REAL ANALYTICITY OF TOPOLOGICAL PRESSURE FOR PARABOLICALLY SEMIHYPERBOLIC GENERALIZED POLYNOMIAL-LIKE MAPS

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Abstract. For arbitrary parabolically semihyperbolic generalized polynomial-like maps f, we prove that on a certain interval, which contains the interval $(0, \mathrm{HD}(J(f)))$, the pressure function $t \mapsto \mathrm{P}(-t \log |f'|)$ is real-analytic. Our results generalize the work of Makarov and Smirnov in [3] and [7].

1. Introduction and Statement of Results

In this paper we consider parabolically semihyperbolic generalized polynomial-like maps f. We show that on a certain interval which contains the interval $(0, \mathrm{HD}(J(f))$, the associated topological pressure $\mathrm{P}(-t \log |f'|)$ is real-analytic as a function in t. Here, $\mathrm{HD}(J(f))$ refers to the Hausdorff dimension of the corresponding Julia set $J(f)$. Roughly speaking, we obtain these results by showing how to associate to f some finitely primitive conformal graph directed Markov system. This then allows to use a result of [5], which states that for this type of Markov system the pressure function is real-analytic in the relevant range. We remark that our paper extends results by Makarov and Smirnov obtained in [3] [7]. Also, note that our method of associating to f a graph directed Markov system is completely different from the method used in [3] and [7]. That is, we do not have to use the construction of Hofbauer towers, nor do we have to introduce a 'new Riemann metric' in order to force some proper expansion.

Before we state our main result more precisely, we introduce some preliminary concepts and notation. Let $U \subset \mathbb{C}$ be an open Jordan domain with smooth boundary, and let $\mathcal{U} := \bigcup_{i \in I} U_i$ be a finite union of Jordan domains which are fully contained in U and which have pairwise disjoint closures. A generalized polynomial-like mapping (abbreviated as GPL-map) is a map

$$f : \mathcal{U} \to U$$

which has a holomorphic extension to an open neighbourhood of $\mathcal{U}$ such that for each $i \in I$ the restriction of this extension to U_i is a surjective branched covering map.
The set of parabolic periodic points of f is defined by

$$\Omega := \{\omega \in U : f^q(\omega) = \omega \ \text{ and } \ (f^q)'(\omega) = 1 \ \text{ for some } \ q \geq 1\}.$$

Without loss of generality, we shall assume that all parabolic periodic points of f are in fact fixed points of f, and that $f'(\omega) = 1$ for each $\omega \in \Omega$ (this is of course achieved by

Research partially supported by the NSF Grant DMS 0100078.

taking a suitable iterate of f, which does not affect our analysis here, since $\mathrm{P}(-t\log|f'|) = \frac{1}{n}\mathrm{P}(-t\log|(f^n)'|)$, for each $n\in\mathbb{N}$).
Also, we define

$$\mathrm{Crit}(f) := \{c : f'(c) = 0\} \quad \text{and} \quad \mathrm{Crit}(J(f)) := J(f)\cap \mathrm{Crit}(f).$$

It will be convenient to split up the index set I in the following way.

- $I_o := \{i\in I : \overline{U_j}\cap \bigcup_{n\geq 1} f^n(\mathrm{Crit}(f)) = \emptyset\}$ ('post-critical free'),
- $I_p := \{i\in I : \Omega\cap\overline{U_i}\neq\emptyset\}$ ('parabolic'),
- $I_c := \{i\in I : U_i\cap \mathrm{Crit}(f)\neq\emptyset\}$ ('critical'),
- $I_r := I\setminus (I_c\cup I_p)$ ('regular').

Furthermore, we define

$$\mathcal{U}_o := \bigcup_{i\in I_o} U_i,\ \mathcal{U}_p := \bigcup_{i\in I_p} U_i,\ \mathcal{U}_c := \bigcup_{i\in I_c} U_i,\ \mathcal{U}_r := \bigcup_{i\in I_r} U_i.$$

Definition. *A GPL-map f is called parabolically semihyperbolic if and only if the following conditions are satisfied.*

$$(\mathbf{a})\ \ I_c\subset I_o,\quad (\mathbf{b})\ \ \overline{\mathcal{U}_o\cup\mathcal{U}_r}\subset U,\quad (\mathbf{c})\ \ \bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))\subset \mathcal{U}_r.$$

Note that in this definition we do not rule out the possibility that $\Omega=\emptyset$. That is, we let the class of semihyperbolic GPL-maps be contained in the class of parabolically semihyperbolic GPL-maps. Also, recall that a GPL-map f is called non-recurrent if for each $c\in\mathrm{Crit}(J(f))$ we have that $U_i\cap\{f^n(c) : n\geq 1\}=\emptyset$, where $i\in I$ is uniquely determined by the fact that $c\in U_i$. Hence, by (a) in the definition above, a parabolically semihyperbolic GPL-map is always non-recurrent. In fact even more can be said, namely that such a GPL-map is parabolically subhyperbolic and critically tame (see [8] for the definitions). Furthermore, we remark that for a parabolically semihyperbolic GPL-map the sets I_o, I_p and I_r are always pairwise disjoint.
Throughout, we shall assume that if $i\in I_p$, then the map $f: U_i\to U$ is a conformal homeomorphism. It then follows from Schwarz's lemma that $\Omega\cap\overline{U_i}$ is a singleton, denoted by ω_i, and that ω_i is contained in the boundary of U_i. Also, let $f_i^{-1}:\overline{U}\to\overline{U_i}$ refer to the inverse branch of f for which $f_i^{-1}(\omega_i)=\omega_i$. By the Denjoy-Wolf theorem, it follows that $f_i^{-n}(z)$ converges to ω_i uniformly, for $z\in U$. Since f_i^{-1} has an analytic extension to an open neighbourhood of ω_i and since $(f_i^{-1})'(\omega_i)=1$, there exists a Taylor expansion of this extension, which for z close to ω_i, and for some fixed $a_i\neq 0$ and $p_i\in\mathbb{N}$, has the form

$$f_i^{-1}(z) = z + a_i(z-\omega_i)^{p_i+1}+\dots .$$

Using this, we obtain that for each compact set $F\subset U$ there exists a constant $C_F\geq 1$ such that for every $n\geq 1$ and for all $z\in F$ we have that (see e.g [1])

$$C_F^{-1} n^{-\frac{p_i+1}{p_i}} \leq |(f_i^{-n})'(z)| \leq C_F n^{-\frac{p_i+1}{p_i}}. \qquad \text{(LBP)}$$

Furthermore, the following 'critical parameters' will be crucial in our analysis. Recall that for $c \in \mathrm{Crit}(J(f))$, the order $q(c)$ of c is determined by the local behaviour of f around c. That is, for z sufficiently close to c we have for f the Taylor expansion

$$f(z) = f(c) + b_0(z-c)^{q(c)} + \dots . \qquad \text{(LBC)}$$

These 'critical parameters' are the following.

$$\chi_0 := \inf\{\limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(z)| : z \in \Omega \cup \omega(\mathrm{Crit}(f))\},$$
$$\chi(c) := \liminf_{k\to\infty} \frac{1}{k} \log \inf_{n\geq 1}\{|(f^k)'(f^n(c))|\},$$
$$\chi_q := \min\left\{\frac{\chi(c)}{q(c)} : c \in \mathrm{Crit}(f)\right\} \text{ and } \chi := \min\{\chi_q, \chi_0\}.$$

For an introduction and discussion of the topological pressure function P for GPL-maps the reader is referred to Section 2. We can now state the main results of this paper as follows.

