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# RIGIDITY OF CONNECTED LIMIT SETS OF CONFORMAL IFS

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ABSTRACT. We consider infinite conformal iterated function systems in the phase space $I\!R^d$ with $d \geq 3$. Let $J$ be the limit set of such a system. Under a mild technical assumption which is always satisfied if the system is finite, we prove that either the Hausdorff dimension of $J$ exceeds 1 or else the closure of $J$ is a proper compact segment of either a geometric circle or a straight line.

## 1. Introduction and preliminaries

In this paper we explore the structure of limit sets $J$ of infinite conformal iterated function systems whose closure is a continuum (compact connected set). Under a natural easily verifiable technical condition (always satisfied if the system is finite), we demonstrate the following dichotomy. Either the Hausdorff dimension of $J$ exceeds 1 or else $\overline{J}$ is a proper compact segment of either a geometric circle or a straight line if $d \geq 3$ or an analytic interval if $d = 2$ (comp. Theorem 1.3). From the viewpoint of conformal dynamics, this result can be thought of as a far going generalization of results originated in [Su] and [Bo] which are formulated in the plane case. The proofs contained there use the Riemann mapping theorem and can be carried out only in the plane. The proof presented in our paper is different and holds in any dimension. The reader is also encouraged to notice an analogy between our result and a series of other papers (see for ex. ([Bo], [FU], [MU2], [Ma], [Pr], [Ru], [Su], [U1], [UV], [Z1], [Z2]) which are aimed toward establishing a similar dichotomy. However, to our knowledge, all these results as those in [Bo] and [Su] were formulated in the plane and used the Riemann mapping theorem, except those in [MU2]. The current result is however much stronger than that in [MU2] and in particular with our present approach the main result of [MU2] can be strengthen as described at the end of this section. Another corollary of our result is the following: if a continuum $C$ in $I\!R^d$ is the self-conformal set generated by finitely many conformal mappings satisfying the open set condition, the Hausdorff 1-measure of $C$ is finite and one of the mappings is a similarity, then the continuum is a line-segment. In particular, this holds if all the maps are similarities, a result obtained early on by Mattila [Ma].

To start the preliminaries, let $I$ be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system $S$ is uniformly contractive. Any such collection $S$ of contractions is called an iterated function system. We are particularly interested in the

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properties of the limit set defined by such a system. We define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\geq 1} I^n$, the space of finite words, and for $\tau \in I^n$, $n \geq 1$, let $\phi_\tau = \phi_{\tau_1} \circ \phi_{\tau_2} \circ \cdots \circ \phi_{\tau_n}$. Let $I^\infty = \{\{\tau_n\}_{n=1}^\infty\}$ be the set of all infinite sequences of elements of $I$. If $\tau \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of $\tau$, we denote by $\tau|_n$ the word $\tau_1\tau_2\ldots\tau_n$. Since given $\tau \in I^\infty$, the diameters of the compact sets $\phi_{\tau|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\tau|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\tau)$, we define the coding map

$$\pi : I^\infty \to X.$$

The main object in the theory of iterated function systems is the limit set defined as follows.

$$J = \pi(I^\infty) = \bigcup_{\tau \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\tau|n}(X)$$

Observe that $J$ satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if $I$ is finite, then $J$ is compact and this property fails for infinite systems. Let $S(\infty)$ be the set of limit points of all sequences $x_i \in \phi_i(X)$, $i \in I'$, where $I'$ ranges over all infinite subsets of $I$. In [MU1] the following has been proved

**Proposition 1.1.** *If* $\lim_{i\in I} \operatorname{diam}(\phi_i(X)) = 0$, *then* $\overline{J} = J \cup \bigcup_{\omega\in I^*} \phi_\omega(S(\infty))$.

An iterated function system $S$ is said to be *conformal* if $X \subset I\!R^d$ for some $d \geq 1$ and the following conditions are satisfied.

- **(1a):** Open Set Condition (OSC). $\phi_i(\mathrm{Int}X) \cap \phi_j(\mathrm{Int}X) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.
- **(1b):** There exists an open connected set $V$ such that $X \subset V \subset I\!R^d$ such that all maps $\phi_i$, $i \in I$, extend to $C^1$ conformal diffeomorphisms of $V$ into $V$. (Note that for $d = 1$ this just means that all the maps $\phi_i$, $i \in I$, are $C^1$ monotone diffeomorphisms, for $d = 2$ the words *conformal* mean holomorphic or antiholomorphic, and for $d \geq 3$, the maps $\phi_i$, $i \in I$ are Möbius transformations. The proof of the last statement can be found in [BP] and [Va] for example, where it is called Liouville's theorem)
- **(1c):** (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, u, \alpha) \subset \mathrm{Int}(X)$ with vertex $x$, the symmetry axis determined by vector $u \in I\!R^d$ of length $l$ and a central angle of Lebesgue measure $\alpha$. Here $\mathrm{Con}(x, u, \alpha) = \{y : 0 < (y - x, u) \leq \cos\alpha ||y - x|| \leq l\}$.
- **(1d):** (Bounded Distortion Property (BDP)). There exists $K \geq 1$ such that

$$|\phi'_\tau(y)| \leq K|\phi'_\tau(x)|$$

for every $\tau \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\tau(x)|$ means the norm of the derivative.

