# BREAKDOWN OF THE REAL-ANALYTICITY OF THE HAUSDORFF DIMENSION IN INFINITE IFS

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Abstract. We show that the Hausdorff dimension of the limit sets associated with analytic families of iterated function systems (IFS) consisting of infinitely many uniformly contracting similitudes does not always depend real-analytically on the underlying parameter, in contradistinction with the case of families of IFS comprising finitely many similitudes. Breakdowns in the real-analyticity often occur when families undergo what we call 'phase transitions'. We give conditions under which such phase transitions are prohibited, and the real-analyticity preserved.

## 1. Introduction

Let $I$ be a countable set, usually called alphabet or index set, and $\Lambda \subseteq \mathbb{C}$ a domain, referred to as parameter space. Let $S = \{f_i : \Lambda \times \mathbb{C} \to \mathbb{C} | \, f_i(\lambda, z) = c_i(\lambda)z + b_i(\lambda)\}_{i \in I}$ be an analytic family of CIFS consisting of locally uniformly contracting similitudes, that is, for each $i \in I$ the functions $c_i, b_i : \Lambda \to \mathbb{C}$ are analytic, and for every $\lambda_0 \in \Lambda$ there is a neighbourhood $\Lambda_0 \subseteq \Lambda$ of $\lambda_0$ such that $C(\lambda_0) := \sup_{\lambda \in \Lambda_0} \sup_{i \in I} |c_i(\lambda)| < 1$ and $B(\lambda_0) := \sup_{\lambda \in \Lambda_0} \sup_{i \in I} |b_i(\lambda)| < \infty$. The latter assumption on the $b_i$'s is always satisfied when $|I| < \infty$, where $|I|$ denotes the cardinality of $I$. This assumption and the one on the $c_i$'s ensure that the limit set $J(\lambda)$, associated with the CIFS $S(\lambda) := \{f_{i,\lambda} : \mathbb{C} \to \mathbb{C} | \, f_{i,\lambda}(z) := f_i(\lambda, z)\}_{i \in I}$, is well defined and locally uniformly bounded in $\lambda$ (see [3]).

In this framework, $S(\lambda)$ is a CIFS if it satisfies the Open Set Condition (OSC) by means of a non-empty open set $U(\lambda) \subseteq \mathbb{C}$ such that (1) $\overline{U(\lambda)}$ is a compact and connected set, (2) $U(\lambda) = \text{int}\, \overline{U(\lambda)}$, and (3) there exist $\alpha, l > 0$ so that for every $z \in \partial \overline{U(\lambda)}$ there is an open cone $\text{Con}(z, u_z, \alpha, l) \subseteq U(\lambda)$ with vertex $z$, direction vector $u_z$, central angle of Lebesgue measure $\alpha$, and altitude $l$. Assumption (3) is called cone condition. Recall that $S(\lambda)$ satisfies the OSC through $U(\lambda)$ if $f_{i,\lambda}(U(\lambda)) \subseteq U(\lambda)$ for every $i \in I$ and $f_{i,\lambda}(U(\lambda)) \cap f_{j,\lambda}(U(\lambda)) = \emptyset$ whenever $i \neq j$.

Finite families of CIFS in which the OSC is satisfied in a strong sense have been previously considered by Astala [1]. Astala used the fact that these families describe a holomorphic motion to prove a long-standing conjecture on quasiconformal maps. General (finite or infinite) families were more recently investigated by Baribeau and Roy [3]. They showed that the closure of the limit set, $\overline{J(\lambda)}$, is a continuous, multianalytic function of $\lambda$.

They further gave conditions under which $J(\lambda)$, and consequently $\overline{J(\lambda)}$, describes a holomorphic motion that respects the dynamics of the CIFS. They also proved that the Hausdorff dimension $h(\lambda)$ of $J(\lambda)$ is the infimum of a collection of positive harmonic functions, and therefore is a continuous, subharmonic function of $\lambda$.

In this short note, we observe that, whereas $h(\lambda)$ is real-analytic when $|I| < \infty$, some breakdowns may occur when $|I| = \infty$. This often happens when a family $S$ undergoes some 'phase transitions'. To explain what we mean by that, we must first briefly recall some basic concepts in the theory of infinite IFS.

