i \leq n$ such that $y_{-i} \neq y'_{-i}$. Can we have two such prehistories both in the same $\hat{B}_j$?

Assume $i \geq 0$ is the largest integer for which $y_{-i} = y'_{-i}$. Denote by $l_0$ the constant of injectivity of $f$, i.e if $f(z) = f(z')$ and $z \neq z'$ $(z, z' \in \Lambda)$, then $d(z, z') > l_0$ ( here we use again that the critical set of $f$ does not intersect $\Lambda$ ). Then for the prehistories $\hat{y}, \hat{y}'$ as above, $d(\hat{f}^{-n}(\hat{y}), \hat{f}^{-n}(\hat{y}')) > l_0 > \delta$, if $\delta$ is small enough. Therefore, the points $\hat{f}^{-n}(\hat{y})$ and $\hat{f}^{-n}(\hat{y}')$ cannot be in the same ball $B_n(\hat{\xi}, \delta), \hat{\xi} \in \hat{E}$. Consequently also $\hat{y}, \hat{y}'$ cannot be in the same $\hat{B}_j$, since $\hat{f}$ is a homeomorphism.

We take now the projections of $\hat{B}_j$ onto $W$, $B_j := \pi(\hat{B}_j) \cap W$. Let $d_j$ denote the diameter of $B_j$ and for all $j \in J$ take $\tilde{B}_j := B(x_j, d_j)$, where $x_j$ is an arbitrary point in $B_j$. In general, by $MB(x, r)$ we shall denote the ball $B(x, Mr)$. From the discussion a few lines above, it can be seen easily that the multiplicity of the cover $\{2\tilde{B}_j\}_j$ of $W$, is at least $d'^n$, if $0 < \delta < \frac{l_0}{6}$. This is true because every point in $W$ has at least $d'^n$ different $n$-prehistories.

We will now extract a subcover of $W$ of multiplicity bounded by some universal constant $C$, coming from the following version of Besicovitch Theorem :

**Theorem ([6]).** *Let $A$ be a bounded set of $\mathbb{R}^n$. For each $x \in A$, a set $H(x)$ is given satisfying the following properties:*

*(a) there exists a fixed number $M > 0$, independent of $x$, and two closed Euclidian balls centered at $x$, $\bar{B}(x, r(x))$ and $\bar{B}(x, Mr(x))$, such that $\bar{B}(x, r(x)) \subset H(x) \subset \bar{B}(x, Mr(x))$;*

*(b) for each $z \in H(x)$, the set $H(x)$ contains the convex hull of the set $\{z\} \cup \bar{B}(x, r(x))$.*

*Then one can select from among $\{H(x)\}_{x \in A}$ a sequence $H_k$ satisfying the following conditions:*

*(i) the set $A$ is covered by $\{H_k\}_k$;*

*(ii) no point of $\mathbb{R}^n$ is in more than $b(n)$ sets $H_k$, and $b(n)$ depends only on the dimension $n$;*

*(iii) the sequence $\{H_k\}_k$ can be split into at most $b(n)$ subfamilies, each of which consists of mutually disjoint elements.*

As an easy observation, notice that if the sets $H(x)$ are convex, then condition (b) of the theorem above is immediately satisfied. Also, part (iii) of the claim of the theorem implies part (ii). However for our purposes, part (ii) will turn out to be enough.

Based on this theorem, we will prove the following covering theorem which will be useful in our context.

**Theorem (Covering Theorem).** *Let $A$ be a bounded set of $\mathbb{R}^n$. Assume that $A$ is covered by a family of balls $\{B(x_i, r_i)\}_{i \in I}$ centered at some points of $A$, where $r_i > 0$, for all $i \in I$. Then there* *exists a cover of $A$ with balls $\{B(x_j, 2r_j)\}_{j\in J}$, where $J \subset I$ and the multiplicity of this cover is bounded by the universal constant $b(n)$.*

For each $x \in A$, choose one ball $B(x_i, r_i)$ containing $x$. Set $H(x) := B(x_i, 2r_i)$ and denote by $r(x)$ the radius $r_i$. Obviously the sets $\{H(x)\}_{x\in A}$ will cover $A$. They are also convex. For every $x \in A$, we have that $B(x, r(x)) \subset H(x) \subset B(x, 3r(x))$. Therefore the assumptions of the previous theorem are satisfied and its direct application ends the proof.

