

# ERGODIC THEORY FOR HOLOMORPHIC ENDOMORPHISMS OF COMPLEX PROJECTIVE SPACES

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ABSTRACT. For every holomorphic endomorphism  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  of a complex projective space  $\mathbb{P}^k, k \geq 1$ , there exists a positive number  $\kappa(f) > 0$  such that if  $\phi : J \rightarrow \mathbb{R}$  is a Hölder continuous function with  $\sup(\phi) - \inf(\phi) < \kappa(f)$ , then  $\phi$  admits a unique equilibrium state  $\mu_\phi$  on  $J$ . This equilibrium state is equivalent to a fixed point of the normalized dual Perron-Frobenius operator. In addition, the dynamical system  $(f, \mu_\phi)$  is K-mixing, whence ergodic. Proving almost periodicity of the corresponding Perron-Frobenius operator is the main technical task of the paper. It requires to produce sufficiently many “good” inverse branches and to control the distortion of the Birkhoff sums of the potential  $\phi$ . In the case when the Julia set  $J$  does not intersect any periodic irreducible variety of the critical set of  $f$ , we have that  $\kappa_f = \log d$ .

## 1. INTRODUCTION

The thermodynamic formalism for holomorphic endomorphisms of the Riemann sphere  $\hat{\mathbb{C}}$  and Hölder continuous potentials, with sufficiently small oscillation, was originated in [DU]. The existence and uniqueness of equilibrium states was proved there, see also [Pr]. The corresponding Perron-Frobenius operator was shown to be almost periodic and the equilibria were shown to be K-mixing. Later ([DPU], [Ha]) more refined mixing and stochastic properties of these equilibria were established. The natural question then arises about the existence and uniqueness of equilibria in the higher dimensional case, namely, for complex projective spaces of an arbitrary dimension. Up to our knowledge, so far, in such generality, only the case of the potential  $\phi$  identically equal to zero has been treated. The existence and uniqueness of its equilibria (e. i. the measure of maximal entropy) were proved in [BD]. Some stochastic properties were established in [CLB] and [Du].

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Our goal in this paper is build the thermodynamic formalism for all holomorphic endomorphisms  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  of all complex projective spaces of an arbitrary dimension  $k \geq 1$ , and, what we would like to emphasize equally strongly, for all Hölder continuous potentials  $\phi : J \rightarrow \mathbb{R}$ , with sufficiently small value (depending only on the endomorphisms  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  and denoted in the sequel by  $\kappa_f$ ) of their oscillation  $\sup(\phi) - \inf(\phi)$ .  $J$  in here and in the sequel, throughout the paper, denotes the Julia set of the map  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ , i. e. the topological support of the measure of maximal entropy. This class of potentials is large indeed: it contains the restrictions to  $J$  of all Hölder continuous functions  $\phi : \mathbb{P}^k \rightarrow \mathbb{P}^k$  such that  $\sup(\phi) - \inf(\phi) < \kappa_f$ . As a matter of fact, if the Julia set  $J$  does not intersect periodic irreducible varieties of the critical set of  $f$ , then we can take  $\kappa_f$  as large as possible, namely equal to  $\log \deg(f)$ . We observe (see Corollary 4.4 that if  $k$ , the dimension of the projective space, is equal to 2, then this intersection consists of finitely many critical periodic orbits only, whence  $\kappa_f$  is easier to estimate.

In order to build the thermodynamic formalism for  $f$  and  $\phi$  we, apart from the methods of algebraic geometry, employ the techniques of ergodic theory, and we cope especially hard with estimating the distortion of the Birkhoff sums of the potential  $\phi$ . This task is entirely absent in the case of the measure of maximal entropy; the distortion is always zero. For more general potentials, the ones we are dealing with, the situation is just opposite; the issue of bounded distortion becomes the central issue of our approach. As a matter of fact, the bounded distortion for, in a sense, most inverse branches, and the existence of sufficiently many “good” inverse branches, are the two main tools used to produce upper and lower bounds of iterates of corresponding Perron-Frobenius operators. This is the main technical theme of the paper. We would like to mention that in the case of the measure of maximal entropy, this issue actually trivializes; the function identically equal to one is then a fixed point of the Perron-Frobenius operator for free.

A basic notion of ergodic theory is that of metric (Kolmogorov-Sinaj) entropy  $h_\mu(f)$  of a probability  $f$ -invariant measure. The basic notion of thermodynamic formalism is that of topological pressure  $P(\phi) = P(f, \phi)$  introduced by D. Ruelle in [Ru1]. We define them both in Section 2. Their alternative definitions and properties can be found for instance in [Wa] and [PU]. The formula relating these two, seemingly independent, concepts is the celebrated Variational Principle stating that

$$(1.1) \quad P(\phi) = \sup \left\{ h_\mu(f) + \int \phi d\mu \right\},$$

where the supremum is taken over all Borel probability  $f$ -invariant measures  $\mu$ . The measures  $\mu$  for which  $h_\mu(f) + \int \phi d\mu = P(\phi)$  are called equilibrium states for the potential  $\phi$ . We prove the following.

**Theorem 1.1.** *For every holomorphic endomorphism  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  of a complex projective space  $\mathbb{P}^k$ ,  $k \geq 1$ , there exists a positive number  $\kappa(f) > 0$  such that if  $\phi : J(\phi) \rightarrow \mathbb{R}$  is a Hölder continuous function with  $\sup(\phi) - \inf(\phi) < \kappa(f)$ , then*

$\phi$  admits a unique equilibrium state  $\mu_\phi$  on  $J$ . This equilibrium state is equivalent to a fixed point of the normalized dual Perron-Frobenius operator. In addition the dynamical system  $(f, \mu_\phi)$  is  $K$ -mixing, whence ergodic. In the case when the Julia set  $J$  does not intersect any periodic irreducible varieties of the critical set of  $f$ , we have that  $\kappa_f = \log d$ .

Due to Yomdin's work [Yo], the existence of equilibria is true for all  $C^\infty$  smooth endomorphisms of compact differentiable manifolds. In fact, Yomdin proved that the entropy function is upper semi-continuous. Our proof of existence of equilibria is entirely different; in particular we do not use upper semi-continuity of the entropy function. With this respect we prove more than merely the existence. We construct an equilibrium as a fixed point of the normalized dual Perron-Frobenius operator. This gives a piece of a valuable information about the structure of this equilibrium and, along with a detailed analysis of iterates of the Perron-Frobenius operator, allows us to conclude the uniqueness of the equilibrium. We do it by showing that the topological pressure function is differentiable. We would like to remark that uniqueness of equilibria is not in general true, and, without appropriate constrains on the potentials, fails even for so smooth maps as rational functions of the Riemann sphere. The  $K$ -mixing property is due to almost periodicity of the corresponding Perron-Frobenius operator. We provide a more detailed description of the allowed oscillation  $\kappa(f)$  in Section 2. We also provide sufficient conditions for  $\kappa(f)$  to be equal to  $\log \deg(f)$ , nearly as good as in [DU]. The proof of Theorem 1.1 contains two additional ingredients we would like to bring reader's attention to. Firstly, we prove a form of uniformly subexponentially slow increase of local degrees of iterates of the map  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ . This fact, being crucial for our entire proof, is interesting itself, and is related to some results of Fabre ([Fa]); in fact it generalizes and simplifies one of his main propositions. Secondly, motivated by the argument of M. Gromov from [Gr], we prove that the topological pressure is not larger than the logarithm of the eigenvalue of the dual to the Perron-Frobenius operator. The proof is based on Lelong's Lemma and makes extensive use of geometry of projective spaces.

Our paper is organized as follows. In Preliminaries, Section 2, we prove an appropriate Hölder continuous extension result, and we define topological pressure and metric entropy. In Section 3, Contracting Inverse Branches, we construct sufficiently many (as it turns out later in the course of the paper) exponentially shrinking inverse branches. We refer to them as good ones. In Section 4 we introduce and discuss the maximally allowed oscillation  $\kappa(f)$  for the potential  $\phi$ , we introduce the corresponding Perron-Frobenius operator, we prove the existence of the "geometric" Gibbs state  $m_\phi$  (formula 4.10), i.e. an eigenmeasure of the dual operator, and the corresponding positive eigenvalue  $\lambda$ . In Section 5, Uniform Bounds of Iterates of the Perron-Frobenius Operator, in a sense a central section of the paper, we establish upper and lower uniform bounds of iterates of the Perron-Frobenius operator. Section 6, Almost Periodicity of the Perron-Frobenius Operator, is devoted to proving almost periodicity of this operator, and its uniform version, needed for the proof of the uniqueness of

the equilibrium state  $\mu_\phi$ . Consequently, we produce a continuous fixed point  $\rho_\phi$  of the Perron-Frobenius operator, and the  $f$ -invariant measure  $\mu_\phi = \rho_\phi m_\phi$  is our candidate for the only equilibrium state of the potential  $\phi$ . Based on almost periodicity of the Perron-Frobenius operator, we establish its spectral properties, and in particular, we show that its iterates converge uniformly. At the end of this section we deduce from this convergence of the Perron-Frobenius operator, K-mixing of the dynamical system  $(f, \mu_\phi)$ , whence its ergodicity. We also prove the decay of correlations. In Section 7, Pressure versus Eigenvalue, developing the idea of Gromov from [Gr], we prove equality of the topological pressure  $P(\phi)$  and the logarithm  $\log \lambda$ . Section 8, Existence and Uniqueness of Equilibrium States, is devoted to proving existence and uniqueness of equilibrium states. The existence part is done by showing that the measure  $\mu_\phi$  is an equilibrium state. The idea of the proof of uniqueness is to show the differentiability of the pressure function and to use the uniform periodicity of the Perron-Frobenius operator established in Section 6. We may therefore state there the main result, existence and uniqueness of equilibrium states, of our paper. The last section of our paper, Local Degree, contains the proof of uniformly subexponentially slow growth of local degrees of iterates of the map  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ .

## 2. PRELIMINARIES

**Lemma 2.1.** *Suppose that  $(X, \rho)$  is a compact metric space and  $F$  is a closed subset of  $X$ . If  $g : F \rightarrow \mathbb{R}$  is a Hölder continuous function with an exponent  $\alpha \in (0, 1)$ , then there exists a Hölder continuous function  $\tilde{g} : X \rightarrow \mathbb{R}$  with the same exponent  $\alpha$  and with the following two properties:  $\tilde{g}|_F = g$ ,  $\sup(\tilde{g}) = \sup(g)$  and  $v_\alpha(\tilde{g}) \leq 2v_\alpha(g)$ .*

*Proof.* Let  $H \geq 0$  be a Hölder constant of the function  $g$ , i.e.  $|g(y) - g(x)| \leq H\rho^\alpha(x, y)$  whenever  $x, y \in F$ . For every  $x \in X$ , put

$$g_*(x) = \inf_{a \in F} \{g(a) + H\rho^\alpha(x, a)\},$$

and set

$$\tilde{g}(x) = g_*(x) - H\rho^\alpha(x, F).$$

If  $x \in F$ , then  $g_*(x) \leq g(x)$  and  $\rho^\alpha(x, F) = 0$ , and consequently,  $\tilde{g}(x) \leq g(x)$ . Also, for every  $a \in F$ ,  $g(a) + H\rho^\alpha(x, a) \geq g(x)$ , and therefore,  $g_*(x) \geq g(x)$ . Thus  $\tilde{g}(x) \geq g(x)$ . Hence  $\tilde{g}(x) = g(x)$ , which means that  $\tilde{g}|_F = g$ . Now, fix two arbitrary points  $x, y \in X$ . Then

$$\begin{aligned} g_*(x) &\leq \inf_{a \in F} \{g(a) + H(\rho(a, y) + \rho(y, x))^\alpha\} \leq \inf_{a \in F} \{g(a) + H(\rho^\alpha(y, a) + \rho^\alpha(x, y))\} \\ &= g_*(y) + H\rho^\alpha(x, y). \end{aligned}$$

Hence  $g_*(x) - g_*(y) \leq H\rho^\alpha(x, y)$ , and changing the roles of  $x$  and  $y$ , we get that  $|g_*(x) - g_*(y)| \leq H\rho^\alpha(x, y)$ . Now,  $\rho(x, F) \leq \rho(x, y) + \rho(y, F)$ , and consequently,

$$\rho^\alpha(x, F) \leq (\rho(x, y) + \rho(y, F))^\alpha \leq \rho^\alpha(x, y) + \rho^\alpha(y, F).$$

Thus, changing also the roles of  $x$  and  $y$ ,  $|H\rho^\alpha(x, F) - H\rho^\alpha(y, F)| \leq H\rho^\alpha(x, y)$ . So,  $|\tilde{g}(x) - \tilde{g}(y)| \leq 2H\rho^\alpha(x, y)$ , meaning that  $\tilde{g}$  is Hölder continuous with the exponent  $\alpha$ .

Obviously,  $\sup(\tilde{g}) \geq \sup(g)$ . To show the opposite inequality, take an arbitrary  $x \in X$  and then take  $b \in F$  such that  $\rho(x, F) = \rho(x, b)$ . Then  $\tilde{g}(x) \leq g(b) + H\rho^\alpha(x, b) - H\rho^\alpha(x, F) = g(b) \leq \sup(g)$ . Thus,  $\sup(\tilde{g}) \leq \sup(g)$ . We are done.  $\square$

A dynamical system  $T : X \rightarrow X$  is called topologically exact if and only if for every non-empty open set  $U$  in  $X$ , there exists  $l \geq 0$  such that  $T^l(U) = X$ . Since it is easy to see that the map  $f : J \rightarrow J$  is locally eventually onto (the union of all forward iterates of every non-empty open subset of  $J$  covers  $J$ ), and since  $J$  is contained in the closure of repelling periodic orbits, we have the following.