Under these assumptions it was shown in [MU1] that the hypothesis of Proposition 1.1 holds and we can change the order of the union and intersection operations to obtain:

$$J = \pi(I^\infty) = \bigcap_{n\geq 1} \bigcup_{|\tau|=n} \phi_\tau(X).$$

In fact throughout the whole paper we will need one additional condition which (comp. [MU1]) can be considered as a strengthening of (BDP).

**(1e):** There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big| |\phi_i'(y)| - |\phi_i'(x)| \Big| \leq L ||\phi_i'|| |y-x|^\alpha.$$

for every $i \in I$ and every pair of points $x, y \in V$.

We remark that in the case $d \geq 3$ the conditions (1d) and (1e) are always satisfied, the latter with $\alpha = 1$.

Let us first collect some geometric consequences of (BDP). We have for all words $\tau \in I^*$ and all convex subsets $C$ of $V$

$$\mathrm{diam}(\phi_\tau(C)) \leq ||\phi_\tau'|| \mathrm{diam}(C) \tag{1.1}$$

and

$$\mathrm{diam}(\phi_\tau(V)) \leq D ||\phi_\tau'||, \tag{1.2}$$

where the norm $||\cdot||$ is the supremum norm taken over $V$ and $D \geq 1$ is a universal constant. Moreover,

$$\mathrm{diam}(\phi_\tau(J)) \geq D^{-1} ||\phi_\tau'|| \tag{1.3}$$

and

$$\phi_\tau(B(x,r)) \supset B(\phi_\tau(x), K^{-1} ||\phi_\tau'|| r), \tag{1.4}$$

for every $x \in X$, every $0 < r \leq \mathrm{dist}(X, \partial V)$, and every word $\tau \in I^*$.

Let us state now an important geometrical feature of conformal systems related to the bounded distortion property. A detailed proof of this fact can be obtained by a slight improvement of Lemma 6 in [MU2].

**Lemma 1.2.** *For every $\beta > 0$ and every $0 < \alpha < \beta$ there exists $\eta > 0$ such that for every $x \in X$, every $u \in I\!R^d$ with $||u|| \leq \eta$ and every $\omega \in I^*$*

$$\phi_\omega(Con(x,u,\alpha)) \subset \mathrm{Con}(\phi_\omega(x), 2\phi_\omega'(x)u, \beta).$$

Let us now recall from [MU1] that a Borel probability measure $m$ is said to be $t$-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \neq j$. It has been proved in [MU1] that if a $t$-conformal measure exists, then $t = h$, the Hausdorff dimension of the limit set $J_S$ of $S$ and this measure is unique. The system $S$ is called regular if a conformal measure exists. The main result of our paper is the following.

**Theorem 1.3.** *If $d \geq 3$, $S = \{\phi_i\}_{i \in I}$ is a conformal IFS, $\overline{J}$ is a continuum (compact connected) and $\dim_H(S(\infty)) < \dim_H(J)$, then either*

**(a):** $\dim_H(J) > 1$ *or*

**(b):** $\overline{J}$ *is a proper compact segment of either a geometric circle or a straight line.*

*In addition, if any one of the maps $\phi_i$ is a similarity mapping, then $\overline{J}$ is a line segment.*

We note that the technical condition in Theorem 1.3 is necessary. Example 5.2 of [MU1] shows that the dichotomy of Theorem 1.3 in general fails if $\dim_H(S(\infty)) \geq \dim_H(J)$. We also mention that having the first part of this theorem proven, the "in addition" part follows immediately from the proof of Lemma 2.5.
We would also like to remark that in the case $d = 2$, for every $i \in I$, $\phi_{ii}$ is a holomorphic map bi-holomorphically conjugate with the linear map $\psi(z) = x_{ii} + \phi'(x_{ii})(z - x_{ii})$ on some neighbourhood $W$ of $x_{ii}$. Proceeding then similarly as in the proof of Theorem 1.3 we could demonstrate the same statement with the segment of the line or the circle replaced by an analytic arc.