Coming back to the actual proof of Theorem 1 we apply the above Covering Theorem for the balls $\tilde{B}_j$, $j \in J$. Hence, we obtain a subcover $2\tilde{B}_k$, where $k$ belongs to a subset $K \subset J$ and such that the multiplicity of this subcover is bounded above by a universal constant $C > 0$ coming from the Covering Theorem. But the multiplicity of the cover $2\tilde{B}_j$, $j \in J$ is larger than or equal to $d'^n$, if $\delta < l_0/6$. However, if $n$ is large enough, the remaining sets $\{2\tilde{B}_j\}_{j\in J\setminus K}$, still cover $W$ and we can extract again, out of them, a subcover of $W$ of multiplicity bounded by $C$. Repeating the procedure and applying the Covering Theorem at each step, one can find at least $C^{-1}d'^n$ such subcovers, each with multiplicity bounded by $C$. But in this case, there will exist a cover $\{2\tilde{B}_s\}_{s\in L}$ of $W$, $L \subset J$, corresponding to a subset $\hat{F}$ of $\hat{E}$ for which :

$$\sum_{\hat{z}\in\hat{F}} |Df|_{E^s_{\hat{z}}}|^t \le e^{n\beta} \cdot d'^n \cdot C(d')^{-n} = C \cdot e^{n\beta} \tag{1}$$

We make now the connection between the $\mathrm{diam} 2\tilde{B}_s$ and $|Df^n|_{E^s_{\hat{z}}}|$ using the bounded distortion property. First observe that in general, $B_j = \pi(\hat{f}^n B_{n+1}(\hat{y}, \delta)) = f^n(\pi(B_{n+1}(\hat{y}, \delta)))$. Also, if $\hat{\xi} \in B_{n+1}(\hat{y}, \delta)$, then $d(\hat{f}^k\hat{\xi}, \hat{f}^k\hat{y}) < \delta$, for $0 \le k \le n$, hence $d(f^k\xi, f^k y) < \delta$. This implies that $\xi \in W^s_\delta(y)$. In this case, we will be able to apply the property of bounded distortion on stable manifolds (see [11]) to conclude that, for a positive constant $A$,

$$\mathrm{diam} 2\tilde{B}_j \le A \cdot |Df^n|_{E^s_{x_j}}|.$$

Applying now inequality (1), and remembering that $\beta < 0$, we get that

$$\sum_{s\in L} (\mathrm{diam} 2\tilde{B}_s)^t \le Const\, e^{n\beta} \le Const.$$

In conclusion, since $t$ has been chosen arbitrarily larger than $t^s_0$, we obtain $HD(W) \le t^s_0$. The proof of the last consequence from the statement is immediate if one uses the properties of the topological pressure from Section 1 and the fact that $P(\phi + c) = P(\phi) + c$, for any continuous real function $\phi$ and any constant $c$.

□

## 3 Lower Estimate for $HD(W^s_\delta(x) \cap \Lambda)$

We are now ready to prove the following.

**Theorem 2.** *Suppose that $\Lambda$ is a basic set of unstable index 1 for the Axiom A holomorphic map $f\colon\ \mathbb{P}^2 \to \mathbb{P}^2$, and $C_f \cap \Lambda = \emptyset$. Then* $HD(W^s_\delta(x) \cap \Lambda) \geq \left|\log \inf\limits_{y \in \Lambda} |Df|_{E^s_y}|\right|^{-1} \cdot h_-(f|_\Lambda)$, *for all* $x \in \Lambda$.

*Proof.* Let $W := W^s_\delta(x) \cap \Lambda$ for a point $x$ in $\Lambda$, and consider $y \in \Lambda$. Let also $\ell$ be a small positive number.

We will prove that there exists an integer $m$ such that $f^{-m}(W) \cap \Lambda$ intersects any local unstable manifold $W^u_{\ell/2}(\hat{y})$, $(\forall)\hat{y} \in \hat{\Lambda}$. Denote by $B(y, \ell/4)$ the $\ell/4$-ball around $y$.

Since $\Lambda$ is a basic set, hence $f|_\Lambda$ is transitive, it follows that $\exists$ $m_1$ integer and $z \in B(y, \ell/4) \cap \Lambda$ such that $f^{m_1}(z) \in B\left(x, \frac{\delta}{2}\right)$. Now take the unstable manifold $W^u_\delta(\hat{f}^{m_1}(z))$, where $\widehat{f^{m_1}(z)}$ is any prehistory of $f^{m_1}(z)$ such that $(f^{m_1}(z))_{-m_1} = z$.