We begin with the definition of $J(\lambda)$. For every $n \in \mathbb{N}$ let $I^n$ be the set of words of length $n$ with letters from the alphabet $I$, and let $I^* := \cup_{n\in\mathbb{N}} I^n$ be the set of finite words. Let also $I^\infty$ be the set of one-sided infinite words. We denote the length of any word $\omega \in I^* \cup I^\infty$ by $|\omega|$. By definition, $|\omega|$ is the unique extended number satisfying $\omega \in I^{|\omega|}$. Given $\omega \in I^* \cup I^\infty$ and $n \in \mathbb{N}$ such that $n < |\omega|$, we denote by $\omega|_n$ the word $\omega_1\omega_2\ldots\omega_n \in I^n$ consisting of the first $n$ letters of $\omega$. We also denote by $\sigma$ the left shift map on $I^\infty$. Moreover, given $\omega \in I^*$ and $\lambda \in \Lambda$, we denote by $f_{\omega,\lambda} : \mathbb{C} \to \mathbb{C}$ the composition $f_{\omega_1,\lambda} \circ f_{\omega_2,\lambda} \circ \cdots \circ f_{\omega_{|\omega|},\lambda}$, and observe that the Jacobian $c_\omega(\lambda)$ of $f_{\omega,\lambda}$ satisfies $c_\omega(\lambda) = \prod_{i=1}^{|\omega|} c_{\omega_i}(\lambda)$.

Now, fix $z \in \mathbb{C}$. For every $\lambda \in \Lambda$, a coding map $\pi_\lambda : I^\infty \to \mathbb{C}$ is defined by $\pi_\lambda(\omega) = \lim_{n\to\infty} f_{\omega|_n,\lambda}(z)$. This coding is well defined and does not depend on the choice of $z$ (see [3]). The limit set for $S(\lambda)$ is $J(\lambda) := \pi_\lambda(I^\infty)$. When $|I| < \infty$, the limit set $J(\lambda)$ is compact and $J(\lambda) = \overline{J(\lambda)}$, but this is generally not true when $|I| = \infty$. Recall also that $J(\lambda)$ satisfies the relation $J(\lambda) = \cup_{i\in I} f_{i,\lambda}(J(\lambda))$, for $f_{i,\lambda}(\pi_\lambda(\omega)) = \pi_\lambda(i\omega)$ for every $i \in I$ and $\omega \in I^\infty$.

Next, we consider the pressure function. For every $\lambda \in \Lambda$ and $t \geq 0$, define

$$\psi(\lambda, t) = \sum_{i\in I} |c_i(\lambda)|^t.$$

The set of finiteness of $\psi(\lambda, \cdot)$ is $F(S(\lambda)) := \{t \geq 0 \,|\, \psi(\lambda, t) < \infty\}$, and its infimum is denoted by $\theta(\lambda) := \inf F(S(\lambda))$.

Recall that the topological pressure for the CIFS $S(\lambda)$ is defined as

$$P(\lambda, t) := \lim_{n\to\infty} \frac{1}{n} \log \psi_n(\lambda, t),$$

where $\psi_n(\lambda, t) := \sum_{\omega\in I^n} |c_\omega(\lambda)|^t$. In the current setting, the pressure takes the simple form

$$P(\lambda, t) = \log \psi(\lambda, t) \tag{1}$$

because $\psi_n(\lambda, t) = \psi(\lambda, t)^n$. In particular, note that $P(\lambda, \cdot)$ is a strictly decreasing function on $F(S(\lambda))$.

Mauldin and Urbanski [6] have distinguished different types of CIFS according to the value of the pressure at $t = \theta(\lambda)$. Following their terminology, the system $S(\lambda)$ is said to be regular if $P(\lambda, \theta(\lambda)) \geq 0$, and irregular otherwise. More precisely, $S(\lambda)$ is called

hereditarily regular if $P(\lambda, \theta(\lambda)) = \infty$;
strongly regular if $0 < P(\lambda, \theta(\lambda)) < \infty$;

and we coin $S(\lambda)$ as being

critically regular when $P(\lambda, \theta(\lambda)) = 0$.

