REAL ANALYTICITY OF HAUSDORFF DIMENSION OF FINER JULIA SETS OF EXPONENTIAL FAMILY

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Abstract. We primarily deal with the mappings $f_\lambda(z) = \lambda e^z$, $\lambda \in (0, 1/e)$. We consider the set $J_r(f_\lambda)$ consisting of those points of the Julia set of f_λ that do not escape to infinity under positive iterates of f_λ. Our ultimate result is that the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is real analytic. In order to prove it we develop the thermodynamic formalism of potentials of the form $-t\log|F_\lambda'|$, where F_λ is the natural map associated with f_λ introduced in [UZd]. It includes appropriately defined topological pressure, Perron-Frobenius operators, geometric and invariant generalized conformal measures (Gibbs states). We show that our Perron-Frobenius operators are quasi-compact and we obtain several other properties of these operators. As a byproduct of our main course of reasoning we prove stochastic properties of the dynamical system generated by F_λ and the invariant Gibbs states μ_t such as the Central Limit Theorem and the exponential decay of correlations. We also indicate how to extend our results to the other hyperbolic islands of the family $f_\lambda(z) = \lambda e^z$, $\lambda \neq 0$.

1. Introduction

In this section we continue our investigations of the subsets $J_r(f_\lambda)$ of the Julia sets of maps $f_\lambda(z) = \lambda e^z$, where λ is in a sufficiently small neighbourhood in $\mathbb{C}$ of the interval $(0, 1/e)$ originated in [UZd]. Our main result is that the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is real analytic. In Section 2 we recall the maps F_λ mapping the strip $P = \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) \leq \pi\}$ into itself, we define appropriate in this context topological pressure of the potentials $-t\log|F_\lambda'|$, $t \geq 0$, corresponding Perron-Frobenius operators and generalized conformal (Gibbs) measures. Using the existence of these measures we prove three basic properties of the Perron-Frobenius operators in Lemmas 2.7, 2.8 and 2.9. We end this section with the proof of the uniqueness and ergodicity of conformal measures. In Section 3 we we show that the the Perron-Frobenius operator satisfies the assumptions of the Ionescu-Tulcea and Marinescu theorem. In particular the Perron-Frobenius operator acting on the space of Hölder continuous functions (with a fixed exponent) is quasi-compact and the full description of its spectral properties is provided in Theorem 3.5. This permits us to prove in section 4 the existence and uniqueness of invariant measures equivalent to conformal measures. We also obtain here finer stochastic properties of the dynamical system generated by the map F and these invariant measures such as the Central Limit Theorem and the exponential decay of correlations. The short 5th section is devoted to the presentation of a proof of an appropriated Bowen's formula (Hausdorff

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dimension of the set $J_r(f_\lambda)$ is equal to the unique zero of the pressure function) which also follows from the results proven in [UZd]. Section 6 of our paper concludes the proof of the real analyticity of the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$, $\lambda \in (0, 1/e)$. We observe that in the same way as in [UZi] one can prove that the function $(\lambda, t) \mapsto \mathcal{L}_{\lambda,t}$ is holomorphic in an appropriate domain of parameters λ and t. We then apply Theorem 3.5 and the perturbation theory to complete the proof. In the last section we show how to modify our arguments for the case of the map f_λ having an attracting periodic cycle.

2. Pressure, Perron-Frobenius Operators and Generalized Conformal Measures

Let

$$P = \{z \in \mathbb{C} \colon -\pi < \mathrm{Im}(z) \le \pi\}.$$

and let

$$\pi_0 : \mathbb{C} \to P$$

be the projection given by $\pi_0(z) = w$ if and only if $w \in P$ and $\exp(z) = \exp(w)$. We define the map $F = F_\lambda : P \to P$ we intend to work with by the formula

$$F(z) = \pi_0(f(z)). \tag{2.1}$$

It is easy to see that there exists a neighbourhood U in $\mathbb{C}$ of the interval $(0, 1/e)$ such that for every $\lambda \in U$ there exists a horizontal strip $P_+ = P_{\lambda+}$ with the following properties.

(a) $P_+ = \{z \in P : \mathrm{Re} z \ge \hat{q}_\lambda\}$ for some $\hat{q}_\lambda > 0$.
(b) $F_\lambda^{-1}(P_{\lambda+}) \subset P_{\lambda+} \cap \{z \in \mathbb{C} \colon -\frac{3}{4}\pi \le \mathrm{Im}(z) \le \frac{3}{4}\pi\}$.
(c) $J(F) := J(f) \cap P \subset P_+$.
(d) $\delta := \frac{1}{2}\mathrm{dist}\Big(\{F^n(0) : n \ge 0\}, P_+\Big) > 0$.

It is also easy to see that for each $\lambda \in U$ the exists $r_\lambda > 0$ such that $B(\lambda, r_\lambda) \subset U$ and there exists a set P_+ satisfying the conditions (a)-(d) for all $\lambda \in B(\lambda, r_\lambda)$.

Our main result is the following .

Theorem 2.1. *The function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$, $\lambda \in U$, is real-analytic.*

We fix now $\lambda \in U$ until Section 6 and we start to deal with topological pressure. For every $t \ge 0$ and every $z \in P_+$ define the lower and upper topological pressure respectively by

$$\underline{\mathrm{P}}_z(t) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} \text{ and } \overline{\mathrm{P}}_z(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t}.$$

Since, by condition (d), all the holomorphic inverse branches of F^n are well-defined on a 2δ-neighbourhood of P_+, it follows from Koebe's distortion theorem that $\underline{\mathrm{P}}_z(t)$ and $\overline{\mathrm{P}}_z(t)$ are independent of z and we denote their respective values by $\underline{\mathrm{P}}(t)$ and $\overline{\mathrm{P}}(t)$. Let

$$\mathrm{P}_z(1,t) := \sum_{x\in F^{-1}(z)} |F'(x)|^{-t} = \sum_{k=-\infty}^{+\infty} |z+2\pi ik|^{-t}$$

Notice that for every $t>1$ the series $\mathrm{P}_z(1,t)$ converges and $||\mathrm{P}(1,t)||_\infty = \sup_{z\in P_+}\{\mathrm{P}_z(1,t)\} < \infty$. Observe now that for every $n\geq 1$ and every $z\in P_+$

$$\begin{aligned}\sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} &= \sum_{w\in F^{-(n-1)}(z)} \sum_{x\in F^{-1}(w)} |(F^{n-1})'(w)|^{-t}|F'(x)|^{-t} \\ &= \sum_{w\in F^{-(n-1)}(z)} |(F^{n-1})'(w)|^{-t} \sum_{x\in F^{-1}(w)} |F'(x)|^{-t} \\ &\leq ||\mathrm{P}(1,t)||_\infty \sum_{w\in F^{-(n-1)}(z)} |(F^{n-1})'(w)|^{-t}.\end{aligned}$$

Therefore, we obtain by induction that $\sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} \leq ||\mathrm{P}(1,t)||_\infty^n$ and consequently

$$\overline{\mathrm{P}}(t) = \overline{\mathrm{P}}_z(t) \leq \log ||\mathrm{P}(1,t)||_\infty \tag{2.2}$$

for all $t>1$. It follows from Hölder's inequality that both functions $\overline{\mathrm{P}}(t)$ and $\underline{\mathrm{P}}(t)$ are convex. Since in addition F is expanding, we get the following.

Proposition 2.2. *Both functions $t\mapsto \overline{\mathrm{P}}(t), \underline{\mathrm{P}}(t)$, $t\in(1,\infty)$, are continuous, strictly decreasing and $\lim_{t\to+\infty}\overline{\mathrm{P}}(t) = -\infty$.*

In this section developing the arguments worked out in [UZd] we construct first conformal measures for the map $F: P\cap J(f)\to P\cap J(f)$. The details are presented for the sake of completeness and the convenience of the reader.