**Proposition 2.2.** *If  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  is a holomorphic endomorphism, then the dynamical system  $f : J \rightarrow J$  is topologically exact.*

Since the set  $J$  is completely invariant, iterating (4.10), and making use of the topological exactness of the map  $f : J \rightarrow J$ , we obtain the following.

**Proposition 2.3.** *The measure  $m_\phi$  is positive on non-empty open subsets of  $J$ ; in other words,  $\text{supp}(m_\phi) = J$ .*

As we have already explained in the introduction, our main result is related to topological pressure and Variational Principle. We want to define and formulate them now. An interested reader is however encouraged to consult [Ru2], [Wa], [Bo] or [PU] for example, to find a comprehensive treatment of these concepts.

Let us begin with recalling the notion of entropy of a measure-theoretic dynamical system. Suppose that  $T : X \rightarrow X$  is a continuous map of a compact metric space  $(X, d)$  and that  $\mu$  is a Borel  $T$ -invariant probability measure on  $X$ .  $T$ -invariance means that if  $A$  is a Borel subset of  $X$  then  $\mu(f^{-1}(A)) = \mu(A)$ . We call  $\mu$  (or  $T$ ) *ergodic* if the only Borel invariant subsets of  $T$  (i.e. satisfying  $T^{-1}(A) = A$ ) are either of measure 0 or 1. Given  $n \geq 0$  we define the metric  $d_n$  on  $X$  by setting

$$d_n(x, y) = \max\{d(T^i(x), T^i(y)) : 0 \leq i \leq n\}.$$

Denote by  $B_n(x, r)$  the open ball in the metric  $d_n$  centered in  $x$  and with radius  $r$ . If the measure  $\mu$  is ergodic, then (see [BK]) for  $\mu$ -a.e. point  $x \in X$  the limit

$$\lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{\log \mu(B_n(x, r))}{n}$$

exists, is independent of  $x$ , and this limit is called the (metric) entropy of the system  $T$  with respect to the measure  $\mu$ . This entropy is denoted by  $h_\mu(T)$ . Roughly speaking it measures the exponential rate of decay of the measure of points that stay  $\varepsilon$ -close to the point  $x$  under forward iterates of  $f$ . Observe that the entropy of any isometry with respect to any invariant measure is always equal to zero, whereas the entropy of the map  $z \mapsto z^d$ ,  $z \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ , with respect to the normalized arc-length measure on  $S^1$ , is equal to  $\log 2$ . Usually a different, more classical approach is undertaken to define the entropy  $h_\mu(T)$  (see [BK], [Wa1], and [HK] for example), the

one chosen here is probably the fastest and, at the same time, it reflects in a better way, the nature of entropy.

In order to introduce topological pressure, we choose one of the fastest methods, and simultaneously the one, we will need in Section 7, Pressure Versus Eigenvalue. For alternative approaches better suited to derive various properties of topological pressure see also the positions quoted above. Consider a continuous mapping  $T : X \rightarrow X$  of a compact metric space  $(X, d)$  and a continuous function  $\phi : X \rightarrow \mathbb{R}$ , called, following physical tradition, a potential. Given  $n \geq 0$  and  $\varepsilon > 0$ , we say that a subset  $F$  of  $X$  is  $(n, \varepsilon)$ -separated if it is separated with respect to the metric  $d_n$ , which means that if  $x$  and  $y$  are two distinct points of  $X$ , then  $d_n(x, y) \geq \varepsilon$ . Fixing now  $\varepsilon > 0$  we consider an arbitrary sequence  $F_n(\varepsilon)$ ,  $n \geq 1$ , of maximal (in the sense of inclusion)  $(n, \varepsilon)$ -separated sets. We then define the topological pressure of the function  $\phi$  with respect to the mapping  $T$  as follows

$$P(T, \phi) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{x \in F_n(\varepsilon)} \exp \sum_{j=0}^{n-1} \phi \circ T^j(x) \right).$$

Topological pressure belongs to topological dynamics, whereas metric entropy is a notion in ergodic theory. The link joining them is given by the following formula called the variational principle (see [Ru2], [Wa], [Wa] and [PU] for example), which we have already mentioned in the introduction.

$$P(T, \phi) = \sup_{\mu} \{h_{\mu}(T) + \int \phi d\mu\},$$

where the supremum is taken over all Borel probability  $T$ -invariant (ergodic) measures of  $X$ .

### 3. CONTRACTING INVERSE BRANCHES

Keeping  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  a holomorphic endomorphism, let  $\text{Crit}(f)$  be the set of all critical points of  $f$ , i. e. the set of such points  $z \in \mathbb{P}^k$  that  $\deg_z f \geq 2$ . We need the following definition.

**Definition 3.1.** *Given an integer  $n \geq 1$  the periodic critical set  $A_n$  is the union of orbits of all irreducible varieties, that are contained in the critical set and are periodic under an iterate  $f^l$  with some  $l \leq n$ . In particular, an orbit of a critical periodic point of period  $l \leq n$  is in the critical periodic set  $A_n$ .*

For all positive integers  $n$  and  $p$  such that  $n > p$  let  $E_n^p$  be the set defined as follows.

**Definition 3.2.**  *$E_n^p$  is the set of all points  $x \in \mathbb{P}^k$  for which there exists a non-negative integer  $i \leq n - 1$  such that  $f^i(x) \in A_p$ .*

Our main result concerning the behavior of the local degree is the following.

**Proposition 3.3.** *For every  $\beta > 0$  there exist  $p = p(\beta)$  and  $N = N(\beta)$  such that for every  $n \geq N$  and for every  $x \notin E_n^p$  we have*

$$\#\{j \leq n : f^j(x) \in C\} \leq \beta n.$$

Because of the very technical and combinatorial nature of the proof of this proposition, and to focus first on the main stream of arguments, we decided to move our presentation of this proof to Section 9, the last section of the paper.

Now, take  $\gamma \in (0, 1)$ . It follows from Proposition 3.3 that there exist two least integers  $q_1(\gamma)$  and  $q_2(\gamma)$  such that if  $q \geq q_2(\gamma)$ ,  $z \in \mathbb{P}^k$ ,  $j \geq q_2(\gamma)$ , and  $f^j(z) \notin A_{q_1(\gamma)}$ , then

$$(3.1) \quad \deg_z(f^j) \leq \gamma^{-j}.$$

Let us record the following obvious observation.

**Lemma 3.4.** *The two functions  $(0, 1)\gamma \mapsto q_1(\gamma), q_2(\gamma)$  are weakly increasing, the function  $(0, 1)\gamma \mapsto A_{q_1(\gamma)}$  is weakly ascending, and, consequently, there exists  $\gamma_* \in (0, 1)$  such that these three functions are constant throughout the interval  $(0, \gamma_*)$ .*

Put

$$A_\gamma := A_{q_1(\gamma)}.$$

We normalize the Fubini-Study metric  $\rho$  on  $\mathbb{P}^k$  so that the area  $A$  of any ball of radius 1 on a projective line is equal to 1. For any two distinct points  $a, b \in \mathbb{P}^k$  denote by  $\Gamma_{a,b}$  the projective line passing through  $a$  and  $b$ . Our main (technical) result in this section is the following.

**Lemma 3.5.** *For every  $\gamma \in (0, 1)$ , every integer  $s \geq 1$ , every integer  $q \geq q_2(\gamma)$ , and every  $\eta > 0$  there exists  $R(\eta) = R(\gamma, s, q; \eta) \in (0, 1)$  such that for every  $z \in \mathbb{P}^k \setminus B(A_\gamma, \eta)$  there exists a dense subset  $D(z) = D(\gamma, s, q, \eta; z)$  of  $B(z, R(\eta)) \subset \mathbb{P}^k$  such that for every  $\xi \in D(z)$  and for all  $n \geq s$ , there exists a family  $W_n(q, \eta, z, \xi)$  of connected components of  $f^{-qn}(B(z, R(\eta)) \cap \Gamma_{z, \xi})$  with the following properties.*

- (a<sub>n</sub>) *If  $V \in W_{n+1}(q, \eta, z, \xi)$ , then  $f^q(V) \in W_n(q, \eta, z, \xi)$ .*
- (b<sub>n</sub>)  *$\max\{\text{diam}(V) : V \in W_n(q, \eta, z, \xi)\} \leq \gamma^{n/2}$ .*
- (c<sub>n</sub>) *There is  $c > 0$  such that if  $n \geq s$ , then*

$$\#(Z_n(q, \eta, z, \xi) \setminus W_n(q, z, \xi)) \leq \gamma^{-sq}(4c\gamma^{-3qn} + (k-1)^{d+1}q)d^{(k-1)q(n+1)},$$

*where  $Z_n(q, \eta, z, \xi)$  is the family of all connected components of all the sets of the form  $f^{-q}(V)$ , where  $V \in W_{n-1}(q, \eta, z, \xi)$ .*

( $d_n$ ) For every  $n \geq s$  and every  $V \in W_n(q, \eta, z, \xi)$ , we have  $V \cap f^q(\text{Crit}(f^q)) = \emptyset$ , and  $f^{qn}|_V$  is at most  $\gamma^{-sq}$ -to-1.

*Proof.* In virtue of (3.1) there exists  $\hat{R}(\eta) > 0$  so small that if  $z \in \mathbb{P}^k \setminus B(A_q, \eta)$ ,  $x \in f^{-j}(z)$ , and if  $V'_x$  is the connected component of  $f^{-q}(B(z, \hat{R}(\eta)))$  containing  $x$ , then

$$(3.2) \quad \deg(f^q|_{V'_x}) \leq \gamma^{-sq}$$

and

$$(3.3) \quad \text{diam}(V'_x) \leq \gamma^{q/2} \quad \text{and} \quad A(V'_x) \leq (4c)^{-1}\gamma^{3q}.$$

Since  $\dim(\text{Crit}(f)) = k - 1$ ,  $\deg(\text{Crit}(f)) = (k - 1)^{d+1}$ , and since for every projective line  $\Gamma$  and every  $l \geq 0$ ,  $\deg(f^{-l}(\Gamma)) = d^{(k-1)l}$ , using Bézout's Theorem, we deduce that there exists  $D(z)$ , a dense subset of  $B(z, \hat{R}(\eta))$ , such that for all  $\xi \in D(z)$  and for all  $l \geq 0$ , we have

$$(3.4) \quad \#(\text{Crit}(f) \cap f^{-l}(\Gamma_{z,\xi})) = (k - 1)^{d+1}d^{(k-1)l}.$$

We shall first construct the sets  $W'_n(q, \eta, z, \xi)$  and  $Z'_n(q, \eta, z, \xi)$ ,  $n \geq s$ , recursively such that the conditions ( $a_n$ ), ( $b'_n$ ), ( $c'_n$ ), and ( $d_n$ ) are satisfied. The condition ( $b'_n$ ) is in here the following.

( $b'_n$ ) If  $V' \in W'_n(q, \eta, z, \xi)$ , then  $\text{Area}(V') \leq \gamma^{n+2qs}/4c$ ,

and ( $c'_n$ ) is the same as ( $c_n$ ) with the number  $\gamma^{-sq}$  disappear. We start the recursion by putting

$$W'_s(q, \eta, z, \xi) = Z'_s(q, \eta, z, \xi),$$

where  $Z'_s(q, \eta, z, \xi)$  is defined to consist of all connected components of all sets of the form  $f^{-sq}(B(z, \hat{R}(\eta)) \cap \Gamma_{z,\xi})$ . Now assume that for some  $n \geq s$  the family  $W'_n(q, \eta, z, \xi)$  has been constructed so that condition ( $b'_n$ ) is satisfied. The inductive step is to construct the family  $W'_{n+1}(q, \eta, z, \xi)$  so that the conditions ( $a_n$ ), ( $b'_{n+1}$ ), ( $c_{n+1}$ ), and ( $d_{n+1}$ ) are satisfied. The family  $W'_{n+1}(q, \eta, z, \xi)$  is defined to consist of all connected components  $V'$  of all the sets  $f^{-q}(G)$  with  $G \in W'_n(q, \eta, z, \xi)$  for which

$$(3.5) \quad \text{Area}(V') \leq (4c)^{-1}\gamma^{(n+1)+2qs} \quad \text{and} \quad V' \cap \bigcup_{j=1}^q f^j(\text{Crit}(f)) = \emptyset.$$

Conditions ( $a_n$ ) and ( $b'_{n+1}$ ) are then automatically satisfied as well as the first part of ( $d_{n+1}$ ). If  $n = s$ , then  $f^{qs}|_V$  is at most  $\gamma^{-sq}$ -to-1 by the choice of  $\hat{R}(\eta)$ , so ( $d_{n+1}$ ) is verified. In order to prove ( $c_{n+1}$ ), let us first estimate from above  $\#(f^{-q(n+1)}(\Gamma_{z,\xi}) \cap f^j(\text{Crit}(f)))$  for all  $j = 1, 2, \dots, q$ . Indeed, if  $x \in f^{-q(n+1)}(\Gamma_{z,\xi}) \cap f^j(\text{Crit}(f))$ , then  $x = f^j(c_x)$  with some  $c_x \in \text{Crit}(f)$ . Thus  $c_x \in f^{-q((n+1)+j)}(\Gamma_{z,\xi})$ , and since the function  $x \mapsto c_x$  is 1-to-1, we conclude from (3.4), that

$$\#(f^{-q(n+1)}(\Gamma_{z,\xi}) \cap f^j(\text{Crit}(f))) \leq (k - 1)^{d+1}d^{(k-1)q(n+1)+j} \leq (k - 1)^{d+1}d^{(k-1)q(n+2)}.$$