From the local product structure, $W^u_\delta(\widehat{f^{m_1}(z)}) \cap W = \xi$. The point $\xi$ has a prehistory $\hat{\xi}$ with $d(\xi_{-m_1}, z) < \ell/4$, $\xi_{-m_1} \in \Lambda$, if $m_1$ is big enough. So, $W^s_{M\cdot\ell}(\xi_{-m_1})$ intersects $W^u_{\ell/2}(\hat{y})$ transversally, for any prehistory $\hat{y}$ of $y$, where $M$ is some constant which does not depend on $y$, giving the maximum inclination of the unstable spaces w.r.t. stable ones. Also, $f^{m_1}(\xi_{-m_1}) \in W$ and $f^{m_1}(W^s_{M\cdot\ell}(\xi_{-m_1}) \cap \Lambda) \subset W$, so $W^s_{M\cdot\ell}(\xi_{-m_1}) \cap \Lambda \subset f^{-m_1}(W) \cap \Lambda$.

We denote $\delta^s(x) := HD(W^s_\delta(x) \cap \Lambda)$ and then take $t > \delta^s(x)$ arbitrary.

Since $f$ is locally bi-Lipschitz near $\Lambda$ we have that $HD(W^s_\delta(x) \cap \Lambda) = HD(f^{-m}(W^s_\delta(x) \cap \Lambda))$. Therefore there will exist $\mathcal{U} = (U_i)_{i \in I}$, an open cover of $f^{-m}(W^s_\delta(x) \cap \Lambda)$ with $mesh(\mathcal{U}) < \eta << \ell$, such that

$$\sum_i (\mathrm{diam}(U_i))^t \leq 1$$

In the sequel we will denote $|Df^k|_{E^s_y}|$ by $|Df^k_s(y)|$ for an arbitrary $k$ positive integer and a point $y$. Now let us consider the preimages of $U_i$; since $\mathrm{diam}(U_i) < \ell$ and $\ell$ is small, it follows that the different preimages of $U_i$ which intersect $\Lambda$, denoted by $U^1_{i,-1}, ..., U^{N_1}_{i,-1}$ are disjoint. Indeed, let us take all the preimages in $\Lambda$, $y^1_{i,-1}, ..., y^{N_1}_{i,-1}$, of a point $y_i$ from $U_i$. If $\ell$ is small enough, then we will have sets $U^1_{i,-1}, ..., U^{N_1}_{i,-1}$ around the points $y^1_{i,-1}, ..., y^{N_1}_{i,-1}$ respectively, such that the restrictions $f|_{U^k_{i,-1}} : U^k_{i,-1} \to U_i$ are homeomorphisms for all $1 \leq k \leq N_1$. This is due to the fact that $\mathcal{C}_f \cap \Lambda = \emptyset$. However by doing this, the diameters of the sets of the form $U^k_{i,-1}$ might increase. If the diameters of the sets $U^k_{i,-1}$ remain still very small, then we can repeat the procedure for each one of them. The preimages of $U^k_{i,-1}$ and of $U^{k'}_{i,-1}$ are disjoint for $k \neq k'$ since the sets $U^k_{i,-1}$ and $U^{k'}_{i,-1}$ are disjoint themselves. Then, if the diameter of $U^k_{i,-1}$ is small enough, its preimages will also be disjoint. Therefore it makes sense in this case to talk about the components of $f^{-1}(U^k_{i,-1})$ and about the components of $f^{-2}(U_i)$.

Hence for $\ell$ small enough and $\mathrm{diam}(U_i) < \ell$, we will take the smallest integer $n_i$ with the property that the diameters of all the components of $f^{-k}(U_i)$ are smaller than $\ell$ and such that there exists a component $U^k_{i,-n_i}$ of $f^{-n_i}(U_i)$ whose diameter is larger than $\ell$.

In this case, the restriction $f^j|_{U^q_{i,-j}} : U^q_{i,-j} \to U_i$ is a homeomorphism, for all components $U^q_{i,-j}$ of $f^{-j}(U_i)$, $1 \leq j < n_i$.

But then, from the Mean Value Inequality there will exist a point $\xi_i$ in a component of the form $U^k_{i,-n_i}$ such that

$$\operatorname{diam}(U_i) \geq \ell |Df_s^{n_i}(\xi_i)| \geq \ell \inf_\Lambda |Df_s^{n_i}|, i \in I.$$

The point $f^{n_i}(\xi_i)$ belongs to $U_i$, for all $i \in I$.