A Borel measure m is called (t,α_t)-conformal (with $t>1$) if for any Borel set $A\subset P$ on which F is injective, we have

$$m(F(A)) = \int_A \alpha_t |F'|^t dm$$

In [UZd] we only considered the case when $\alpha_t = 1$. Let

$$\tilde{P}_M = \left\{ z\in P_+ : -\frac{3}{4}\pi < \mathrm{Im} z < \frac{3}{4}\pi \text{ and } \mathrm{Re} z\leq M \right\}.$$

Consider the preimage $F^{-1}(\tilde{P}_M)$. This set is a union of infinitely many topological disks Q_i contained in the strip $-\frac{\pi}{2} < \mathrm{Im} z < \frac{\pi}{2}$ and

$$\overline{Q_i}\cap\overline{Q_j} = \emptyset.$$

Now, we consider the finite family of disks Q_i^M, which are contained in $\tilde{P}_M$. In this way we obtain the finite iterated function system:

$$\{\phi_i : \tilde{P}_M \to Q_i^M\},$$

where ϕ_i are appropriate holomorphic branches of F^{-1}. Let J_M be the limit set of this system and let m_M be its unique $(t, \exp(\mathrm{P}_M(t)))$-conformal measure, where

$$\mathrm{P}_M(t) \le \overline{\mathrm{P}}(t) < \infty \tag{2.3}$$

is the standard topological pressure for the repeller $F|_{J_M} : J_M \to J_M$ and the potential $-t\log|F'|$.

Remark 2.3. *We have $J_M \subset J_{M+1}$ for all M large enough. In order to see this, take Q_i^M and let Q_i^{M+1}, be the preimage of $\tilde{P}_{M+1}$ under the same holomorphic branch F_*^{-1} of F^{-1}. Then, obviously, $Q_i^{M+1} \supset Q_i^M$. Since $F(Q_i^{M+1} \setminus Q_i^M) \subset \{z \in \tilde{P}_{M+1} : M < \mathrm{Re}z \le M+1\}$ and since the derivative of F_*^{-1} on $\{z \in \tilde{P}_{M+1} : M < \mathrm{Re}z \le M+1\}$ is bounded from above by $C_1 M^{-1}$, we conclude that* $\mathrm{diam}(Q_i^{M+1} \setminus Q_i^M) \le C_2 M^{-1}$ *for some appropriate constants C_1 and C_2. Since $Q_i^M \subset \{\mathrm{Re}z \le M\}$, this implies that*

$$Q_i^{M+1} \subset \{\mathrm{Re}z \le M+1\}$$

for all M large enough. Hence, each $Q_i^{M+1} \supset Q_i^M$ is (see the definition) used in the construction of J_{M+1}. Thus, the corresponding limit set J_{M+1} contains J_M.

Proposition 2.4. *If $t > 1$, then the sequence of measures m_M, $M \in \mathbb{N}$ is tight, i.e. for every $\epsilon > 0$ there exists M so large that for every N*

$$m_N(\{z \in P_+ : \mathrm{Re}z > M\}) < \epsilon.$$

Proof. Fix $\epsilon > 0$, $M > 0$ and $N \ge \hat{q}$. We shall estimate separately the measure m_N of two sets, which cover $\{z \in P_+ : \mathrm{Re}z > M\}$. First, we have

$$m_N(\{x \in J_N : \mathrm{Re}F(x) \ge M\}) = \sum_{k\in\boldsymbol{Z}} m_N(\{x \in J_N : f(x) \in [M,N] \times (-\pi,\pi] + 2k\pi i).$$

If $x \in J_N$ and $f(x) \in [M,N] \times [-\pi,\pi] + 2k\pi i\}$, then

$$|F'(x)| = |f(x)| \ge \frac{1}{2}(M + \pi|k|) \ge \frac{1}{2}(M + |k|)$$

which gives

$$\begin{aligned} m_N(\{x : \mathrm{Re}F(x) \ge M\}) &\le 2\sum_{k=0}^{\infty} m_N(\{x : M \le \mathrm{Re}x \le N\}) \cdot \frac{2^t e^{-\mathrm{P}_N(t)}}{(M+k)^h} \\ &\le 2^{t+1} e^{-\mathrm{P}_N(t)} \sum_{k=0}^{\infty} \frac{1}{(M+k)^h}, \end{aligned} \tag{2.4}$$

If $N \le M$, then

$$m_N(\{z \in P_+ : \mathrm{Re}z > M\}) = 0. \tag{2.5}$$

Since, by Remark 2.3, the sequence $N \mapsto \mathrm{P}_N(t)$ is eventually non-decreasing, we see that $\gamma(t) = \sup_N\{-\mathrm{P}_N(t)\} < \infty$. If $N > M$, then it follows from this along with (2.4) and (2.3) that

$$m_N(\{x : \mathrm{Re}F(x) \geq M\}) \leq \frac{2^{t+1}e^{-\mathrm{P}_N(t)}}{t-1}M^{1-t} \leq \frac{2^{t+1}\gamma(t)}{t-1}M^{1-t}. \tag{2.6}$$

Keeping $N > M$ we now estimate the measure of the second set. Namely

$$m_N(\{x : M < \mathrm{Re}x < N \text{ and } \mathrm{Re}F(x) < M\}).$$

If $\mathrm{Re}x > M$, then $|f(x)| > \lambda e^M$ and therefore $|\mathrm{Im}f(x)| \geq \sqrt{\lambda^2 e^{2M} - M^2}$. Hence $|F'(x)| = |f'(x)| = |f(x)| \geq |\mathrm{Im}f(x)| \geq \sqrt{\lambda^2 e^{2M} - M^2}$. Thus,

$$\begin{aligned} m_N(\{x : M < \mathrm{Re}x < N \text{ and } \mathrm{Re}F(x) < M\} &\leq const\,\gamma(t) \sum_{k \geq \sqrt{\lambda^2 e^{2M} - M^2}}^{\infty} k^{-t} \\ &\leq const\frac{\gamma(t)}{t-1}e^{M(1-t)}. \end{aligned} \tag{2.7}$$

Combining this along with (2.5) and (2.6) we obtain

$$m_N(\{x : \mathrm{Re}x > M\}) < \epsilon$$

for all N and all M large enough. ■

Since, by Prop 2.4 the sequence m_N is tight, it follows from Prochorov's theorem that there exists a subsequence m_{N_i} weakly convergent to some limit probability measure $m = m_t$. This is the measure we are looking for. Recall that

$$J(F) = P \cap J(f).$$

The proof of the following theorem requires only minor obvious modifications of the proof of Theorem 3.4 in [UZd].

Theorem 2.5. *The measure m is (t, α_t)-conformal, where $\alpha_t = \lim_{N\to\infty} e^{\mathrm{P}_N(t)}$ and $m(J(F)) = 1$.*

Let $C_b(P_+)$ be the Banach space of all bounded continuous complex-valued functions on P_+. Notice that for each $z \in P_+$ the series

$$\sum_{k=-\infty}^{+\infty} |z + 2\pi ik|^{-t}$$

converges if and only if $\mathrm{Re}t > 1$. This enables us to define for these t's the Perron-Frobenius operator $\mathcal{L} = \mathcal{L}_t : C_b(P_+) \to C_b(P_+)$ by the formula

$$\mathcal{L}_t g(z) = \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} g(x) = \sum_{k=-\infty}^{+\infty} |z + 2\pi ik|^{-t} g(z_k), \tag{2.8}$$

where z_k is the only point of the intersection $P_+ \cap f^{-1}(z+2\pi ik)$. It immediately follows from (2.8) that

$$\mathcal{L}_t g(z) \le ||\mathcal{L}_t 1\!\!1||_\infty ||g||_\infty \text{ and } \lim_{\mathrm{Re} z \to +\infty} \mathcal{L}_t 1\!\!1(z) = 0. \tag{2.9}$$

Notice also that $\mathcal{L}_t : C_b(P_+) \to C_b(P_+)$ is a bounded operator and its norm is equal to $||\mathrm{P}(1,t)||_\infty$. Assume from now on throughout this section that $t \in (1,\infty)$ and consider the dual operator $\mathcal{L}_t^* : C_b^*(P_+) \to C_b^*(P_+)$ given by the formula $\mathcal{L}_t^*\mu(g) = \mu(\mathcal{L}_t g)$. A straighforward calculation (see Proposition 2.2 in [DU1] for example, where the finiteness of the partition can be replaced by its countability) shows the following.