Theorem 1.1. *Let f be a parabolically semihyperbolic GPL-map. For each positive t for which $\mathrm{P}(t) > -\chi t$ we have that the topological pressure function* P *is real-analytic.*

Note that if f has parabolic elements, then $\chi_0 = \chi = 0$, and $\mathrm{P}(t) = 0$ for all $t \geq \mathrm{HD}(J(f))$ (see section 2). Hence, in this case the theorem exactly states that P is real-analytic in the interval $(0, \mathrm{HD}(J(f)))$. However, if f has no parabolic elements, then the parameters t satisfying the assumptions of the theorem may also be larger than $\mathrm{HD}(J(f))$. Therefore, we have the following immediate corollary.

Corollary 1.2. *For a parabolically semihyperbolic GPL-map f, the associated topological pressure function is real-analytic in the interval $(0, \mathrm{HD}(J(f)))$.*

We remark that these results generalize the results in [3] and [7] mainly in two ways. First, our results do not require that there is exactly only one critical point contained in the Julia set. Secondly, we also allow the presence of parabolic periodic points. Another important difference in comparision with [3] and [7] is that we use a completely different method, which is based primarily on the progress on graph directed Markov systems obtained in [2], [5] and [9].

Acknowledgement: We should like to thank the Mathematics Department at the University of Bourgundy in Dijon for its warm hospitality and excellent working conditions while discussing the material of this paper. The second author extends these thanks to the Technical University of Clausthal where part of the research on this article has been conducted.

2. **Review of topological pressure and conformal Gibbs states**

Let us first recall the classical definition of pressure, and formulate the variational principle. Let $T : X \to X$ be a continuous map of a compact metric space (X, d) into itself. For $x, y \in X$ and $n \geq 0$, the metric d_n is defined by

$$d_n(x, y) := \max\{d(T^i(x), T^i(y)) : 0 \leq i \leq n-1\}.$$

For $\epsilon > 0$, a set $F \subset X$ is called (n, ϵ)-separated if it is separated with respect to the metric d_n, that is if $d_n(x, y) \geq \epsilon$ for all distinct $x, y \in F$. With $(F_n(\epsilon))_{n\in\mathbb{N}}$ denoting a sequence of maximal (in the sense of inclusion) (n, ϵ)-separated sets, the topological pressure of the continuous potential function $\phi : X \to I\!\!R$ is defined by

$$\mathrm{P}(T, \phi) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{x\in F_n(\epsilon)} \exp \sum_{j=0}^{n-1} \phi \circ T^j(x) \right).$$

Note that the notion of topological pressure belongs to topological dynamics, whereas measure theoretical entropy $h_\mu(T)$ represents an important concept in ergodic theory. The link between these two notions is given by the following so called variational principle

$$\mathrm{P}(T, \phi) = \sup\{h_\mu(T) + \int \phi \, d\mu\},$$

where the supremum is taken with respect to all T-invariant (ergodic) Borel probability measures μ supported on X.

In the more general case of a GPL-map f, if there are critical points in the Julia set, then for $t \geq 0$, the potential $-t \log |f'|$ is neither continuous nor bounded. Hence, a priori it is not clear how to adapt the above definition of pressure to this more general situation. However, for arbitrary rational maps F. Przytycki suggested in [6] several ways to extend the concept of topological pressure associated with the potential $-t \log |f'|$. We now recall these suggestions in the setting of a GPL-map f, and for $t \geq 0$.

(1) *Variational pressure.*

$$\mathrm{P_V}(t) := \sup\{h_\mu(f) - t \int \log |f'| d\mu\},$$

where the supremum is taken with respect to all ergodic f-invariant measures supported on $J(f)$.

(2) *Hyperbolic variational pressure.*

$$\mathrm{P_{HV}}(t) := \sup\{h_\mu(f) - t \int \log |f'| d\mu\},$$

where the supremum is taken with respect to all ergodic f-invariant measures supported on $J(f)$ such that the Lyapunov exponent is positive, i.e. such that $\int \log |f'| d\mu > 0$.

(3) *Hyperbolic pressure.*

$$\mathrm{P_H}(t) := \sup\{\mathrm{P}(f|_X, -t \log |f'|)\},$$

where the supremum is taken with respect to all f-invariant hyperbolic subsets X of $J(f)$ such that some iterate of $f|_X$ is topologically conjugate to a subshift of finite type.

(Recall that a forward invariant compact set $X \subset J(f)$ is called hyperbolic if there exists $n \geq 1$ such that $|(f^n)'(x)| > 1$, for each $x \in X$).

(4) *DU-pressure.*

$$\mathrm{P}_{\mathrm{DU}}(t) := \sup\{\mathrm{P}(f|_{K(V)}, -t\log|f'|)\},$$

where the supremum is taken with respect to all open subsets V of $J(f)$ for which $J(f) \cap \mathrm{Crit}(f) \subset V$, and where $K(V) := J(f) \setminus \bigcup_{n\geq 0} f^{-n}(V)$. Note that $K(V)$ is compact, f-invariant and disjoint from $\mathrm{Crit}(f)$.

(5) *Conformal pressure.*

$$\mathrm{P}_{\mathrm{C}}(t) := \log \lambda(t),$$

where $\lambda(t)$ is defined as the infimum of the set of all positive λ for which there exists a Borel probability measure m with the property $d(m \circ f)/dm = \lambda |f'|^t$.

(6) *Point pressure.*

$$\mathrm{P}_z(t) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in f^{-n}(z)} |(f^n)'(x)|^{-t}$$

for $z \in G$, where $G \subset \overline{\mathbb{C}} \setminus \bigcup_{n\geq 0} f^n(\mathrm{Crit}(f))$ has the property that $\mathrm{HD}(\overline{\mathbb{C}} \setminus G) = 0$ and that $\mathrm{P}_z(t) = \mathrm{P}_w(t)$ for all $z, w \in G$. Note that the existence of such a set G has been obtained in [6].

Since, as we already mentioned, a parabolically semihyperbolic GPL-map is in particular non-recurrent, the following theorem can clearly be applied to the situation of a parabolically semihyperbolic GPL-map. For the proof of this theorem we refer to the proof of Theorem 2.6 in [8] (cf. also [6]).