Hence,

$$\#(f^{-q(n+1)}(\Gamma_{z,\xi}) \cap \bigcup_{j=1}^q f^j(\text{Crit}(f))) \leq (k-1)^{d+1} q d^{(k-1)q(n+2)}.$$

Thus, the number of elements from  $Z'_n(q, \eta, z, \xi)$  that fail to satisfy the second condition of (3.5), is bounded above by  $(k-1)^{d+1} q d^{(k-1)q(n+2)}$ . Since  $\text{Area}(f^{-q(n+1)}(\Gamma_{z,\xi})) = d^{(k-1)q(n+1)}$ , the number of elements from  $Z'_{n+1}(q, \eta, z, \xi)$  that fail to satisfy the first condition of (3.5), is bounded above by

$$4c\gamma^{-(n+1)-2q} d^{(k-1)q(n+1)} \leq 4c\gamma^{-3q(n+1)} d^{(k-1)q(n+1)}.$$

Thus the condition  $(c_{n+1})$  for  $W'$  and  $Z'$  is established, and the inductive construction of the families  $W'_n(q, \eta, z, \xi)$  and  $Z'_n(q, \eta, z, \xi)$  satisfying conditions  $(a_n), (b'_n), (c_n)$ , and  $(d_n)$  is complete. Now, decreasing  $\hat{R}_q$  appropriately (the smaller radius will be called  $R_q$ ) we shall check that condition  $(b_n)$  also holds. Let  $0 < R(\eta) \leq \hat{R}(\eta)$  be sufficiently small as specified later in the course of the proof. For every  $n \geq q$  define  $W_n(q, \eta, z, \xi)$  to consist of all connected components  $V$  of all elements of  $W'_n(q, \eta, z, \xi)$  intersected with  $f^{-n}(B(z, R_q) \cap \Gamma_{z,\xi})$ . Since each element of  $W'_n(q, \eta, z, \xi)$  contains at least one and at most  $\gamma^{-q}$  elements of  $W_n(q, \eta, z, \xi)$ , item  $(c_n)$  follows immediately from the corresponding statement for  $W'_n$ . Conditions  $(a_n)$  and  $(d_n)$  also follow from corresponding statements for families  $W'_n$ . We are to show that  $(b_n)$  holds. We shall specify the value of  $R_q$  at this step. First, fix a positive integer

$$M > \# \left( \Gamma_{z,\xi} \cap \bigcup_{j=1}^q f^j(\text{Crit}(f)) \right)$$

Then fix an integer  $a > 1$  such that  $\gamma^{sq} \log a > 1$ , and let  $0 < R_q < \hat{R}_q$  be so small that

$$0 < \frac{R_q}{\hat{R}_q} < a^{-(M+1)},$$

Now, for all  $p = 0, 1, \dots, M$ , consider the annuli

$$A_p = (B(z, a^{p+1}R_q) \setminus B(z, a^pR_q)) \cap \Gamma_{z,\xi}.$$

By the choice of  $M$ , there exists at least one annulus in this collection, that does not intersect the set  $\bigcup_{j=1}^q f^j(\text{Crit}(f))$ . Let us keep the notation  $A_p$  for this specified annulus. Set

$$D' = B(z, \hat{R}(\eta)) \cap \Gamma_{z,\xi}, \quad D = B(z, R_q) \cap \Gamma_{z,\xi},$$

$$D_1 = B(z, a^p R(\eta)) \cap \Gamma_{z,\xi}, \quad D_2 = B(z, a^{p+1} R_q) \cap \Gamma_{z,\xi}$$

(so  $D \subset D_1 \subset D_2 \subset D'$ ). As above, let  $V \in W_n$  be a connected component of  $f^{-n}(D)$ . Then, let  $V_1$  be the connected component of  $f^{-qn}(D_1)$  containing  $V$ , let  $V_2$  be the connected component of  $f^{-n}(D_2)$  containing  $V_1$ , and, as above, let  $V'$  be the connected component of  $f^{-n}(D')$  containing  $V_2$  (thus  $V \subset V_1 \subset V_2 \subset V'$ ). Clearly,  $V' \cap f^{-qn}(A_p)$  is a union of at most  $\gamma^{-qs}$  of some annuli, say,  $A'_j$ , the modulus of each

annulus  $A'_j$  is bounded below by  $\gamma^{qs} \log a$ , and, after appropriate rearrangement of indices  $j$ ,

$$V_2 \setminus V_1 = \bigcup_{j=1}^m A'_j,$$

with some  $m \leq \gamma^{-sq}$ . Since the modulus of every annulus  $A'_j$  in  $V_2 \setminus V_1$  is larger than  $\gamma^{qs} \log a > 1$ , we have

$$1 > \frac{1}{\text{mod}(A'_j)} = \sup_{\rho} \frac{(\inf_l \text{length}_{\rho}(l))^2}{\text{Area}_{\rho}(A'_j)} \geq \frac{(\text{length}(l))^2}{\text{Area}(A'_j)},$$

where the supremum is taken over all measurable Riemannian metrics on  $A'_j$  and the infimum is taken over all closed piecewise-smooth curves that separate both components of the boundary of  $R_q$ . The values  $\text{length}(l)$  and  $\text{Area}(A)$  respectively denote the length and the area calculated with respect to the Fubini-Study metric. Thus for every annulus  $A'_j$  there exists a curve  $l_j$  in this family such that

$$\text{length}(l_j) \leq \sqrt{\text{Area}(A'_j)} \leq \sqrt{\text{Area}(V_2)} \leq \sqrt{\text{Area}(V')}.$$

We claim that this implies the following.

$$(3.6) \quad \text{diam}(V) < 2\sqrt{c}\sqrt{\text{Area}(V')} \gamma^{-qs}.$$

Indeed, one can enlarge  $V_1$  so that the boundary of this modified domain is exactly the union of curves  $l_1, \dots, l_m$ . Let us keep the notation  $V_1$  for this modified domain. Put  $Q = \sqrt{c}\sqrt{\text{Area}(V')}$ . Then consider the following two cases. Either

(a) there exists  $x \in V_1$  such that  $d(x, l_i) > Q$  for all  $i = 1, \dots, m$ .

or

(b) for every  $x \in V_1$  there exists  $l_x \in \{l_1, \dots, l_m\}$  such that  $d(x, l_x) < Q$ .

In the case (a), take  $x \in V_1$ , satisfying (b), and let

$$U := V_1 \cap B(x, Q) \subset B(x, Q).$$

Then  $U$  is a closed algebraic variety in  $B(x, Q)$  and, using Lelong's Lemma ([McM]), we get

$$\text{Area}(U \cap B(x, Q)) > \frac{1}{c}Q^2.$$

But  $\text{Area}(U \cap B(x, Q)) \leq \text{Area}(V') = \frac{1}{c}Q^2$ . This is a contradiction implying that the case (a) never occurs. In the case (b) we get that

$$V_1 \subset \bigcup_{i=1}^m B(l_i, Q) = B(l_1, Q) \cup \bigcup_{i=2}^m B(l_i, Q)$$

for every  $1 \leq j \leq m - 1$ . Since the set  $V_1$  is connected and both sets in the above union are open, they must intersect, say  $B(l_2, Q) \cap B(l_1, Q) \neq \emptyset$ . Thus, proceeding

by induction and after permuting the sets  $B(l_i, Q)$  if necessary, we can require that

$$B(l_{j+1}, Q) \cap \bigcup_{i=1}^j B(l_i, Q) \neq \emptyset$$

Therefore, if  $x \in B(l_1, Q)$  and  $y \in B(l_j, Q)$  then  $\text{dist}(x, y) \leq jQ + j \sup \text{length}(l_i)$ . This implies that

$$\text{diam}(V_1) \leq mQ + m \sup \text{length}(l_i) \leq 2\sqrt{c}\sqrt{\text{Area}(V')}\gamma^{-qs}.$$

As  $V \subset V'$ , the formula (3.6) is thus proved. But, by our condition  $(b'_n)$  on the area of  $V'$ , we can now write

$$\text{diam}(V) \leq 2\sqrt{c} \frac{\gamma^{\frac{n}{2}} \gamma^{qs}}{\sqrt{c} \cdot 2} \gamma^{-qs} = \gamma^{\frac{n}{2}}.$$

So,  $(b_n)$  is established and we are done.  $\square$

We would like to note that, although we do not use explicitly in the above proof the geometric distortion lemma from [BD], this lemma has motivated our approach here. We directly used Lelong's inequality instead.

#### 4. POTENTIAL'S OSCILLATION

For every  $\gamma \in (0, 1)$  let

$$A_{J,\gamma} = A_\gamma \cap J.$$

Since the Julia set  $J$  is backward and forward invariant and since the set  $A_\gamma$  fails to be backward invariant, there exists a least integer  $q_3(\gamma) \geq 1$  such that

$$(4.1) \quad \deg(f^{q_3(\gamma)}|_{A_{J,\gamma}}) \leq d^{q_3(\gamma)k} - 1.$$

Note that like  $g_1$  and  $g_2$  the function  $g_3 : (0, 1) \rightarrow \mathbb{N}$  is weakly increasing and is constant throughout the interval  $(0, \gamma_*)$ . For every  $\gamma \in (0, 1)$  set,

$$(4.2) \quad g_\gamma := \max \left\{ \frac{1}{q_3(\gamma)} \log_d \left( \deg(f^{q_3(\gamma)}|_{A_{J,\gamma}}) \right), k - 1 \right\} < k.$$

Now, for every  $\kappa \in (0, \log d)$  let

$$(4.3) \quad \gamma_\kappa = \exp \left( \frac{1}{5}(\kappa - \log d) \right).$$

It then follows from above that the function  $(0, \log d) \ni \kappa \mapsto g_{\gamma_\kappa}$  is weakly increasing and takes on a constant value (in  $(0, k)$ ) throughout the interval  $(0, \log(d\gamma_*^4))$ . Consequently, the function  $(0, \log d) \ni \kappa \mapsto k - g_{\gamma_\kappa}$  is weakly decreasing and takes on a constant value (in  $(0, k)$ ) throughout the interval  $(0, \log(d\gamma_*^4))$ . Now we can define the maximal oscillation of our potentials. Namely,

$$(4.4) \quad \kappa_f := \sup \{ \kappa \in (0, \log d) : (k - g_{\gamma_\kappa}) \log d > \kappa \} \in (0, \log d]$$

and

$$(4.5) \quad g_f := k - \kappa_f \in [k - 1, k).$$

Let us record now the following obvious observation.

**Lemma 4.1.** *If  $A_q \cap J = \emptyset$  for all  $q \geq 1$ , then*

$$(4.6) \quad \kappa_f = \log d.$$

Sticking to this issue, we would like to discuss at this moment the case when the dimension  $k = 2$ . Although then the set  $A_q \cap J$  does not have to be empty, nevertheless the task to estimate  $\kappa_f$  reduces to looking at finitely many periodic points only. Indeed, we recall the following lemma from [FS] (Lemma 7.9).

**Lemma 4.2.** *Suppose that  $g : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  is a holomorphic map of degree  $d$  and that  $g$  maps a compact complex hypersurface  $Z$  into itself and that  $Z$  is contained in the critical set of  $g$ . Then*

$$\text{dist}(f(z), Z) = o(\text{dist}(z, Z)).$$

We shall prove the following.

**Lemma 4.3.** *Suppose that  $g : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  is a holomorphic map of degree  $d$ . If  $D \subset C$  is an irreducible component of the critical set  $C$ , and  $D$  is periodic under  $f$  ( $f^l D = D$  for some  $l \geq 1$ ), then  $D$  does not intersect the Julia set  $J$ .*

*Proof.* Indeed, let  $z \in J$  and let  $U$  be an arbitrary neighborhood of  $z$ . It then follows from the construction of the maximal measure in [BD] that  $\bigcup_{n \geq 0} f^n(U) = \mathbb{P}^2 \setminus E$  where  $E$  is a (possibly empty) exceptional set, i.e. the largest totally invariant algebraic subset of the critical set. Applying Lemma 4.3, we conclude that if  $D$  is a periodic irreducible component of the critical set  $C$  then there exists a neighborhood of  $D$  which is mapped into itself under  $f$ . Therefore,  $D \cap J = \emptyset$ . We are done.  $\square$

For every periodic point  $z$  of  $f$  let  $p(z) \geq 1$  be the least integer such that  $f^{p(z)}(z) = z$ . Denote by  $\text{Per}(f)$  the set of all periodic points of  $f$ . As a corollary of Lemma 4.3 we get the following.

