RECURRENCE RATES FOR LOOSELY MARKOV DYNAMICAL SYSTEMS

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Abstract. The concept of loosely Markov dynamical systems is introduced. It is shown that for these systems the recurrence rates and poinwise dimensions coincide. The systems generated by hyperbolic exponential maps, arbitrary rational functions of the Riemann sphere, and measurable dynamical systems generated by infinite conformal iterated function systems are all checked to be loosely Markov.

1. Introduction and Preliminaries

Let (X,ρ) be a metric space, let μ be a Borel probability measure on X and let $T: X \to X$ be a μ-invariant measurable map. Given $x \in X$ one defines lower and upper pointwise dimensions of μ at the point x respectively as follows.

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ and } \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Given in addition $r > 0$ one defines

$$\tau_r(x) = \inf\{n \geq 1 : T^n(x) \in B(x,r)\} = \inf\{n \geq 1 : \rho(T^n(x), x) < r\}.$$

The lower and upper recurrence rates of x are respectively defined as

$$\underline{R}(x) = \liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \text{ and } \overline{R}(x) = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r}.$$

Let $\mathrm{HD}(\mu)$ be the Hausdorff dimension of the measure μ ($\inf\{\mathrm{HD}(Y)\}$ taken over all Borel sets Y with $\mu(Y) = 1$). It is well known that if X is a subset of a Euclidean space and ρ is the corresponding Euclidean metric, then $\mathrm{HD}(\mu) = \mathrm{esssup}(\underline{d}_\mu)$. The pointwise dimensions and recurrence rates seem to be apriori absolutely unrelated. The former are purely geometric notions whereas the latter are rather dynamical. However M. Boshernitzan in his pioneering work [4] has been able to prove the remarkable inequality $\underline{R}(x) \leq \mathrm{HD}(\mu)$ for μ-a.e. $x \in X$. Since then a number of papers devoted to find closer relations (under various additional assumptions) between recurrence rates and dimensions have appeared. In the remarkable paper [2] the authors have provided a sufficient condition for $\underline{R}(x) = \underline{d}_\mu(x)$ and $\overline{R}(x) = \overline{d}_\mu(x)$ to hold μ-a.e. Their condition involved uniform expanding property and a kind of a strong mixing property with respect to the reference partition. We also provide a sufficient condition for these two equalities to hold. In contrast to [2] we do not assume any expanding property and instead of assuming a mixing condition with respect to a reference partition, we impose an assumption on the rate of convergence of the Perron-Frobenius operator associated to the measure μ. We do not have to therefore care too much about a good partition, and in particular, about mixing with respect to such a partition. Our general condition, Theorem 2.1 applies for example to the dynamical systems such as hyperbolic exponential maps, arbitrary rational functions of the Riemann sphere and measurable dynamical systems generated by infinite conformal iterated function systems. Let us now introduce and describe some technical notions and results needed in the proof of Theorem 2.1. Given two points $x \in X$ and $y \in B(x,r)$, the return time of the point y into X is defined as

$$\tau_r(y,x) = \min\{k \geq 1 : \rho(T^k(y),x) < r\}.$$

For each $x \in X$ and all $r, \varepsilon > 0$, consider the set

$$A_\varepsilon(x,r) = \{y \in B(x,r) : \tau_r(y,x) \leq \mu(B(x,r))^{-1+\varepsilon}\}. \tag{1.1}$$

The measure μ is said to have long return time with respect to T if

$$\liminf_{r\to 0} \frac{\log \mu\Big(A_\varepsilon(x,r)\Big)}{\log \mu\Big(B(x,r)\Big)} > 1$$

for μ-a.e. $x \in X$ and all $\varepsilon > 0$ sufficiently small. The following fact has been proved in [3].

Proposition 1.1. *Let X be a subset of a Euclidean space, let μ be a Borel probability measure on X and let $T : X \to X$ be a μ-invariant measurable map. If μ has long return time and $\underline{d}_\mu(x) > 0$ for μ-a.e. $x \in X$, then*

$$\underline{d}_\mu(x) = \underline{R}(x) \text{ and } \overline{d}_\mu(x) = \overline{R}(x)$$

for μ-a.e. $x \in X$.

A Borel probability measure on a Euclidean space $I\!R^d$ is said to be weakly diametrically regular if for μ-a.e. $x \in I\!R^d$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $r < \delta$, then

$$\mu\Big(B(x,2r)\Big) \leq \mu\Big(B(x,r)\Big) r^{-\varepsilon}.$$

The following result also comes from [3].

Proposition 1.2. *Every Borel probability measure on a Euclidean space is weakly diametrically regular.*

The return time $\tau(A)$ of a set A into itself is defined as

$$\tau(A) = \min\{n \geq 1 : T^n(A) \cap A \neq \emptyset\}.$$

Given a partition α of X and a point $x \in X$, denote by $\alpha(x)$ the only element of α containing x. Given in addition an integer $n \geq 1$, the refined partition $\alpha \vee T^{-1}(\alpha) \vee \ldots \vee T^{n-1}(\alpha)$ is denoted by α_n. The basic property joining the above notions is captured by the following result proven in [11].

Proposition 1.3. *Suppose that $T : Y \to Y$ is a measurable transformation preserving an ergodic probability measure μ. If α is a finite or countable partition of Y with positive entropy $h_\mu(T,\alpha)$, then*

$$\liminf_{n\to\infty} \frac{\tau(\alpha_n(x))}{n} \geq 1$$

for μ-a.e. $x \in Y$.

2. Loosely Markov Maps

Let (X,ρ) be a metric space and let $f : X \to X$ be a Borel map which is at most countable-to-one. Let μ be a Borel probability f-invariant measure on X. Suppose that f is non-singular with respect to μ, i.e. that $\mu(f(A)) = 0$ if $\mu(A) = 0$. Consider J_μ, the Jacobian of the map f with respect to the measure μ. The Perron-Frobenius operator $\mathcal{L}_\mu$ of f with respect to the measure is given by the formula

$$\mathcal{L}_\mu(g)(x) = \sum_{y\in f^{-1}(x)} J_\mu^{-1}(y)g(y).$$

Since the measure μ is f-invariant, $\mathcal{L}_\mu(\mathbb{1}) = \mathbb{1}$. Suppose that $J_\mu : X \to (0,+\infty)$ is a bounded locally Hölder continuous function with an exponent $\xi > 0$. Denote the class of all such functions by H_ξ. Denote also by $C_b(X)$ the Banach space of all bounded real-valued functions on X endowed with the supremum norm $||\cdot||$. H_ξ becomes a Banach space when equipped with the norm $||\cdot||_\xi$ defined as follows.