Theorem 2.1. *For a non-recurrent map f and for each $t \geq 0$, all types of pressure functions defined in (1)-(6) above coincide. Their common value will be denoted by* $\mathrm{P}(t)$. *Furthermore, the following holds.*

(a) *If $0 \leq t < \mathrm{HD}(J(f))$, then $\mathrm{P}(t) > 0$.*
(b) *If $\Omega = \emptyset$, then $\mathrm{P}(t) < 0$ for all $t > \mathrm{HD}(J(f))$.*
(c) *If $\Omega \neq \emptyset$, then $\mathrm{P}(t) = 0$ for all $t \geq \mathrm{HD}(J(f))$.*

Finally, we collect a few facts concerning conformal Gibbs states which will be relevant throughout. For the proofs of these facts we refer to [8] where they were in fact proven for non-recurrent rational functions. We remark that it is straight forward to adapt these proofs to the setting of a non-recurrent GPL-map.
Recall that for a given $t \geq 0$ a Borel probability measure m_t supported on $J(f)$ is called t-conformal Gibbs state if f is non-singular with respect to m_t and

$$\frac{d(m_t \circ f)}{dm_t} = \mathrm{e}^{\mathrm{P}(t)} |f'|^t.$$

In view of the definition of conformal pressure, it follows from Theorem 2.1 that for a non-recurrent GPL-map and for each $t \geq 0$ there exists a t-conformal Gibbs state m_t. Now, the following result is an immediate consequence of the definition of a t-conformal Gibbs state and the fact that $m_t\left(\bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f))\right) = 0$.

Proposition 2.2. *If f is a non-recurrent GPL-map f and $\mathrm{P}(t) > -\chi_0 t$, then we have that $m_t\left(\overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}\right) = 0$. (We remark that this conclusion can hence be drawn in particular for $t \in (0, \mathrm{HD}(J(f)))$).*

Furthermore, we have the following important result, which guarantees that even in the parabolic situation there exists a finite invariant measure equivalent to a conformal Gibbs state.

Theorem 2.3. ([8]) *Let m_t be a t-conformal Gibbs state for a non-recurrent GPL-map f such that $m_t\left(\overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}\right) = 0$. Then up to a multiplicative constant there exists a unique f-invariant, σ-finite measure μ_t which is conservative, ergodic and equivalent to m_t. Moreover, if $t \in [0, \mathrm{HD}(J(f)))$ then μ_t is a finite measure (which hence can always be assumed to be a probability measure).*

3. Conformal graph directed Markov system and GPL-maps

Before we state the main result of this section, we first recall the definition of a conformal graph directed Markov system, abbreviated as CGDM-system.
Recall from [5] that the combinatorical spine of a graph directed Markov system is represented by a directed multigraph (V, E, i, t, A), consisting of a finite set V of vertices, a countable set E of directed edges, two functions $i, t : E \to V$, and a transition matrix $A : E \times E \to \{0, 1\}$. Here, $i(e)$ refers to the initial vertex and $t(e)$ to the terminal vertex of an edge $e \in E$. In our special context here, the matrix $A = (A_{uv})$ has the property that $A_{uv} = 1$ if and only if $t(u) = i(v)$. We then define the corresponding symbolic space as follows.

$$\mathcal{E} := \{(e_1, e_2, \dots) \in E^{\infty} : A_{e_i e_{i+1}} = 1 \text{ for all } i \geq 1\}.$$

Additionally, assume that we are given a set $\{X_v : v \in V\}$ of non-empty compact connected subsets X_v of $\mathbb{C}$, and a set $\Phi = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e \in E}$ of univalent contractions, all with some fixed Lipschitz constant $0 < s < 1$, which have respective conformal extensions from an open connected neighbourhood $W_{t(e)}$ of $X_{t(e)}$ to an open connected neighbourhood $W_{i(e)}$ of $X_{t(e)}$. If Φ satisfies additionally the 'open set condition' and the 'cone condition' (see [5], section 4.2), then Φ is called a CGDM-system. The limit set J_Φ of Φ is then defined as follows. For an arbitrary $\tau = (\tau_1, \tau_2, \dots) \in \mathcal{E}$ and $n \geq 1$, let

$$\phi_{\tau|_n} := \phi_{\tau_1} \circ \ldots \circ \phi_{\tau_n} : X_{t(\tau_n)} \to X_{\tau(e_1)}.$$

Since Φ consists of s-Lipschitz contractions, the intersection $\bigcap_{n\geq 1} \phi_{\tau|_n}\left(X_{t(\tau_n)}\right)$ is a singleton, which we denote by $\pi(\tau)$. In this way we obtain a map $\pi : \mathcal{E} \to \bigcup_{v\in V} X_v$, and we let

$$J_\Phi := \pi(\mathcal{E}).$$

The following proposition will turn out to be crucial in our analysis of the analytic properties of the pressure function. Note that in the proof of this proposition we introduce some notation which will also be used in the following section. Also, in here 'finitely primitive of order 2' refers to that for each pair $u, v \in V_f$ there exist $a, b \in E_f$ such that $i(a) = u, t(b) = v$ and $A_{a,b} = 1$ (cf. [5]).

Proposition 3.1. *For a parabolically semihyperbolic GPL-map f there exists a CGDM-system Φ_f which is finitely primitive of order 2, such that $J_{\Phi_f} \subset J(f)$ and*

$$J_{\Phi_f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n\geq 0} f^{-n}(\Omega \cup \bigcap_{k\geq 0} f^{-k}(\mathcal{U}_r)).$$

Proof. For the proof it is sufficient to show how to associate to f a CGDM-system. For this we define $U_{(i,j)} := f_j^{-1}(U_i)$, for each $(i,j) \in (I_p \times I_r) \cup (I_p \times I_p \setminus \{diag.\})$. Here $\{diag.\}$ denotes the diagonal in $I_p \times I_p$, and $f_j^{-1} : U \to U_j$ refers the inverse of the map $f|_{U_j}$. Using condition (c) in the definition of a parabolically semihyperbolic GPL-map, it follows that

$$U_{(i,j)} \cap \bigcup_{n\geq 1} f^n(\mathrm{Crit}(f)) = \emptyset. \tag{3.1}$$