Mauldin and Urbanski proved that the Hausdorff dimension $h(\lambda)$ of the limit set $J(\lambda)$ is entirely determined by the graph of the pressure function. Indeed, if $S(\lambda)$ is regular, then $h(\lambda)$ is the unique zero of the pressure function $P(\lambda, \cdot)$. If not, then $h(\lambda) = \theta(\lambda)$. In view of (1) this means that we have the following.

**Theorem 1.** *The (linear) CIFS $S(\lambda)$ defined above is regular if and only if the equation $\psi(\lambda, t) = 1$ has a real solution. If such a solution exists, then it is equal to the Hausdorff dimension $h(\lambda)$ of the limit set $J(\lambda)$ of $S(\lambda)$.*

## 2. Phase transitions

We now describe precisely what we mean by 'phase transition' and give some concrete examples of this phenomenon.

**Definition 1.** We say that a family $S$ undergoes a phase transition at $\lambda_0 \in \Lambda$ if there is a parameter $\lambda$ in every neighbourhood of $\lambda_0$ at which $S(\lambda)$ is of a different type than $S(\lambda_0)$.

More precisely, given that $S(\lambda_0)$ is of type $T_0$, we say that a family $S$ experiences a phase transition at $\lambda_0$ from a $T_0$-system to type $T$ systems ($T \neq T_0$) if there is a continuous path $\gamma : [0,1] \to \Lambda$ originating from $\lambda_0 = \gamma(0)$ such that $S(\gamma(t))$ has type $T$ for every $t > 0$.

**Example 1.** (Phase transitions from irregular to hereditarily regular systems; breakdown of the real-analyticity of the Hausdorff dimension function)

Let $I = \{(n,k) \in \mathbb{N} \times \mathbb{N} \,|\, k \leq 2^{n^2-1}\}$ and $S(\lambda) = \{f_{n,k,\lambda} : \mathbb{C} \to \mathbb{C} \,|\, (n,k) \in I\}$, where $f_{n,k,\lambda}(z) = c_{n,k}(\lambda) z + b_{n,k}(\lambda)$, with $c_{n,k}(\lambda) = \lambda^{n^{3/2}} 2^{-(n^2+n)}$ and the $b_{n,k}$'s to be specified later. Then

$$\psi(\lambda, t) = \frac{1}{2} \sum_{n \in \mathbb{N}} \exp\Big(n^2(1-t)\log 2 + n^{3/2} t \log|\lambda| - nt \log 2\Big).$$

One deduces from this that $\theta(\lambda) = 1$ for every $\lambda$. Moreover, $\psi(\lambda, \theta(\lambda)) \leq 1/2$ whenever $|\lambda| \leq 1$, the equality prevailing if and only if $|\lambda| = 1$. Hence $S(\lambda)$ is irregular and $h(\lambda) = \theta(\lambda) = 1$ for every $|\lambda| \leq 1$. However, $\psi(\lambda, \theta(\lambda)) = \infty$ when $|\lambda| > 1$. Thus, $S(\lambda)$ is hereditarily regular and $h(\lambda) > \theta(\lambda) = 1$ for every $|\lambda| > 1$.

We are therefore in the presence of phase transitions from irregular to hereditarily regular systems at every point of the unit circle $C(0,1)$. This is accompanied by a breakdown of the real-analyticity of the Hausdorff dimension function on $C(0,1)$, as this function is constant (equal to 1 on

$\overline{\mathbb{D}(0,1)}\backslash\{0\})$ but non-constant on every neighbourhood of any point of $C(0,1)$.

Note that it is possible to find analytic functions (in fact, even constant functions) $b_{n,k}$'s such that the OSC is satisfied for every $|\lambda| < \sqrt{2}$ through $U(\lambda) = \mathbb{D}(0,1)$, the open unit disk. Indeed, the sum of the Jacobians of the $f_{n,k,\lambda}$'s is $\psi(\lambda, 2)$, and it turns out that $\psi(\lambda, 2) < 1$ whenever $|\lambda| < \sqrt{2}$.