$$||g||_\xi = v_\xi(g) + ||g||_\infty,$$

where

$$v_\xi(g) = \sup\left\{\frac{|g(y)-g(x)|}{\rho^\xi(y,x)} : x \in X,\ y \in B(x,\delta)\setminus\{x\}\right\}$$

and δ is small enough. Suppose that the Perron-Frobenius operator $\mathcal{L}_\mu$ preserves $C_b(X)$ and H_ξ. Assuming from now on that X is Borel subset of a Euclidean space and ρ is the standard Euclidean metric, the dynamical system (f,μ) is said to be loosely Markov provided that the following conditions are satisfied. There exists a continuous function $C_1 : X \to (0,+\infty)$ such that

(a) There exists $\gamma \in (0,1)$ such that for all $g \in \mathrm{H}_\xi$, all $n \geq 0$, and all $x \in X$

$$|\mathcal{L}_\mu^n(g)(x) - \mu(g)| \leq C_1(x)\gamma^n ||g||_\xi$$

(b) For μ-a.e $x \in X$ there exists $\chi(x) > 0$ and a countable partition α of X (by Borel sets) such that $\mathrm{h}_\mu(f,\alpha) > 0$ and

$$\alpha_n(x) \supset B(x, \exp(-\chi(x)n))$$

for μ-a.e. $x \in X$ and all $n \geq 0$ large enough.

(c) $0 < \underline{d}_\mu(x) \leq \overline{d}_\mu(x) < \infty$ for μ-a.e $x \in X$.

We shall now prove a general theorem (which in some sense corresponds to Theorem 6 in [2]) such that all the concluding results stated in the next sections will follow from it after some appropriate preparations.

Theorem 2.1. *If (f,μ) is a loosely Markov dynamical system, then*

$$\underline{R}(x) = \underline{d}\mu(x) \text{ and } \overline{R}(x) = \overline{d}\mu(x)$$

for μ-a.e. $x \in X$.

Proof. Fix a Borel set $W \subset X$ such that $\mu(W) = 1$ and conditions (b) and (c) are satisfied for all $x \in W$. Combining condition (b) and Proposition 1.3, we see that

$$\liminf_{r\to 0} \frac{\tau(B(x,r))}{-\log r} = \liminf_{n\to\infty} \frac{\tau\Big(B(x,\exp(-\chi(x)n)\Big)}{\chi(x)n} \ge \liminf_{n\to\infty} \frac{\tau(\alpha_n(x))}{\chi(x)n} \ge \frac{1}{\chi(x)}.$$

This implies that

$$B(x,r) \cap f^{-k}(B(x,r)) = \emptyset \tag{2.1}$$

for all $r > 0$ small enough and all $k \le -\frac{1}{2\chi}\log r$, where $\chi := \chi(x)$. It follows from condition (c) that for every $s > 0$ sufficiently small (depending on x), we have

$$s^{2\overline{d}_\mu(x)} \le \mu(B(x,s)) \le s^{\underline{d}_\mu(x)/2}. \tag{2.2}$$

Put $B_r = B(x,r)$ for all $r \in (0,1)$. Given in addition $\kappa > 0$ define the function $\zeta_{r,\kappa} : [0,+\infty) \to [0,1]$ by the following formula.

$$\zeta_{r,\kappa}(t) = \begin{cases} 1 & \text{if } 0 \le t \le r \\ r^{-\kappa}(r + r^\kappa - t) & \text{if } r \le t \le r + r^\kappa \\ 0 & \text{if } t \ge r + r^\kappa. \end{cases}$$

Define then $g_{r,\kappa} : X \to [0,1]$ by putting

$$g_{r,\kappa}(z) = \zeta_{r,\kappa}(\rho(z,x)).$$

Since the function $z \mapsto \rho(z,x)$ is Lipschitz continuous with the Lipschitz constant equal to 1, and since $\zeta_{r,\kappa}$ is Lipschitz continuous with the Lipschitz constant equal to $r^{-\kappa}$, the composite function $g_{r,\kappa} : X \to [0,1]$ is Lipschitz continuous with the Lipschitz constant equal to $r^{-\kappa}$. In particular $g_{r,\kappa} \in \mathrm{H}_\xi$ and $||g_{r,\kappa}||_\xi \le r^{-\kappa}$. We shall show now that the measure μ has long return time at the point x. Since $C_1 : X \to (0,+\infty)$ is continuous, we may assume $r > 0$ to be so small that that $C_1(z) \le 2C_1(x)$ for all $z \in B_r$. It follows from condition (a) of the loosely Markov property and Proposition 1.2 that for every $k \geq 0$

$$\begin{aligned}\mu\Big(B(x,r)\cap f^{-k}(B(x,r))\Big) &\leq \int_{B_r}\mathcal{L}_\mu^k\Big(1\!\!1_{B_r}\Big)d\mu \leq \int_{B_r}\mathcal{L}_\mu^k\Big(g_{r,\kappa}\Big)d\mu \\ &\leq \int_{B_r}\Big(\mu(g_{r,\kappa})+C_1\gamma^k||g_{r,\kappa}||_\xi\Big)d\mu \\ &\leq \int_{B_r}\Big(\mu(g_{r,\kappa})+2C_1(x)r^{-\kappa}\gamma^k\Big)d\mu.\end{aligned} \tag{2.3}$$

Assuming that $r\in(0,1)$ is sufficiently small and using (2.2), we get with $\beta=-\frac{1}{4\chi}\log\gamma$ and some universal constant $C_2>0$, that

$$r^{-\beta}\sum_{k=-\frac{1}{2\chi}\log r}^{2\frac{\log r}{\log\gamma}}\gamma^k \leq C_2r^{-\beta}r^{-\frac{1}{2\chi}\log\gamma}=C_2r^{-\frac{1}{4\chi}\log\gamma}\leq C_2\mu(B_r)^{-\frac{\log\gamma}{8\chi\overline{d}_\mu(x)}}$$

and

$$\mu(g_{r,\beta})\leq\mu\Big(B_{r+r^\beta}\Big)\leq\mu\Big(B_{2r^{\min\{1\beta\}}}\Big)\leq 2^{\frac{\underline{d}_\mu(x)}{2}}r^{\frac{1}{2}\underline{d}_\mu(x)\min\{1,\beta\}}\leq 2^{\frac{\underline{d}_\mu(x)}{2}}\mu(B_r)^{\frac{\underline{d}_\mu(x)\min\{1,\beta\}}{4\overline{d}_\mu(x)}}.$$