Proposition 2.6. *For every $t > 1$, $\mathcal{L}_t^* m_t = \alpha_t m_t$.*

Fix $t > 1$, put $\mathcal{L} = \mathcal{L}_t$ and define

$$\mathcal{L}_0 = \alpha_t^{-1}\mathcal{L}.$$

For every $x \in I\!\!R$ let

$$P_x = \{z \in P_+ : \mathrm{Re} z \le x\}$$

and let K_x be the distortion constant for holomorphic inverse branches of iterates of F on P_x, i.e.

$$\frac{|(F_*^{-n})'(w)|}{|(F_*^{-n})'(w)|} \le K_x$$

for all $n \ge 1$, all holomorphic inverse branches of F^n defined on 2δ-neighbourhood of P_+ and all $z, w \in P_x$. We shall prove the following.

Lemma 2.7. $Q = \sup_n\{||\mathcal{L}_0^n(1\!\!1)||_\infty\} < \infty$.

Proof. Fix $x \ge \hat{q}$ so large that for every $w \in P_x^c$

$$\alpha_t^{-1} \sum_{k=-\infty}^{+\infty} |w + 2\pi ik|^{-t} \le 1. \tag{2.10}$$

We shall prove by induction that for every $n \ge 0$,

$$||\mathcal{L}_0^n(1\!\!1)||_\infty \le \frac{K_x^t}{m(P_x)}.$$

And indeed, for $n = 0$ this estimate is immediate. So, suppose that it holds for some $n \ge 0$ and let $\hat{z}_{n+1} \in P_+$ be such a point that $\mathcal{L}_0^{n+1}(1\!\!1)(\hat{z}_{n+1}) = ||\mathcal{L}_0^{n+1}(1\!\!1)||_\infty$ (such a point exists due to (2.9)). If $\hat{z}_{n+1} \in P_x$, then

$$1 = \int \mathcal{L}_0^{n+1}(1\!\!1)dm \ge \int_{P_x} \mathcal{L}_0^{n+1}(1\!\!1)dm \ge K_x^{-t}||\mathcal{L}_0^{n+1}(1\!\!1)||_\infty m(P_x)$$

and consequently $||\mathcal{L}_0^{n+1}(1\!\!1)||_\infty \le K_x^t\Big(m(P_x)\Big)^{-1}$. If $\hat{z}_{n+1} \notin P_x$, then it follows from (2.10) and the inductive assumption that

$$\begin{aligned}||\mathcal{L}_0^{n+1}(1\!\!1)||_\infty &= \mathcal{L}_0^{n+1}(1\!\!1)(\hat{z}_{n+1}) = \sum_{k=-\infty}^{+\infty} \mathcal{L}_0^n(1\!\!1)\Big((\hat{z}_{n+1})_k\Big)\alpha_t^{-1}|\hat{z}_{n+1}+2\pi ik|^{-t} \\ &\le \sum_{k=-\infty}^{+\infty} \alpha_t^{-1}||\mathcal{L}_0^n(1\!\!1)||_\infty|\hat{z}_{n+1}+2\pi ik|^{-t} \le K_x^t\Big(m(P_x)\Big)^{-1}\alpha_t^{-1}\sum_{k=-\infty}^{+\infty}|\hat{z}_{n+1}+2\pi ik|^{-t} \\ &\le K_x^t\Big(m(P_x)\Big)^{-1}.\end{aligned}$$

We are done. ■

Lemma 2.8. *There exists $x_0 \ge \hat{q}$ such that for every $x \ge x_0$*

$$\inf_{n\ge 0}\sup_{z\in P_x}\{\mathcal{L}_0^n(1\!\!1)(z)\} \ge \frac{1}{4}.$$

Proof. Let Q come from Lemma 2.7. Let x_0 be so large that $m(P_{x_0}^c) \le 1/(4Q)$. Suppose for the contrary that $\mathcal{L}_0^n(1\!\!1)(z) \le 1/(4Q)$ for all $n \ge 0$ and all $z \in P_{x_0}$. Then

$$1 = \int \mathcal{L}_0^n(1\!\!1)dm = \int_{P_{x_0}} \mathcal{L}_0^n(1\!\!1)dm + \int_{P_{x_0}^c} \mathcal{L}_0^n(1\!\!1)dm \le \frac{1}{4}m(P_{x_0}) + Qm(P_{x_0}^c) \le \frac{1}{4} + Q\frac{1}{4Q} = \frac{1}{2}.$$

This contradiction finishes the proof. ■

As an immediate consequence of this lemma (and Koebe's distortion theorem), we get the following.

Lemma 2.9. *For every $x \in [\hat{q}, +\infty)$ we have*

$$\inf_{n\ge 0}\inf_{z\in P_x}\{\mathcal{L}_0^n(1\!\!1)(z)\} \ge \frac{1}{4}\Big(\max\{K_x, K_{x_0}\}\Big)^{-t}.$$

We shall prove the following.

Proposition 2.10. *For every $t > 1$ we have $\underline{\mathrm{P}}(t) = \overline{\mathrm{P}}(t) = \log\alpha_t$.*

Proof. It follows from Lemma 2.7 that $\mathcal{L}_t^n(1\!\!1)(z) \le Q\alpha_t^n$ for every $z \in P_+$. Hence

$$\overline{\mathrm{P}}(t) = \overline{\mathrm{P}}_z(t) = \limsup_{n\to\infty}\frac{1}{n}\mathcal{L}^n(1\!\!1)(z) \le \log\alpha_t.$$

In view of Lemma 2.9, $\mathcal{L}_t^n(1\!\!1)(x_0) \ge K_{x_0}^{-t}\alpha_t^n$ and therefore

$$\underline{\mathrm{P}}(t) = \underline{\mathrm{P}}_z(t) = \liminf_{n\to\infty}\frac{1}{n}\mathcal{L}_t^n(1\!\!1)(z) \ge \log\alpha_t.$$

We are done. ■

Denote the common value of $\underline{\mathrm{P}}(t)$ and $\overline{\mathrm{P}}(t)$ by $\mathrm{P}(t)$.

Let

$$I_\infty(F) = \{z \in P_+ : \lim_{n\to\infty} F^n(z) = \infty\},$$

i.e. $I_\infty(F)$ is the set of points escaping to infinity under forward iterates of F. Analogously define

$$I_\infty(f) = \{z \in P_+ : \lim_{n\to\infty} f^n(z) = \infty\}.$$

Recall that

$$J_r(F) = J(F) \setminus I_\infty(f) \quad \text{and} \quad J_r(f) = J(f) \setminus I_\infty(F)$$

Notice that $I_\infty(f) \cap P = I_\infty(F)$. Let $m = m_t$ be the $(t, e^{\mathrm{P}(t)})$-conformal measure constructed in Theorem 2.5 (due to Proposition 2.10 $\alpha_t = e^{\mathrm{P}(t)}$). We shall prove the following.

Proposition 2.11. *There exists $M > 0$ such that for m-a.e. x*

$$\liminf_{n\to\infty} \mathrm{Re} F^n(x) \le M.$$

In particular, $m(I_\infty(F)) = 0$ or equivalently $m(J_r(F)) = 1$.