**Example 2.** (Simultaneous phase transitions from critically regular to irregular and hereditarily regular systems; breakdown of the real-analyticity of the Hausdorff dimension function)

Substitute the restriction $k \leq 2^{n^2}$ for $k \leq 2^{n^2-1}$ in the previous example. Then

$$\psi(\lambda, t) = \sum_{n\in\mathbb{N}} \exp\Big(n^2(1-t)\log 2 + n^{3/2} t \log|\lambda| - nt\log 2\Big).$$

This means that $\psi(\lambda, t)$ is simply twice what it was in the previous example. Hence $\theta(\lambda) = 1$ for every $\lambda$, and $S(\lambda)$ is hereditarily regular when $|\lambda| > 1$, as before. This time, however, $\psi(\lambda, \theta(\lambda)) \leq 1$ for every $|\lambda| \leq 1$, with equality precisely when $|\lambda| = 1$. This means that $S(\lambda)$ is critically regular when $|\lambda| = 1$, and irregular when $|\lambda| < 1$.

We thereby witness simultaneous phase transitions at every point of $C(0,1)$, from critically regular systems to irregular ones (as $\lambda$ penetrates into $\mathbb{D}(0,1)$) and hereditarily regular ones (as $\lambda$ leaves $\overline{\mathbb{D}(0,1)}$). Once again, this goes with a breakdown in the real-analyticity of the Hausdorff dimension function, since $h(\lambda) = \theta(\lambda) = 1$ for every $|\lambda| \leq 1$, while $h(\lambda) > \theta(\lambda) = 1$ for every $|\lambda| > 1$.

**Example 3.** (Phase transitions from strongly regular to hereditarily regular systems)

Replace in the original example the restriction $k \leq 2^{n^2-1}$ by $k \leq 2^{n^2+1}$. Then

$$\psi(\lambda, t) = 2\sum_{n\in\mathbb{N}} \exp\Big(n^2(1-t)\log 2 + n^{3/2} t \log|\lambda| - nt\log 2\Big).$$

Observe that $\psi(\lambda, t)$ is just 4 times what it was originally. This means that, as before, $\theta(\lambda) = 1$ for every $\lambda$, and $S(\lambda)$ is hereditarily regular when $|\lambda| > 1$. Now, $1 < \psi(\lambda, \theta(\lambda)) \leq 2$ for every $\delta < |\lambda| \leq 1$ and some $\delta > 0$. This implies that $S(\lambda)$ is strongly regular in the annulus $\delta < |\lambda| \leq 1$.

**Example 4.** (Phase transitions from hereditarily regular to strongly regular systems)

Replace in the original example the $c_{n,k}$'s with $c_{n,k}(\lambda) = \lambda^n 2^{-n^2}$. Then

$$\psi(\lambda, t) = \frac{1}{2}\sum_{n\in\mathbb{N}} \exp\Big(n^2(1-t)\log 2 + nt\log|\lambda|\Big).$$

As earlier, $\theta(\lambda) = 1$ for every $\lambda$. Moreover, $\psi(\lambda, \theta(\lambda)) = \infty$ for each $|\lambda| \geq 1$, and thereby $S(\lambda)$ is hereditarily regular in this region. However, $S(\lambda)$ is strongly regular whenever $2/3 < |\lambda| < 1$, for $\psi(\lambda, \theta(\lambda)) = |\lambda|/(2(1 - |\lambda|))$.

**Example 5.** (Simultaneous phase transitions from critically regular to irregular and strongly regular systems)

Set $I = \mathbb{N}$ and choose a sequence of positive real numbers $\{c_n\}_{n\in\mathbb{N}}$ so that $\sum_{n\in\mathbb{N}} c_n = 1$ and $\sum_{n\in\mathbb{N}} c_n^t = \infty$ for every $0 \leq t < 1$. Let $S(\lambda) = \{f_{n,\lambda} : \mathbb{C} \to \mathbb{C} | \, n \in \mathbb{N}\}$ consist of the similarities $f_{n,\lambda}(z) = c_n(\lambda)z + b_n(\lambda)$, where $c_n(\lambda) = \lambda c_n$ and the $b_n$'s will be specified later. Then

$$\psi(\lambda, t) = |\lambda|^t \sum_{n\in\mathbb{N}} c_n^t.$$

As in all the examples given so far, $\theta(\lambda) = 1$ for every $\lambda$. Furthermore, $\psi(\lambda, \theta(\lambda)) = |\lambda|$. This shows that $S(\lambda)$ is critically regular when $|\lambda| = 1$, irregular when $|\lambda| < 1$ and strongly regular when $|\lambda| > 1$.