Let $V_f := I_o \cup (I_p \times I_r) \cup (I_p \times I_p \setminus \{diag.\})$ be the set of vertices. The conformal univalent contractions of our system are given as follows. By (3.1) and the definition of the set I_o, we have that for each $v \in V_f$ the holomorphic inverse branches of any iterate of f are well-defined on U_v. Hence, for $v \in V_f$ and $n \geq 1$ we consider all holomorphic inverse branches $f_*^{-n} : U_v \to U$ of f^n for which $f_*^{-n}(U_v) \subset U_w$ for some $w \in V_f$, and for which $f^k(f_*^{-n}(U_v)) \cap (\bigcup_{s\in V} U_s) = \emptyset$ for all $1 \leq k < n$. In this situation we write $\phi_e : U_{t(e)} \to U_{i(e)}$ instead of $f_*^{-n} : U_v \to U_w$, where $t(e) = v$ and $i(e) = w$. Also, we define $N(e) := n$. Now, let

$$\Phi_f := \{\phi_e : \overline{U_{t(e)}} \to \overline{U_{i(e)}}\}_{e\in E_f},$$

where E_f is some countable auxiliary set parametrizing the family Φ_f. Note that the set V_f of vertices is finite, whereas in general the set E_f of edges is infinite. We denote by $\mathcal{E}_f$ the corresponding symbolic space. Since $\overline{U_v} \cap \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))} = \emptyset$, it follows that for each $v \in V_f$ there exists an open connected simply connected set $\overline{U_v} \subset W_v \subset U$ such that if $e \in E_f$ and $t(e) = v$, then ϕ_e has a univalent holomorphic extension to W_v and $\phi_e(W_v) \subset U_{i(e)}$ (for later use, we also introduce accordingly W and $\mathcal{W}_o := \bigcup_{i\in I_o} W_i$). Since for each $i \in I_p$ we have that $\bigcap_{n\geq 0} f^{-n}(J(f) \cap \overline{U_i}) = \{\omega_i\}$, we immediately obtain from the construction of Φ_f that

$$J_{\Phi_f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n\geq 0} f^{-n}(\Omega \cup \bigcap_{k\geq 0} f^{-k}(\mathcal{U}_r)).$$

We remark that the cone condition is satisfied, since for each $v \in V$ the boundaries of the disc $\overline{U_v}$ is smooth. Also, the open set condition follows immediately from the construction of Φ_f, noting that the elements of Φ_f are inverse branches of forward iterates of f. Finally, since for each pair $u, v \in V$ there exist $a, b \in E_f$ such that $i(b) \in I_o$ and such that $i(a) = u$, $t(b) = v$ and $A_{a,b} = 1$, it follows that the system Φ_f is finitely primitive of order 2.

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4. Real analyticity of the topological pressure

In this section we give the proof of our main result in this paper. From now on we shall always assume that f is a parabolically semihyperbolic GPL-map.

Lemma 4.1. *For each positive t for which $\mathrm{P}(t) > -\chi_0 t$, there exists $0 < \rho < 1$ such that for all $n \geq 1$ we have*

$$m_t\left(\bigcap_{j=0}^{n} f^{-j}(\mathcal{U}_r)\right) \ll \rho^n.$$

Proof. Fix $q \geq 1$, and consider the set

$$\mathcal{U}_r^{(q)} := \mathcal{U}_r \cap f^{-1}(\mathcal{U}_r) \cap \ldots \cap f^{-q}(\mathcal{U}_r).$$

Since the map $f : U_j \to U$ is univalent for each $j \in I_r$, it follows by induction that there exist finitely many, say k_q, holomorphic inverse branches of f^q, denoted by $f_1^{-q} : U \to \mathcal{U}_r, \ldots, f_{k_q}^{-q} : U \to \mathcal{U}_r$, such that

$$\mathcal{U}_r^{(q)} = \bigcup_{j=1}^{k_q} f_j^{-q}(\mathcal{U}_r). \tag{4.1}$$

Hence, for any arbitrary set $A \subset \mathcal{U}_r$ it follows that

$$\mathcal{U}_r^{(q)} \cap f^{-q}(A) = \bigcup_{j=1}^{k_q} f_j^{-q}(A), \tag{4.2}$$

and by conformality of m_t we have for each $j \in \{1, 2, \ldots, k_q\}$ that

$$m_t(f_j^{-q}(A)) \leq m_t(A) e^{-\mathrm{P}(t)q} \sup_{z \in A}\{|(f_j^{-q})'(z)|\}^t \leq m_t(A) e^{-\mathrm{P}(t)q} \sup_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\}^t, \tag{4.3}$$

as well as

$$m_t(f_j^{-q}(\mathcal{U}_r)) \geq m_t(\mathcal{U}_r) e^{-\mathrm{P}(t)q} \inf_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\}^t. \tag{4.4}$$

Now, on $\mathcal{U}_r$ we can apply Koebe's distortion theorem, that is there exists a constant $K \geq 1$ such that

$$\sup_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\} \leq K \inf_{z \in \mathcal{U}_r}\{|(f_j^{-q})'(z)|\}.$$

Therefore, (4.3) and (4.4) imply that

$$m_t(f_j^{-q}(A)) \leq \frac{K^t}{m_t(\mathcal{U}_r)} m_t(A) m_t(f_j^{-q}(\mathcal{U}_r)).$$

Combining this estimate with (4.1) and (4.2), it follows that

$$m_t(\mathcal{U}_r^{(q)} \cap f^{-q}(A)) \leq \frac{K^t}{m_t(\mathcal{U}_r)} m_t(\mathcal{U}_r^{(q)}) m_t(A). \tag{4.5}$$

Let $\mathcal{U}_r^{(\infty)} := \bigcap_{j\geq 0} f^{-j}(\mathcal{U}_r) = \bigcap_{q\geq 1} \mathcal{U}_r^{(q)}$, and observe that $f^{-1}(\mathcal{U}_r^{(\infty)}) \supset \mathcal{U}_r^{(\infty)}$. By ergodicity of μ_t, we hence have that $\mu_t(\mathcal{U}_r^{(\infty)}) \in \{0,1\}$. Now, since $\mu_t(\mathcal{U}_o) > 0$, and since $\mathcal{U}_r \subset U \setminus \mathcal{U}_o$, we have $\mu_t(\mathcal{U}_r) < 1$, which then implies that $\mu_t(\mathcal{U}_r^{\infty}) = 0$. Since $\{\mathcal{U}_r^{(q)}\}_{q=1}^{\infty}$ is a descending sequence of sets, we conclude that $\lim_{q\to\infty} \mu_t(\mathcal{U}_r^{(q)}) = 0$, and hence that $\lim_{q\to\infty} m_t(\mathcal{U}_r^{(q)}) = 0$. Therefore, we can choose $q \geq 1$ sufficiently large such that $K^t m_t(\mathcal{U}_r^{(q)})/m_t(\mathcal{U}_r) \leq 1/2$. Inserting this observation into (4.5), we obtain that for any arbitrary $A \subset \mathcal{U}_r$ we have that

$$m_t(\mathcal{U}_r^{(q)} \cap f^{-q}(A)) \leq \frac{1}{2}\, m_t(A). \tag{4.6}$$

In order to finish the proof, we use (4.6) and observe that for every $k \geq 1$ we have that

$$m_t\left(\bigcap_{j=0}^{qk} f^{-j}(\mathcal{U}_r)\right) = m_t\left(\mathcal{U}_r^{(q)} \cap f^{-q}\left(\bigcap_{j=0}^{q(k-1)} f^{-j}(\mathcal{U}_r)\right)\right) \leq \frac{1}{2} m_t\left(\bigcap_{j=0}^{q(k-1)} f^{-j}(\mathcal{U}_r)\right).$$