Since in the finite case the set $S(\infty)$ is empty, we get immediately from Theorem 1.3 the following.

**Corollary 1.4.** *If $d \geq 2$, $S = \{\phi_i\}_{i \in I}$ is a finite conformal IFS and $\overline{J}$ is a continuum, then either*

**(a):** $\dim_H(J) > 1$ *or*

**(b):** $\overline{J}$ *is a proper compact segment of either a geometric circle or a straight line.*

*In addition, if any one of the maps $\phi_i$ is a similarity mapping, then $\overline{J}$ is a line segment.*

We note that with the methods of this paper one can strengthen the theorem placed on p.88 of [MU2] which concerns conformal repellers, by replacing the words "smooth Jordan curve" by geometric circle if $d \geq 3$ and a real-analytic Jordan curve if $d = 2$.

## 2. **Proof of Theorem 1.3**

The proof of this theorem will consist of several steps. First of all we assume from now on throughout the entire paper that the assumptions of Theorem 1.3 are satisfied and $\dim_H(J) = 1$. Our goal is to show that then the item (b) is satisfied. Since $\dim_H(S(\infty)) < \dim_H(J) = 1$ and $\overline{J}$ is a continuum, using Proposition 1.1, we conclude that $\mathcal{H}^1(J) > 0$. It therefore follows from Theorem 4.16 in [MU1] that the system $S$ is regular. Let $m$ be the corresponding 1-dimensional measure. By Lemma 4.2 in [MU1] and since $\dim_H(S(\infty)) < \dim_H(J) = 1$ the 1-dimensional Hausdorff measure $\mathcal{H}^1$ on $\overline{J}$ is absolutely continuous with respect to $m$ and $\frac{d\mathcal{H}^1}{dm}$

is uniformly bounded away from infinity. So, $\overline{J}$ is a continuum whose $\mathcal{H}^1$ measure is finite. Therefore, the following fact follows from [EH] and [Wh].

**Lemma 2.1.** *$\overline{J}$ is a locally arcwise connected continuum.*

Given $x \in I\!R^d$, $\theta \in I\!PI\!R^d$, and $\gamma > 0$, we put

$$\mathrm{Con}(x,\theta,\gamma) = \mathrm{Con}(x,\eta,\gamma) \cup \mathrm{Con}(x,-\eta,\gamma),$$

where $\eta \in I\!R^d$ is a representative of $\theta \in I\!PI\!R^d$. We recall that a set $Y$ has a tangent in the direction $\theta \in I\!PI\!R^d$ at a point $x \in Y$ if for every $\gamma > 0$

$$\lim_{r\to 0} \frac{\mathcal{H}^1\Big(Y \cap (B(x,r) \setminus \mathrm{Con}(x,\theta,\gamma))\Big)}{r} = 0.$$

Since we will consider only tangents of 1-sets (the set $\overline{J}$ above, this definition coincides with the definition given on p. 31 of [Fa]). Following [MU2] we say that a set $Y$ has a strong tangent in the direction $\theta \in I\!PI\!R^d$ at a point $x$ provided for each $0 < \beta \le 1$ there is some $r > 0$ such that $Y \cap B(x,r) \subset \mathrm{Con}(x,\theta,\beta)$. In [MU2] we have proved the following.

**Theorem 2.2.** *If $Y$ is locally arcwise connected at a point $x$ and $Y$ has a tangent $\theta$ at $x$, then $Y$ has strong tangent $\theta$ at $x$.*

We call a point $\tau \in I^\infty$ transitive if $\omega(\tau) = I^\infty$, where $\omega(\tau)$ is the $\omega$-limit set of $\tau$ under the shift transformation $\sigma : I^\infty \to I^\infty$. We denote the set of these points by $I_t^\infty$ and put

$$J_t = \pi(I_t^\infty).$$

We call the $J_t$ the set of transitive points of $J$ and notice that for every $\tau \in I_t^\infty$, the set $\{\pi(\sigma^n\tau) : n \ge 0\}$ is dense in $J$ or ($\overline{J}$ if this is the space under consideration).