**Corollary 4.4.** *If  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  is a holomorphic map of degree  $d$ , then the set*

$$W := \{z : \deg_z f > d\} = \{z : \deg_z f \geq d + 1\}$$

*is finite and*

$$\kappa_f \geq \log d - \max_{z \in W \cap \text{Per}(f) \cap J} \left\{ \frac{1}{p(z)} \log \deg_z(f^{p(z)}) \right\}.$$

*In particular, if  $W \cap J = \emptyset$ , then  $\kappa_f = \log d$ .*

As was indicated in the introduction, our general assumption is that  $\phi : J \rightarrow \mathbb{R}$  is a Hölder continuous function and

$$(4.7) \quad \sup(\phi) - \inf(\phi) < \kappa_f.$$

Let us take the first fruits of this assumption. First of all fix two positive numbers  $\alpha$  and  $\beta$  such that

$$(4.8) \quad \sup(\phi) - \inf(\phi) < \alpha < \beta < \kappa_f.$$

Set

$$(4.9) \quad \theta = \frac{\beta - \alpha}{2} > 0.$$

We consider the dynamical system  $f : J \rightarrow J$ . Let  $C(J)$  denote the Banach space of all complex-valued continuous functions on  $J$  endowed with the supremum norm. For every  $g \in C(J)$  define  $\mathcal{L}_\phi g$  by the formula

$$\mathcal{L}_\phi g(z) = \sum_{x \in f^{-1}(z)} e^{\phi(x)} g(x),$$

where the inverse images of critical values of  $f$  are counted with multiplicities. Then  $\mathcal{L}_\phi g \in C(J)$  and the linear operator  $\mathcal{L}_\phi g : C(J) \rightarrow C(J)$  is bounded.  $\mathcal{L}_\phi$  is called the Perron-Frobenius (transfer) operator associated to the potential  $\phi$ . Now consider the dual operator  $\mathcal{L}_\phi^* : C^*(J) \rightarrow C^*(J)$  defined by the formula  $\mathcal{L}_\phi^* \mu(g) = \mu(\mathcal{L}_\phi g)$ . Let  $M_J$  be the set of all Borel probability measures on  $J$ . The map

$$\mu \mapsto \frac{\mathcal{L}_\phi^* \mu}{\mathcal{L}_\phi^* \mu(\mathbb{1})}, \quad \mu \in M_J,$$

is well-defined and continuous. Since  $M_J$  is convex and compact (in the weak-\* topology), this map has a fixed point in virtue of Schauder-Tichonov Theorem. Denote this fixed point by  $m_\phi$  and set  $\lambda = \mathcal{L}_\phi m_\phi(\mathbb{1})$ . Then

$$(4.10) \quad \mathcal{L}_\phi^* m_\phi = \lambda m_\phi.$$

Hence, using (4.7), we get

$$\begin{aligned} \lambda &= \int \mathcal{L}_\phi \mathbb{1} dm_\phi = \int \sum_{x \in f^{-1}(z)} e^{\phi(x)} dm_\phi \geq \int d^k \exp(\inf(\phi)) dm_\phi = d^k \exp(\inf(\phi)) \\ &= \exp(k \log d + \inf(\phi)) > \exp(\sup(\phi) - \alpha + k \log d). \end{aligned}$$

Equivalently,

$$(4.11) \quad \sup(\phi) - \log \lambda < \alpha - k \log d.$$

Therefore,

$$(4.12) \quad \beta - \log d + \log \lambda - \sup(\phi) - (k - 1) \log d > \beta - \alpha = 2\theta$$

$$(4.13) \quad \sup(\phi) - \log \lambda < (1 - k) \log d. \quad \text{NOT USED YET}$$

We also get from (4.11) that

In view of (4.12) there exists an integer  $q_0 \geq 1$  so large that

$$(4.14) \quad \log \lambda - \sup(\phi) - (k-1) \log d - q_0^{-1}(k-1) \log d > \theta. \quad \text{NOT USED YET}$$

### 5. UNIFORM BOUNDS OF ITERATES OF THE PERRON-FROBENIUS OPERATOR

In view of (4.12) and (4.3) there exists  $q_* \geq \max\{q_1(\gamma_\beta), q_2(\gamma_\beta), q_3(\gamma_\beta)\}$  so large that for all  $q \geq q_*$  and all  $n \geq 1$ , we have

$$(5.1) \quad \gamma_\beta^{-sq} (4c\gamma_\beta^{-3qn} + (k-1)^{d+1}q) \exp \left( qn \left( (k-1) \log d + \frac{k-1}{qn} \log d + \sup(\phi) - \log \lambda \right) \right) \leq e^{-\theta qn}.$$

We also assume  $q_*$  to be so large that  $q \geq q_*$  so large that

$$(5.2) \quad (1 - e^{-q\theta})^{-1} (1 + (1 - e^{-q\theta})^{-1} \lambda^{-q} e^{q \sup(\phi)} d^{qk}) e^{-\theta sq} \leq \frac{1}{4}.$$

Now set

$$\begin{aligned} A_* &:= A_{\gamma_\beta}, \\ A_J^* &= A_{\gamma_\beta} \cap J. \end{aligned}$$

and

$$g_* = g_{\gamma_\beta}.$$

Fix an arbitrary integer

$$q \geq q_* \max\{q_1(\gamma_\beta), q_2(\gamma_\beta), q_3(\gamma_\beta)\}$$

which is an integral multiple of  $q_3(\gamma_\beta)$ . Apply Lemma 3.5 with this  $q$  and  $\gamma := \gamma_\beta$ . By (4.2) there exists  $\Delta_q^* > 0$  so small that

$$(5.3) \quad \deg(f^q : B(A_J^*, \Delta_q^*) \cap f^{-q}(B(A_J^*, \Delta_q^*)) \rightarrow B(A_J^*, \Delta_q^*)) \leq d^{q_*q}.$$

Since the Julia set  $J$  is backward and forward invariant, there exists  $\Delta_q^{(1)} \in (0, \Delta_q^*/2)$  so small that

$$(5.4) \quad f^{-q}(B(A_J^*, \Delta_q^{(1)})) \cap B(A_*, \Delta_q^*) \subset B(A_J^*, \Delta_q^*),$$

and then  $\Delta_q^{(2)} \in (0, \Delta_q^{(1)}/2)$  so small that

$$(5.5) \quad f^{-q}(B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(2)})) \cap B(A_*, \Delta_q^*) \subset B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2).$$

Since  $\{B(A_J^*, \Delta_q^{(1)}), B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(2)})\}$  is an open cover of the compact set  $A_*$ , there exists  $\Delta_q \in (0, \Delta_q^{(2)})$  such that

$$(5.6) \quad B(A_*, 2\Delta_q) \subset B(A_J^*, \Delta_q^{(1)}) \cup B(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(2)}).$$

Fix  $\tau \in \mathbb{R}$  so small that  $\tau < \sup(\phi)$  and

$$(5.7) \quad \lambda^{-1} d^k e^\tau \leq e^{-\theta}.$$

Define the function  $\phi_q : J \cup \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) \rightarrow \mathbb{R}$  by the following formula.

$$\phi_q(z) = \begin{cases} S_q \phi(z) & \text{if } z \in J, \\ q\tau & \text{if } z \in \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) \end{cases}$$

This function is well defined since  $J \cap \overline{B}(A_* \setminus B(A_*^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) = \emptyset$ . Clearly, it is Hölder continuous and  $\sup(\phi_q) = \sup(\phi)$ . Let  $\tilde{\phi}_q : \mathbb{P}^k \rightarrow \mathbb{R}$  be the Hölder continuous extension, produced in Lemma 2.1, of the function  $\phi_q : J \cup \overline{B}(A_* \setminus B(A_*^*, \Delta_q^{(1)}), \Delta_q^{(1)}/2) \rightarrow \mathbb{R}$ . Remember that  $\sup(\tilde{\phi}_q) = \sup(\phi_q)$ , and that  $\tilde{\phi}_q$  has the same Hölder exponent as  $\phi_q$  and  $\phi$ . Denote this exponent by  $\omega$  and the  $\omega$ -variation of  $\tilde{\phi}_q$  by  $H_q$ , which, by Lemma 2.1, is bounded by the double  $\omega$ -variation of  $\phi_q$ . for every  $g : \mathbb{P}^k \rightarrow \mathbb{R}$  let

$$S_n g = \sum_{j=0}^{n-1} g \circ f^{qj}.$$

It then follows from Lemma 3.5(b<sub>n</sub>) that for every  $n \geq 1$ , every  $V \in W_n(q, \Delta, z, \xi)$ , and all  $x, y \in V$ , we have

$$\begin{aligned} |S_n \tilde{\phi}_q(x) - S_n \tilde{\phi}_q(y)| &\leq \sum_{j=0}^{n-1} |\tilde{\phi}_q(f^{qj}(x)) - \tilde{\phi}_q(f^{qj}(y))| \\ (5.8) \quad &\leq \sum_{j=0}^{n-1} H_q \rho^\omega(f^{qj}(x), f^{qj}(y)) \leq H_q \sum_{j=0}^{n-1} H_q \gamma^{\frac{n-j}{2}\omega} \\ &\leq H_q \sum_{j=0}^{\infty} \gamma^{\frac{\omega}{2}j} \\ &= H_q (1 - \gamma^{\omega/2})^{-1}. \end{aligned}$$

Hence, for all  $n \geq q$ ,

$$(5.9) \quad \tilde{C}_q^{-1} \leq \frac{\lambda^{-qn} \exp(S_n \tilde{\phi}_q(x))}{\lambda^{-qn} \exp(S_n \tilde{\phi}_q(y))} \leq \tilde{C}_q,$$

where  $\tilde{C}_q = \exp(H_q (1 - \gamma^{\omega/2})^{-1})$ . In this section we will need more auxiliary Perron-Frobenius operators. First, define  $\mathcal{L}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \rightarrow C(\mathbb{P}^k)$  by the formula

$$\mathcal{L}_{\tilde{\phi}_q} g(z) = \sum_{x \in f^{-q}(z)} e^{\tilde{\phi}_q(x)} g(x),$$

where the summation is taken over all the points of  $f^{-1}(z)$  counted with multiplicities. Similarly, as in Preliminaries,  $\mathcal{L}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \rightarrow C(\mathbb{P}^k)$ , is a bounded linear operator acting on  $C(\mathbb{P}^k)$ . It is also called the Perron-Frobenius (transfer) operator associated to the potential  $\tilde{\phi}$ . Define the operators  $\hat{\mathcal{L}}_{\tilde{\phi}_q} : C(\mathbb{P}^k) \rightarrow C(\mathbb{P}^k)$  and  $\hat{\mathcal{L}}_\phi : C(J) \rightarrow C(J)$  by the formulas

$$\hat{\mathcal{L}}_\phi = \lambda^{-1} \mathcal{L}_\phi \quad \text{and} \quad \hat{\mathcal{L}}_{\tilde{\phi}_q} = \lambda^{-q} \mathcal{L}_{\tilde{\phi}_q},$$

Our goal now is to prove some sufficiently good uniform upper and lower bounds on the iterates  $\hat{\mathcal{L}}_\phi^n$ ,  $n \geq 0$ . This will be done inductively. Fix

$$0 < \eta \leq \Delta_q.$$

For every  $n \geq 0$ , set

$$\hat{\mathcal{L}}_q^n \mathbb{1} = \hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1}|_{B^c(A_*, \Delta_q)} \quad \text{and} \quad \hat{\mathcal{L}}_{J,q}^n \mathbb{1} = \hat{\mathcal{L}}_{\tilde{\phi}}^{qn} \mathbb{1}|_{B^c(A_J^*, \Delta_q)}.$$

Set  $R_q = R(\eta) > 0$ . For every  $n \geq q$ , every  $z \in \mathbb{P}^k \setminus B(A_*, \eta)$ , every  $\xi \in D(z)$ , and every  $w \in \Gamma_{z,\xi} \cap B(z, R_q)$ , set

$$G_{\tilde{\phi}_q, z, \xi}^{(n)}(w) = \sum_{x \in f^{-qn}(w) \cap \cup W_n(q, \Delta_q, z, \xi)} \lambda^{-qn} \exp(\tilde{\phi}_q(x)),$$

and

$$B_{\tilde{\phi}_q, z, \xi}^{(n)}(w) = \sum_{x \in f^{-qn}(w) \cap \cup (Z_n(q, \Delta_q, z, \xi) \setminus W_n(q, \Delta_q, z, \xi))} \lambda^{-qn} \exp(S_n \tilde{\phi}_q(x)).$$

It follows from (5.9) that

$$(5.10) \quad \tilde{C}_q \leq \frac{G_{\tilde{\phi}_q, z, \xi}^{(n)}(w)}{G_{\tilde{\phi}_q, z, \xi}^{(n)}(z)} \leq \tilde{C}_q.$$

It also follows from Lemma 3.5( $c_n$ ) and ( $d_n$ ), and from (5.1) that

$$(5.11) \quad \begin{aligned} B_{\tilde{\phi}_q, z, \xi}^{(n)}(w) &\leq \lambda^{-qn} \exp(\sup(\tilde{\phi}_q)) \#(Z_n(q, \Delta_q, z, \xi) \setminus W_n(q, \Delta_q, z, \xi)) \\ &\leq \gamma_\beta^{-sq} \exp(qn(\sup(\phi) - \log \lambda)) (4c\gamma_\beta^{-3qn} + (k-1)^{d+1}q) d^{(k-1)q(n+1)} \\ &= \gamma_\beta^{-sq} (4c\gamma_\beta^{-3qn} + (k-1)^{d+1}q) \exp\left(qn\left((k-1)\log d + \frac{k-1}{qn} \log d + \right. \right. \\ &\quad \left. \left. + \sup(\phi) - \log \lambda\right)\right) \\ &\leq e^{-\theta qn}. \end{aligned}$$

for every  $n \geq s$ , every  $z \in \mathbb{P}^k \setminus B(A_*, \eta)$ , every  $\xi \in D(z)$ , and every  $w \in \Gamma_{z,\xi} \cap B(z, R_q)$ . Now, let  $L_\infty(\overline{B}(A_*, \Delta_q))$  be the Banach space of all real-valued bounded functions on  $\overline{B}(A_*, \Delta_q)$ . For every  $h \in L_\infty(\overline{B}(A_*, \Delta_q))$  and every  $z \in \overline{B}(A_*, \Delta_q)$ , let

$$(5.12) \quad \hat{\mathcal{L}}_* g(z) = \sum_{x \in f^{-q}(z) \cap \overline{B}(A_*, \Delta_q)} \lambda^{-q} \exp(\tilde{\phi}_q(x)) g(x).$$