By way of induction, this gives that

$$m_t\left(\bigcap_{j=0}^{qk} f^{-j}(\mathcal{U}_r)\right) \leq \left(\frac{1}{2}\right)^k,$$

which also holds for $k = 0$. Now let $n \geq 1$ be given, and write $n = qk + r$, for $0 \leq r < q$ and $k \geq 0$. It follows that

$$m_t\left(\bigcap_{j=0}^{n} f^{-j}(\mathcal{U}_r)\right) \leq m_t\left(\bigcap_{j=0}^{qk} f^{-j}(\mathcal{U}_r)\right) \leq \left(\frac{1}{2}\right)^k \leq \left(\frac{1}{2}\right)^{\frac{n}{q}-1} = 2\left(\left(\frac{1}{2}\right)^{\frac{1}{q}}\right)^n.$$

■

As an immediate consequence we derive the following corollary, which shows that for certain values of t the sets $J(f)$ and J_{Φ_f} coincide m_t-almost everywhere on $\mathcal{U}_o$.

Corollary 4.2. *For each positive t for which $\mathrm{P}(t) > -\chi_0 t$, we have that $m_t(J_{\Phi_f} \cap \mathcal{U}_o) = m_t(\mathcal{U}_o) > 0$.*

Proof. By Proposition 3.1 we have $J_{\Phi_f} \cap \mathcal{U}_o = J(f) \cap \mathcal{U}_o \setminus \bigcup_{n\geq 0} f^{-n}(\Omega \cup \bigcap_{k\geq 0} f^{-k}(\mathcal{U}_r))$. Since $\mathrm{P}(t) > -\chi_0 t$, Proposition 2.2 implies that m_t has no atoms. Finally, by Lemma 4.1 we have that $m_t(\bigcap_{k\geq 0} f^{-k}(\mathcal{U}_r)) = 0$. Combining these three observations, the statement of the corollary follows. ■

Lemma 4.3. *For each $c \in \mathrm{Crit}(J(f))$ we have $\chi(c) > 0$.*

Proof. For every $q \geq 1$, let $f_1^{-q}, \dots, f_{k_q}^{-q}$ be the holomorphic inverse branches which we already considered at the beginning of the proof of Lemma 4.1. By Vitali's theorem, the family $\{f_i^{-q} : q \geq 1, 1 \leq i \leq k_q\}$ is normal, and since $J(f) \subset \overline{U}$, this implies that each point of accumulation of this family is a function equal to some constant. Hence, each point of accumulation of the family of derivatives of these functions is the constant function equal to zero. Thus, there exists $q \geq 1$ such that $|(f_i^{-q})'(z)| \leq 1/2$, for all $1 \leq i \leq k_q$, $z \in \mathcal{U}_r$. Also note that, since $\bigcup_{k \geq 1} f^k(\mathrm{Crit}(f)) \subset \mathcal{U}_r$, we have for each pair $n, l \geq 1$ that there exists $j \in \{1, 2, \dots, k_l\}$ such that $f_j^{-l}(f^{n+l}(c)) = f^n(c)$. Now, let $l \geq 1$ be fixed such that $l = sq + r$, for $0 \leq r < q$ and $s \geq 0$. For each $n \geq 1$ we then have by the chain rule

$$|(f^l)'(f^n(c))| \geq 2^s M_q \geq 2^{\frac{l}{q}-1} M_q = \left(2^{1/q}\right)^l M_q/2,$$

where we have set $M_q := \min_{0 \leq j < q}\{\inf\{|(f^j)'(z)| : z \in \mathcal{U}_r\}\}$, which is strictly positive due to the fact that $\mathrm{Crit}(f) \cap \overline{\mathcal{U}_r} = \emptyset$. Hence, it follows that $\inf_{j \geq 1}\{|(f^l)'(f^j(c))|\} \geq \left(2^{1/q}\right)^l M_q/2$ for every $l \geq 1$, which implies that $\chi(c) \geq (\log 2)/q > 0$. ■

Lemma 4.4. *For each positive t for which $\mathrm{P}(t) > -\chi_q t$, there exists $l \geq 1$ such that for each Borel set $A \subset U$ we have*

$$m_t(f^{-1}(A)) \ll (m_t(A))^{1/l}.$$

Proof. Using the t-conformality of m_t, it follows that the assertion holds for all Borel sets $A \subset U$ such that $A \cap \bigcup_{c \in \mathrm{Crit}(J(f))} B(f(c), \delta) = \emptyset$, for some fixed positive δ. Hence, from now on let a Borel set $A \subset B(f(c), \delta)$ be fixed, for some $c \in \mathrm{Crit}(J(f))$, with $m_t(A) > 0$ and where $\delta < \mathrm{dist}(\mathcal{U}_r, \partial U)/2$ is chosen sufficiently small (which will be specified during the proof). Let $f_c^{-1}(A)$ be the intersection of $f^{-1}(A)$ with the component of $f^{-1}(B(f(c), \delta))$ which contains c. Also, for $n \geq 1$ we define

$$\lambda_n(c) := |(f^n)'(f(c))|,$$

and let $A(w, r, R) := \{z \in \mathbb{C} : r \leq |z - w| < R\}$ denote the annulus centred at $w \in \mathbb{C}$ of inner radius r and outer radius R.

The structure of the proof is as follows. We show that for each pair $s, k \geq 1$ we have that

(i) $m_t(B(c, (\delta \lambda_{sk}(c)^{-1})^{1/q(c)})) \ll \lambda_{sk}(c)^{-t/q(c)} e^{-\mathrm{P}(t)sk}$.

For this we slice $B(f(c), \delta)$ into annuli and define the 'stopping time'

$$u := \sup\{n \geq 0 : m_t(A \cap A(f(c), \delta \lambda_{sn}(c)^{-1}, \delta)) \leq \lambda_{sn}(c)^{-t} e^{-\mathrm{P}(t)sn}\}.$$

We show that u is a finite number, and by combining this with the estimate in (i), we obtain

(ii) $m_t(f_c^{-1}(A)) \ll \lambda_{su}(c)^{-t/q(c)} e^{-\mathrm{P}(t)su}$.

Finally, we prove the following two facts, which then finishes the proof of the proposition.

(iii) $\lambda_{su}(c)^{-t/q(c)} e^{-\mathrm{P}(t)su} \leq (\lambda_{su}(c)^{-t} e^{-\mathrm{P}(t)su})^{1/l}$, for some $l > 0$ and for s sufficiently large.

(iv) $\lambda_{su}(c)^{-t}e^{-\mathrm{P}(t)su} \ll m_t\Big(A\cap A(f(c),\delta\lambda_{s(u+1)}(c)^{-1},\delta)\Big)$ $\quad(\le m_t(A))$.