Combining these two last formulas, (2.3) and (2.2), we obtain

$$\begin{aligned}&\sum_{k=-\frac{1}{2\chi}\log r}^{2\frac{\log r}{\log\gamma}}\mu\Big(B_r\cap f^{-k}(B_r)\Big)\leq \\ &\leq \sum_{k=-\frac{1}{2\chi}\log r}^{2\frac{\log r}{\log\gamma}}\int_{B_r}\Big(\mu(g_{r,\beta})+2C_1(x)r^{-\beta}\gamma^k\Big)d\mu \\ &\leq\left(\left(\frac{2}{\log\gamma}+\frac{1}{2\chi}\right)\log(r)\mu(g_{r,\beta})+2C_1(x)r^{-\beta}\sum_{k=-\frac{1}{2\chi}\log r}^{2\frac{\log r}{\log\gamma}}\gamma^k\right)\mu(B_r) \\ &\leq\left(\left(\frac{2}{\log\gamma}+\frac{1}{2\chi}\right)\log(r)2^{\frac{\underline{d}_\mu(x)}{2}}\mu(B_r)^{\frac{\underline{d}_\mu(x)\min\{1,\beta\}}{4\overline{d}_\mu(x)}}+2C_1(x)C_2\mu(B_r)^{-\frac{\log\gamma}{8\chi\overline{d}_\mu(x)}}\right)\mu(B_r) \\ &\leq\Big(\mu(B_r)^{\frac{\underline{d}_\mu(x)\min\{1,\beta\}}{8\overline{d}_\mu(x)}}+2C_1(x)C_2\mu(B_r)^{-\frac{\log\gamma}{8\chi\overline{d}_\mu(x)}}\Big)\mu(B_r) \\ &\leq(1+2C_1(x)C_2)\mu(B_r)^{1+\theta},\end{aligned} \tag{2.4}$$

where

$$\theta=\frac{1}{8\overline{d}_\mu(x)}\min\left\{\underline{d}_\mu(x)\min\{1,\beta\},-\frac{\log\gamma}{\chi}\right\}.$$

Using (2.2) we also get

$$r^{-1}\sum_{k=2\frac{\log r}{\log\gamma}}^{\mu(B_r)^{-1+\varepsilon}}\gamma^k\leq C_3r^{-1}\gamma^{2\frac{\log r}{\log\gamma}}=C_3r^{-1}\exp\left(2\frac{\log r\log\gamma}{\log\gamma}\right)=C_3r\leq C_3\mu(B_r)^{\frac{1}{2\overline{d}_\mu(x)}}$$

with some universal constant $C_3 > 0$. Applying (2.3) and Proposition 1.2, we thus get for all $r > 0$ small enough that

$$\begin{aligned}
\sum_{k=2\frac{\log r}{\log\gamma}}^{\mu(B_r)^{-1+\varepsilon}} \mu\Big(B_r \cap f^{-k}(B_r)\Big) &\le \sum_{k=2\frac{\log r}{\log\gamma}}^{\mu(B_r)^{-1+\varepsilon}} \int_{B_r} \Big(\mu(g_{r,1}) + 2C_1(x)r^{-1}\gamma^k\Big)d\mu \\
&\le \left(\mu(B_r)^{-1+\varepsilon}\mu(g_{r,1}) + 2C_1(x)r^{-1}\sum_{k=2\frac{\log r}{\log\gamma}}^{\mu(B_r)^{-1+\varepsilon}}\gamma^k\right)\mu(B_r) \\
&\le \left(\mu(B_r)^{-1+\varepsilon}\mu(B_{2r}) + 2C_1(x)C_3\mu(B_r)^{\frac{1}{2\overline{d}_\mu(x)}}\right)\mu(B_r) \\
&= \left(\mu(B_r)^{-1+\varepsilon}\mu(B_r)r^{-\varepsilon/2} + 2C_1(x)C_3\mu(B_r)^{\frac{1}{2\overline{d}_\mu(x)}}\right)\mu(B_r) \\
&= \left(\mu(B_r)r^{\varepsilon/2} + 2C_1(x)C_3\mu(B_r)^{\frac{1}{2\overline{d}_\mu(x)}}\right)\mu(B_r) \\
&\le (1 + 2C_1(x)C_3)\mu(B_r)^{1+\eta},
\end{aligned}$$

where $\eta = \frac{1}{2}\min\{\varepsilon, 1/\overline{d}_\mu(x)\}$. Combining now this, (1.1), (2.1) and (2.4), we obtain for all $r \in (0,1)$ sufficiently small that

$$\begin{aligned}
\mu(A_\varepsilon(x,r)) &\le \sum_{k=0}^{\mu(B_r)^{-1+\varepsilon}} \mu\Big(B_r \cap f^{-k}(B_r)\Big) \le (1+2C_1(x)C_2)\mu(B_r)^{1+\theta} + (1+2C_1(x)C_3)\mu(B_r)^{1+\eta} \\
&\le 2C_1(x)(1+C_2+C_3)\mu(B_r)^{1+\min\{\theta,\eta\}}.
\end{aligned}$$

Thus

$$\frac{\log\Big(\mu(A_\varepsilon(x,r))\Big)}{\log(\mu(B_r))} \ge \frac{\log\Big(2C_1(x)(1+C_2+C_3)\Big)}{\log(\mu(B_r))} + (1+\min\{\theta,\eta\})$$

and consequently

$$\liminf_{r\to 0} \frac{\log\Big(\mu(A_\varepsilon(x,r))\Big)}{\log(\mu(B_r))} \ge (1+\min\{\theta,\eta\}) > 1.$$

Hence, the measure μ has long return time and the theorem follows by applying Proposition 1.1. ■

3. Rational Functions of the Riemann Sphere $\overline{\mathbb{C}}$

In this short section $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an arbitrary rational function of degree ≥ 2. Denote by $J(f)$ the Julia set of the function f. A function $\phi : J(f) \to \mathbb{R}$ is assumed to be Hölder continuous and to satisfy inequality $\mathrm{P}(\phi) > \sup(\phi)$, where $\mathrm{P}(\phi)$ is the topological pressure of ϕ (see [17] and [13] for its definition and a fairly thorough exposition of its properties). It was shown in [5] that in this situation there exists a unique equilibrium state (see also [17] and [13]) μ_ϕ of

ϕ and $f : J(f) \to J(f)$. Our aim is to show that the dynamical system $(J(f), \mu_\phi)$ is loosely Markov. And indeed, it was proved in [13] (see Theorem 9.4.2) that

$$\underline{d}_{\mu_\phi}(x) = \overline{d}_{\mu_\phi}(x) = \mathrm{HD}(\mu_\phi) = \frac{\mathrm{h}_{\mu_\phi}}{\chi_{\mu_\phi}} \tag{3.1}$$

for μ_ϕ-a.e. $x \in J(f)$. In particular, condition (c) of the definition of a loosely Markov map is satisfied. Condition (b) was established in the proof of Theorem 9.4.2 from [13]. Finally, condition (a) is exactly the content of Theorem 10 in [8]. Hence, applying Theorem 2.1 and using (3.1), we get the following.