The phase transitions that take place on $C(0, 1)$ are, for similar reasons as before, accompanied by a breakdown in the real-analyticity of $h(\lambda)$.

Note that it is possible to find analytic functions (in fact, even constant functions) $b_n$'s such that the OSC is satisfied for $|\lambda| < 1/\sqrt{\sum_{n\in\mathbb{N}} c_n^2}$.

We now present an example where the function $\theta$ is not constant.

**Example 6.** Replace in the original example the expression for the functions $\{c_{n,k}\}$ by $c_{n,k}(\lambda) = \lambda^{\frac{1}{4}n^{3/2}} 2^{-(\lambda n^2+n)}$, and let $\Lambda = \{\lambda \in \mathbb{C} | \, Re(\lambda) > 1/2\}$. Then

$$\psi(\lambda, t) = 2 \sum_{n\in\mathbb{N}} \exp\Big(n^2(1 - tRe(\lambda)) \log 2 + \frac{1}{4} n^{3/2} t \log |\lambda| - nt \log 2\Big).$$

In this case, $\theta(\lambda) = 1/Re(\lambda)$. Moreover, $\psi(\lambda, 1/Re(\lambda)) = \infty$ for every $|\lambda| > 1$. This means that $S(\lambda)$ is hereditarily regular whenever $|\lambda| > 1$. However, $\psi(\lambda, 1/Re(\lambda)) < \infty$ for every $|\lambda| \leq 1$. Observe further that $\psi(\lambda, 1/Re(\lambda)) = 2 \sum_{n\in\mathbb{N}} 2^{-n/Re(\lambda)}$ on $\Lambda \cap C(0, 1)$ and is hence a continuous increasing function of $Re(\lambda)$ there, with $2/3 < \psi(\lambda, 1/Re(\lambda)) \leq 2$ and equality when $\lambda = 1$. It is also easy to see that the two complex conjugate parameters $\lambda_c, \overline{\lambda_c} \in \Lambda \cap C(0, 1)$ such that $\psi(\lambda, 1/Re(\lambda)) = 1$ are determined by the relation $Re(\lambda_c) = \ln 2/\ln 3$. (Henceforth, we assume that $\lambda_c$ has a positive imaginary part.) The CIFS $S(\lambda_c)$ and $S(\overline{\lambda_c})$ are critically regular, and there are phase transitions at $\lambda_c$ and $\overline{\lambda_c}$ to hereditarily regular systems as $\lambda$ leaves $\overline{I\!D(0, 1)}$, to irregular systems as $\lambda$ moves away from these points along $C(0, 1)$ in the direction of $1/2 + i\,(\pm\sqrt{3}/2)$, respectively, and to strongly regular systems as $\lambda$ moves along $C(0, 1)$ in the direction of 1. There are also phase transitions at every point of the arc $(\lambda_c, \overline{\lambda_c})$ from strongly regular to hereditarily regular systems, and transitions at every point of the arcs $(1/2 + i\,\sqrt{3}/2, \lambda_c)$ and $(1/2 - i\,\sqrt{3}/2, \overline{\lambda_c})$ from irregular to hereditarily regular systems as $\lambda$ quits $\overline{I\!D(0, 1)}$.

The restriction $Re(\lambda) > 1/2$ ensures that $\psi(\lambda, 2) < 1$ on $\Lambda$, and consequently that constant functions $b_{n,k}$'s can be found so that $S(\lambda)$ satisfies the OSC with $U(\lambda) = I\!\!D(0,1)$.

We will observe in the next section that phase transitions from hereditarily or strongly regular systems to irregular or critically regular systems are forbidden when the function $\theta$ is continuous.

Although we are not aware of instances of transitions from irregular to critically or strongly regular systems, there does not seem to be any obstruction to these.

## 3. Conditions for the preservation of the real-analyticity

Fix $\lambda_0 \in \Lambda$. For each $\lambda \in \Lambda$, let

$$p(\lambda) = \inf_{i\in I} \left| \frac{c_i(\lambda)}{c_i(\lambda_0)} \right| \qquad \text{and} \qquad q(\lambda) = \sup_{i\in I} \left| \frac{c_i(\lambda)}{c_i(\lambda_0)} \right|.$$

It is then clear that

$$p(\lambda)^t \psi(\lambda_0, t) \le \psi(\lambda, t) \le q(\lambda)^t \psi(\lambda_0, t). \tag{2}$$

Thus, if $q(\lambda) < \infty$, then $\psi(\lambda_0, t) < \infty$ implies that $\psi(\lambda, t) < \infty$. Therefore $\theta(\lambda) \le \theta(\lambda_0)$.