**Lemma 2.3.** *If $\overline{J}$ has a strong tangent at a point $x = \pi(\tau)$, $\tau \in I^\infty$, then $\overline{J}$ has a strong tangent at every point $\overline{\pi(\omega(\tau))}$.*

*Proof.* Suppose on the contrary that $\overline{J}$ does not have a strong tangent at some point $y \in \overline{\pi(\omega(\tau))}$. Let $\theta \in I\!PI\!R^d$ be the tangent direction of $\overline{J}$ at $x$ and let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers such that $\lim_{k\to\infty} \pi(\sigma^{n_k}\tau) = y$. Passing to a subsequence, we may assume that

$$\lim_{k\to\infty} \frac{\left(\phi^{-1}_{\omega|_{n_k}}\right)'(x)}{|\left(\phi^{-1}_{\omega|_{n_k}}\right)'(x)|}\theta = \xi$$

for some $\xi \in I\!PI\!R^d$. Since $\overline{J}$ does not have a strong tangent at $y$, there exists $0 < \beta \le 1$ such that for every $r > 0$

$$\overline{J} \cap B(y,r) \setminus \overline{J} \cap \mathrm{Con}(y,\xi,\beta) \ne \emptyset.$$

Then

$$\overline{J} \cap B(\pi(\sigma^{n_k}\tau),r) \setminus \overline{J} \cap \mathrm{Con}(\pi(\sigma^{n_k}\tau),\xi_k,\beta/2) \ne \emptyset \tag{2.1}$$

for all $k$ large enough where

$$\xi_k = \frac{\left(\phi^{-1}_{\omega|_{n_k}}\right)'(x)}{|\left(\phi^{-1}_{\omega|_{n_k}}\right)'(x)|}\theta.$$

But in view of Lemma 1.2 applied for $\phi^{-1}_{\omega|_{n_k}}$ we see that for all $r > 0$ small enough the following holds.

$$\begin{aligned}
\phi_{\omega|_{n_k}}\Big(B(\pi(\sigma^{n_k}\tau), r) \setminus \mathrm{Con}(\pi(\tau), \xi_k, \beta/2)\Big) \subset \\
&\subset B\Big(x, r||\phi'_{\omega|_{n_k}}||\Big) \setminus \mathrm{Con}\left(x, \frac{\phi'_{\omega|_{n_k}}(\pi(\sigma^{n_k}\tau))}{|\phi'_{\omega|_{n_k}}(\pi(\sigma^{n_k}\tau))|}\xi_k, \frac{\beta}{4}\right) \\
&= B\Big(x, r||\phi'_{\omega|_{n_k}}||\Big) \setminus \mathrm{Con}(x, \theta, \beta/4).
\end{aligned}$$

Since in view of (2.1), $\overline{J} \cap \phi_{\omega|_{n_k}}\Big(B(\pi(\sigma^{n_k}\tau), r) \setminus \mathrm{Con}(\pi(\sigma^{n_k}\tau), \xi_k, \beta/2)\Big) \neq \emptyset$, we conclude that for every $k$ large enough, $\overline{J} \cap \Big(B\Big(x, r||\phi'_{\omega|_{n_k}}||\Big) \setminus \mathrm{Con}(x, \theta, \beta/4)\Big) \neq \emptyset$. Since $\lim_{k\to\infty} ||\phi'_{\omega|_{n_k}}|| = 0$, this implies that $\theta$ is not the strong density direction of $\overline{J}$ at $x$. This contradiction finishes the proof. ■

**Corollary 2.4.** *The continuum $\overline{J}$ has a strong tangent at every point.*

*Proof.* Since $\mathcal{H}^1(\overline{J}) < \infty$, in view of Corollary 3.15 from [Fa], $\overline{J}$ has a tangent at $\mathcal{H}^1$-a.e. point in $\overline{J}$, and therefore at a set of points of positive $m$ measure. Since $m(J_t) = 1$, there thus exists at least one transitive point $x$ in $J$ having a tangent of $J$. By Theorem 2.2 and Lemma 2.1, $\overline{J}$ has a strong tangent at $x$, and it then follows from Lemma 2.3 that $\overline{J}$ has a strong tangent at every point. The proof is complete. ■

Now, the following lemma finishes the proof.

**Lemma 2.5.** *Suppose that $\phi : \overline{I\!R^d} \to \overline{I\!R^d}$, $d \geq 3$, is a conformal diffeomorphism that has an attracting fixed point $a$ ($\phi(a) = a$, $|\phi'(a)| < 1$). Suppose that a compact connected set $M$ has a strong tangent at $a$, that $\phi(M) \subset M$ and that $\lim_{n\to\infty} \phi^n(x) = a$ for all $x \in M$. Then $M$ is a segment of a $\phi$-invariant line or circle. If $\phi$ is affine ($\phi(\infty) = \infty$), then the former possibility holds.*