Theorem 3.1. *If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational function of degree ≥ 2 and if $\phi : J(f) \to \mathbb{R}$ is a Hölder continuous potential such that $\mathrm{P}(\phi) > \sup(\phi)$, then for μ_ϕ-a.e. $z \in J(f)$, the number $R_{\mu_\phi}(z)$ exists and moreover*

$$R_{\mu_\phi}(z) = \mathrm{HD}(\mu_\phi) = \frac{h_{\mu_\phi}}{\chi_{\mu_\phi}}.$$

4. Hyperbolic Exponential Maps

In this section we consider the maps $f_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$, given by the formula

$$f_\lambda(z) = \lambda e^z.$$

We assume that f_λ is hyperbolic, that is that f_λ has an attracting periodic cycle (this is always the case if $\lambda \in \mathbb{C}$ is in a sufficiently small neighborhood of the interval $(0, 1/e)$. We will be interested in the recurrence rates for the mapping F_λ, canonically associated with f_λ, which carries essentially all significant information about the dynamics of f_λ and is more suitable for measure theoretic considerations. So, the equivalence relation "$\sim$" on $\mathbb{C} \times \mathbb{C}$ is defined as follows: $z \sim w$ if and only if $w - z \in 2\pi i\mathbb{Z}$. The map f_λ projects down to the holomorphic map F_λ of the cylinder $Q = \mathbb{C}/\sim$ such that if $\Pi : \mathbb{C} \to Q$ denotes the canonical projection, then $f_\lambda \circ \Pi = \Pi \circ F_\lambda$. From now on we will skip the subscript λ and will simply write f and F. Let

$$\delta = \frac{1}{2} \min \left\{ \frac{1}{2}, \mathrm{dist}\Big(J(F), \{F^n(\Pi(0)) : n \geq 0\}\Big) \right\}$$

Since the map $f : J(f) \to J(f)$ is hyperbolic, we see that $\delta > 0$. For every $n \geq 1$ and every $v \in J(F)$ the map

$$F_v^{-n} : B(F^n(v), 2\delta) \to P$$

is then defined to be the holomorphic inverse branche of F^n on $B(F^n(v), 2\delta)$ sending $F^n(v)$ to v. Let ϕ be a real-valued Hölder continuous function defined on some Euclidean R-neighbourhood of the Julia set $J(F) \subset \mathbb{C}$, $R \in (0, \delta/2)$. Hölder continuous means here that

$$\exists_{(\alpha>0)} \forall_{r \in (0,R)} \exists_{H_r > 0} \text{ if } |y - x| \leq r, \text{ then } |\phi(y) - \phi(x)| \leq H_r |y - x|^\alpha.$$

One can prove (see [15]) that for every $z \in J(F)$ the following limit exists and is independent of the point z.

$$\mathrm{P}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in F^{-n}(z)} \exp\Big(S_n\phi(x)\Big).$$

The number $\mathrm{P}(\phi)$ is called the topological pressure of the potential ϕ. The following easy technical fact is also established in [15].

Lemma 4.1. *For every α-Hölder function $\phi : J(F) \to \mathbb{C}$ there exists a constant $L_\phi > 0$ such that*

$$|S_n\phi(F_v^{-n}(y)) - S_n\phi(F_v^{-n}(x))| \le L_\phi |y-x|^\alpha$$

for all $n \ge 1$, all $x, y \in J(F)$ with $|x-y| \le \delta$ and all $v \in F^{-n}(x)$. In particular

$$\exp\Big(S_n\phi(F_v^{-n}(y)\Big) \le T \exp\Big(S_n\phi(F_v^{-n}(x)\Big),$$

where $T = \exp\Big(L\delta^\alpha\Big)$.

A Hölder continuous function $\phi : J(F) \to \mathbb{R}$ is called 1^+-tame if there exists $\kappa > 1$ such that

$$A_\phi := \sup\{|\phi(z) + \kappa \mathrm{Re} z| : z \in J(F)\} < +\infty.$$

The following three basic facts have been proved in [15].

Theorem 4.2. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{C}$ is a 1^+-tame potential, then there exists a unique Borel probability measure m_ϕ on $J(F)$ such that*

$$m_\phi\Big(F_v^{-n}\Big(B(z,\delta)\Big) = \int_{B(z,\delta)} \exp\Big(S_n\phi\Big(F_v^{-n}(w)\Big) - \mathrm{P}(\phi)n\Big) dm_\phi(w)$$

for all $n \ge 1$, all $z \in J(F)$ and all $v \in F^{-n}(z)$.

Theorem 4.3. *There exists a unique Borel probability F-invariant measure μ_ϕ absolutely continuous with respect to m_ϕ. In addition μ_ϕ is ergodic, equivalent to m_ϕ and the Radon-Nikodym derivative $d\mu_\phi/dm_\phi$ has a continuous everywhere (on J(F)) positive and uniformly bounded version.*

Theorem 4.4. *The invariant measure μ_ϕ is an equilibrium state of the potential ϕ, that is $\mathrm{P}(\phi) = h_{\mu_\phi} + \int \phi d\mu_\phi$.*

The following two more technical facts have been also proved in [15].

Theorem 4.5. *The entropy $h_{\mu_\phi}(F)$ is positive.*

and

Theorem 4.6. *The Lyapunov exponent $\chi_{\mu_\phi} = \int \log|F'|d\mu_\phi$ is finite.*

We shall verify that the dynamical system (F, μ_F) is loosely Markov starting with the following.

Theorem 4.7. *If $f : \mathbb{C} \to \mathbb{C}$ is a hyperbolic exponential map and $\phi : J(F) \to \mathbb{C}$ is a 1^+-tame potential, then for μ_ϕ-a.e. $z \in J(F)$, the local dimension $d_{\mu_\phi}(z)$ exists and is equal to $h_{\mu_\phi}/\chi_{\mu_\phi}$. In particular*

$$\mathrm{HD}(\mu_\phi) = \frac{h_{\mu_\phi}}{\chi_{\mu_\phi}}.$$

Proof. In view of Birkhoff's ergodic theorem there exists a Borel set $X \subset J(F)$ such that $\mu_\phi(X) = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} \log|(F^n)'(x)| = \chi_{\mu_\phi} \text{ and } \lim_{n\to\infty} \frac{1}{n} S_n\phi(x) = \int \phi d\mu_\phi \tag{4.1}$$

for every $x \in X$. Fix $x \in X$ and $\varepsilon > 0$. There then exists $k \geq 1$ such that

$$\left|\frac{1}{n} \log|(F^n)'(x)| - \chi_{\mu_\phi}\right| < \varepsilon \tag{4.2}$$

for every $n \geq k$. Fix $r \in (0, \delta)$ and let $n = n(r) \geq 0$ be the largest integer such that

$$B(x, r) \subset F_x^{-n}\Big(B(F^n(x), \delta)\Big). \tag{4.3}$$

Then $B(x, r)$ is not contained in $F_x^{-(n+1)}(B(F^{n+1}(x), \delta))$ and it follows from the $\frac{1}{4}$-Koebe's distortion theorem that

$$r \geq \frac{1}{4}\delta|(F^{n+1})'(x)|^{-1}. \tag{4.4}$$

Taking $r > 0$ sufficiently small, we may assume that $n \geq k$. Applying Theorem 4.2 and utilizing (4.3) along with Lemma 4.1, we get that

$$\begin{aligned} m_\phi(B(x,r)) &\leq \int_{B(F^n(x),\delta)} \exp\Big(S_n\phi \circ F_x^{-n} - \mathrm{P}(\phi)n\Big) dm_\phi \\ &\leq T\exp\Big(S_n\phi(x) - \mathrm{P}(\phi)n\Big) m_\phi\Big(B(F^n(x), \delta)\Big) \leq T\exp\Big(S_n\phi(x) - \mathrm{P}(\phi)n\Big). \end{aligned}$$