Proof. Put

$$Y_M = \{z \in P : \mathrm{Re} z > M\}.$$

Let $B \subset Y_M$ be an arbitrary Borel set. We shall estimate from above the measure $m(B \cap F^{-1}(B))$. We have

$$m(B \cap F^{-1}(B)) \le m(F^{-1}(B)) = \sum_{k\in\mathbf{Z}} m(x : f(x) \in B + 2k\pi i).$$

If $f(x) \in B + 2k\pi i$, then

$$|F'(x) = |f'(x)| = |f(x)| > (M^2 + k^2)^{\frac{1}{2}}.$$

Thus

$$m\Big(\{x : F(x) \in B\}\Big) \le 2\sum_{k=0}^{\infty} m(B) \cdot \frac{e^{-\mathrm{P}(t)}}{(M^2+k^2)^{\frac{t}{2}}} \le const\, m(B) M^{1-t}.$$

Therefore, in particular, one gets

$$m(B \cap F^{-1}(B)) < \frac{C}{M^{t-1}} m(B) \tag{2.11}$$

for every Borel set $B \subset Y_M$ and for some constant C independent of M and B. Since $B \cap F^{-1}(B) \subset Y_M$, one can now use the estimate (2.11) to get inductively

$$m\Big(B \cap F^{-1}(B) \cap \cdots \cap F^{-n}(B)\Big) \le (CM^{1-t})^n m(B)$$

This implies that for all M large enough

$$m(\bigcap_{n=0}^{\infty} F^{-n}(Y_M)) = 0$$

and consequently

$$m(\bigcup_{k=0}^{\infty} F^{-k}(\bigcap_{n=0}^{\infty} F^{-n}(Y_M))) = 0.$$

The proof is finished. ■

Let us show now that the estimates used in Proposition 2.11 and Proposition 2.4 lead to the following.

Corollary 2.12.

$$m(Y_M) \le Ce^{(1-t)M}$$

for some constant C and all $M \ge 0$ large enough.

Proof. It follows from the proof of Proposition 2.11 that

$$m\Big(\{x \in Y_M : F(x) \in Y_M\}\Big) < m(Y_M)CM^{1-t}$$

and from the proof of Proposition 2.4 (formula (2.7) with m_N replaced by m that

$$m\Big(\{x \in Y_M : \mathrm{Re}F(x) \le M\}\Big) \le Ce^{(1-t)M}.$$

These two sets cover the whole set Y_M. The first inequality says that (for all M sufficiently large) the first set covers less than, say, one half of the measure of Y_M. Thus,

$$m(Y_M) \le 2m\Big(\{x \in Y_M : \mathrm{Re}F(x) \le M\}\Big) \le 2Ce^{(1-t)M}$$

and the proof is complete. ■

Theorem 2.13. *The $(t, e^{\mathrm{P}(t)})$-conformal measure $m = m_t$ is a unique (t,β)-conformal measure for F with $t > 1$. In addition it is ergodic with respect to each iterate of F.*

Proof. Suppose that ν is a (t, β^j)-conformal measure for F^j ($j \ge 1$) with some $t > 1$ and $\beta > 0$. The same proof as in the case of the measure m shows that $\nu(I_\infty(F)) = 0$. Let $J_{r,N}(F)$ be the subset of $J_r(F)$ defined as follows: $z \in J_{r,N}(F)$ if the trajectory of z under F^j has an accumulation point in $\{\mathrm{Re}z < N\}$. Obviously, $\bigcup_N J_{r,N}(F) = J_r(F)$ and by Proposition 2.11 there exists $M' > 0$ such that $\nu(J_{r,M}(F)) = m(J_{r,M'}(F)) = 1$.

Fix $z \in J_{r,N}(F)$ and let us recall that $\delta = \mathrm{dist}(\{P_+, \{f^n(0)\}\})/2$. Then there exist $y \in J(F)$ such that $\mathrm{Re}y < N$ and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that $y = \lim_{k\to\infty} F^{jn_k}(z)$. Considering for k large enough the sets $F_z^{-jn_k}(B(y,\delta))$ and $F_z^{-jn_k}(B(y,\delta/K))$, where $F_z^{-jn_k}$ is the holomorphic inverse branch of F^{n_k} defined on $B(y, 2\delta)$ and sending $F^{jn_k}(z)$ to z, using conformality of measures m and ν along with Koebe's distortion theorem we easily deduce that

$$B_N(\nu)^{-1}\beta^{-jn_k}|(F^{jn_k})'(z)|^{-t} \le \nu\Big(B_N(z, c|(F^{jn_k})'(z)|^{-1})\Big) \le B_N(\nu)\beta^{-jn_k}|(F^{jn_k})'(z)|^{-t} \tag{2.12}$$

for all $k \geq 1$ large enough, where $c \asymp 1$, $K \geq 1$ is the constant appearing in the Koebe's distortion theorem and ascribed to the scale $1/2$, $B_N(\nu)$ is some constant depending on ν and N. Let M be fixed as above. Fix now E, an arbitrary bounded Borel set contained in $J_r(F)$ and let $E' = E \cap J_{r,M'}(F)$. Since m is regular, for every $x \in E'$ there exists a radius $r(x) > 0$ of the form from (2.12) (and the corresponding number $n(x) = n_k(x)$ for an appropriate k) such that

$$m\Big(\bigcup_{x \in E'} B(x, r(x)) \setminus E'\Big) \leq \epsilon. \tag{2.13}$$

Now by the Besicovič theorem (see [G]) we can choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^{\infty}$ with $r(x_i) \leq \epsilon$ and $jn(x_i) \geq \epsilon^{-1}$, from the cover $\{B(x, r(x))\}_{x \in E'}$ of E', of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore, assuming $e^{\mathrm{P}(t)} < \beta$ and using (2.12) along with (2.13), we obtain

$$\begin{aligned}
\nu(E) = \nu(E') &\leq \sum_{i=1}^{\infty} \nu(B(x_i, r(x_i)))\beta^{-jn(x_i)} \leq B_M(\nu) \sum_{i=1}^{\infty} r(x_i)^t \beta^{-jn(x_i)} \\
&\leq B_M(\nu) B_M(m) \sum_{i=1}^{\infty} m(B(x_i, r(x_i)))\beta^{-jn(x_i)} e^{\mathrm{P}(t)jn(x_i)} \\
&\leq B_M(\nu) B_M(m) C m\Big(\bigcup_{i=1}^{\infty} B(x_i, r(x_i))\Big)\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{jn(x_i)} \\
&\leq B_M(\nu) B_M(m) C m\Big(\bigcup_{i=1}^{\infty} B(x_i, r(x_i))\Big)\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{\epsilon^{-1}} \\
&\leq C B_M(\nu) B_M(m)\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{\epsilon^{-1}}(\epsilon + m(E')) \\
&= C B_M(\nu) B_M(m)\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{\epsilon^{-1}}(\epsilon + m(E)).
\end{aligned} \tag{2.14}$$

Hence letting $\epsilon \searrow 0$ we obtain $\nu(E) = 0$ and consequently $\nu(J(F)) = 0$ which is a contradiction. We obtain a similar contradiction assuming that $\beta < e^{\mathrm{P}(t)}$ and replacing in (2.14) the roles of m and ν. Thus $\beta = e^{\mathrm{P}(t)}$ and letting $\epsilon \searrow 0$ again, we obtain from (2.14) that $\nu(E) \leq CB_M(\nu)B_M(m)m(E)$. Exchanging m and ν, we obtain $m(E) \leq CB_M(\nu)B_M(m)\nu(E)$. These two conclusions along with the already mentioned fact that $m(J_r(F)) = \nu(J_r(F)) = 1$, imply that the measures m and ν are equivalent with Radon-Nikodym derivatives bounded away from zero and infinity.

Let us now prove that any $(t, e^{\mathrm{P}(t)})$-conformal measure ν is ergodic with respect to F^j. Indeed, suppose to the contrary that $F^{-j}(G) = G$ for some Borel set $G \subset J(F)$ with $0 < m(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(F)\setminus G}$

$$\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(F)\setminus G}(B) = \frac{\nu(B \cap J(F) \setminus G)}{\nu(J(F) \setminus G)}$$

would be $(t, e^{j\mathrm{P}(t)})$-conformal for F^j and mutually singular. This contradiction finishes the proof. ■

3. Finer Properties of Perron-Frobenius Operators

Fix $\alpha \in (0,1]$. Given $g \in C_b(P_+)$ let

$$v_\alpha = \inf\{L \geq 0 : |g(y) - g(x)| \leq L|y-x|^\alpha \text{ for all } x,y \in P_+ \text{ with } |y-x| \leq \delta\},$$

where $\delta > 0$ was defined in item (d) at the beginning of Section 2 and let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty.$$

Clearly the space

$$\mathrm{H}_\alpha(P_+) = \{g \in C_b(P_+) : ||g||_\alpha < \infty\}$$

equipped with the norm $||\cdot||_\alpha$ is a Banach space densly contained in $C_b(P_+)$ (with respect to the $||\cdot||_\infty$ norm). We shall prove the following.