*Proof.* Since $a$ is an attracting fixed point of $\phi$, there exists a radius $r > 0$ so small that $\phi^{-1}\Big(\overline{R}^d \setminus B(a, r)\Big) \subset \overline{R}^d \setminus B(a, r)$, where $\overline{R}^d$ is the Alexandrov compactification of $I\!R^d$ done by adding the point at infinity. Since $\overline{R}^d \setminus B(a, r)$ is a topological closed ball, in view of Brouwer fixed point theorem there exists a fixed point $b$ of $\phi^{-1}$ in $\overline{R}^d \setminus B(a, r)$. Hence $b$ is also a fixed point of $\phi$ and $b \neq a$. Then the map

$$\psi = i_{b,1} \circ \phi \circ i_{b,1}$$

($i_{b,1}$ equals identity if $b = \infty$) fixes $\infty$ which means that this map is affine, and $w = i_{b,1}(a)$ is an attracting fixed point of $\psi$. In addition $\psi(\tilde{M}) \subset \tilde{M}$, where $\tilde{M} = i_{b,1}(M)$, $w \in \tilde{M}$, and $\tilde{M}$ has a strong tangent at $w$. Let $l$ be the line through $w$ determined by the strongly tangent direction of $\tilde{M}$ at $w$. Since $\psi(w) = w$, since $\psi(l)$ is a straight line through $w$ and since $\psi(\tilde{M}) \subset \tilde{M}$, we conclude that $\psi(l) = l$. Suppose now that $\tilde{M}$ is not contained in $l$. Consider $x \in \tilde{M} \setminus l$. Then for every $n \geq 0$

$$\psi^n(x) \in \psi(\tilde{M}) \setminus \psi(l) \subset \tilde{M} \setminus l$$

and since the map $\psi$ is conformal and affine

$$\angle(\psi^n(x) - w, l) = \angle(\psi^n(x - w), \psi^n(l)) = \angle(x - w, l).$$

Since $\lim_{n\to\infty} \psi^n(x) = w$, we therefore conclude that $l$ is not a strongly tangent line of $\tilde{M}$ at $w$. This contradiction shows that $\tilde{M} \subset l$. Since in addition $\tilde{M}$ is a continuum, it is a segment of $l$. We are done. ■

And indeed to conclude the proof of Theorem 1.3 it suffices to pick an arbitrary index $i \in I$ (affine if exists) and to put $\phi = \phi_i$, $M = \overline{J}$ and $a = x_i$, the only attracting fixed point of $\phi_i$ belonging to $J$.

## References


[Bo] R. Bowen, Hausdorff dimension of quasi-circles, Pub. Math. IHES 50 (1979).

[Fa] K. Falconer, The geometry of fractal sets, Cambridge University Press, 1986.

[FU] A. Fisher, M. Urbański, On invariant line fields, Bull. London Math. Soc. 32, no. 5 (2000), 555-570.

[Ma] P. Mattila, On the structure of self-similar fractals, Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 189-192.

[MU1] R. D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc. (3) 73 (1996) 105-154.

[MU2] R. D. Mauldin, M. Urbański, Jordan curves as repellors, Pacific Journal of Math. 166 (1994), 85-97.

[Ma] V. Mayer, Comparing measures and invariant line fields, Preprint 1999.

[Pr] F. Przytycki, On holomorphic perturbations of $z \to z^n$, Bull. Polish Acad. Sci. Math. 24(1986), 127-132.

[PUZ] F. Przytycki, M. Urbański, and A. Zdunik, Harmonic, Gibbs and Hausdorff measures on repellors for holomorphic maps, I, Ann. Math. 130(1989), 1-40.

[Ru] D. Ruelle, Repellors for real analytic maps, Ergod. Th. & Dyn. Sys. 2(1982), 99-107.

[Su] D. Sullivan, Seminar on conformal and hyperbolic geometry by D.P.Sullivan (Notes by M. Baker and J. Seade), preprint IHES (1982).

[U1] M. Urbański, On the Hausdorff dimension of a Julia set with a rationally indifferent periodic point, Studia Math. 97(1991), 167-188.

[U3] M. Urbański, Rigidity of multi-dimensional conformal iterated function systems, Preprint Warwick 2000.

[UV] M. Urbański, A. Volberg, A rigidity theorem in complex dynamics, in Fractal Geometry and Stochastics, Progress in Probability 37, Birkhäuser Verlag (1995).

[Wh] G.T. Whyburn, Concerning continua or finite degree and local separating points, Amer. J. Math., 57 (1935), 11-18.

[Z1] A. Zdunik, Parabolic orbifolds and the dimension of the maximal measure for rational maps, Invent. Math. 99(1990), 627-649.

[Z2] A. Zdunik, Harmonic measure versus Hausdorff measures on repellors for holomorphic maps, Trans. Amer. Math. Soc. 326(1991), 633-652.

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