Since all unstable manifolds of points in $\Lambda$, of size $\frac{\ell}{2}$ intersect the set $f^{-m}(W^s_\delta(x) \cap \Lambda)$, and since $(U_i)_{i \in I}$ cover $f^{-m}(W^s_\delta(x) \cap \Lambda)$, it follows that $\Lambda$ is the union of the inverse balls $B^-_{n_i}(f^{n_i}(\xi_i), 2\ell)$, $i \in I$. For each $U_i$ we have only one corresponding inverse ball $B^-_{n_i}(f^{n_i}(\xi_i), 2\ell)$ and $(\operatorname{diam}(U_i))^t \geq \ell^t (\inf_\Lambda |Df_s|)^{tn_i}$, so

$$1 \geq \sum_i (\operatorname{diam}(U_i))^t \geq \sum_i \exp(tn_i \log \inf_\Lambda |Df_s|)$$

But then, from the fact that the sets $B^-_{n_i}(f^{n_i}(\xi_i), 2\ell)$, $i \in I$, cover $\Lambda$, it follows that $H_-(t \left| \log \inf_\Lambda |Df_s| \right|) \leq 0$, hence $t \left| \log \inf_\Lambda |Df_s| \right| \geq h_-(\ell)$. Since $t$ was chosen arbitrarily bigger than $\delta^s(x)$, and $\ell$ was chosen arbitrarily larger than 0, we get that $\delta^s(x) \geq h_-(f|_\Lambda) \left| \log \inf_\Lambda |Df_s| \right|^{-1}$, for all $x$ in $\Lambda$.

□

**Remark 1:**

Let us notice that both Theorems 1 and 2 work for the case of an open, surjective, Axiom A $\mathcal{C}^\infty$ map $f$ on a real compact manifold $M$. In addition we also need the conformality hypothesis for $f$ or at least the (real) dimension of the stable tangent space should be one.

The proofs in this more general case follow closely the ones already displayed.

We preffered to state and prove the theorems in the case of a holomorphic map on the complex projective space $\mathbb{P}^2$ since the question first appeared to us in this setting, vis-a-vis the papers by Fornaess-Sibony ([5]) and Verjovsky-Wu ([17]).

**Remark 2:**

We conclude this section with the remark that $h_-(f|_\Lambda) \neq h_-(\hat{f}|_{\hat{\Lambda}})$ and that the value of $h_-$ is not stable under perturbations.

Indeed, let us take $\Lambda$, a basic set for an Axiom $A$ endomorphism $f$ without cycles among the basic sets of its saddle part $S_1$. Then, if $f_\varepsilon$ is sufficiently close to $f$, $f_\varepsilon$ will have a basic set $\Lambda_\varepsilon$ close to $\Lambda$ in the Hausdorff metric. And we know from the Stability Theorem ([14]) that there exists a homeomorphism $\hat{g}\colon\ \hat{\Lambda} \to \hat{\Lambda}_\varepsilon$ which commutes with a surjection $g\colon\ \hat{\Lambda} \to \Lambda_\varepsilon$

$$\begin{array}{ccc} \hat{\Lambda} & \xrightarrow{\hat{g}} & \hat{\Lambda}_\varepsilon \\ p\big\downarrow & \searrow & \big\downarrow p \\ \Lambda & g & \Lambda_\varepsilon \end{array}$$

$p$ is the canonical projection.

We know that $h(f|_\Lambda) = h(\hat{f}|_{\hat{\Lambda}}) = h(\hat{f}_\varepsilon|_{\hat{\Lambda}_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon})$, so the value of the entropy remains constant under small perturbations. However $h_-(f|_\Lambda) \neq h_-(\hat{f}|_{\hat{\Lambda}})$ in general. For example when $f(z,w) = (z^2+c, w^2), |c|$ small, $\Lambda = \{z_0\} \times J_{w^2} = \{z_0\} \times S^1$ and we have $h_i(f|_\Lambda) = 0$ as we showed in the Remark from §1 (since $\Lambda$ is a circle); this implies $h_-(f|_\Lambda) = 0$. But at the same time Proposition 2 implies that $h_-(\hat{f}|_{\hat{\Lambda}}) = h_i(\hat{f}|_{\hat{\Lambda}}) = h(\hat{f}|_{\hat{\Lambda}})$ since $\hat{f}\colon\ \hat{\Lambda} \to \hat{\Lambda}$ is a homeomorphism. Combining this with $h(\hat{f}|_{\hat{\Lambda}}) = h(f|_\Lambda) = \log 2$ shows $h_-(f|_\Lambda) \neq h_-(\hat{f}|_{\hat{\Lambda}})$.