Lemma 3.1. *If $t > 1$ and $\alpha \in (0,1]$, then there exists a constant $C > 0$ such that*

$$||\mathcal{L}_0^n g||_\alpha \leq \frac{1}{2}||g||_\alpha + C||g||_\infty$$

for all $n \geq 1$ large enough and every $g \in \mathrm{H}_\alpha(P_+)$.

Proof. Fix $g \in \mathrm{H}_\alpha(P_+)$ and $x,y \in \mathrm{P}_+$ with $|y-x| \leq \delta$. For every $n \geq 1$ let $\{F_v^{-n} : B(x,2\delta) \to \mathbb{C}\}_{v \in V_n}$ be the family of all holomorphic inverse branches of F^n defined on the 2δ-neighbourhood of P_+ restricted to $B(x,2\delta)$. Since $|F'(z)| = e^{\mathrm{Re}F(z)}$ it easily follows from Koebe's distortion theorem that there are constants $L > 0$ and $\beta < 1$ such that for every $n \geq 0$, every $z \in P_+$ and every $v \in V_n$, $|(F_v^{-n})'(z)| \leq L\beta^n$. But the key technical fact for our proof is the following version of distortion estimate, whose proof can be found for example in [MU].

$$||(F_v^{-n})'(y)|^t - |(F_v^{-n})'(x)|^t| \leq M_t|(F_v^{-n})'(x)|^t|y-x|$$

for every $n \geq 1$, every $v \in V_n$ and some constant M_t. Using at the last step Lemma 2.7, we can then estimate as follows.

$$
\begin{aligned}
&|\mathcal{L}_0^n g(y) - \mathcal{L}_0^n g(x)| = \\
&= \left| \sum_{v\in V_n} e^{-\mathrm{P}(t)n} g(F_v^{-n}(y))|(F_v^{-n})'(y)|^t - \sum_{v\in V_n} e^{-\mathrm{P}(t)n} g(F_v^{-n}(x))|(F_v^{-n})'(x)|^t \right| \\
&= \Big| \sum_{v\in V_n} e^{-\mathrm{P}(t)n} g(F_v^{-n}(y))\Big(|(F_v^{-n})'(y)|^t - |(F_v^{-n})'(x)|^t\Big) \\
&+ \sum_{v\in V_n} e^{-\mathrm{P}(t)n}|(F_v^{-n})'(x)|^t\Big(g(F_v^{-n}(y)) - g(F_v^{-n}(x))\Big)\Big| \\
&\leq \sum_{v\in V_n} e^{-\mathrm{P}(t)n} g(F_v^{-n}(y)) \left||(F_v^{-n})'(y)|^t - |(F_v^{-n})'(x)|^t\right| \\
&+ \sum_{v\in V_n} e^{-\mathrm{P}(t)n}|(F_v^{-n})'(x)|^t \left|g(F_v^{-n}(y)) - g(F_v^{-n}(x))\right| \\
&\leq \sum_{v\in V_n} e^{-\mathrm{P}(t)n}||g||_\infty M_t|(F_v^{-n})'(x)|^t|y-x| + \sum_{v\in V_n} e^{-\mathrm{P}(t)n}|(F_v^{-n})'(x)|^t v_\alpha(g)|F_v^{-n}(y) - F_v^{-n}(x)|^\alpha \\
&\leq M_t||g||_\infty|y-x|^\alpha \sum_{v\in V_n} e^{-\mathrm{P}(t)n}|(F_v^{-n})'(x)|^t + v_\alpha(g)\sum_{v\in V_n} e^{-\mathrm{P}(t)n}|(F_v^{-n})'(x)|^t L\beta^{\alpha n}|y-x|^\alpha \\
&= \Big(M_t||g||_\infty \mathcal{L}_0^n(1\!\!1)(x) + L\beta^{\alpha n}v_\alpha(g)\mathcal{L}_0^n(1\!\!1)(x)\Big)|y-x|^\alpha \\
&\leq \Big(\beta^{\alpha n}LQ||g||_\alpha + M_tQ||g||_\infty\Big)|y-x|^\alpha
\end{aligned}
$$

Therefore $v_\alpha(\mathcal{L}_0^n g) \leq \beta^{\alpha n}LQ||g||_\alpha + M_tQ||g||_\infty$ and consequently, using Lemma 2.7 again, we get

$$||\mathcal{L}_0^n g||_\alpha \leq \beta^{\alpha n}LQ||g||_\alpha + M_tQ||g||_\infty + ||\mathcal{L}_0^n g||_\infty \leq \beta^{\alpha n}LQ||g||_\alpha + Q(M_t+1)||g||_\infty.$$

Taking now $n \geq 1$ so large that $\beta^{\alpha n}LQ \leq 1/2$ finishes the proof. ∎

Remark 3.2. *As a byproduct of the calculations performed in the proof of Lemma 3.1 we obtain that* $\mathcal{L}_t(\mathrm{H}_\alpha(P_+)) \subset \mathrm{H}_\alpha(P_+)$ *(and* $\mathcal{L}_t : \mathrm{H}_\alpha(P_+)) \to \mathrm{H}_\alpha(P_+)$ *is bounded) for all complex* t *with* $\mathrm{Re} t > 1$.

In order to apply the theorem of Ionescu-Tulcea and Marinescu we also need the following.

Lemma 3.3. *If* B *is a bounded subset of* $\mathrm{H}_\alpha(P_+)$ *(with the* $||\cdot||_\alpha$ *norm), then* $\mathcal{L}_0(B)$ *is a pre-compact subset of* $C_0(P_+)$ *(with the* $||\cdot||_\infty$ *norm).*

Proof. Fix an arbitrary sequence $\{g_n\}_{n=1}^\infty \subset B$. Since, by Lemma 3.1, the family $\mathcal{L}_0(B)$ is equicontinuous, it follows from Ascoli's theorem that we can choose from $\{\mathcal{L}_0(g_n)\}_{n=1}^\infty$ an infinite subsequence $\{\mathcal{L}_0(g_{n_j})\}_{j=1}^\infty$ converging uniformly on compact subsets of P_+ to a function

$\phi \in C(P_+)$. Fix now $\epsilon > 0$. Since B is a bounded subset of $C_b(P_+)$, it follows from (2.9) that there exists $T \ge \hat{q}$ such that $|\mathcal{L}_0 g(z)| \le \epsilon/2$ for all $g \in B$ and all $z \in P_+$ with $\mathrm{Re} z \ge T$. Hence

$$|\phi(z) \le \epsilon/2 \tag{3.1}$$

for all $z \in P_+$ with $\mathrm{Re} z \ge T$. Thus $|\mathcal{L}_0(g_{n_j})(z) - \phi(z)| \le \epsilon$ for all $j \ge 1$ and all $z \in P_+$ with $\mathrm{Re} z \ge T$. In addition, there exists $p \ge 1$ such that $|\mathcal{L}_0(g_{n_j})(z) - \phi(z)| \le \epsilon$ for every $j \ge p$ and every $z \in P_T$. Therefore $|\mathcal{L}_0(g_{n_j})(z) - \phi(z)| \le \epsilon$ for all $j \ge p$ and all $z \in P_+$. This means that $||\mathcal{L}_0(g_{n_j}) - \phi||_\infty \le \epsilon$ for all $j \ge p$. Letting $\epsilon \searrow 0$ we conclude from this and from (3.1) that $\mathcal{L}_0(g_{n_j})$ converges uniformly on P_+ to $\phi \in C_b(P_+)$. We are done. ■

Combining Lemma 2.7, Lemma 3.1 and Lemma 3.3, we see that the assumptions of Theorem 1.5 in [IM] are satisfied with Banach spaces $\mathrm{H}_\alpha(P_+) \subset C_b(P_+)$ and the bounded operator $\mathcal{L}_0 : C_b(P_+) \to C_b(P_+)$ preserving $\mathrm{H}_\alpha(P_+)$. It gives us the following.