Similarly, if $p(\lambda) > 0$, then $\psi(\lambda, t) < \infty$ implies that $\psi(\lambda_0, t) < \infty$. Hence, in this case, $\theta(\lambda) \ge \theta(\lambda_0)$.

In particular, if both conditions are simultaneously satisfied, then $\psi(\lambda, t) = \infty$ if and only if $\psi(\lambda_0, t) = \infty$ for every $t$, and $\theta(\lambda) = \theta(\lambda_0)$. If this turns out to be the case for every $\lambda \in \Lambda$, then the function $\theta$ is constant on $\Lambda$. This is the case in Example 5. Although these conditions are sufficient for the constancy of $\theta$, they are not necessary, as the first four examples in the previous section show.

### 3.1. **The irregular case.**

**Lemma 1.** *Assume that the function $\theta$ is constant. Suppose also that $S(\lambda_0)$ is irregular for some $\lambda_0 \in \Lambda$. If $q(\lambda) < \psi(\lambda_0, \theta)^{-1/\theta}$ for some $\lambda \in \Lambda$, then $S(\lambda)$ is irregular.*

This follows immediately from the rightmost inequality in (2). In particular, if the above condition holds on an open neighbourhood $\Lambda_0$ of $\lambda_0$, then $S(\lambda)$ is irregular on that neighbourhood, and $h = \theta$ on $\Lambda_0$. This is the case in Example 5 with $\lambda_0 \in \Lambda_0 = I\!\!D(0,1)$.

### 3.2. **The strongly and hereditarily regular cases.**

**Lemma 2.** *Assume that $\theta$ is continuous and that $S(\lambda_0)$ is strongly or hereditarily regular for some $\lambda_0 \in \Lambda$. Then there is a neighbourhood $\Lambda_0$ of $\lambda_0$ on which every $S(\lambda)$ is either strongly or hereditarily regular.*

*Proof.* Indeed, as $\psi(\lambda_0,\theta(\lambda_0)) > 1$, there is a finite subset $F$ of $I$ such that $\sum_{i\in F}|c_i(\lambda_0)|^{\theta(\lambda_0)} > 1$. It follows from the continuity of the $c_i$'s that there exists $\delta > 0$ so that $\psi(\lambda,\theta(\lambda)) \geq \sum_{i\in F}|c_i(\lambda)|^{\theta(\lambda)} > 1$ for every $\lambda \in \mathbb{D}(\lambda_0,\delta) \subseteq \Lambda$. □

This explains the absence of phase transitions from hereditarily or strongly regular systems to irregular or critically regular systems when $\theta$ is continuous.

As in the irregular case, we can be a little more specific when $\theta$ is constant.

**Lemma 3.** *Suppose that $\theta$ is constant.*

*If $S(\lambda_0)$ is strongly regular and for some $\lambda$ it holds that $p(\lambda) > \psi(\lambda_0,\theta)^{-1/\theta}$, then $S(\lambda)$ is either strongly regular or hereditarily regular.*

*On the other hand, if $S(\lambda_0)$ is hereditarily regular and $p(\lambda) > 0$ for some $\lambda$, then $S(\lambda)$ is hereditarily regular.*

This is a straightforward consequence of the leftmost inequality in (2).

Finally, we give conditions under which the Hausdorff dimension $h(\lambda)$ of the limit set $J(\lambda)$ depends real-analytically on the parameter $\lambda$ in families of strongly or hereditarily regular systems. The proof of this fact shares some common parts with the proof of Theorem 6.3 in [7]. It is however much simpler and elementary since, due to (1), we use the much simpler formula $\psi(\lambda,t) = 1$, rather than $P(\lambda,t) = 0$, to determine the Hausdorff dimension. Moreover, we do not have to check the analyticity of some Banach space-valued functions, and instead of applying quasi-compactness of associated Perron-Frobenius operators along with the perturbation theory for linear operators, we merely use the implicit function theorem.