If we consider the example discussed in Section 4 of a perturbation $f_\varepsilon$ such that $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism, we see that $h_-(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$. However $h_-(f) = 0$ since $\Lambda$ is a quasicircle, hence the value of $h_-$ is not stable under perturbation.

## 4 Examples and applications

First we study a large class of maps obtained as perturbations of $(z^2+c, w^2)$ $(0 \neq |c|$ small) and identify a set of elements of this class which are injective on their respective basic sets.

**Theorem 3.** *Given the map $f_\varepsilon(z,w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, there exist small positive constants $c(a,b,d,e)$ and $\varepsilon(a,b,c,d,e)$ such that, for $b \neq 0$, $0 \neq |c| < c(a,b,d,e)$ and $0 < \varepsilon < \varepsilon(a,b,c,d,e)$ we have that $f_\varepsilon$ is injective on its basic set $\Lambda_\varepsilon$ close to $p_0(c) \times S^1$ (where $p_0(c)$ is the attracting fixed point for $z^2+c$).*

*Proof.* Assume that $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ for two points $(z,w),(z',w') \in \Lambda_\varepsilon$

$$\Rightarrow z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 = z'^2 + a\varepsilon z' + b\varepsilon w' + c + d\varepsilon z'w' + e\varepsilon w'^2$$
$$\Rightarrow (z^2 - z'^2) + \varepsilon[a(z-z') + b(w-w') + d(zw - z'w') + e(w^2 - w'^2)] = 0.$$

Assume $w \neq w'$. Then, $w' = -w$ since $w^2 = w'^2$

$$\Rightarrow (z-z')(z+z'+\varepsilon a) + \varepsilon[2bw + dw(z+z')] = 0.$$

$$\Rightarrow (z-z')(z+z'+\varepsilon a) = -\varepsilon[2bw + dw(z+z')]. \tag{2}$$

Let $p_0(c)$ be the fixed attracting point for $z \to z^2 + c$, $0 \neq |c|$ small. And consider $\alpha := \sup_{(z,w)\in\Lambda_\varepsilon} |z - p_0(c)|$. Let $(z_0, w_0)$ a point in $\Lambda_\varepsilon$, where the previous supremum is attained in $\Lambda_\varepsilon$. Hence $\alpha = |z_0 - p_0(c)|$, and $p_0^2(c) + c = p_0(c)$. Now, we can find a point $(z,w) \in \Lambda_\varepsilon$ such that

$$\begin{aligned} f_\varepsilon(z,w) = (z_0,w_0) &\Rightarrow z_0 = z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 \\ &\Rightarrow z_0 - p_0(c) = z^2 - p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \\ &= (z - p_0(c))^2 + 2zp_0(c) - 2p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \Rightarrow \\ &\alpha \leq \alpha^2 + 2|p_0(c)|\alpha + K\varepsilon, \end{aligned}$$

with $K$, a positive constant depending on the parameters of the map. Now $\alpha^2+\alpha(2|p_0(c)|-1)+K\varepsilon \geq 0$, and since $\alpha \ll 1$ (since $\Lambda_\varepsilon$ is very close to $\{p_0(c)\} \times S^1$), we obtain (for some constant $K'$):

$$0 \leq \alpha \leq \frac{2K\varepsilon}{1-2|p_0(c)|+\sqrt{(1-2|p_0(c)|)^2-4K\varepsilon}} \leq K' \cdot \varepsilon.$$

From (3) one gets $2\alpha(2|p_0(c)|+\varepsilon|a|) > |2bw+dw(z+z')|$ for $\varepsilon < \varepsilon(a,b,c,d,e)$, since $z, z'$ are both $\varepsilon$-close to $p_0(c)$.