Theorem 3.4. *If $t > 1$ then there exist finite numbers $\gamma_1, \dots, \gamma_p \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, finitely many bounded finitely dimensional operators $Q_1, \dots, Q_p : \mathrm{H}_\alpha(P_+) \to \mathrm{H}_\alpha(P_+)$ and an operator $S : \mathrm{H}_\alpha(P_+) \to \mathrm{H}_\alpha(P_+)$ such that*

$$\mathcal{L}_0^n = \sum_{i=1}^{p} \gamma_i^n Q_i + S^n$$

for all $n \ge 1$,

$$Q_i^2 = Q_i,\ Q_i \circ Q_j = 0, (i \ne j),\ Q_i \circ S = S \circ Q_i = 0$$

and

$$||S^n||_\alpha \le C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \ge 1$. In particular all numbers $\gamma_1, \dots, \gamma_p$ are isolated eigenvalues of the operator $\mathcal{L}_0 : \mathrm{H}_\alpha(P_+) \to \mathrm{H}_\alpha(P_+)$.

In fact, we can prove more.

Theorem 3.5. *If $t > 1$ then we have the following.*

(a) *The number 1 is a simple isolated eigenvalue of the operator $\mathcal{L}_0 : \mathrm{H}_\alpha(P_+) \to \mathrm{H}_\alpha(P_+)$.*
(b) *The eigenspace of the eigenvalue 1 is generated by nowhere vanishing function $\phi \in \mathrm{H}_\alpha(P_+)$ such that $\int \phi dm = 1$ and $\lim_{\mathrm{Re} z \to +\infty} \phi(z) = 0$.*
(c) *The number 1 is the only eigenvalue of modulus 1.*
(d) *With $S : \mathrm{H}_\alpha(P_+) \to \mathrm{H}_\alpha(P_+)$ as in Theorem 3.4, we have*

$$\mathcal{L}_0 = Q_1 + S,$$

where $Q_1 : \mathrm{H}_\alpha(P_+) \to \mathbb{C}\phi$ is a projector on the eigenspace $\mathbb{C}\phi$, $Q_1 \circ S = S \circ Q_1 = 0$ and

$$||S^n||_\alpha \le C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \ge 1$.

Proof. Let us show that 1 is an eigenvalue of $\mathcal{L}_0$ and let us identify the eigenfunction claimed in part (b). And indeed, in view of Lemma 3.1, $||\mathcal{L}_0^n(1\!\!1)||_\alpha \le C_1$ for some constant $C_1 > 0$ and all $n \ge 0$. Thus,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} \mathcal{L}_0^j(1\!\!1) \right\|_\alpha = \left\| \mathcal{L}_0 \left(\sum_{j=0}^{n-1} \mathcal{L}_0^j(1\!\!1) \right) \right\|_\alpha \le C_1$$

for every $n \ge 1$. Therefore, it follows from Lemma 3.3 that there exists a strictly increasing sequence of positive integers $\{n_k\}_{k\ge1}$ such that the sequence $\left\{ \frac{1}{n_k} \sum_{j=1}^{n_k} \mathcal{L}_0^j(1\!\!1) \right\}_{k\ge1}$ converges in the Banach space $C_b(P_+)$ to a function $\phi : P_+ \to I\!\!R$. Obviously, $||\phi||_\alpha \le C_1$ and, in particular $\phi \in \mathrm{H}_\alpha$. Since m is a fixed point of the operator conjugate to $\mathcal{L}_0$, $\int \mathcal{L}_0^j(1\!\!1) dm = 1$ for every $j \ge 0$. Consequently, $\int \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}_0^j(1\!\!1) dm = 1$ for every $n \ge 1$. So, applying Lebesgue's dominated convegence theorem along with Lemma 2.7, we conclude that $\int \psi d\tilde{m} = 1$. It immediately follows from Lemma 2.9 that $\phi > 0$ throughout P_+. Since $\phi = \mathcal{L}_0\phi$, it follows from (2.9) that $\lim_{\mathrm{Re} z \to +\infty} \phi(z) = 0$. Thus, in order to complete the proof of the items (a), (b), (c) (that 1 is an isolated eigenvalue of $\mathcal{L}_0 : \mathrm{H}_\alpha(P_+) \to \mathrm{H}_\alpha(P_+)$ follows from Theorem 3.4) it suffices to show that if $\beta \in S^1$ is an eigenvalue of $\mathcal{L}_0 : \mathrm{H}_\alpha(P_+) \to \mathrm{H}_\alpha(P_+)$ and ψ is its eigenfunction, then $\beta = 1$ and $\psi \in C\!\!\!\!I\,\phi$. But this can be done in exactly the same way as the proof of Theorem 35(ii) in [DU2]. The item (d) is now an immediate consequence of Theorem 3.4 and items (a), (b) and (c). ■

4. Invariant Measures

The following theorem immediately follows from Theorem 3.5, Proposition 2.11 and Theorem 2.13.

Theorem 4.1. *If $t > 1$, then the measure $\mu = \mu_t = \phi m_t$ is F-invariant, ergodic with respect to each iterate of F and equivalent with the measure m_t. In particular $\mu(J_r(F)) = 1$.*

Due to Theorem 3.5 the F-invariant measure μ has much finer stochastic properties than ergodicity of all iterates of F. Here these follow.

Theorem 4.2. *The dynamical system (F, μ) is metrically exact i.e., its Rokhlin natural extension is a K-system.*

The proof of this fact is the same as the proof of Corollary 37 in [DU2]. The next two theorems are standard consequences of Theorem 3.5 (see [DPU] and [PU] for example). Let g_1 and g_2 be real square-μ integrable functions on $J_r(F)$. For every positive integer n the n-th correlation of the pair g_1, g_2, is the number

$$C_n(g_1, g_2) := \int g_1 \cdot (g_2 \circ F^n)\, d\mu - \int g_1\, d\mu \int g_2\, d\mu.$$

provided the above integrals exist. Notice that due to the F-invariance of μ we can also write

$$C_n(g_1,g_2)=\int (g_1-Eg_1)\Big((g_2-Eg_2)\circ F^n\Big)\,d\mu,$$

where we write $Eg=\int g\,d\mu$. We have the following.

Theorem 4.3. *There exists $C\ge 1$ and $\rho<1$ such that for all $g_1\in \mathrm{H}_\alpha(P_+), g_2\in L^1(\mu)$*

$$C_n(g_1,g_2)\le C\rho^n\|g_1-Eg_1\|_\alpha\|g_2-Eg_2\|_{L^1}.$$

Let $g:J_r(F)\to R$ be a square-integrable function. The limit

$$\sigma^2(g)=\lim_{n\to\infty}\frac{1}{n}\int\Big(\sum_{j=0}^{n-1} g\circ F^j-nEg\Big)^2\,d\mu$$

is called asymptotic variance or dispersion, provided it exists.