For (i), first note that by Koebe's distortion theorem we have for all $n \ge 1$ that

$$m_t(B(f(c),\delta\lambda_n(c)^{-1})) \asymp \lambda_n(c)^{-t}e^{-\mathrm{P}(t)n}.$$

Using this observation and the fact that $|(f_c^{-1})'(z)| \asymp |z-f(c)|^{-(1-1/q(c))}$ for $z \neq f(c)$ close to $f(c)$ (where f_c^{-1} refers to an inverse branch of f defined on some neighbourhood of z which maps z close to c), it follows for each pair $s,k \ge 1$ that

$$\begin{aligned}
m_t(B(c,(\delta\lambda_{sk}(c)^{-1})^{1/q(c)})) &= \sum_{j=k}^{\infty} m_t(A(c,(\delta\lambda_{s(j+1)}(c)^{-1})^{1/q(c)},(\delta\lambda_{sj}(c)^{-1})^{1/q(c)}))\\
&\asymp \sum_{j=k}^{\infty} m_t\Big(f_c^{-1}(A(f(c),\delta\lambda_{s(j+1)}(c)^{-1},\delta\lambda_{sj}(c)^{-1}))\Big)\\
&\asymp \sum_{j=k}^{\infty} \lambda_{sj}(c)^{\left(1-\frac{1}{q(c)}\right)t}e^{-\mathrm{P}(t)}m_t\Big(A(f(c),\delta\lambda_{s(j+1)}(c)^{-1},\delta\lambda_{sj}(c)^{-1})\Big)\\
&\le e^{-\mathrm{P}(t)}\sum_{j=k}^{\infty} \lambda_{sj}(c)^{\left(1-\frac{1}{q(c)}\right)t}m_t\Big(B(f(c),\delta\lambda_{sj}(c)^{-1})\Big)\\
&\asymp \sum_{j=k}^{\infty} \lambda_{sj}(c)^{\left(1-\frac{1}{q(c)}\right)t}\lambda_{sj}(c)^{-t}e^{-\mathrm{P}(t)sj} = \sum_{j=k}^{\infty}\lambda_{sj}(c)^{-t/q(c)}e^{-\mathrm{P}(t)sj}\\
&= \lambda_{sk}(c)^{-t/q(c)}e^{-\mathrm{P}(t)sk}\left(1+\sum_{j=k+1}^{\infty}\left(\frac{\lambda_{sj}(c)}{\lambda_{sk}(c)}\right)^{-t/q(c)}e^{-\mathrm{P}(t)s(j-k)}\right).
\end{aligned}$$

Hence, we are left with to show that the sum in the latter expression is bounded from above. For this recall that, since $\chi(c) > -q(c)\mathrm{P}(t)/t$, there exists $\kappa > 0$ and some s depending on c, such that $\frac{1}{v}\log|(f^v)'(f^n(c))| \ge -q(c)\mathrm{P}(t)/t+\kappa$ for all for all $v \ge s, n \ge 1$. Thus, for every $j \ge k+1$ we have that

$$\begin{aligned}
\log\left(\frac{\lambda_{sj}(c)}{\lambda_{sk}(c)}\right) &= \sum_{i=k}^{j-1}\log\lambda_{s(i+1)}(c)-\log\lambda_{si}(c) = \sum_{i=k}^{j-1}\log\left(\frac{\lambda_{s(i+1)}(c)}{\lambda_{si}(c)}\right)\\
&= \sum_{i=k}^{j-1}\log|(f^s)'(f^{si}(c))| \ge \left(\frac{-q(c)\mathrm{P}(t)}{t}+\kappa\right)s(j-k).
\end{aligned}$$

It follows that

$$\left(\frac{\lambda_{sj}(c)}{\lambda_{sk}(c)}\right)^{-t/q(c)}e^{-\mathrm{P}(t)s(j-k)} \le e^{-(-q(c)\mathrm{P}(t)+t\kappa)s(j-k)/q(c)}e^{-\mathrm{P}(t)s(j-k)} = \exp\left(-\frac{t\kappa s}{q(c)}(j-k)\right),$$

which completes the proof of the statement in (i) above.

Since $\lim_{n\to\infty} m_t(A\cap A(f(c),\delta\lambda_{sn}(c)^{-1},\delta) = m_t(A\cap B(f(c),\delta)) > 0$, in order to see that the 'stopping time' u is finite it is sufficient to show that

$$\lim_{n\to\infty}\lambda_{sn}(c)^{-t}e^{-\mathrm{P}(t)sn} = 0.$$

If $\mathrm{P}(t) \geq 0$, then this is an immediate consequence of Lemma 4.3.
If $\mathrm{P}(t) < 0$, then $t\chi(c) > -q(c)\mathrm{P}(t)$ implies that $t \log |(f^v)'(f(c))| > -vq(c)\mathrm{P}(t)$, for each v sufficiently large. This gives that $\lambda_v(c)^{-t} e^{-\mathrm{P}(t)v} < e^{\mathrm{P}(t)v(q(c)-1)}$, and since $q(c) \geq 2$, the result follows.

For (ii), we combine (i) and the finiteness of u and obtain

$$\begin{aligned} m_t(f_c^{-1}(A)) &= m_t\Big(f_c^{-1}(A \cap \overline{B(f(c), \delta\lambda_{su}(c)^{-1})})\Big) + m_t\Big(f_c^{-1}(A \cap A(f(c), \delta\lambda_{su}(c)^{-1}), \delta))\Big) \\ &\leq m_t\Big(f_c^{-1}(\overline{B(f(c), \delta\lambda_{su}(c)^{-1})})\Big) + m_t\Big(f_c^{-1}(A \cap A(f(c), \delta\lambda_{su}(c)^{-1}), \delta))\Big) \\ &\ll m_t\left(\overline{B\left(c, (\delta\lambda_{su}(c)^{-1})^{1/q(c)}\right)}\right) + (\delta\lambda_{su}(c)^{-1})^{\left(\frac{1}{q(c)}-1\right)t} e^{-\mathrm{P}(t)} m_t(A \cap A(f(c), \delta\lambda_{su}(c)^{-1}, \delta)) \\ &\leq K_c \lambda_{su}(c)^{-t/q(c)} e^{-\mathrm{P}(t)su} + (\delta\lambda_{su}(c)^{-1})^{\left(\frac{1}{q(c)}-1\right)t} e^{-\mathrm{P}(t)su} \lambda_{su}(c)^{-t} \\ &\asymp \lambda_{su}(c)^{-t/q(c)} e^{-\mathrm{P}(t)su}, \end{aligned}$$

where we have set $K_c := \left(1 + \sum_{j=k+1}^{\infty} \exp(-t\kappa s(j-k)/q(c))\right)$.