**Theorem 2.** *Assume that $S(\lambda_0)$ is strongly or hereditarily regular and that for some $R > 0$,*

$$Q := \sup_{\lambda\in\mathbb{D}(\lambda_0,R)} q(\lambda) < \infty.$$

*Then there exists a neighbourhood $\Lambda_0 \subseteq \Lambda$ of $\lambda_0$ on which the Hausdorff dimension function $h : \Lambda_0 \to (0,\infty)$ is real-analytic.*

*Proof.* For every $k \in I$ consider the function $\psi_k : \mathbb{D}(\lambda_0,R) \to \mathbb{C}$ given by the formula

$$\psi_k(\lambda) = \frac{c_k(\lambda)}{c_k(\lambda_0)}.$$

The assumption on the $q(\lambda)$'s implies that $||\psi_k||_\infty \leq Q$ for all $k \in I$. It follows from Cauchy's inequalities that $|c_k'(\lambda)| \leq 2|c_k(\lambda)|R^{-1}$ for all $k \in I$ and all $\lambda \in \mathbb{D}(\lambda_0,R/2)$. Put $r = R/8Q$. Then for every $k \in I$ and all $\lambda \in \mathbb{D}(\lambda_0,2r)$ we have that

$$\begin{aligned}|\psi_k(\lambda)-1| &= \frac{|c_k(\lambda)-c_k(\lambda_0)|}{|c_k(\lambda_0)|} = \frac{\left|\int_{\lambda_0}^{\lambda} c_k'(z)dz\right|}{|c_k(\lambda_0)|} \leq \frac{\int_{\lambda_0}^{\lambda}|c_k'(z)||dz|}{|c_k(\lambda_0)|} \\ &\leq \frac{\int_{\lambda_0}^{\lambda}2R^{-1}|c_k(z)||dz|}{|c_k(\lambda_0)|} \leq 2R^{-1}Q|\lambda-\lambda_0| \leq \frac{1}{2}.\end{aligned}$$

Let $\log : I\!D(1,1) \to \mathbb{C}$ be the branch of the logarithm uniquely determined by the condition that $\log(1) = 0$. Let $M = \sup\{|\log(z)| : z \in I\!D(1,1/2)\} < \infty$. Then for every $k \in I$ the composition $\log \psi_k : I\!D(\lambda_0, 2r) \to \mathbb{C}$ is well defined and

$$|\log \psi_k(\lambda)| \le M$$

for all $\lambda \in I\!D(\lambda_0, 2r)$. Now, each function $\log \psi_k : I\!D(\lambda_0, 2r) \to \mathbb{C}$ being holomorphic, let

$$\log \psi_k(\lambda) = \sum_{n=0}^{\infty} a_n(k)(\lambda - \lambda_0)^n$$

be its Taylor series expansion on $I\!D(\lambda_0, 2r)$. By Cauchy's inequalities,

$$|a_n(k)| \le \frac{M}{r^n} \tag{3}$$

for all $n \ge 0$. For every $\lambda = x + iy \in I\!D(\lambda_0, r)$ we have

$$\begin{aligned}
\mathrm{Re}\log \psi_k(\lambda) &= \mathrm{Re}\left(\sum_{n=0}^{\infty} a_n(k)\big((x - \mathrm{Re}\lambda_0) + (y - \mathrm{Im}\lambda_0)i\big)^n\right) \\
&= \sum_{p,q=0}^{\infty} \mathrm{Re}\left(a_{p+q}(k)\begin{pmatrix} p+q \\ q \end{pmatrix} i^q\right)(x - \mathrm{Re}\lambda_0)^p (y - \mathrm{Im}\lambda_0)^q \\
&= \sum_{p,q=0}^{\infty} m_{p,q}(k)(x - \mathrm{Re}\lambda_0)^p (y - \mathrm{Im}\lambda_0)^q,
\end{aligned}$$

where, due to (3), $|m_{p,q}(k)| \le |a_{p+q}(k)|2^{p+q} \le Mr^{-(p+q)}2^{p+q}$. Hence, the real-valued function $\mathrm{Re}\log\psi_k$ extends by the same power series expansion $\sum_{p,q=0}^{\infty} m_{p,q}(k)(x - \mathrm{Re}\lambda_0)^p(y - \mathrm{Im}\lambda_0)^q$ to a complex-valued analytic function defined on the polydisk $I\!D_{\mathbb{C}^2}(\lambda_0, r/4)$ and denoted by the same symbol $\mathrm{Re}\log\psi_k$. In addition,

$$|\mathrm{Re}\log\psi_k| \le 4M \quad \text{on} \quad I\!D_{\mathbb{C}^2}(\lambda_0, r/4).$$