Obviously,  $\hat{\mathcal{L}}_* g(z)$  is a linear operator acting on  $L_\infty(\overline{B}(A_*, \Delta_q))$ . If  $z \in \overline{B}(A_J^*, \Delta_q)$ , then it follows from (5.3), (5.4), (4.4), and (4.12) that,

$$(5.13) \quad \begin{aligned} \hat{\mathcal{L}}_* \mathbb{1}(z) &\leq \lambda^{-q} d^{g_* q} e^{\sup(\tilde{\phi}_q)} \leq \exp(q(\sup(\phi) - \log \lambda + g_* \log d)) \\ &< \exp(\sup(\phi) - \log \lambda + k \log d - \beta) \\ &< e^{-2\theta q} < e^{-\theta q}. \end{aligned}$$

If, on the other hand,  $z \in \overline{B}(A_* \setminus B(A_J^*, \Delta_q^{(1)}), \Delta_q^{(2)})$ , then it follows from (5.7), and the definitions of  $\phi_q$  and  $\tilde{\phi}_q$ , that

$$(5.14) \quad \hat{\mathcal{L}}_* \mathbb{1}(z) \leq \lambda^{-q} d^{kq} e^{q\tau} \leq e^{-\theta q}.$$

Both (5.13) and (5.14) along with (5.6), imply that

$$\|\hat{\mathcal{L}}_*\| = \|\hat{\mathcal{L}}_* \mathbb{1}\|_\infty \leq e^{-\theta q}.$$

Consequently, for all  $n \geq 0$  and all  $z \in \overline{B}(A_*, \Delta_q)$ ,

$$(5.15) \quad \hat{\mathcal{L}}_*^n \mathbb{1}(z) \leq e^{-\theta q n}.$$

Now, fix  $z \in \mathbb{P}^k \setminus B(A_*, \eta)$ ,  $\xi \in D(z)$ , and  $w \in \Gamma_{z,\xi} \cap B(z, R_q)$ . Set

$$Z_j = Z_j(q, \eta, z, \xi) \quad \text{and} \quad W_j = W_j(q, \eta, z, \xi).$$

We then have

$$(5.16) \quad \begin{aligned} \hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1}(w) &= \sum_{j=s}^n \sum_{x \in f^{-qj}(w) \cap B^c(A_*, \Delta_q) \cap (\cup(Z_j \setminus W_j))} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x)) \hat{\mathcal{L}}_q^{n-j} \mathbb{1}(x) + \\ &+ \sum_{j=s}^n \sum_{x_1 \in \Lambda_1(w)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2 \in \Lambda_2(x_1)} \lambda^{-qi} \exp(S_i \tilde{\phi}_q(x_2)) \cdot \\ &\cdot \sum_{x_3 \in \Lambda_3(x_2)} \lambda^{-q} e^{\tilde{\phi}_q(x_3)} \hat{\mathcal{L}}_q^{n-(j+i+1)} \mathbb{1}(x_3) + G_{\tilde{\phi}_q, z, \xi}^{(n)}(w), \end{aligned}$$

where

$$\Lambda_1(w) = f^{-qj}(w) \cap B(A_*, \Delta_q) \cap (\cup(Z_j \setminus W_j)),$$

$$\Lambda_2(x_1) = f^{-qi}(x_1) \cap \bigcap_{l=0}^i f^{-ql}(B(A_*, \Delta_q)),$$

$$\Lambda_3(x_2) = f^{-q}(x_2) \cap B^c(A_*, \Delta_q).$$

Denote the first summand in (5.16) by  $\Sigma_1^{(n)}(w)$  and the second one by  $\Sigma_2^{(n)}(w)$ . We will estimate each of them separately. Set for all  $l \geq s$ ,

$$M_l^*(\tilde{\phi}_q) = \max \left\{ \left\| \hat{\mathcal{L}}_q^j \mathbb{1} \right\|_\infty : s \leq j \leq l \right\}$$

and

$$M_l^*(\phi) = \max \left\{ \left\| \hat{\mathcal{L}}_\phi^{qj}(\mathbb{1}) \right\|_\infty : s \leq j \leq l \right\}.$$

Because of (5.11), we have

$$\begin{aligned}
(5.17) \quad \Sigma_1^{(n)}(w) &\leq \sum_{j=s}^n \sum_{x \in f^{-qj}(w) \cap B^c(A_*, \Delta_q) \cap \cup(Z_j \setminus W_j)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x)) M_{n-1}^*(\tilde{\phi}_q) \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \sum_{j=s}^n B_{\tilde{\phi}_q, z, \xi}^{(j)}(w) \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \sum_{j=s}^n e^{-\theta q j} \\
&\leq (1 - e^{-q\theta})^{-1} e^{-\theta s q} M_{n-1}^*(\tilde{\phi}_q).
\end{aligned}$$

Because of (5.11) and (5.15) we have

$$\begin{aligned}
(5.18) \quad \Sigma_2^{(n)}(w) &\leq \sum_{j=s}^n \sum_{x_1 \in \Lambda_1(w)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2 \in \Lambda_2(x_1)} \lambda^{-qi} \exp(S_i \tilde{\phi}_q(x_2)) \sum_{x_3 \in \Lambda_3(x_2)} \lambda^{-q} e^{\tilde{\phi}_q(x_3)} M_{n-1}^*(\tilde{\phi}_q) \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \sum_{j=s}^n \sum_{x_1 \in \Lambda_1(w)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2 \in \Lambda_2(x_1)} \lambda^{-qi} \exp(S_i \tilde{\phi}_q(x_2)) \hat{\mathcal{L}}_{\phi_q} \mathbb{1}(x_2) \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \left\| \hat{\mathcal{L}}_{\phi_q} \right\| \sum_{j=s}^n \sum_{x_1 \in \Lambda_1(w)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2 \in \Lambda_2(x_1)} \lambda^{-qi} \exp(S_i \tilde{\phi}_q(x_2)) \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \left\| \hat{\mathcal{L}}_{\phi_q} \right\| \sum_{j=s}^n \sum_{x_1 \in \Lambda_1(w)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} \tilde{\mathcal{L}}_*^i \mathbb{1}(x_1) \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \left\| \hat{\mathcal{L}}_{\phi_q} \right\| \sum_{j=s}^n \sum_{x_1 \in \Lambda_1(w)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x_1)) \sum_{i=0}^{n-j} e^{-\theta q i} \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \left\| \hat{\mathcal{L}}_{\phi_q} \right\| \sum_{j=s}^n \sum_{x_1 \in \Lambda_1(w)} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x_1)) (1 - e^{-q\theta})^{-1} \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \left\| \hat{\mathcal{L}}_{\phi_q} \right\| (1 - e^{-q\theta})^{-1} \sum_{j=s}^n B_{\tilde{\phi}_q, z, \xi}^{(j)}(w) \\
&\leq M_{n-1}^*(\tilde{\phi}_q) \left\| \hat{\mathcal{L}}_{\phi_q} \right\| (1 - e^{-q\theta})^{-1} \sum_{j=s}^{\infty} e^{-\theta q j} \\
&\leq (1 - e^{-q\theta})^{-2} \left\| \hat{\mathcal{L}}_{\phi_q} \right\| e^{-\theta q s} M_{n-1}^*(\tilde{\phi}_q) \\
&\leq \lambda^{-q} e^{\sup(\phi_q)} d^{qk} (1 - e^{-q\theta})^{-2} e^{-\theta q s} M_{n-1}^*(\tilde{\phi}_q) \\
&\leq (\lambda^{-1} e^{\sup(\phi)} d^k)^q (1 - e^{-q\theta})^{-2} e^{-\theta q s} M_{n-1}^*(\tilde{\phi}_q)
\end{aligned}$$

Combining (5.16), (5.17) and (5.18) together, and then making use of (5.2), we get for all  $n \geq 1$ , all  $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$ , all  $\xi \in D(z)$ , and all  $w \in \Gamma_{z,\xi} \cap B(z, R_q)$  that,

$$(5.19) \quad \begin{aligned} \hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1}(w) &\leq (1 - e^{-q\theta})^{-1} e^{-\theta sq} M_{n-1}^*(\tilde{\phi}_q) + (\lambda^{-1} e^{\sup(\phi)} d^k (1 - e^{-q\theta})^{-2}) e^{-\theta sq} M_{n-1}^*(\tilde{\phi}_q) + \\ &\quad + G_{\tilde{\phi}_q, z, \xi}^{(n)}(w) \leq \\ &\leq (1 - e^{-q\theta})^{-1} (1 + (1 - e^{-q\theta})^{-1} \lambda^{-q} e^{q \sup(\phi)} d^{qk}) e^{-\theta sq} M_{n-1}^*(\tilde{\phi}_q) + G_{\tilde{\phi}_q, z, \xi}^{(n)}(w) \\ &\leq G_{\tilde{\phi}_q, z, \xi}^{(n)}(w) + \frac{1}{4} M_{n-1}^*(\tilde{\phi}_q). \end{aligned}$$

If  $z \in J \setminus B(A_*, \eta)$  and  $w = z$ , we thus get

$$(5.20) \quad \hat{\mathcal{L}}_{\tilde{\phi}}^{qn} \mathbb{1}(w) \leq G_{\tilde{\phi}_q, w, \xi}^{(n)}(w) + \frac{1}{4} M_{n-1}^*(\phi)$$

for all  $n \geq s$  and all  $\xi \in D(z)$ . Now, it follows from (5.10) that

$$(5.21) \quad \tilde{C}_q^{-1} \leq \frac{G_{\tilde{\phi}_q, w, \xi}^{(n)}(w)}{G_{\tilde{\phi}_q, w, \xi}^{(n)}(z)} \leq \tilde{C}_q.$$

In view of (4.10), we get for every  $z \in J$  that

$$1 = \int \hat{\mathcal{L}}_{\tilde{\phi}}^q \mathbb{1} dm_{\tilde{\phi}} \geq \int_{B(z, R_q)} \hat{\mathcal{L}}_{\tilde{\phi}}^q \mathbb{1} dm_{\tilde{\phi}} \geq \hat{C}_q^{-1} \hat{\mathcal{L}}_{\tilde{\phi}}^q \mathbb{1}(x)$$

with some  $x \in J \cap B(z, R_q)$ , where  $\hat{C}_q^{-1} = \inf\{m_{\tilde{\phi}}(B(y, R_q/2)) : y \in J\}$  is positive in virtue of Proposition 2.3. Hence  $\hat{\mathcal{L}}_{\tilde{\phi}}^q \leq \hat{C}_q$ . Since the function  $\hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1} : \mathbb{P}^k \rightarrow \mathbb{R}$  is continuous, there exists  $T \in (0, R_q - \rho(z, x))$  such that  $\hat{\mathcal{L}}_{\tilde{\phi}_q}^n(\mathbb{1})(w) \leq 2\hat{C}_q$  for all  $w \in B(x, T) \subset B(z, R_q)$ . Now assume that  $z \in J \setminus B(A_J^*, \Delta_q)$ . Since the set  $D(z)$  is dense in  $B(z, R_q)$ , there thus exists  $y \in D(z) \cap B(x, T)$  such that  $\hat{\mathcal{L}}_{\tilde{\phi}_q}^n(\mathbb{1})(y) \leq 3\hat{C}_q$ . So,  $G_{\tilde{\phi}_q, z, y}^{(n)}(y) \leq 3\hat{C}_q$ . Along with (5.21), this implies that

$$G_{\tilde{\phi}_q, z, y}^{(n)}(z) \leq \tilde{C}_q G_{\tilde{\phi}_q, z, y}^{(n)}(y) \leq C_q := 3\hat{C}_q \tilde{C}_q.$$

Inserting this into (5.20), we get

$$(5.22) \quad \hat{\mathcal{L}}_{\tilde{\phi}}^n \mathbb{1}(z) \leq C_q + \frac{1}{4} M_{n-1}^*(\phi)$$

for all  $n \geq s$  and all  $z \in J \setminus B(A_*, \eta)$ . Thus, (remember that  $\eta \leq \Delta_q$ ),

$$(5.23) \quad M_n^*(\phi) \leq C_q + \frac{1}{4} M_{n-1}^*(\phi).$$

Now, we can prove by induction the following.