Theorem 4.4. *If $g\in \mathrm{H}_\alpha(\mathrm{P}_+)$, $\alpha\in(0,1)$, then $\sigma^2(g)$ exists and, if $\sigma^2(g)>0$, then the sequence of random variables $\{g\circ F^n\}_{n=0}^\infty$ with respect to the probability measure μ satisfies the Central Limit Theorem, i.e.*

$$\mu\left(\left\{x\in J_r(F):\frac{\sum_{j=0}^{n-1} g\circ F^j-nEg}{\sqrt{n}}<r\right\}\right)\to\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^r e^{-t^2/2\sigma^2}\,dt.$$

5. Bowen's Formula

For every $n\ge 1$ let

$$A_n=\{z\in P_+:\hat q+n-1\le \mathrm{Re}z\le \hat q+n\}$$

We shall prove the following

Lemma 5.1. *If $t>1$ then, $\int\log|F'|d\mu_t<+\infty$.*

Proof. Since $\phi:P_+\to(0,+\infty)$ is bounded (even more $\lim_{\mathrm{Re}z\to+\infty}\phi(z)=0$), applying Corollary 2.12, we obtain

$$\begin{aligned}\int\log|F'|d\mu &\preceq \int\log|F'|dm=\sum_{n=1}^\infty\int_{A_n}\log|F'|dm\\ &\le\sum_{n=1}^\infty m(\Big(Y_{\hat q+n-1}\Big)\log\Big(|\lambda|\exp(\hat q+n)\Big)\\ &\le\sum_{n=1}^\infty C\exp\Big((1-t)(\hat q+n-1)\Big)\Big(\log|\lambda|+\hat q+n\Big)<+\infty.\end{aligned}$$

It is well-known (see [PU] for example) that $h_M = \mathrm{HD}(J_M)$ is the unique zero of the pressure function $t \mapsto \mathrm{P}_M(t)$. Since $\mathrm{HD}(J_r(F)) = \lim_{n\to\infty} \mathrm{HD}(J_M)$ (see [UZd]) and since $\mathrm{HD}(J_r(F)) > 1$ (also see [UZd]), there exists M so large that $h_M = \mathrm{HD}(J_M) > 1$. Then $\mathrm{P}(h_M) \geq \mathrm{P}_M(h_M) > 0$. Since, in addition $\mathrm{P}(h_M) < \infty$, we conclude from Proposition 2.2 and Proposition 2.10 that there exists a unique $h > 1$ such that $\mathrm{P}(h) = 0$. So, the measure m_h is h-conformal in a sense of [UZd]. It therefore follows from Theorem 4.5 of this paper that $h = \mathrm{HD}(J_r(F))$. Below we provide a direct independent proof of this result.

Theorem 5.2. $\mathrm{HD}(J_r(F)) = h$, *the unique zero of the pressure function* $t \mapsto \mathrm{P}(t)$.

Proof. Given $k \geq 1$ let

$$X_k = \{z \in P_+ : \liminf_{n\to\infty} \mathrm{Re}(F^n(z)) < k\}.$$

Choose an arbitrary point $z \in P_k$. Fix $t > h$ and $n \geq 1$ so large that

$$\frac{1}{j} \log \sum_{x\in F^{-j}(z)} |(F^j)'(x)|^{-t} \leq \frac{1}{2}\mathrm{P}(t) \quad (\text{notice that } \mathrm{P}(t) < 0).$$

for all $j \geq n$. Since

$$X_k \subset \bigcup_{j=n}^{\infty} \bigcup_{x\in F^{-j}(z)} F_x^{-j}(P_k),$$

where F_x^{-j} is the holomorphic branch of F^j defined on the 2δ-neighbourhood of P_+ and sending z to x, we conclude that

$$\mathrm{H}^t(X_k) \leq \lim_{n\to\infty} \sum_{j=n}^{\infty} \sum_{x\in F^{-j}(z)} K_k^t |(F^j)'(x)|^{-t} \mathrm{diam}^t(P_k) \leq K_k^t \mathrm{diam}^t(P_k) \lim_{n\to\infty} \sum_{j=n}^{\infty} \exp(\frac{1}{2}\mathrm{P}(t)j) < 0$$

since $\mathrm{P}(t)/2 < 0$. Hence $\mathrm{HD}(X_k) \leq t$. Since $J_r(F) = \bigcup_{k=0}^{\infty} X_k$, this implies that $\mathrm{HD}(J_r(F)) \leq t$. Letting now $t \searrow h$, we conclude that $\mathrm{HD}(J_r(F)) \leq h$. In order to prove the opposite inequality fix $\epsilon > 0$. Since $\mu(J_r(F)) = 1$ and since μ is ergodic F-invariant, it follows from Birkhoff's ergodic theorem and Jegorov's theorem that there exist a Borel set $Y \subset J_r(F)$ and the integer $k \geq 1$ such that $\mu(Y) \geq \frac{1}{2}$ and for every $x \in Y$ and every $n \geq k$

$$\left|\frac{1}{n} \log |(F^n)'(x)| - \chi\right| < \epsilon, \tag{5.1}$$

where $\chi = \int \log |F'| d\mu$ is finite due to Lemma 5.1. Put $\nu = m_{|Y}$. Given $x \in Y$ and $0 < r < \delta$ let $n \geq 0$ be the largest integer such that

$$B(x,r) \subset F_x^{-n}(B(F^n(x),\delta)). \tag{5.2}$$

Then $B(x,r)$ is not contained in $F_x^{-(n+1)}(B(F^{n+1}(x),\delta))$ and applying Koebe's distortion theorem we get

$$r \geq K^{-1}\delta |(F^{n+1})'(x)|^{-1}. \tag{5.3}$$

Taking $r > 0$ sufficiently small, we may assume that $n \geq k$. Combining now (5.2) along with $\mathrm{P}(h) = 0$, Koebe's Distortion Theorem and (5.3), we obtain

$$m(B(x,r)) \leq m(F_x^{-n}(B(F^n(x),\delta))) \asymp |(F_x^{-n})'(x)|^h m(B(F^n(x),\delta))$$
$$\preceq |(F_x^{-n})'(x)|^h \preceq r^h \frac{|(F^{n+1})'(x)|^h}{|(F^n)'(x)|^h}$$

Employing now (5.1), we thus get

$$\mu(B(x,r)) \leq r^h e^{(\chi+\epsilon)(n+1)} e^{-(\chi-\epsilon)n} \simeq r^h e^{2\epsilon n} \tag{5.4}$$

But we deduce from (5.1) and (5.3) that $r \succeq e^{-(\chi+\epsilon)(n+1)}$ and therefore $e^{2\epsilon n} \succeq r^{-\frac{2\epsilon}{\chi+\epsilon}}$. This and (5.4) imply that $\nu(B(x,r)) \leq m(B(x,r)) \preceq r^{h-\frac{2\epsilon}{\chi+\epsilon}}$. Consequently $\mathrm{HD}(J_r(F)) \geq \mathrm{HD}(\nu) \geq h - \frac{2\epsilon}{\chi+\epsilon}$ and letting $\epsilon \to 0$ we finally obtain $\mathrm{HD}(J_r(F)) \geq h$. We are done. ■

6. Real Analyticity of the Hausdorff Dimension

In this section we prove Theorem 2.1, our main result in this paper. Fix $\lambda_0 \in U$. Similarly as Theorem 2.8 in [UZi] we can demonstrate that the function $(\lambda,t) \mapsto \mathcal{L}_{\lambda,t} \in L(H_1(P_+))$, $(\lambda,t) \in B(\lambda_0, r_{\lambda_0}) \times \{z \in \mathbb{C} : \mathrm{Re} z > 1\}$ is holomorphic, where $L(H_1(P_+))$ is the space of all linear bounded self operators of $H_1(P_+)$. The main modification is that instead of $f(i\omega)$ and $g(i\omega)$ we take $|(F_x^{-1})'(w)|^t$ and $g(F_x^{-1}(w))$ for $z \in P_+$, $w \in B(z,\delta)$ and $x \in F^{-1}(z)$. So, z corresponds to the word ω and x to i. Since, in view of Theorem 3.5, for every $(\lambda,t) \in B(\lambda_0, r_{\lambda_0}) \times (1,\infty)$ the operator $\mathcal{L}_{\lambda,t} : H_1(P_+) \to H_1(P_+)$ has a simple isolated eigenvalue $e^{\mathrm{P}_\lambda(t)}$, applying the perturbation theory for linear operators (see [Ka]), we conclude that there exists an open set $W \subset B(\lambda_0, r_{\lambda_0}) \times \{z \in \mathbb{C} : \mathrm{Re} z > 1$ containing $\lambda_0 \times (1,+\infty)$ and an analytic function $\alpha : W \to \mathbb{C}$ ascribing simple isolated eigenvalues for operators $\mathcal{L}_{(\lambda,t)}$ and such that $\alpha(\lambda,t) = e^{\mathrm{P}_\lambda(t)}$ for all $\lambda \in B := W \cap p_2^{-1}((1,+\infty))$, where $p_2 : C \times \mathbb{C} \to \mathbb{C}$ is the projection onto the second coordinate. Therefore the function $(\lambda,t) \mapsto \mathrm{P}_\lambda(t)$, $(\lambda,t) \in B$ is real-analytic. Since, by Theorem 5.2, $\mathrm{P}_\lambda(h_\lambda) = 0$, where $h_\lambda = \mathrm{HD}(J_{\lambda,r}(F))$, in order to conclude the proof it suffices to show that $\frac{\partial \mathrm{P}_\lambda(t)}{\partial t}(\lambda,t) \neq 0$ for all $(\lambda,t) \in B$. So fix $(\lambda,t) \in B$. Let C and β have the same meaning as in the proof of Lemma 3.1. Fix $z \in P_+$. Since for every $u \geq 0$ and every $n \geq 1$

$$\sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-(t+u)} = \sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-u} |(F_\lambda^n)'(x)|^{-t} \leq C C^{-u} \beta^{-un} \sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-t},$$

we conclude that $\mathrm{P}_\lambda(t+u) - \mathrm{P}_\lambda(t) \leq -u \log \beta$ which implies that $\frac{\partial \mathrm{P}_\lambda(t)}{\partial t}(\lambda,t) \leq -\log\beta < 0$. We are done. ■