But for $(z,w) \in \Lambda_\varepsilon$, one has $|w|=1$, and $|z+z'|$ is $\varepsilon$-close to $2|p_0(c)|$ . Therefore, if $b \neq 0$, and $|c|$ is small enough in comparison to $|b|$, then $|p_0(c)|$ becomes so small that $|2bw+dw(z+z')| > |b| > 0$. Hence $2K'\varepsilon(2|p_0(c)|+\varepsilon|a|) \geq |b| > 0$, which is a contradiction if $|c|$ is small enough (since $|c|$ small will imply that $|p_0(c)|$ is also small, and we can always reduce $\varepsilon$ accordingly).

Hence, we proved that $w' = w$. Then, from $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ it follows that $z^2 - z'^2 = -\varepsilon[a(z-z')+dw(z-z')]$. If $z \neq z'$, we would then get $z+z' = -\varepsilon(a+dw)$.

But $z, z'$ are both $\varepsilon$ -close to $p_0(c) \neq 0$, so if we choose $\varepsilon < \varepsilon(a,b,c,d,e)$ appropiately, then $|z+z'| > |p_0(c)| > 0$. On the other hand, if $|a+dw| \neq 0$, we can take $\varepsilon$ small enough such that $\varepsilon|a+dw| < |p_0(c)|$, which gives a contradiction. If $|a+dw| = 0$, then we get a contradiction again since $p_0(c) \neq 0$.

In conclusion $z = z', w = w'$ and in consequence $f_\varepsilon|_{\Lambda_\varepsilon}\colon\ \Lambda_\varepsilon \to \Lambda_\varepsilon$ is an injective map.

□

**Corollary 1.** *In the setting of Theorem 3, if $f_\varepsilon(z,w)$ is injective on $\Lambda_\varepsilon$ and $|c|$ is small enough, then $h_-(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$ and $HD(W^s_\delta(x) \cap \Lambda_\varepsilon) > \frac{\log 2}{\log|\frac{1+\sqrt{1-4c}}{2c}|}$.*

*In addition, in this case $\Lambda_\varepsilon$ is not a graph and in particular not a Jordan curve.*

*Proof.* If $f_\varepsilon|_{\Lambda_\varepsilon}$ is injective, then it is a homeomorphism of $\Lambda_\varepsilon$ and hence $h_-(f_\varepsilon|_{\Lambda_\varepsilon}) = h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon})$ . But

$$h(f_\varepsilon|_{\Lambda_\varepsilon}) = h(\hat{f}_\varepsilon) = h(\hat{f}) = h(f) = \log 2$$

In the above, $\hat{f}$ represents the lifting of $f$ to a homeomorphism of $\hat{\Lambda}$ and we can use the conjugacy between $\hat{f}$ and $\hat{f}_\varepsilon$ ([14]); also the last equality is due to the fact that $f|_\Lambda$ is just the map $w \to w^2$ whose entropy is $\log 2$.

The estimate for the stable dimension in our claim is proved by using Theorem 2 and the fact that $\left|\inf_{y\in\Lambda} |Df|_{E^s_y}|\right| > |p_0(c)|$, where $p_0(c) = \frac{2c}{1+\sqrt{1-4c}}$ is the unique fixed attracting point of the map $z^2+c$, for $|c|$ small.

The last consequence to show is that the basic set $\Lambda_\varepsilon$ is not a graph.

Indeed, if $\Lambda_\varepsilon$ were a graph, the Nitecki-Przytycki Theorem in Section 1 would imply that $h_i(f_\varepsilon|_{\Lambda_\varepsilon})$ is zero, but as we saw above, $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$.

□

In the case of diffeomorphisms the stable dimension depends continuously on the map, even real analytically in the case of Hénon maps ([17]). However this property is not present anymore for holomorphic endomorphisms.

Indeed, from Corollary 1, if $f_\varepsilon$ is a perturbation of $f(z,w) = (z^2+c, w^2)$, such that $f_\varepsilon$ is injective on its basic set $\Lambda_\varepsilon$, then $HD(W^s_\delta(x_\varepsilon) \cap \Lambda_\varepsilon)$ is bounded below by a constant which does not depend on small $\varepsilon$.