For (iii), recall that $\frac{1}{v} \log |(f^v)'(f(c))| \geq -q(c)\mathrm{P}(t)/t + \kappa$ for all for all $v \geq s$. Hence, by choosing $l(c)$ sufficiently large such that $\kappa > \mathrm{P}(t)q(c)(1-q(c))/(t(l(c)-q(c))$, it follows that

$$(l(c) - q(c)) \frac{\log \lambda_{su}(c)}{su} \geq \left(\frac{-q(c)\mathrm{P}(t)}{t} + \kappa\right)(l(c) - q(c)) \geq \frac{q(c)\mathrm{P}(t)}{t}(1 - l(c)).$$

An elementary rearrangement then gives

$$\lambda_{su}(c)^{-t/q(c)} e^{-\mathrm{P}(t)su} \leq \left(\lambda_{su}(c)^{-t} e^{-\mathrm{P}(t)su}\right)^{1/l(c)}.$$

By defining $l := \max\{l(c) : c \in \mathrm{Crit}(J(f))\}$, the statement in (iii) follows.

Finally for (iv), the finiteness of u gives

$$m_t(A \cap A(f(c), \delta\lambda_{s(u+1)}(c)^{-1}, \delta) > \lambda_{s(u+1)}(c)^{-t} e^{-\mathrm{P}(t)s(u+1)} \geq e^{-\mathrm{P}(t)s} ||f'||^{-st} \lambda_{su}(c)^{-t} e^{-\mathrm{P}(t)su},$$

which completes the proof of the proposition. ■

We now pass to the CGDM-system Φ_f associated with the GPL-map f. For this the reader is asked to recall the construction and notation given in Section 3.
For each $t \geq 0, s \in \mathbb{R}$ and $e \in E_f$ we define the potential $g_{t,s}^{(e)} : W_{t(e)} \to \mathbb{R}$ for $x \in W_{t(e)}$ by

$$g_{t,s}^{(e)}(x) := t \log |\phi_e'(x)| - sN(e).$$

We shall now see that for suitably chosen s and t the family $G_{t,s} := \{g_{t,s}^{(e)} : e \in E_f\}$ is a summable Hölder family of functions, where Hölder refers to the fact that for some $\gamma > 0$ we have (cf. [2], [5])

$$\sup_{n \geq 1} \sup_{(\tau_1, \tau_2, \dots) \in \mathcal{E}_f} \sup_{z,w \in U_{t(\tau_n)}} |g_{t,s}^{(\tau_1)}(\phi_{\tau_2,\dots,\tau_n}(z)) - g_{t,s}^{(\tau_1)}(\phi_{\tau_2,\dots,\tau_n}(w))| e^{\gamma(n-1)} < \infty.$$

Lemma 4.5. *For each positive u such that $\mathrm{P}(u) > -\chi u$, there exists $\delta > 0$ such that $G_{t,s}$ is a summable Hölder family of functions, for each $t \in (u-\delta, u+\delta)$ and $s > \mathrm{P}(u) - \delta$.*

Proof. Using Koebe's distortion theorem, it is straight forward to see that $G_{t,s}$ is a Hölder family of functions, for each $t \geq 0$ and $s \in I\!R$ (see [5] paragraph 4.2, Lemma 2.2). In order to prove that $G_{t,s}$ is summable, put $Z^{(n)} := \{e \in E_f : N(e) = n\}$ and define

$$R_n := \bigcup_{e \in Z^{(n)}} \phi_e(U_{t(e)}) \quad \text{for} \quad n \geq 1.$$

We first observe that if there are no parabolic elements then we have for $n > 1$ that $R_n \subset f^{-1}\left(\bigcap_{j=0}^{n-2} f^{-j}(\mathcal{U}_r)\right)$ (for $n = 1$, we have $R_1 \subset \mathcal{U}_o$), and hence Lemma 4.1 and Lemma 4.4 imply

$$m_u(R_n) \leq m_u\left(f^{-1}\left(\bigcap_{j=0}^{n-2} f^{-j}(\mathcal{U}_r)\right)\right) \ll \left(m_u\left(\bigcap_{j=0}^{n-2} f^{-j}(\mathcal{U}_r)\right)\right)^{1/l} \ll \rho^{n/l}. \tag{4.7}$$

If there are parabolic points then $\chi_0 = 0$, and consequently the condition $\mathrm{P}(u) > -\chi u$ implies that $\mathrm{P}(u) > 0$. For $e \in Z^{(n)}$ we have that there exists $1 \leq k \leq n$ such that $f^j(U_{i(e)}) \subset \mathcal{U}_r$ for all $k \leq j < n$ and such that $f^j(U_{i(e)}) \subset U_i$ for all $1 \leq j < k$ and for some $i \in I_p$. Using Lemma 4.1, Lemma 4.4 and (LBP), we obtain in this situation, with some fixed β such that $\max\{e^{-\mathrm{P}(u)}, \rho\} < \beta < 1$,

$$\begin{aligned} m_u(R_n) &\ll (m_u(f(R_n)))^{1/l} \ll \left(\sum_{k=1}^{n} e^{-k\mathrm{P}(u)} \sum_{i \in I_p} k^{-\frac{p_i+1}{p_i}u} m_u\left(\bigcap_{j=1}^{n-k} f^{-j}(\mathcal{U}_r)\right)\right)^{1/l} \\ &\ll \left(e^{-k\mathrm{P}(u)} \rho^{n-k}\ \mathrm{card}(I_p)\right)^{1/l} \ll \beta^{n/l}. \end{aligned}$$

Combining this estimate and (4.7), we conclude that no matter if there are parabolic points or not, there exists $\alpha > 0$ such that for all $n \geq 1$ we have

$$m_u(R_n) \ll e^{-\alpha n}. \tag{4.8}$$

Using the definition of the measure m_u along with Koebe's distortion theorem, we now immediately have for all $n \geq 1$ that

$$\sum_{e \in Z^{(n)}} \sup_{z \in \overline{U_{i(e)}}} (|(f^n)'(z)|^{-u}) e^{-\mathrm{P}(u)n} \ll e^{-\alpha n}.$$

Observe that the family $\{\phi_e : W_{t(e)} \to U_{i(e)}\}_{e \in E_f}$ is normal, and hence all its limit functions are constant. This implies that

$$\Delta_1 := \sup_{e \in E_f} \sup_{z \in \overline{U_{t(e)}}} |\phi_e'(z)| < \infty.$$