So, the function $\zeta_k : I\!D_{\mathbb{C}^2}(\lambda_0, r/4) \to \mathbb{C}$, given by the formula

$$\zeta_k(\lambda) = \mathrm{Re}\log\psi_k(\lambda) + \log|c_k(\lambda_0)|,$$

extends the function $\log|c_k| : I\!D_{\mathbb{C}}(\lambda_0, r/4) \to \mathbb{R}$ to the polydisk $I\!D_{\mathbb{C}^2}(\lambda_0, r/4)$. For every $t \in \mathbb{C}$ we have

$$\begin{aligned}
|\exp\big(t\zeta_k(\lambda)\big)| &= \exp\mathrm{Re}\Big(t\mathrm{Re}\log\psi_k(\lambda) + t\log|c_k(\lambda_0)|\Big) \\
&= \exp\Big(\mathrm{Re}\big(t\mathrm{Re}\log\psi_k(\lambda)\big)\Big) \cdot |c_k(\lambda_0)|^{\mathrm{Re}\,t} \\
&\le \exp\big(|t\mathrm{Re}\log\psi_k(\lambda)|\big)|c_k(\lambda_0)|^{\mathrm{Re}\,t} \\
&\le \exp\big(|t||\mathrm{Re}\log\psi_k(\lambda)|\big)|c_k(\lambda_0)|^{\mathrm{Re}\,t} \\
&\le \exp(4M|t|)|c_k(\lambda_0)|^{\mathrm{Re}\,t}.
\end{aligned}$$

Hence the series

$$\tilde{\psi}(\lambda,t)=\sum_{k\in I}\exp\bigl(t\zeta_k(\lambda)\bigr)$$

is uniformly absolutely convergent on all compact subsets of $I\!\!D_{\mathbb{C}^2}(\lambda_0,r/4)\times\{t\in\mathbb{C}:\mathrm{Re}\,t>\theta(\lambda_0)\}$ and defines an analytic function on this latter set. Moreover, the restriction of $\tilde{\psi}$ to the set $I\!\!D_{\mathbb{C}}(\lambda_0,r/4)\times(\theta(\lambda_0),\infty)$ coincides with $\psi(\lambda,t)$. We thus have for all $(\lambda,t)\in I\!\!D_{\mathbb{C}^2}(\lambda_0,r/4)\times\{t\in\mathbb{C}:\mathrm{Re}\,t>\theta(\lambda_0)\}$ that

$$\frac{\partial\tilde{\psi}}{\partial t}(\lambda,t)=\sum_{k\in I}\exp\bigl(t\zeta_k(\lambda)\bigr)\zeta_k(\lambda).$$

From the above we conclude that the function $\psi: I\!\!D_{\mathbb{C}}(\lambda_0,r/4)\times(\theta(\lambda_0),\infty)$ is real-analytic and

$$\frac{\partial\psi}{\partial t}(\lambda,t)=\sum_{k\in I}|c_k(\lambda)|\log|c_k(\lambda)|<0. \tag{4}$$

Since the CIFS $S(\lambda_0)$ is either strongly or hereditarily regular, there exists a unique $h(\lambda_0)\in\mathbb{R}$ such that $\psi(\lambda_0,h(\lambda_0))=1$. According to Theorem 1, $h(\lambda_0)$ coincides with the Hausdorff dimension of the limit set $J(\lambda_0)$ of $S(\lambda_0)$. It then follows from the implicit function theorem for real-analytic functions and from (4) that there exists $\delta\in(0,r/4)$ and a real-analytic function $h: I\!\!D_{\mathbb{C}}(\lambda_0,\delta)\to\mathbb{R}$ such that $\psi(\lambda,h(\lambda))=1$. Invoking Theorem 1 again, we see that $h(\lambda)$ is the Hausdorff dimension of the limit set $J(\lambda)$. □

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