On the other hand $HD(W^s_\delta(x) \cap \Lambda) = 0$, if $\Lambda$ is the basic set of $f$ and $x \in \Lambda$. Therefore we get a contradiction, since in any neighbourhood of $f$ we were able to find a map having stable dimension bounded below by a constant not depending on $\varepsilon$. This will be formulated in the following:

**Corollary 2.** *There exist Axiom A holomorphic maps $f$ on $\mathbb{P}^2$ which have basic sets $\Lambda$, such that if $f_\varepsilon$ is an $\varepsilon$- perturbation of $f$ and $\Lambda_\varepsilon$ is the basic set of $f_\varepsilon$ close to $\Lambda$, then $HD(W^s_\delta(x_\varepsilon) \cap \Lambda_\varepsilon)$ does not converge to $HD(W^s_\delta(x) \cap \Lambda)$, when $\varepsilon \to 0$ and $x_\varepsilon \in \Lambda_\varepsilon, x \in \Lambda, x_\varepsilon \to x$. Hence the dependence of the stable dimension on the map is not continuous in general.*

Now we ask what can be said about the number of preimages that a point in the basic set $\Lambda$ can have in $\Lambda$, in order to shed more light on the upper estimate from Theorem 1.

**Lemma 1.** *Let $f$ be a holomorphic map on $\mathbb{P}^2$, and $\Lambda$ one of its basic sets of infinite cardinality, such that $C_f \cap \Lambda = \emptyset$, where $C_f$ is the critical set of $f$. Then the set $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ is open with respect to the induced topology on $\Lambda$.*

*Proof.* If $C_f \cap \Lambda = \emptyset$, then there exists $\delta > 0$ such that, if $f(z_1) = f(z_2), z_1, z_2 \in \Lambda, z_1 \neq z_2$, then $d(z_1, z_2) > \delta$. Now assume that the statement of our lemma is not true; we can assume also that all the points we are working with are not isolated (as a matter of fact $\Lambda$ cannot have any isolated points since $f|_\Lambda$ is transitive and $\Lambda$ is infinite, but this is not necessary here). Then there would exist $x \in \Lambda$ such that $f^{-1}(x) \cap \Lambda = \{z\}$ and there would exist a sequence $y_n \in \Lambda$, $y_n \to x$, with $\{z_n^1, z_n^2\} \subset f^{-1}(y_n) \cap \Lambda, z_n^1 \neq z_n^2, \forall n$. From the discussion above, it follows that $d(z_n^1, z_n^2) > \delta$. But since $\Lambda$ is compact, there exists a subsequence of $(z_n^1)_n$, denoted also by $(z_n^1)_n$ such that $z_n^1 \to \xi^1 \in \Lambda$.

By taking yet another subsequence, one can assume also that $z_n^2 \to \xi^2 \in \Lambda$.

Since $f(z_n^1) = y_n$ and $y_n \to x$, we have $f(\xi^1) = x$ and similarly $f(\xi^2) = x$. But $d(z_n^1, z_n^2) > \delta$, hence $d(\xi^1, \xi^2) \geq \delta$. Therefore we get a contradiction with the fact that the set $f^{-1}(x) \cap \Lambda$ has only one element. So the set of points $x \in \Lambda$ having only one preimage in $\Lambda$, is open in the induced topology. □

**Lemma 2.** *In the same setting as in Lemma 1 and assuming also that there exists a neighbourhood $U$ of $\Lambda$ such that $f^{-1}(\Lambda) \cap U = \Lambda$, it follows that the set $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ is closed.*

*Proof.* Assume that the set of points with only one preimage in $\Lambda$ is not closed. Then, there would exist $x \in \Lambda$ having two preimages $y_1, y_2$ in $\Lambda$ and a sequence $x_n \in \Lambda, x_n \to x$ with $f^{-1}(x_n) \cap \Lambda = \{z_n\}$. By taking eventually a subsequence, one can suppose that $d(y_1, z_n) > \delta_0, \forall n$, for some positive constant $\delta_0$. As observed before, the set $\Lambda$ does not have any isolated points since $f|_\Lambda$ is transitive and $\Lambda$ was assumed infinite (otherwise the problem would be trivial if $\Lambda$ finite). Let now $V$ be the open ball around $y_1$ of radius $\frac{\delta_0}{2}$. Since $f^{-1}(U) \cap \Lambda = \Lambda$ and since $\Lambda$ has no isolated points, it follows that $x_n \in f(V)$ for $n$ large enough. So, there exists $\xi_n \in V$ such that $f(\xi_n) = x_n$. Also, since we assumed $d(y_1, z_n) > \delta_0$, it is clear that $z_n \neq \xi_n$. Hence, this implies that $\{\xi_n, z_n\} \subset f^{-1}(x_n) \cap \Lambda$, which is a contradiction with our assumption. Therefore the set $\{x \in \Lambda, \mathrm{Card}\{f^{-1}(x) \cap \Lambda\} = 1\}$ is closed in $\Lambda$. We are done. $\square$

Lemmas 1, 2 imply the following corollaries.