**Lemma 5.1.** *There exists a constant  $Q_q^+ > 0$  such that  $\|\hat{\mathcal{L}}_{\tilde{\phi}}^{qn}\|_{\infty} \leq Q_q^+$  for all  $n \geq 0$ .*

*Proof.* Put  $Q_q^* = \max\{\frac{4}{3}C_q, M_{s-1}^*(\phi)\}$ . We shall show first by induction that  $M_n^*(\phi) \leq Q_q^*$  for every  $n \geq s-1$ . The case  $n = s-1$  is obvious. So, suppose that  $n \geq s$  and  $M_{n-1}^*(\phi) \leq Q_q^*$ . We then get by (5.23) that

$$(5.24) \quad M_n^*(\phi) \leq C_q + \frac{1}{4}Q_q^* \leq \frac{3}{4}Q_q^* + \frac{1}{4}Q_q^* = Q_q^*.$$

The inductive proof is complete, and in fact (5.24) holds for all  $n \geq 0$ . Now, fix  $n \geq 0$  and  $w \in B(A_*, \Delta_q)$ . It then follows from (5.15) that

$$(5.25) \quad \begin{aligned} \hat{\mathcal{L}}_{\tilde{\phi}_q}^n(w) &= \sum_{j=0}^n \sum_{x \in f^{-qj}(w) \cap \cap_{l=0}^j f^{-ql}(B(A_*, \Delta_q))} \lambda^{-j} \exp(S_j \tilde{\phi}_q(x)) \sum_{y \in f^{-q}(x_2) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\tilde{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} \mathbb{1}(y) \\ &\leq M_{n-1}^*(\tilde{\phi}_q) \sum_{j=0}^n \sum_{x \in f^{-qj}(w) \cap \cap_{l=0}^j f^{-ql}(B(A_*, \Delta_q))} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x)) \sum_{y \in f^{-q}(x_2) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\tilde{\phi}_q(y)} \\ &\leq M_{n-1}^*(\tilde{\phi}_q) \|\tilde{\mathcal{L}}_{\phi_q} \mathbb{1}\|_\infty \sum_{j=0}^n \sum_{x \in f^{-qj}(w) \cap \cap_{l=0}^j f^{-ql}(B(A_*, \Delta_q))} \lambda^{-qj} \exp(S_j \tilde{\phi}_q(x)) \\ &\leq M_{n-1}^*(\tilde{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q \sum_{j=0}^n \hat{\mathcal{L}}_*^j(w) \\ &\leq M_{n-1}^*(\tilde{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q \sum_{j=0}^n e^{-\theta q j} \\ &\leq (\lambda^{-1} d^k e^{\sup(\phi)})^q (1 - e^{-q\theta})^{-1} M_{n-1}^*(\tilde{\phi}_q). \end{aligned}$$

Moreover, if  $w \in B(A_*, \Delta_q) \cap J$ , then  $M_{n-1}^*(\tilde{\phi}_q)$  can be replaced by  $M_{n-1}^*(\phi)$ , and then along with (5.24), this estimate gives,

$$\hat{\mathcal{L}}_\phi^{qn}(w) \leq (\lambda^{-1} d^k e^{\sup(\phi)})^q (1 - e^{-q\theta})^{-1} Q_q^*.$$

We are therefore done by setting

$$Q_q^+ = Q_q^* \max\left\{1, (\lambda^{-1} d^k e^{\sup(\phi)})^q (1 - e^{-q\theta})^{-1}\right\}.$$

□

**Lemma 5.2.** *There exists a constant  $Q_q^- > 0$  such that  $\hat{\mathcal{L}}_\phi^{qn} \mathbb{1}(z) \geq Q_q^-$  for all  $n \geq 0$  and all  $z \in J$ .*

*Proof.* Fix  $n \geq s$ . Take  $z_n \in J \setminus B(A_*, \Delta_q)$  such that  $\hat{\mathcal{L}}_q^n \mathbb{1}(z_n) = M_n^*(\phi)$ . It then follows from (5.20) that

$$M_n^*(\phi) \leq G_{\tilde{\phi}_q, z_n, \xi}^{(n)}(z_n) + \frac{1}{4}M_{n-1}^*(\phi) \leq G_{\tilde{\phi}_q, z_n, \xi}^{(n)}(z_n) + \frac{1}{4}M_n^*(\phi).$$

for every  $\xi \in D_q(z_n)$ . Therefore,

$$(5.26) \quad G_{\tilde{\phi}_q, z_n, \xi}^{(n)}(z_n) \geq \frac{3}{4} M_n^*(\phi)$$

But  $\int \hat{\mathcal{L}}_\phi^{qn} \mathbb{1} dm_\phi = \int \mathbb{1} dm_\phi = 1$ , and so, there exists a point  $y_n \in J$  such that  $\hat{\mathcal{L}}_\phi^{qn} \mathbb{1}(y_n) \geq 1$ . If  $y_n \in J \setminus B(A_*, \Delta_q)$ , then we get that

$$M_n^*(\phi) \geq \hat{\mathcal{L}}_\phi^{qn} \mathbb{1}(y_n) \geq 1.$$

Otherwise, it follows from (5.25) that

$$\begin{aligned} 1 &\leq \hat{\mathcal{L}}_\phi^{qn} \mathbb{1}(y_n) \\ &\leq (\lambda^{-1} d^k e^{\sup(\phi)})^q (1 - e^{-q\theta})^{-1} M_{n-1}^*(\tilde{\phi}_q). \\ &\leq (\lambda^{-1} d^k e^{\sup(\phi)})^q (1 - e^{-q\theta})^{-1} M_n^*(\tilde{\phi}_q) \end{aligned}$$

Thus,

$$M_n^*(\phi) \geq (\lambda d^{-k} e^{-\sup(\phi)})^q (1 - e^{-q\theta}).$$

In either case,

$$M_n^*(\phi) \geq M := \min \left\{ 1, (\lambda d^{-k} e^{-\sup(\phi)})^q (1 - e^{-q\theta}) \right\}.$$

Hence, by (5.26),

$$G_{\tilde{\phi}_q, z_n, \xi}^{(n)}(z_n) \geq \frac{3}{8} M$$

Thus, using (5.10) we obtain for every  $\xi \in D_q(z_n)$ , that

$$G_{\tilde{\phi}_q, z_n, \xi}^{(n)}(\xi) \geq 3(8\hat{C}_q)^{-1} M.$$

Consequently, we get for every  $\xi \in D_q(z_n)$ , that

$$\hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1}(\xi) \geq G_{\tilde{\phi}_q, z_n, \xi}^{(n)}(\xi) \geq 3(8\hat{C}_q)^{-1} M.$$

Since  $\hat{\mathcal{L}}_{\tilde{\phi}_q}^n \mathbb{1}$  is continuous and  $D_q(z_n)$  is dense in  $B(z_n, R_q)$ , this inequality extends to all  $\xi \in B(z_n, R_q)$ . Since, by Proposition 2.2, the map  $f^q : J \rightarrow J$  is topologically exact, there exists  $l \geq 1$  such that  $f^{ql}(B(z_n, R_q) \cap J) = J$  for all  $n \geq 1$ . Hence, for every  $x \in J$  and every  $n \geq l + s$ , there exists  $\xi \in B(z_n, R_q) \cap J$  such that  $f^{ql}(\xi) = x$ . Therefore,

$$\hat{\mathcal{L}}_\phi^{qn} \mathbb{1}(x) \geq \lambda^{-ql} \exp(ql \inf(\phi)) \hat{\mathcal{L}}_{\tilde{\phi}_q}^{qn-l} \mathbb{1}(\xi) \geq 3(4\hat{C}_q)^{-1} M \lambda^{-ql} \exp(ql \inf(\phi)).$$

We are therefore done.  $\square$

## 6. ALMOST PERIODICITY OF THE PERRON-FROBENIUS OPERATOR

As an immediate consequence of (5.21) and Lemma 5.1,

$$(6.1) \quad G_{\tilde{\phi}_q, z, \xi}^{(n)}(w) \leq \tilde{C}_q Q_q^+$$

for all  $z \in J \setminus B(A_*, \eta)$ ,  $\xi \in D(z)$  and all  $w \in B(z, R_q) \cap \Gamma_{z, \xi}$ .

**Lemma 6.1.** *There exists a Hölder continuous function  $\hat{\phi}_q : \mathbb{P}^k \rightarrow \mathbb{R}$  with the following properties.*

- (a) *There exists a neighborhood  $U \subset \mathbb{P}^k$  of  $J$  such that  $\hat{\phi}_q|_U = \tilde{\phi}_q$ . In particular,  $\hat{\phi}_q|_J = \sum_{j=0}^{q-1} \phi \circ f^j$ .*
- (b)  *$\hat{\phi}_q \leq \tilde{\phi}_q$  throughout  $\mathbb{P}^k$ .*
- (c)  *$\hat{Q}_q := \sup_{n \geq 0} \{ \|\hat{\mathcal{L}}_{\hat{\phi}}^n \mathbb{1}\|_{\infty} \} < +\infty$ .*

*Proof.* Assume without loss of generality that

$$R_q \leq \frac{1}{2} \Delta_q.$$

Consider two sets

$$B_q = B(J \setminus B(A_*, \Delta_q/2), R_q)$$

and

$$(\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q).$$

We shall show that

$$(6.2) \quad B(J, R_q) \cap ((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)) = \emptyset.$$

Indeed, suppose for the contrary that there is some  $z \in B(J, R_q) \cap ((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q))$ . Then  $z \in B(J, R_q)$  and  $z \notin B(J \setminus B(A_*, \Delta_q/2), R_q)$ . Hence,  $z \in B(J \cap B(A_*, \Delta_q/2), R_q)$ . So,  $z \in B(A_*, \frac{1}{2}\Delta_q + R_q) \subset B(A_*, \Delta_q)$ . This contradiction finishes the proof of (6.2).

Since  $f^q(J) = J$ , there exists  $\varepsilon_q \in (0, R_q/2)$  such that

$$\overline{B}(J, \varepsilon_q) \cap \left( (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q) \cup f^{-q}((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)) \right) = \emptyset.$$

Fix  $t > 0$  so large that

$$(6.3) \quad e^{-t}(\lambda^{-2} d^{2k} e^{\sup(\phi)})^q (1 + e^{-q\theta} (1 - e^{-q\theta})^{-1}) \leq \frac{1}{4}.$$

and

$$-t \leq \inf \left\{ \tilde{\phi}_q(z) : z \in (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q) \cup f^{-q}((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)) \right\}.$$

So, by Lemma 2.1 there exists a Hölder continuous function  $\hat{\phi}_q : \mathbb{P}^k \rightarrow \mathbb{R}$  such that  $\hat{\phi}_q|_{\overline{B}(J, \varepsilon_q)} = \tilde{\phi}_q$ ,  $\hat{\phi}_q$  restricted to  $(\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q) \cup f^{-q}((\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q))$  is

equal to  $-t$  and  $\sup(\hat{\phi}_q) \leq \max\left\{\sup(\tilde{\phi}_q|_{\overline{B}(J, \varepsilon_q)}), -t\right\} \leq \sup(\tilde{\phi}_q)$ . Conditions (a) and (b) are satisfied by the very definition of  $\hat{\phi}$  with  $U = B(J, \varepsilon_q)$ . Put

$$M_n^{(1)} = \sup\{\hat{\mathcal{L}}_{\hat{\phi}_q}^n \mathbb{1}(z) : z \in B_q\}, \quad M_n^{(2)} = \sup\{\hat{\mathcal{L}}_{\hat{\phi}_q}^n \mathbb{1}(z) : z \in (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)\}.$$

and

$$M_n^*(\hat{\phi}_q) = \sup\{\hat{\mathcal{L}}_{\hat{\phi}_q}^n \mathbb{1}(z) : z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)\} = \max\{M_n^{(1)}, M_n^{(2)}\}.$$

Fix now an arbitrary  $z \in J \setminus B(A_*, \Delta_q/2)$ ,  $\xi \in D(z)$ , and  $w \in \Gamma_{z, \xi} \cap B(z, R_q)$ . Since  $\hat{\phi}_q \leq \tilde{\phi}$ , it then follows from (5.19), applied with  $\eta = \Delta_q/2$ , that

$$\hat{\mathcal{L}}_{\hat{\phi}_q}^n(\mathbb{1})(w) \leq G_{\tilde{\phi}_q, z, \xi}^{(n)}(w) + \frac{1}{4}M_{n-1}^*(\hat{\phi}_q).$$

Applying (6.1) we thus further get

$$\hat{\mathcal{L}}_{\hat{\phi}_q}^n(\mathbb{1})(w) \leq \tilde{C}_q Q_q^+ + \frac{1}{4}M_{n-1}^*(\hat{\phi}_q).$$

Since the union

$$\bigcup \{\Gamma_{z, \xi} \cap B(z, R_q) : z \in J \setminus B(A_*, \Delta_q/2), \xi \in D(z)\}$$

is dense in  $B(J \setminus B(A_*, \Delta_q/2), R_q) = B_q$ , we thus get that

$$(6.4) \quad M_n^{(1)} \leq \tilde{C}_q Q_q^+ + \frac{1}{4}M_{n-1}^*(\hat{\phi}_q).$$