Applying now (4.4) and (4.2), we obtain

$$
\begin{aligned}
\frac{\log m_\phi(B(x,r))}{\log r} &\ge \frac{\log T + S_n\phi(x) - \mathrm{P}(\phi)n}{\log r} \ge \frac{\log T + S_n\phi(x) - \mathrm{P}(\phi)n}{\log\delta - \log 4 - \log|(F^{n+1})'(x)|} \\
&\ge \frac{\log T + S_n\phi(x) - \mathrm{P}(\phi)n}{\log\delta - \log 4 - (\chi_{\mu_\phi} - \varepsilon)(n+1)}.
\end{aligned}
$$

Dividing now the numerator and the denominator of the last quotient by $n = n(r)$, letting $r \searrow 0$ (which implies that $n(r) \to \infty$) and using the second part of (4.1, we therefore get that

$$
\underline{d}_{m_\phi}(x) \ge \frac{-\int \phi d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}.
$$

Since, by Theorem 4.3, the measures μ_ϕ and m_ϕ are equivalent with positive continuous Radon-Nikodym derivatives , we obtain for all $x \in$ that

$$
\underline{d}_{\mu_\phi}(x) \ge \frac{-\int \phi d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}. \tag{4.5}
$$

For every $s > 0$, let $J_s = \{z \in J(F) : |\mathrm{Re}(z)| \le s\}$. Take M so large that $\mu_\phi(J_M) > 0$. Since the measure m_ϕ is positive on non-empty open subsets of $J(F)$, we get that

$$
W := \inf\{m_\phi(B(z,\delta) : z \in J_M\} > 0.
$$

In view of ergodicity of the measure μ_ϕ and Birkhoff's ergodic theorem, there exists a Borel set $Y \subset X$ such that $\mu_\phi(Y) = 1$ and

$$
\lim_{n\to\infty} \frac{1}{n} S_n\Big(1\!\!1_{J_M}\Big)(x) = \mu_\phi(J_M) > 0
$$

for all $x \in Y$. In particular, if $\{n_j\}_{j=1}^\infty$ is the unbounded increasing sequence of all integers $n \ge 1$ such that $F^n(x) \in J_M$, then

$$
\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1. \tag{4.6}
$$

Keep $x \in Y$ and let $l \ge 0$ be the least integer such that

$$
B(x,r) \supset F_x^{-i}\Big(B(F^i(x),\delta)\Big)
$$

for all $i \ge l$. Taking $r > 0$ small enough, we may assume that $l > \max\{k, n_1\}$. There then exists a unique $j \ge 2$ such that

$$
n_{j-1} < l \le n_j. \tag{4.7}
$$

Also $F_x^{-(l-1)}\Big(B(F^{l-1}(x),\delta)\Big)$ is not contained in $B(x,r)$, and it therefore follows from Koebe's distortion theorem that

$$
r \le K\delta|(F^{l-1})'(x)|^{-1}. \tag{4.8}
$$

It follows from the definition of l and formula (4.7) along with Lemma 4.1 that

$$\begin{aligned}
m_\phi(B(x,r)) &\geq m_\phi\Big(F_x^{-n_j}\Big(B(F^{n_j}(x),\delta)\Big)\Big) = \int_{B(F^{n_j}(x),\delta)} \exp\Big(S_{n_j}\phi\circ F_x^{-n_j} - \mathrm{P}(\phi)n_j\Big)dm_\phi \\
&\geq T^{-1}\exp\Big(S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j\Big)m_\phi\Big(B(F^{n_j}(x),\delta)\Big) \\
&\geq WT^{-1}\exp\Big(S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j\Big).
\end{aligned}$$

Applying now (4.2), (4.4) and (4.7), we obtain

$$\begin{aligned}
\frac{\log m_\phi(B(x,r))}{\log r} &\leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log r} \leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log(K\delta) - \log|(F^{l-1})'(x)|} \\
&\leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log(K\delta) - (\chi_{\mu_\phi}+\varepsilon)(l-1)} \leq \frac{\log W - \log T + S_{n_j}\phi(x) - \mathrm{P}(\phi)n_j}{\log(K\delta) - (\chi_{\mu_\phi}+\varepsilon)n_{j-1}}.
\end{aligned}$$

Dividing now the numerator and the denominator of the last quotient by n_{j-1} letting $r\searrow 0$ (which implies that $n_{j-1}\to\infty$) and using the second part of (4.1) along with (4.6), we therefore get that

$$\overline{d}_{m_\phi}(x) \leq \frac{-\int\phi d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}.$$

Since, by Theorem 4.3, the measures μ_ϕ and m_ϕ are equivalent with positive continuous Radon-Nikodym derivatives , we obtain for all $x\in Y$ that

$$\overline{d}_{\mu_\phi}(x) \leq \frac{-\int\phi d\mu_\phi + \mathrm{P}(\phi)}{\chi_{\mu_\phi}}.$$

Since, by Theorem 4.4, $\mathrm{P}(\phi) - \int\phi d\mu_\phi = \mathrm{h}_{\mu_\phi}$, combining this inequality with (4.5), completes the proof. ■

We shall prove now the existence of a partition with the properties required by condition (b) of the definition of loosely Markov maps.

Lemma 4.8. *If $f:\mathbb{C}\to\mathbb{C}$ is hyperbolic exponential map and $\phi:J(F)\to\mathbb{C}$ is a 1^+-tame potential, then there exists a countable partition α of $J(F)$ (consisting of Borel sets) such that $h_{\mu_\phi}(T,\alpha)>0$ and $\alpha_n(x)\supset B(x,e^{-\kappa(x)n})$ for μ_ϕ-a.e. $x\in J(F)$, some constant $\chi(x)>0$ and all $n\geq 1$ large enough.*

Proof. It was proven in [16] that if μ is a Borel probability measure on the cylinder Q, then for every $\xi>0$ there exists a partition α of Q with $\mathrm{diam}(\alpha)\leq\xi$ and such for every $\beta>0$

$$\sum_{n=0}^{\infty}\sum_{A\in\alpha}\mu\Big(B(\partial A, e^{-\beta n})\Big) < \infty. \tag{4.9}$$

Let us apply now this fact with $\mu = \mu_\phi$. Since $\mathrm{h}_{\mu_\phi} > 0$, there exists $\xi > 0$ so small that $\mathrm{h}_{\mu_\phi}(T, \alpha) > 0$. Since the measure μ_ϕ is F-invariant, it follows from (4.9) that

$$\sum_{n=0}^{\infty}\sum_{A\in\alpha}\mu\Big(F^{-n}\Big(B(\partial A, e^{-\beta n})\Big)\Big) = \sum_{n=0}^{\infty}\sum_{A\in\alpha}\mu\Big(B(\partial A, e^{-\beta n})\Big) < \infty. \tag{4.10}$$