**Corollary 3.** *In the same setting as in Lemma 2, and assuming also that the basic set $\Lambda$ is connected, one has that, if there exists a point $x \in \Lambda$ with only one preimage in $\Lambda$, then every point in $\Lambda$ has exactly one preimage in $\Lambda$.*

**Corollary 4.** *In the same setting as in Lemma 2, and assuming also that $\Lambda$ is connected, the number of preimages that a point from $\Lambda$ has in $\Lambda$ is constant.*

As can be seen from the proofs of Lemmas 1, 2, this statement is true in a more general setting: $f$ does not have to be necessarily holomorphic for example. A natural question is how we can check which basic sets are connected. In fact it turns out that it is enough to perturb a map with a basic set which we know to be connected in order to obtain other such examples.

**Lemma 3.** *In the same setting as in Lemma 1, if $\Lambda$ is connected and $g$ is sufficiently close to $f$, with its corresponding basic set $\Lambda_g$, then $\Lambda_g$ is connected as well.*

*Proof.* We know from a result of Przytycki ([14]) that there exists a conjugacy at the level of inverse limits, between $\hat{\Lambda}$ and $\widehat{\Lambda_g}$ if $g$ is close enough to $f$. Also an easy result from topology shows that $\Lambda$ is connected if and only if $\hat{\Lambda}$ is connected. Hence if we assumed $\Lambda$ to be connected, it follows also that $\hat{\Lambda}$, $\widehat{\Lambda_g}$, and $\Lambda_g$ are all connected.

$\square$

Corollary 3 and Lemma 3, together with Theorems 1 and 2 give now information about the stable dimension of perturbations of the map $(z^2 + c, w^2)$.

**Corollary 5.** *Let $g$ be a perturbation of the map $(z^2 + c, w^2)$, where $0 \neq |c|$ small, and consider the basic set $\Lambda_g$, close to $p_0(c) \times S^1$. Assume also that, if $g|_{\Lambda_g}$ is not a homeomorphism, then there exists a neighbourhood $U$ of $p_0(c) \times S^1$ such that $g^{-1}(\Lambda_g) \cap U = \Lambda_g$. Then we are in one of the following two cases:*

*(a) $g|_{\Lambda_g}$ is a homeomorphism. In this case we have the following inequality*

$$\frac{\log 2}{|\log \inf_{y \in \Lambda_g} |Df|_{E^s_y}||} \leq HD(W^s_\delta(x) \cap \Lambda_g) \leq \frac{\log 2}{|\log \sup_{\xi \in \Lambda_g} |Df|_{E^s_\xi}||},$$

*(b) Or $g|_{\Lambda_g}$ is not a homeomorphism. Then $HD(W^s_\delta(x) \cap \Lambda_g) = 0$.*

*Proof.* Let us prove first item (a). Since $g$ is a homeomorphism on $\Lambda_g$, the Theorems 1 and 2 imply the required inequalities for $HD(W^s_\delta(x) \cap \Lambda_g)$.

We prove now item (b). Since the basic set $\Lambda = \{p_0(c)\} \times S^1$ is connected, Lemma 3 implies that $\Lambda_g$ is also connected if $g$ is a small perturbation of the map $(z^2+c, w^2)$. Hence, from Corollary 4, the number of preimages that a point can have in $\Lambda_g$ is constant and will be denoted by $d'$. To find this constant we notice that, since $h(g|_{\Lambda_g}) = \log 2$ and $h(g|_{\Lambda_g}) \geq \log d'$ (from §1), we get $1 \leq d' \leq 2$. So, either every point of $\Lambda_g$ has exactly one preimage in $\Lambda_g$, or else every point from $\Lambda_g$ has exactly two different preimages in this set. But the condition $g^{-1}(\Lambda_g) \cap U = \Lambda_g$ prevents $d'$ from being equal to 1 in this case. Therefore, every point in $\Lambda_g$ has exactly two preimages in $\Lambda_g$. Thus, Theorem 1 gives that $HD(W^s_\delta(x) \cap \Lambda_g) = 0$. $\square$

Obviously the same reasoning can be applied to other perturbations of maps $f$ on $\mathbb{P}^2$ for which $C_f \cap \Lambda = \emptyset$, (see for example [5] for more examples, like product maps, solenoids, etc.)

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