Therefore, for fixed $u > 0$ there exists $0 < \delta < \min\{u, \frac{\alpha}{2}, \frac{\alpha}{4}|\log \Delta_1|^{-1}, \frac{\alpha}{4}|\log \Delta_2|^{-1}\}$, where we have put $\Delta_2 := \sup_{e \in E_f} \sup_{z \in \overline{U_{i(e)}}} |f'(z)|$. With this choice of δ we obtain for each $t \in (u-\delta, u+\delta)$ and $s > \mathrm{P}(u) - \delta$ that

$$\sum_{e \in Z^{(n)}} \sup_{z \in \overline{U_{i(e)}}} (|(f^n)'(z)|^{-t}) e^{-sn} \leq \sum_{e \in Z^{(n)}} \sup_{z \in \overline{U_{i(e)}}} (|(f^n)'(z)|^{-u})\, e^{-\mathrm{P}(u)n} e^{\delta n} \ \max\{\Delta_1^{\delta}, \Delta_2^{n\delta}\}$$
$$\ll e^{-\alpha n} e^{\frac{\alpha}{2} n} e^{\frac{\alpha}{4} n} = e^{-\frac{\alpha}{4} n}.$$

■

For the following lemma recall that the topological pressure $\mathcal{P}$ associated with the family $G_{t,s}$ is given by (cf. [2], [5])

$$\mathcal{P}(t,s) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\tau_1, \ldots, \tau_n) \in E_f^{(n)}} \sup_{z \in U_{t(\tau_n)}} \exp\left(g_{t,s}^{(\tau_n)}(z) + \sum_{i=1}^{n-1} g_{t,s}^{(\tau_i)}(\phi_{\tau_{i+1}, \ldots, \tau_n}(z)) \right),$$

where we have set $E_f^{(n)} := \{(\tau_1, \ldots, \tau_n) \in E_f^n : A_{\tau_j \tau_{j+1}} = 1 \text{ for all } j = 1, 2, \ldots, n-1\}$. Also, associated with $G_{t,s}$ there exists a unique $G_{t,s}$-conformal probability measure $m_{t,s}$ supported on J_{Φ_f}. That is, for each $n \geq 1$ and $\tau = (\tau_1, \ldots, \tau_n) \in E_f^{(n)}$ we have for every Borel set $A \subset U_{t(\tau_n)}$ that

$$m_{t,s}(\phi_\tau(A)) = \int_A \exp\left(g_{t,s}^{(\tau_n)}(z) + \sum_{i=1}^{n-1} g_{t,s}^{(\tau_i)}(\phi_{\tau_{i+1}, \ldots, \tau_n}(z)) - n\mathcal{P}(t,s) \right) dm_{t,s}(z).$$

Lemma 4.6. *For each positive t for which $\mathrm{P}(t) > -\chi t$, we have $\mathcal{P}(t, \mathrm{P}(t)) = 0$. Furthermore, for every $n \geq 1$ and for each $\tau = (\tau_1, \ldots, \tau_n) \in E_f^{(n)}$ we have that*

$$m_{t,\mathrm{P}(t)}(\phi_\tau(U_{t(\tau_n)})) \asymp m_t(\phi_\tau(U_{t(\tau_n)})),$$

with comparability constants not depending on n and τ.

Proof. By conformality of m_t and $m_{t,s}$, we have for each $n \geq 1$ that

$$\begin{aligned} m_t(\phi_\tau(U_{t(\tau_n)})) &= \int_{U_{t(\tau_n)}} |\phi_\tau'(z)|^t e^{-\mathrm{P}(t) \sum_{j=1}^n N(\tau_j)} dm_t(z) \\ &\asymp ||\phi_\tau'||^t e^{-\mathrm{P}(t) \sum_{j=1}^n N(\tau_j)} m_t(U_{t(\tau_n)}) \\ &\asymp e^{n\mathcal{P}(t,\mathrm{P}(t))} ||\phi_\tau'||^t e^{-\mathrm{P}(t) \sum_{j=1}^n N(\tau_j)} e^{-n\mathcal{P}(t,\mathrm{P}(t))} \\ &\asymp e^{n\mathcal{P}(t,\mathrm{P}(t))} m_{t,\mathrm{P}(t)}(\phi_\tau(U_{t(\tau_n)})). \end{aligned}$$

Therefore, if on the one hand $\mathcal{P}(t, \mathrm{P}(t)) > 0$ then $m_{t,\mathrm{P}(t)}(J_{\Phi_f}) = 0$, which contradicts $m_{t,\mathrm{P}(t)}(J_{\Phi_f}) = 1$. On the other hand, if $\mathcal{P}(t, \mathrm{P}(t)) < 0$ then we obtain $m_t(J_{\Phi_f}) = 0$, which is also a contradiction. Thus, it follows that $\mathcal{P}(t, \mathrm{P}(t)) = 0$, which gives the lemma. ■

Proof of the Theorem 1.1.
Using Lemma 4.5 and applying Theorem 2.6.12 of [5] (or alternatively [2] Theorem 6.4), we have for each positive u with $\mathrm{P}(u) > -\chi u$ that there exists $\delta > 0$ such that $\mathcal{P}$ is real-analytic on $(u-\delta, u+\delta) \times (\mathrm{P}(u)-\delta, \mathrm{P}(u)+\delta)$ in both variables t and s. In order to prove that P is real-analytic on $(u-\delta, u+\delta)$, we employ the implicit function theorem, showing that P is the unique real-analytic function which satisfies $\mathcal{P}(t, \mathrm{P}(t)) = 0$ for all $t \in (u-\delta, u+\delta)$. For this it is now sufficient to verify that for all $t \in (u-\delta, u+\delta)$ we have

$$\frac{\partial \mathcal{P}(t,s)}{\partial s}\Big|_{(t,\mathrm{P}(t))} \quad \text{exists and is strictly negative.} \tag{4.9}$$

Denote the measure $m_{t,\mathrm{P}(t)}$ by ν_t. Proposition 3.1, Lemma 4.5 and Lemma 4.6 guarantee that Theorem 3.7 of [5] is applicable. This gives that the measure ν_t has a lift $\tilde{\nu}_t$ to the symbolic space $\mathcal{E}_f$, and that there exists a measure $\tilde{\mu}_t$ in the measure class of $\tilde{\nu}_t$ which is invariant under the shift map on the space $\mathcal{E}_f$, and whose Radon-Nikodym derivative with respect to $\tilde{\nu}_t$ is bounded away from zero and infinity. We can now apply Proposition 2.6.13 of [5] (or alternatively [2] Proposition 6.5), which gives

$$\frac{\partial \mathcal{P}(t,s)}{\partial s}\Big|_{(t,\mathrm{P}(t))} = -\int N d\tilde{\mu}_t. \tag{4.10}$$

Using the estimate in (4.8) and the second part of Lemma 4.6 we then compute

$$\int N d\tilde{\mu}_t \asymp \int N d\tilde{\nu}_t = \int N d\nu_t = \sum_{n\geq 1} n\nu_t(R_n) \asymp \sum_{n\geq 1} n m_t(R_n) \ll \sum_{n\geq 1} n e^{-\alpha n} < \infty, \tag{4.11}$$

where after the first equality sign we treated the function N slightly informally as defined on the limit set J_{Φ_f}. Combining (4.10) and (4.11), and using the fact that the function N is strictly positive, we derive (4.9), which then completes the proof of Theorem 1.1. ■

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