For every  $j \geq 0$  put

$$\Lambda_j = \bigcap_{i=0}^j f^{-qi}(B(A_*, \Delta_q)).$$

Using the definition of  $\hat{\phi}_q$  and (5.15), we get for all  $w \in \mathbb{P}^k$  that,

$$\begin{aligned}
(6.5) \quad & \hat{\mathcal{L}}_{\hat{\phi}_q}^n(\mathbb{1})(w) = \\
& = \sum_{x \in f^{-q}(w) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}_{\hat{\phi}_q}^{n-1}(\mathbb{1})(x) + \\
& \quad + \sum_{j=1}^n \sum_{y \in f^{-qj}(w) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(y)) \sum_{x \in f^{-q}(y) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}_{\hat{\phi}_q}^{n-(j+1)}(\mathbb{1})(x) + \\
& \quad + \sum_{x \in f^{-n}(w) \cap \Lambda_{n-1}} \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) \\
& \leq \sum_{x \in f^{-q}(w) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\sup(\hat{\phi}_q)} \hat{\mathcal{L}}_{\hat{\phi}_q}^{n-1}(\mathbb{1})(x) + \\
& \quad + M_{n-1}^*(\hat{\phi}_q) \sum_{j=1}^n \sum_{y \in f^{-qj}(w) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(y)) \sum_{x \in f^{-q}(y) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(x)} + \\
& \quad + \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}_*^{n-1}(\mathbb{1})(x) \\
& \leq (\lambda^{-1} d^k e^{\sup(\phi)})^q M_{n-1}^*(\hat{\phi}_q) + \\
& \quad + M_{n-1}^*(\hat{\phi}_q) \sum_{j=1}^n \sum_{y \in f^{-qj}(w) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(y)) (\lambda^{-1} d^k e^{\sup(\phi)})^q + \\
& \quad + (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
& \leq (\lambda^{-1} d^k e^{\sup(\phi)})^q M_{n-1}^*(\hat{\phi}_q) + M_{n-1}^*(\hat{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q \sum_{j=1}^n \hat{\mathcal{L}}_*^j(\mathbb{1})(w) \\
& \quad + (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
& \leq (\lambda^{-1} d^k e^{\sup(\phi)})^q M_{n-1}^*(\hat{\phi}_q) + M_{n-1}^*(\hat{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q \sum_{j=1}^n e^{-\theta qj} + \\
& \quad + (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-\theta q(n-1)} \\
& \leq (\lambda^{-1} d^k e^{\sup(\phi)})^q M_{n-1}^*(\hat{\phi}_q) + M_{n-1}^*(\hat{\phi}_q) (\lambda^{-1} d^k e^{\sup(\phi)})^q e^{-q\theta} (1 - e^{-q\theta})^{-1} + \\
& \quad + (\lambda^{-1} d^k e^{\sup(\phi)})^q \\
& \leq T M_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1} d^k e^{\sup(\phi)})^q,
\end{aligned}$$

where  $T = (\lambda^{-1}d^k e^{\sup(\phi)})^q (1 + e^{-q\theta}(1 - e^{-q\theta})^{-1})$ . Therefore, using the definition of  $\hat{\phi}_q$  and (6.3), we get for every  $w \in (\mathbb{P}^k \setminus B_q) \setminus B(A_*, \Delta_q)$ , that

$$\begin{aligned} \hat{\mathcal{L}}_{\hat{\phi}_q}^n(\mathbb{1})(w) &= \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} \hat{\mathcal{L}}_{\hat{\phi}_q}^{n-1}(\mathbb{1})(x) \\ &\leq \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{\hat{\phi}_q(x)} (TM_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q) \\ &= (TM_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q) \sum_{x \in f^{-q}(w)} \lambda^{-q} e^{-s} \\ &= (\lambda^{-1}d^k)^q e^{-s} (TM_{n-1}^*(\hat{\phi}_q) + (\lambda^{-1}d^k e^{\sup(\phi)})^q) \\ &\leq \frac{1}{4}M_{n-1}^*(\hat{\phi}_q) + \frac{1}{4}. \end{aligned}$$

Thus,

$$M_n^{(2)} \leq \frac{1}{4} + \frac{1}{4}M_{n-1}^*(\hat{\phi}_q).$$

Combining this with (6.4), we get

$$M_n^*(\hat{\phi}_q) \leq \max \left\{ \frac{1}{4}, \tilde{C}_q Q_q^+ \right\} + \frac{1}{4}M_{n-1}^*(\hat{\phi}_q).$$

Now we can prove in the same standard way as in the proof of Lemma 5.1 that

$$M^* := \sup_{n \geq 1} \{M_n^*(\hat{\phi}_q)\} < +\infty.$$

So, applying (6.5). we get for every  $n \geq 1$  that

$$\|\hat{\mathcal{L}}_{\hat{\phi}_q}^n \mathbb{1}\|_\infty \leq TM^* + (\lambda^{-1}d^k e^{\sup(\phi)})^q.$$

We are done. □

We will need the following strengthening of the distortion property (5.8).

**Lemma 6.2.** *For every  $\varepsilon > 0$  and  $\eta \in (0, \Delta_q]$  there exists  $\delta_1 > 0$  such that*

$$|S_n \hat{\phi}_q(x) - S_n \hat{\phi}_q(y)| \leq \varepsilon$$

for all  $n \geq s$ ,  $z \in \mathbb{P}^k \setminus B(A_*, \eta)$ ,  $\xi \in D(z)$ ,  $V \in W_n(q, \eta, z, \xi)$  and all  $x, y \in V$  with  $\rho(f^{qn}(x), f^{qn}(y)) \leq \delta_1$ .

*Proof.* Let  $\hat{H} > 0$  be the Hölder constant of the Hölder continuous function  $\hat{\phi}_q : \mathbb{P}^k \rightarrow \mathbb{R}$  produced in Lemma 6.1. Take  $k_s \geq 1$  so large that  $\hat{H} \sum_{j=k_s+1}^{\infty} \gamma^{\frac{\alpha}{2}j} \leq \varepsilon/2$ . Since all the functions  $S_j \hat{\phi}_q$ ,  $j = s, s+1, \dots, s+k_s-1$ , are continuous (and there are only finitely many of them) it suffices to prove the lemma for all  $n \geq s+k_s$ . Take  $\delta_2 > 0$  so small that

$$|\hat{\phi}_q(b) - \hat{\phi}_q(a)| \leq \frac{\varepsilon}{2k_s},$$

whenever  $\rho(a, b) \leq \delta_2$ . By Lemma 3.5( $b_n$ ) there exists  $\delta_1 > 0$  so small that for all  $n \geq s + k_s$  and every  $n - k_s \leq j \leq n - 1$ , we have

$$\rho(f^{qj}(x), f^{qj}(y)) \leq \delta_2$$

whenever  $z, \xi, V, x, y$  are as in the hypothesis of the lemma. Applying Lemma 3.5( $b_n$ ) again, with  $n, z, \xi, V, x, y$  as in the hypothesis of the lemma, we get

$$\begin{aligned} |S_n \hat{\phi}_q(x) - S_n \hat{\phi}_q(y)| &\leq \\ &\leq \sum_{j=0}^{n-(k_s+1)} |\hat{\phi}_q(f^{qj}(x)) - \hat{\phi}_q(f^{qj}(y))| + \sum_{j=n-k_s}^{n-1} |\hat{\phi}_q(f^{qj}(x)) - \hat{\phi}_q(f^{qj}(y))| \\ &\leq \sum_{j=0}^{n-(k_s+1)} \hat{H} \gamma^{\frac{n-j}{2}\alpha} + \sum_{j=n-k_s}^{n-1} \frac{\varepsilon}{2k_s} \\ &\leq \hat{H} \sum_{i=k_q+1}^{\infty} \gamma^{\frac{\alpha}{2}i} + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

We are done.  $\square$

Recall that a bounded linear operator  $L : B \rightarrow B$  acting on a Banach space  $B$  is called almost periodic if and only if for every  $x \in B$ , the closure  $\overline{\{L^n(x) : n \geq 0\}}$  is compact in  $B$ . We shall prove the following.

**Proposition 6.3.** *The Perron-Frobenius operator  $\hat{\mathcal{L}}_{\hat{\phi}_q} : C(\mathbb{P}^k) \rightarrow C(\mathbb{P}^k)$  is almost periodic.*

*Proof.* Fix  $\varepsilon > 0$ . Then take  $s \geq 1$  so large that

$$(6.6) \quad 2(1 - e^{-q\theta})^{-2}(1 - e^{-q\theta} + (\lambda^{-1} e^{\sup(\phi)} d^k)^q) e^{-\theta sq} < \varepsilon.$$

Fix a function  $g \in C(\mathbb{P}^k)$ . Generalizing the functions  $\Sigma_1^{(n)}(w)$  and  $\Sigma_2^{(n)}(w)$ , for every  $n \geq q$ , every  $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$ , every  $\xi \in D(z)$ , and every  $w \in \Gamma_{z, \xi} \cap B(z, R_q)$ , set

$$\Sigma_1^{(n)}(g)(w) = \sum_{j=s}^n \sum_{x \in f^{-qj}(w) \cap B^c(A_*, \Delta_q) \cap (\cup(Z_j \setminus W_j))} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \hat{\mathcal{L}}_q^{n-j}(g)(x)$$

and

$$\begin{aligned} \Sigma_2^{(n)}(g)(w) &= \\ &= \sum_{j=s}^n \sum_{x_1 \in \Lambda(w)} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x_1)) \sum_{i=0}^{n-j} \sum_{x_2 \in \Lambda_2(x_1)} \lambda^{-qi} \exp(S_i \hat{\phi}_q(x_2)) \sum_{x_3 \in \Lambda_2(x_2)} \lambda^{-q} e^{\hat{\phi}_q(x_3)} \hat{\mathcal{L}}_q^{n-(j+i+1)}(g)(x_3). \end{aligned}$$

We then have by (5.16), (5.17), (5.18), and (6.6), that

(6.7)

$$\begin{aligned}
& |\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| = \\
& = |(\Sigma_1^{(n)}(g)(w) - (\Sigma_1^{(n)}(g)(z)) + (\Sigma_2^{(n)}(g)(w) - (\Sigma_2^{(n)}(g)(z)) + \\
& \quad + (G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(w) - G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(z))| + \\
& \leq |\Sigma_1^{(n)}(g)(w) - (\Sigma_1^{(n)}(g)(z))| + |\Sigma_2^{(n)}(g)(w) - (\Sigma_2^{(n)}(g)(z))| \\
& \leq |\Sigma_1^{(n)}(g)(w)| + |\Sigma_1^{(n)}(g)(z)| + |\Sigma_2^{(n)}(g)(w)| + |\Sigma_2^{(n)}(g)(z)| + \\
& \quad + |G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(w) - G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(z)| \\
& \leq |\Sigma_1^{(n)}(w)| \|g\|_\infty + |\Sigma_1^{(n)}(z)| \|g\|_\infty + |\Sigma_2^{(n)}(w)| \|g\|_\infty + |\Sigma_2^{(n)}(z)| \|g\|_\infty + \\
& \quad + |G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(w) - G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(z)| \\
& \leq 2(1 - e^{-q\theta})^{-1} e^{-\theta sq} \hat{Q}_q \|g\|_\infty + 2(\lambda^{-1} d^k e^{\sup(\phi)})^q (1 - e^{-q\theta})^{-2} e^{-\theta sq} \hat{Q}_q \|g\|_\infty + \\
& \quad + |G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(w) - G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(z)| \\
& \leq 2(1 - e^{-q\theta})^{-2} (1 - e^{-q\theta} + (\lambda^{-1} e^{\sup(\phi)} d^k)^q) \hat{Q}_q e^{-\theta sq} \|g\|_\infty + \\
& \quad + |G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(w) - G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(z)| \\
& \leq \hat{Q}_q \|g\|_\infty \varepsilon + |G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(w) - G_{\hat{\phi}_q, z, \xi}^{(n)}(g)(z)|.
\end{aligned}$$

Denote  $G_{\hat{\phi}_q, z, \xi}^{(n)}(g)$  by  $G_n(g)$ . As before, set

$$Z_j = Z_j(q, \eta, z, \xi) \quad \text{and} \quad W_j = W_j(q, \eta, z, \xi).$$

Fix  $\xi \in D(z)$ ,  $n \geq 1$ , and  $V$ , an arbitrary connected component in  $W_n$ . Since the map  $f^{qn}|_V : V \rightarrow \Gamma_{z, \xi} \cap B(z, R_q)$  is proper, the degree of this map is well defined and is constant throughout  $V$ . By Lemma 3.5( $d_n$ ) this degree is bounded above by  $\gamma^{-sq}$ . Let  $\hat{f}_\xi^{-qn}(w)$  be the collection of all points  $x$  from  $f^{-qn}(w) \cap \cup W_n$ , each repeated according to the local degrees. Let  $\sigma$  be an arbitrary bijection from  $\hat{f}_\xi^{-qn}(z)$  to  $\hat{f}_\xi^{-qn}(w)$  respecting all components  $V \in W_n$ . Put  $\delta_3 = \min\{R_q, \delta_1, \delta_2\}$ , where  $\delta_1$  and  $\delta_2$  come from Lemma 6.2. Using Lemma 6.2, and Lemma 6.1, we thus get for all