Fix now $\varepsilon > 0$. It follows from Birkhoff's ergodic theorem, (4.10), and the Borel-Cantelli lemma that there exists a Borel set $X_\varepsilon \subset J(F)$ such that $\mu_\phi(X_\varepsilon) = 1$ and for every $x \in X_\varepsilon$ there exists $n(x) \geq 1$ such that $F^n(x) \notin B(\partial A, e^{-\beta n})$ and

$$\left|\frac{1}{n}\log|(F^n)'(x)| - \chi_{\mu_\phi}\right| < \varepsilon \tag{4.11}$$

for all $n \geq n(x)$. The former property means that

$$B(F^n(x), e^{-\varepsilon n}) \subset \alpha(F^n(x)). \tag{4.12}$$

So, fix $x \in X_\varepsilon$ and $n \geq n(x)$. There obviously exists $\gamma(x) > 0$ so large that

$$F^q\Big(B(x, e^{-\gamma(x)n}\Big) \subset F^q\Big(B(x, e^{-\gamma(x)n(x)}\Big) \subset \alpha(F^q(x)). \tag{4.13}$$

for all integers $q = 0, 1, \dots, n(x)$. It follows from the $\frac{1}{4}$-Koebe's distortion theorem that

$$F_x^{-n}\Big(B(F^n(x), \delta)\Big) \supset B\left(x, \frac{1}{4}\delta|(F^n)'(x)|^{-1}\right).$$

Consequently, all the maps F^q, $0 \leq q \leq n$, restricted to the ball $B\left(x, \frac{1}{4}\delta|(F^n)'(x)|^{-1}\right)$ are univalent. If now $q \in \{n(x), n(x)+1, \dots, n\}$, then applying (4.11), we get

$$B\Big(x, 8^{-1}\delta\exp\Big(-(\chi_{\mu_\phi} + 3\varepsilon)n\Big)\Big) \subset B\Big(x, 8^{-1}\delta|(F^n)'(x)|^{-1}\Big),$$

and it therefore follows from Koebe's distortion theorem along with (4.11) that,

$$\begin{aligned}
F^q\Big(B\Big(x, 8^{-1}\delta\exp\Big(-(\chi_{\mu_\phi}+3\varepsilon)n\Big)\Big)\Big) &\subset B\Big(F^q(x), 8^{-1}\delta K|(F^q)'(x)|\exp\Big(-(\chi_{\mu_\phi}+3\varepsilon)n\Big)\Big) \\
&\subset B\Big(F^q(x), 8^{-1}\delta K\exp\Big((\chi_{\mu_\phi}+\varepsilon)q\Big)\exp\Big(-(\chi_{\mu_\phi}+3\varepsilon)n\Big)\Big) \\
&\subset B\Big(F^q(x), 8^{-1}\delta K\exp\Big(\chi_{\mu_\phi}(q-n)\exp(-2\varepsilon n)\Big) \\
&\subset B\Big(F^q(x), 8^{-1}\delta K)\exp(-2\varepsilon n)\Big) \subset B\Big(F^q(x), \exp(-\varepsilon n)\Big) \subset B\Big(F^q(x), \exp(-\varepsilon q)\Big),
\end{aligned} \tag{4.14}$$

where the second last inclusion sign was written assuming that n is large enough (depending on ε). Since $\chi_{\mu_\phi} > 0$, we can find $\varepsilon > 0$ so small that $\chi_{\mu_\phi} + 3\varepsilon \leq 2\chi_{\mu_\phi}$. Then, for all $n \geq n(x)$ large enough we have that $8^{-1}\delta K\exp\Big(-(\chi_{\mu_\phi}+3\varepsilon)n\Big) \geq \exp\Big(-3\chi_{\mu_\phi}n\Big)$. It therefore follows from (4.14) that

$$F^q\Big(B\Big(x, \exp\Big(-3\chi_{\mu_\phi}n\Big)\Big) \subset B\Big(F^q(x), \exp(-\varepsilon q)\Big)$$

for all n large enough and all $q \in \{n(x), n(x)+1, \dots, n\}$. Combining this along with (4.12) and (4.13), we see that

$$F^q\Big(B\Big(x, e^{-\kappa(x)n}\Big)\Big) \subset \alpha(F^q(x))$$

for all $n \geq 1$ and all $0 \leq q \leq n$, where $\kappa(x) = \max\{\gamma(x), 3\chi_{\mu_\phi}\}$. Hence, $B\Big(x, e^{-\kappa(x)n}\Big) \subset \alpha_n(x)$ and we are done. ■

We are now in position to conclude the main result of this section.

Theorem 4.9. *If $f : \mathbb{C} \to \mathbb{C}$ is a hyperbolic exponential map and $\phi : J(F) \to \mathbb{C}$ is a 1^+-tame potential, then for μ_ϕ-a.e. $z \in J(F)$, $R_{\mu_\phi}(z)$ exists and moreover*

$$R_{\mu_\phi}(z) = \mathrm{HD}(\mu_\phi) = \frac{h_{\mu_\phi}}{\chi_{\mu_\phi}}.$$

Proof. We shall verify that the dynamical system (F, μ_ϕ) is loosely Markov. Indeed, condition (a) is guaranteed by appropriate results from [15]. Condition (b) is just Lemma 4.8, and condition (c) was established in Theorem 4.7. Thus the proof is completed by applying Theorem 2.1 along with Theorem 4.7. ■

5. Conformal Infinite Iterated Function Systems

Let us describe first the setting of conformal (infinite) iterated function systems introduced in [9]. Let I be a countable index set or alphabet with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric space X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$, for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We define the limit set, J, of this system as the image of the coding space under a coding map as follows. Let I^n denote the space of words of length n, I^∞ the space of infinite sequences of symbols in I, $I^* = \bigcup_{n\geq 1} I^n$ and for $\omega \in I^n$, $n \geq 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \geq 1$, converge to zero and since they form a decreasing family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = J_S = \pi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the Open Set Condition (OSC) if there exists a nonempty open set $U \subset X$ (in the topology of X) such that

$\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. (We do not exclude the possibility that $\overline{\phi_i(U)} \cap \overline{\phi_j(U)} \neq \emptyset$.)

An iterated function system S satisfying the Open Set Condition is said to be conformal if $X \subset I\!R^d$ for some $d \geq 1$ and the following conditions are satisfied.

(1a): $U = \mathrm{Int}_{I\!R^d}(X)$.