7. General hyperbolic case

In this section we show how to modify our arguments for the case of the map f_λ having an attracting periodic cycle. We decided to write the details of the proof for the particular case of the attracting fixed point because the dynamics is very simple in this case. On the other hand, the extension of the arguments for the general hyperbolic case is rather straightforward, but it requires some extra information about the structure of the Julia set (see [BD]). The construction of (t, α_t) conformal measures can be generalized to this case exactly in the same way as it was indicated in [UZd] (where the case $\alpha_t = 1$ was cosidered). For reader's convenience we sketch it briefly: In what follows we rely on the description given in [BD], we also use the notation of this paper. We assume that $z_0, z_1, \dots, z_n = z_0$ is an attracting cycle of $f = f_{\lambda_0}$. Assume that the singular value 0 is contained in the domain A_1, the immediate basin of attraction of z_1. The topological disc $B_{n+1} \subset A_1$ is chosen so that $0, z_1 \in B_{n+1}$, $\mathrm{cl} f^n(B_{n+1}) \subset B_{n+1}$. Then B_n is defined as $B_n = f^{-1}(B_{n+1})$. Notice that B_n contains some half-plane $\mathrm{Re} z < -M$ and $z_0 \in B_n$. The sets $B_{n-1}, B_{n-2}, \dots, B_1$ are defined as consecutive preimages of of B_n along the appropriate branch of f^{-i} (so that $z_i \in B_{n-i}$). Notice that for $i > n$ B_{n-i} is a simply-connected unbounded set (a finger-in the notation of [BD]). The set B_0 is a complement of a union of infinitely many "fingers" F_i (obviously, $F_i = F_0 + 2k\pi i$) (see Fig 3. in [BD]). Here, $P + 2k\pi i$ is the image of P under the map $z \to z + 2k\pi i$. To build an appropriate dynamics, we fix one component ("finger") F_0 of the complement of B_0. Let

$$P = F_0 \setminus \pi^{-1}(\bigcup_{i=1}^{n-1} B_i)$$

where π is the natural projection $\pi : \bigcup F_i \to F_0$. If $n = 1$ then we have simply $P = F_0$. So,

$$f(P) \supset \bigcup_k (P + 2k\pi i)$$

and, actually, modifying the set P slightly, we can require that $f(P) \supset \mathrm{cl} \bigcup_k (P + 2k\pi i)$.

Now, $F : P \cap f^{-1}(\pi^{-1}(P)) \to P$ is defined as $F = \pi \circ f$. Let

$$J(F) = \{z \in P : F^n \text{ is defined for all n}\}$$

One can easily see that

$$J(f) \cap P = J(F).$$

Now, the construction of (t, α_t) conformal measures as well as the definiton of topological pressure can be extended to this case in a straightforward way; the strip P_+ should be replaced by the domain P. In order to study the dependence of $\mathrm{HD}(J_r)$ on a parameter, we need to modify our construction to obtain a domain which is backward invariant not only with respect to f_{λ_0} but also with respect to f_λ for λ close to λ_0. First, we choose a topological disc $\tilde{B}_1$ with smooth boundary such that $0, z_1 \in \tilde{B}_1$ and $\mathrm{cl} f^n(\tilde{B}_1) \subset \tilde{B}_1$ (e.g. $\tilde{B}_1 = B_{n+1}$). Take a slightly smaller topological disc B such that $\mathrm{cl} f^n(\tilde{B}_1) \subset B \subset \mathrm{cl} B \subset \tilde{B}_1$. Next, $\tilde{B}_n$ is defined as $f^{-1}(B)$. $\tilde{B}_n$ contains some half-plane $\mathrm{Re} z < -M$, $z_n \in \tilde{B}_n$ and $\tilde{B}_n$ is invariant by the shift $z \mapsto z + 2\pi i$. Moreover, if $n > 2$, one can find disjoint topological discs $\tilde{B}_2, \dots, \tilde{B}_{n-1}$ such that $z_i \in \tilde{B}_i$, $(i = 2, \dots, n-1)$ and $\mathrm{cl} f(\tilde{B}_i) \subset \tilde{B}_{i+1}$, $i = 1, \dots, n-1$ (and, obviously, $\mathrm{cl} f(\tilde{B}_n) \subset \tilde{B}_1$ by the construction). Now, let $\pi : \mathbb{C} \to Q$ be the natural projection of $\mathbb{C}$ onto the cylinder Q given by the relation $z \approx w$ iff $z - w = 2k\pi i$ for some $k \in Z$ and let

$$\tilde{P} = \{x \in Q : \pi^{-1}(x) \cap (\tilde{B}_1 \cup \tilde{B}_2 \dots \cup \tilde{B}_n) = \emptyset\}$$

The map $\tilde{F} = \tilde{F}_{\lambda_0} : \tilde{P} \to Q$, $\tilde{F} = \pi \circ f \circ \pi^{-1}$ is well-defined because f is periodic with period $2\pi i$. Also, the derivative $\tilde{F}'$ has its natural meaning. Moreover, we have $\mathrm{cl}\tilde{F}^{-1}(\tilde{P}) \subset \tilde{P}$. Let $J(\tilde{F}) = \{x \in \tilde{P} : \text{all iterates } \tilde{F}^k(x) \text{ are defined}\}$. It is easy to see that $\pi^{-1}(J(\tilde{F})) = J(f)$. Notice that, given λ_0 and the domain $\tilde{P} = \tilde{P}(\lambda_0)$ constructed as above, we can find a neighbourhood U of λ_0 such that for all $\lambda \in U$ we have $\mathrm{cl}\tilde{F}_\lambda^{-1}(\tilde{P}) \subset \tilde{P}$. This enables us to define the Perron-Frobenius operator $\mathcal{L} = \mathcal{L}_t : C_b(\tilde{P}) \to C_b(\tilde{P})$ and to study its dependence on the parameter λ exactly in the same way as it was done in Section 2. Notice that we have removed the neighbourhood of 0, so for every t with $\mathrm{Re}t > 1$ and every $x \in \tilde{P}$ the sum $\sum_{w:\tilde{F}(w)=x} \frac{1}{|\tilde{F}'|^t(w)}$ is finite. We omit the details and summarize the results in the following.

Theorem 7.1. *The results of Sections 2-6 hold for every λ such that f_λ has an attracting periodic orbit with the domain P_+ replaced by $\tilde{P}$.*

In particular, we have

Theorem 7.2. *Let $\lambda_0 \in \mathbb{C}$ be such that the map $f_{\lambda_0}(z) = \lambda_0 \exp(z)$ has an attracting periodic orbit. Then there exists a neighbourhood U of λ_0 in which the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is real-analytic.*

Remark 7.3. *Actually, to build a conformal measure one can work only with the domain $\tilde{P}$ and the map $\tilde{F}$ defined on the annulus Q (but the construction becomes slightly more delicate). We decided to use the domain P (depending on λ) because this allows us to repeat word by word the construction given for the fixed point case.*

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