(1b): There exists an open connected set V such that $X \subset V \subset I\!R^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V. (Note that for $d = 1$ this just means that all the maps ϕ_i, $i \in I$, are C^1 monotone diffeomorphisms, for $d \geq 2$ the words C^1 conformal mean holomorphic or antiholomorphic, and for $d > 2$ the maps ϕ_i, $i \in I$ are Möbius transformations. The proof of the last statement can be found in [1] for example, where it is called Liouville's theorem.)

(1c): There exist $\gamma, l > 0$ such that for every $x \in X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.

(1d): Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets C of V

$$\mathrm{diam}(\phi_\omega(C)) \leq ||\phi'_\omega||\mathrm{diam}(C) \tag{5.1}$$

and

$$\mathrm{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||, \tag{5.2}$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \geq 1$ is a constant depending only on V. Moreover,

$$\mathrm{diam}(\phi_\omega(X)) \geq D^{-1}||\phi'_\omega|| \tag{5.3}$$

and

$$\phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r) \tag{5.4}$$

for every $x \in X$, every $0 < r \leq \mathrm{dist}(X, \partial V)$, and every word $\omega \in I^*$.

Let $\sigma : I^\infty \to I^\infty$ be the shift mapping, i.e. cutting off the firs coordinate. In order to define the dynamical systems we want to deal with in this section we state first the following special case of Theorem 4.4.1 proven in [10].

Theorem 5.1. *If μ is a Borel shift-invariant ergodic probability measure on I^∞, then*

$$\mu\circ\pi^{-1}\Big(\phi_\omega(X)\cap\phi_\tau(X)\Big)=0 \tag{5.5}$$

for all incomparable words $\omega,\tau\in I^*$.

It follows from this theorem that for μ-a.e. $x\in J$ there exists exactly one element $i\in I$ such that $x\in\phi_i(J)$. Putting

$$f(x)=\phi_i^{-1}(x)$$

we obtain a dynamical system $f:J\to J$ defined μ-a.e. on J. Given a subset Y of I^∞ and $i\in I$, put

$$[iY]=\{i\omega\in I^\infty:\omega\in Y\}.$$

Using shift-invariantness of the measure μ, we get for every Borel set $A\subset J$ that

$$\begin{aligned}\mu\circ\pi^{-1}(f^{-1}(A))&=\mu\circ\pi^{-1}\left(\bigcup_{i\in I}\phi_i(A)\right)=\mu\left(\bigcup_{i\in I}\pi^{-1}(\phi_i(A))\right)=\mu\left(\bigcup_{i\in I}[i\pi^{-1}(A)\right)\\&=\mu(\sigma^{-1}(\pi^{-1}(A)))=\mu\circ\pi^{-1}(A).\end{aligned}$$

This means that the measure $\mu\circ\pi^{-1}$ is f-invariant and we can consider the metric dynamical system $(f,\mu\circ\pi^{-1})$. The Lyapunov characteristic exponent $\chi_{\mu\circ\pi^{-1}}(f)$ of this system is defined by the formula

$$\chi_{\mu\circ\pi^{-1}}(f)=-\int_{I^\infty}\log|\phi'_{\omega_1}(\pi(\sigma(\omega)))|d\mu\circ\pi^{-1}(\omega)>0.$$

Notice that we could have defined this quantity in terms of the dynamical system $(f,\mu\circ\pi^{-1})$ as this system and the shift map (σ,μ) are isomorphic via the projection map $\pi:I^\infty\to J$. In order to relate the recurrence rates and pointwise dimensions of the dynamical system $(f,\mu\circ\pi^{-1})$, we need to restrict ourselves to a smaller, though fairly large and geometrically significant, class of shift-invariant measures μ generated by the summable Hölder families of functions. Following [14], [6] and [10], we define these families and measures as follows. Fix $\beta>0$ and let $G=\{g^{(i)}:X\to I\!R:i\in I\}$ be a family of continuous functions. For each $n\geq 1$ put

$$V_n(G)=\sup_{\omega\in I^n}\sup_{x,y\in X}\{|g^{(\omega_1)}(\phi_{\sigma(\omega)}(x))-g^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\}\mathrm{e}^{\beta(n-1)},$$

and assume that

$$V_\beta(G)=\sup_{n\geq 1}\{V_n(G)\}<\infty.$$

The collection G is then called a Hölder family of functions (of order β). If in addition

$$\sum_{i\in I}\mathrm{e}^{\sup(g^{(i)})}<\infty,$$

then the family G is called a summable Hölder family of functions of order β. Throughout this section the family F is assumed to be summable Hölder of some order $\beta>0$. Following the classical thermodynamic formalism, we defined the topological pressure of F by setting

$$\mathrm{P}(G)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{|\omega|=n}\exp\left(\sup_X\sum_{j=1}^n g^{\omega_j}\circ\phi_{\sigma^j\omega}\right).$$

Notice that the limit indeed exists since the logarithm of the partition function

$$Z_n(G)=\sum_{|\omega|=n}\exp(\sup(S_\omega(G)))$$

is subadditive, where

$$S_\omega(G)=\sum_{j=1}^n g^{(\omega_j)}\circ\phi_{\sigma^j\omega}.$$

Moreover

$$\mathrm{P}(G)=\inf_{n\ge 1}\left\{\frac{1}{n}\log Z_n(G)\right\}.$$

Now, a Borel probability measure m_G is said to be G-conformal provided it is supported on the limit set J, for every Borel set $A\subset X$

$$m_G(\phi_\omega(A))=\int_A\exp\Big(S_\omega(G)-\mathrm{P}(G)|\omega|\Big)\,dm_G,\quad\forall\omega\in I^* \tag{5.6}$$

and

$$m_G(\phi_\omega(X)\cap\phi_\tau(X))=0 \tag{5.7}$$

for all incomparable $\omega,\tau\in I^*$. In [14] and [6], comp. [10]), we have proved the following.

Theorem 5.2. *If G is a summable Hölder family of functions, then there exists exactly one G-conformal measure m_G. In addition, there exists a unique shift-invariant Borel probability measure $\tilde\mu_G$ on I^∞ such that the measure $\mu_G=\tilde\mu_G\circ\pi^{-1}$ on J is absolutely continuous with respect to m_G. In addition, the Radon-Nikodym derivative $d\mu_G/dm_G$ is uniformly bounded away from zero and infinity.*

The measure μ_G is called the Gibbs state of the Hölder family G. It follows from Theorem 5.2 and the statements above that we may consider the dynamical system (f,μ_G). Our aim is to verify that this dynamical system is loosely Markov. We start with the following.

Proposition 5.3. *If G is a summable Hölder family of functions with finite Lyapunov exponent χ_{μ_G}, then*

$$\lim_{r\to 0}\frac{\log\mu_G(B(x,r))}{\log r}=\frac{h_{\mu_G}(f)}{\chi_{\mu_G}}=\mathrm{HD}(\mu_G).$$

for μ_G-a.e. $x\in J$.

Proof. Define the function $\rho : I^\infty \to I\!R$ by the formula

$$\rho(\omega) = \liminf_{r \to 0} \frac{\log m_G(B(x,r))}{\log r}.$$

We shall show for every $\omega \in I^\infty$, we have

$$\rho(\sigma\omega) \geq \rho(\omega). \tag{5.8}$$

Indeed, using G-conformality of the measure μ_G, we get for every $\omega \in I^\infty$ and every $r \in (0, \mathrm{dist}(X, \partial V)$ that

$$m_G\Big(B(\pi(\omega), ||\phi'_{\omega_1}||r)\Big) \geq m_G\Big(\phi_{\omega_1}\Big(B(\pi(\sigma\omega), r)\Big)\Big) = \int_{B(\pi(\sigma\omega),r)} \exp\Big(g^{(\omega_1)}(\pi(\omega)) - \mathrm{P}(G)\Big) dm_G$$
$$\geq e^{-\mathrm{P}(G)} \exp\Big(\inf\{g^{(\omega_1)}\} m_G\Big(B(\pi(\sigma\omega), r)\Big).$$

Hence, (5.8) follows. Since the measure $\tilde{\mu}_G$ is ergodic, it follows from Birkhoff's ergodic theorem that the function $\rho : I^\infty \to I\!R$ is constant $\tilde{\mu}_G$-a.e., say $\rho(\omega) = \hat{\rho}$ for some $\hat{\rho} > 0$ and all $\omega \in Y \subset I^\infty$ with $\tilde{\mu}_G(Y) = 1$. Put now

$$\underline{\rho}(\omega) = \liminf_{r \to 0} \frac{\log \mu_G(B(x,r))}{\log r}.$$

Since, by Theorem 5.2, the measures μ_G and m_G are equivalent with bounded Radon-Nikodym derivatives, $\underline{\rho} = \rho$, and consequently, $\underline{\rho} = \hat{\rho}$ for all $\omega \in Y$. Since $\mathrm{HD}(\mu_G) = \mathrm{esssup}(\underline{\rho})$ and, since in view of Theorem 4.4.2 in [10] (comp. Theorem 3.1) in [14]), $\mathrm{HD}(\mu_G) = \mathrm{h}_{\mu_G}(f)/\chi_{\mu_G}$, we therefore conclude that

$$\liminf_{r \to 0} \frac{\log \mu_G(B(x,r))}{\log r} = \frac{\mathrm{h}_{\mu_G}(f)}{\chi_{\mu_G}} = \mathrm{HD}(\mu_G).$$

for all $\omega \in Y$. Since it was demonstrated in the proof of Theorem 4.4.2 in [10] (comp. Theorem 3.1) that

$$\limsup_{r \to 0} \frac{\log \mu_G(B(x,r))}{\log r} \leq \frac{\mathrm{h}_{\mu_G}(f)}{\chi_{\mu_G}}$$

for $\tilde{\mu}_G$-a.e $\omega \in I^\infty$, we therefore conclude that

$$\lim_{r \to 0} \frac{\log \mu_G(B(x,r))}{\log r} = \frac{\mathrm{h}_{\mu_G}(f)}{\chi_{\mu_G}} = \mathrm{HD}(\mu_G)$$

for $\tilde{\mu}_G$-a.e $\omega \in I^\infty$. We are done. ■

The iterated function system $\{\phi_i\}_{i \in I}$ is said to satisfy the strong open set condition if $J \cap \mathrm{Int}_{I\!R^d}(X) \neq \emptyset$. In the case when the alphabet I is finite, the the strong open set condition is always satisfied (see [12]) perhaps with a different seed set X. The main result of this section is the following.

Theorem 5.4. *Suppose that $\{\phi_i\}_{i\in I}$ is a conformal iterated function system satisfying the strong open set condition. Suppose also that G is a summable Hölder family of functions with finite Lyapunov exponent χ_{μ_G}. If (f,μ_G) is the corresponding dynamical system, then for μ_G-a.e. $z\in J$, the number $R_{\mu_G}(z)$ exists and moreover*

$$R_{\mu_G}(z)=\mathrm{HD}(\mu_G)=\frac{h_{\mu_G}(f)}{\chi_{\mu_G}}.$$

Proof. We shall verify that the dynamical system (f,μ_G) is loosely Markov. Item (a) follows immediately from Theorem 2.4.6 in [10] and the relation $L_\phi(1\!\!1)(\omega)=\tilde L_\phi(1\!\!1)(\pi\omega)$ established in the middle of the page 128 in [7]. In order to check item (b) put $\chi=\chi_{\mu_G}$. Let $W_1\subset I^\infty$ be the set of all sequences $\omega\in I^\infty$ such that

$$\lim_{n\to\infty}\frac{1}{n}\log|\phi'_{\omega_i}(\pi(\sigma^{i+1}\omega))|=-\chi. \tag{5.9}$$

By Birkhoff's ergodic theorem $\tilde\mu_G(W_1)=1$. The strong open set condition implies the existence of a finite word $\eta\in I^*$ such that $\phi_\eta(X)\subset\mathrm{Int}X$. Let $r=\mathrm{dist}(\phi_\eta(X),\partial X)$. It follows from Birkhoff's ergodic theorem that there exists a Borel set $W_2\subset W_1$ such that $\tilde\mu_G(W_2)=1$ and for every $\omega\in W_2$

$$\lim_{k\to\infty}\frac{n_{k+1}(\omega)}{n_k(\omega)}=1, \tag{5.10}$$

where $\{n_k(\omega)\}_{k=1}^\infty$ is the infinite sequence of all consecutive integers $n\ge 1$ such that $\sigma^n(\omega)\in\{\eta\}\times I^\infty$. In particular $\mu_G(\pi(W_2))=1$. The family

$$\alpha=\{\phi_i(X)\}_{i\in I}.$$

is a partition of J. Fix $x\in\pi(W_2)$, $x=\pi(\omega)$, where $\omega\in W_2$. Using the bounded distortion property and looking at (5.9) and (5.10), we see that for all $n\ge 1$ large enough, $|\phi'_{\omega|_n}(x)|\ge r^{-1}\exp(-2\chi n)$ and $n_{k+1}\le 2n_k$, where $k\ge 1$ is uniquely determined by the requirement that $n_k\le n<n_{k+1}$. We then have

$$\begin{aligned}\alpha_n(x)&=\phi_{\omega|_n}(X)\supset\phi_{\omega|_{n_{k+1}}}(X)\supset\phi_{\omega|_{n_{k+1}}}\Big(B\Big(\pi(\sigma^{n_{k+1}}\omega,r\Big)\Big)\supset B\Big(\pi(\omega),\exp(-2\chi n_{k+1}\Big)\\&\supset B\Big(\pi(\omega),\exp(-4\chi n_k)\Big)\supset B\Big(\pi(\omega),\exp(-4\chi n)\Big).\end{aligned}$$

So, condition (b) of the definition of loosely Markov systems is also satisfied. Condition (c) follows immediately from Proposition 5.3. Thus the proof is completed by applying Theorem 2.1 along with Proposition 5.3. We are done. ■

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