$n \geq s$ , all  $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$ , all  $\xi \in D(z)$ , and all  $w \in \Gamma_{z,\xi} \cap B(z, \delta_3)$ , that

$$\begin{aligned}
& |G_n(g)(w) - G_n(g)(z)| = \\
& = \left| \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \left( \lambda^{-qn} \exp(S_n \hat{\phi}_q(\sigma(x))) g(\sigma(x)) - \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) g(x) \right) \right| \\
& \leq \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \left| \exp(S_n \tilde{\phi}(\sigma(x))) - \exp(S_n \tilde{\phi}(x)) \right| |g(\sigma(x))| + \\
& \quad + \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \exp(S_n \tilde{\phi}(x)) |g(\sigma(x)) - g(x)| \\
& \leq \|g\|_\infty \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \max \left\{ \exp(S_n \tilde{\phi}(\sigma(x))), \exp(S_n \tilde{\phi}(x)) \right\} |S_n \tilde{\phi}(\sigma(x)) - S_n \tilde{\phi}(x)| + \\
& \quad + \varepsilon \|g\|_\infty \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \exp(S_n \hat{\phi}_q(x)) + \\
& \leq \|g\|_\infty \left( \varepsilon \sum_{x \in \hat{f}_\xi^{-qn}(z) \cap \cup W_n} \lambda^{-qn} \max \left\{ \exp(S_n \tilde{\phi}(\sigma(x))), \exp(S_n \tilde{\phi}(x)) \right\} + \varepsilon \hat{\mathcal{L}}_{\hat{\phi}_q} \mathbf{1}(z) \right) \\
& \leq \varepsilon \|g\|_\infty (\max \{ \hat{\mathcal{L}}_{\hat{\phi}_q} \mathbf{1}(z), \hat{\mathcal{L}}_{\hat{\phi}_q} \mathbf{1}(w) \} + \hat{Q}_q) \\
& \leq 2\hat{Q}_q \|g\|_\infty \varepsilon
\end{aligned}$$

Combining this with (6.7) we get that,

$$(6.8) \quad |\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq \hat{Q}_q \|g\|_\infty \varepsilon + 2\hat{Q}_q \|g\|_\infty \varepsilon = 3\hat{Q}_q \|g\|_\infty \varepsilon$$

for all  $n \geq s$ , all  $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$ , all  $\xi \in D(z)$ , and all  $w \in \Gamma_{z,\xi} \cap B(z, \delta_3)$ . Since the set  $\bigcup_{\xi \in D(z)} \Gamma_{z,\xi} \cap B(z, \delta)$  is dense in  $B(z, \delta)$  and since the functions  $\hat{\mathcal{L}}_{\hat{\phi}_q}^n g$  are continuous, we conclude that

$$(6.9) \quad |\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq 4\hat{Q}_q \varepsilon \|g\|_\infty$$

for all  $n \geq s$ , all  $z \in \mathbb{P}^k \setminus B(A_*, \Delta_q)$ , and all  $w \in B(z, \delta_3)$ .

Now suppose that  $z \in B(A_*, \Delta_q)$ . Recall that given  $j \geq 0$  we denoted

$$\Lambda_j = \bigcap_{l=0}^j f^{-ql}(B(A_*, \Delta_q)).$$

According to (5.25), we have

$$(6.10) \quad \begin{aligned} \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z) &= \sum_{j=0}^{s-1} \sum_{x \in f^{-qj}(z) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \sum_{y \in f^{-q}(x) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y) + \\ &+ \sum_{j=s}^n \sum_{x \in f^{-qj}(z) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \sum_{y \in f^{-q}(x) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y). \end{aligned}$$

Now assume that  $n \geq 2s$ . Applying (6.9) we see that there exists  $\delta_4 \in (0, \delta_3]$  so small that if  $x, x' \in \mathbb{P}^k$  and  $\rho(x', x) < \delta_4$ , then

$$(6.11) \quad \begin{aligned} &\left| \sum_{j=0}^{s-1} \sum_{x \in f^{-qj}(z) \cap \Lambda_j} \sum_{y \in f^{-1}(x) \cap B^c(A_*, \Delta_q)} \left( \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \lambda^{-1} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y) - \right. \right. \\ &\quad \left. \left. - \lambda^{-qj} \exp(S_j \hat{\phi}_q(x')) \lambda^{-1} e^{\hat{\phi}_q(y')} \hat{\mathcal{L}}_q^{n-j-1} g(y') \right) \right| \leq \\ &\leq \hat{Q}_q \|g\|_{\infty} \varepsilon. \end{aligned}$$

Now, take  $\delta_5 \in (0, \delta_4)$  so small that for all  $j = 0, 1, \dots, s-1$  and all  $a, b \in \mathbb{P}^k$  with  $\rho(a, b) < \delta$  there exists a bijection  $\tau_{a,b}^j : f^{-qj}(a) \rightarrow f^{-qj}(b)$  such that  $\tau_{a,b}^j \circ \tau_{b,a}^j = \text{Id}$  and  $\rho(\tau_{a,b}^j(x), x) < \delta_4$  for all  $x \in f^{-qj}(a)$ . If now  $z \in B(A_*, \Delta_q)$  and  $w \in B(z, \delta_5)$ ,

then looking at (6.10) we can write,

$$\begin{aligned}
(6.12) \quad & \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z) = \\
& = \sum_{\substack{0 \leq j \leq s-1 \\ x \in f^{-qj}(z) \cap \Lambda_j \\ y \in f^{-q}(x) \cap B^c(A_*, \Delta_q)}} \left( \lambda^{-qj} \exp(S_j \hat{\phi}_q(\tau_{z,w}^j(x))) \lambda^{-q} \exp(\hat{\phi}_q(\tau_{x, \tau_{z,w}^j}^1(y))) \hat{\mathcal{L}}_q^{n-j-1} g(\hat{\phi}_q(\tau_{x, \tau_{z,w}^j}^1(y))) - \right. \\
& \quad \left. - \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \lambda^{-q} \exp(\hat{\phi}_q(y)) \hat{\mathcal{L}}_q^{n-j-1} g(y) \right) \\
& + \sum_{\substack{0 \leq j \leq s-1 \\ x \in f^{-qj}(z) \cap \Lambda_j \\ y \in f^{-q}(\tau_{z,w}^j) \setminus \tau_{x, \tau_{z,w}^j}^1(f^{-q}(x) \cap B^c(A_*, \Delta_q))}} \left( \lambda^{-qj} \exp(S_j \hat{\phi}_q(\tau_{z,w}^j(x))) \lambda^{-q} \exp(\hat{\phi}_q(y)) \hat{\mathcal{L}}_q^{n-j-1} g(y) - \right. \\
& \quad \left. - \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \lambda^{-q} \exp(\hat{\phi}_q(\tau_{\tau_{z,w}^j}^1(y))) \hat{\mathcal{L}}_q^{n-j-1} g(\tau_{\tau_{z,w}^j}^1(y)) \right) \\
& + \sum_{\substack{0 \leq s-1 \\ x \in (f^{-qj}(w) \cap \Lambda_j) \setminus \tau_{z,w}^j(f^{-qj}(z) \cap \Lambda_j) \\ y \in f^{-q}(x) \cap B^c(A_*, \Delta_q)}} \left( \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \lambda^{-q} \exp(\hat{\phi}_q(y)) \hat{\mathcal{L}}_q^{n-j-1} g(y) - \right. \\
& \quad \left. - \lambda^{-qj} \exp(S_j \hat{\phi}_q(\tau_{w,z}^j(x))) \lambda^{-q} \exp(\hat{\phi}_q(\tau_{x, \tau_{w,z}^j}^1(y))) \hat{\mathcal{L}}_q^{n-j-1} g(\tau_{x, \tau_{w,z}^j}^1(y)) \right) \\
& + \Sigma_2^*(g)(w) - \Sigma_2^*(g)(z),
\end{aligned}$$

where  $\Sigma_2^*(g)(z)$  is a subsum of

$$\Sigma_3^*(g)(z) := \sum_{j=s}^n \sum_{x \in f^{-qj}(z) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \sum_{y \in f^{-q}(x_2) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y),$$

and likewise,  $\Sigma_2^*(g)(w)$  is a subsum of

$$\Sigma_3^*(g)(w) := \sum_{j=s}^n \sum_{x \in f^{-qj}(w) \cap \Lambda_j} \lambda^{-qj} \exp(S_j \hat{\phi}_q(x)) \sum_{y \in f^{-q}(x_2) \cap B^c(A_*, \Delta_q)} \lambda^{-q} e^{\hat{\phi}_q(y)} \hat{\mathcal{L}}_q^{n-j-1} g(y).$$

The same calculation as in (5.25) shows that

$$(6.13) \quad |\Sigma_2^*(g)(w)| \leq |\Sigma_2^*(|g|)(w)| \leq |\Sigma_3^*(|g|)(w)| \leq \hat{Q}_q \|g\|_\infty \varepsilon,$$

and similarly,

$$(6.14) \quad |\Sigma_2^*(g)(z)| \leq |\Sigma_2^*(|g|)(z)| \leq |\Sigma_3^*(|g|)(z)| \leq \hat{Q}_q \|g\|_\infty \varepsilon,$$

Denote the first three differences in (6.12) by  $\Sigma_1^*(g)(w) - \Sigma_1^*(g)(z)$ . In view of (6.11) we have

$$|\Sigma_1^*(g)(w) - \Sigma_1^*(g)(z)| \leq 3\hat{Q}_q \|g\|_\infty \varepsilon$$

Combining this along with (6.13), (6.14), and applying (6.12), we thus get that

$$|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq 5\hat{Q}_q \|g\|_\infty \varepsilon$$

for all  $n \geq 2q$ , all  $z \in B(A_*, \Delta_q)$ , and all  $w \in B(z, \delta_5)$ . In turn, combining this with (6.9), we obtain

$$(6.15) \quad |\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^g n(z)| \leq 5\hat{Q}_q \|g\|_\infty \varepsilon$$

for all  $n \geq 2q$  and all  $z, w \in \mathbb{P}^k$  with  $\rho(z, w) < \delta_5$ . Clearly, there exists  $\delta \in (0, \delta_5)$  (independent of  $g$ ) so small that

$$|\hat{\mathcal{L}}_{\hat{\phi}_q}^n g(w) - \hat{\mathcal{L}}_{\hat{\phi}_q}^n g(z)| \leq \hat{Q}_q \|g\|_\infty \varepsilon$$

for all  $n = 0, 1, \dots, 2q - 1$  and all  $z, w \in \mathbb{P}^k$  with  $\rho(z, w) < \delta$ . Along with (6.15). Thus, the family  $(\hat{\mathcal{L}}_{\hat{\phi}_q}^n g)_{n=0}^\infty$  is equicontinuous. Hence, invoking also Lemma 6.1(c), it follows from Arzela-Ascoli's theorem that the family  $(\hat{\mathcal{L}}_{\hat{\phi}_q}^n g)_{n=0}^\infty$  is relatively compact. We are done.  $\square$

As an immediate consequence of this proposition and the fact that every function  $g \in C(J)$  extends continuously to a  $\mathbb{P}^k$  with the same supremum norm, we obtain the following.

**Proposition 6.4.** *The Perron-Frobenius operator  $\hat{\mathcal{L}}_{\hat{\phi}_q} : C(J) \rightarrow C(J)$  is almost periodic.*

As a direct consequence of Lemma 5.1 and Lemma 5.2, we get the following.

**Lemma 6.5.** *There exist constants  $Q_+ > 0$  and  $Q_- > 0$  such that  $\|\hat{\mathcal{L}}_\phi^n\|_\infty \leq Q_+$  for all  $n \geq 0$ , and  $\hat{\mathcal{L}}_\phi^{qn} \mathbf{1}(z) \geq Q_-$  for all  $n \geq 0$  and all  $z \in J$ .*

and

**Proposition 6.6.** *The Perron-Frobenius operator  $\hat{\mathcal{L}}_\phi : C(J) \rightarrow C(J)$  is almost periodic.*

It follows from this proposition that the sequence  $(\frac{1}{n} \sum_{j=0}^{n-1} \hat{\mathcal{L}}_\phi^j \mathbf{1})_0^\infty$  is pre-compact, and it is easy to see that any of its limit points  $\rho_\phi$  is a fixed point of the operator  $\hat{\mathcal{L}}_\phi$  and its integral against the measure  $m_\phi$  is equal to 1. Therefore, we have the following.

**Proposition 6.7.** *There exists a continuous function  $\rho_\phi : J \rightarrow [0, +\infty)$  with the following properties:  $\hat{\mathcal{L}}_\phi \rho_\phi = \rho_\phi$ ,  $\int \rho_\phi dm_\phi = 1$ ,  $Q_- \leq \inf(\rho_\phi) \leq \sup(\rho_\phi) \leq Q_+$ . In particular,  $\mu_\phi = \rho_\phi m_\phi$  is a Borel probability  $f$ -invariant measure equivalent to  $m_\phi$ .*

An important ingredient in the proof that  $\mu_\phi$  is a unique equilibrium state of the potential  $\phi : J \rightarrow \mathbb{R}$  is the fact that for every continuous function  $g : J \rightarrow \mathbb{C}$ , the iterates  $\hat{\mathcal{L}}_\phi^n g$  converge uniformly to  $(\int g dm_\phi) \rho_\phi$ . We are going to prove it now. The proof requires some preparations. We start with the following.

**Lemma 6.8.** *Let  $\mathcal{F}$  be a  $\sigma$ -algebra of subsets of some set  $X$ . Suppose that  $\mu_1$  and  $\mu_2$  are some equivalent probability measures on  $\mathcal{F}$  with uniformly bounded Radon-Nikodym derivatives. If  $\mathcal{B}$  is a sub  $\sigma$ -algebra of  $\mathcal{F}$  and  $E_{\mu_2}(g|\mathcal{B}) = E_{\mu_1}(g|\mathcal{B})$  for every  $\mathcal{F}$ -measurable function  $g \in L^1(\mu_1) = L^1(\mu_2)$ , then the Radon-Nikodym derivative  $\frac{d\mu_2}{d\mu_1}$  is  $\mathcal{B}$ -measurable.*

*Proof.* Put  $\rho = \frac{d\mu_2}{d\mu_1}$ . Seeking contradiction suppose that  $\rho$  is not  $\mathcal{B}$ -measurable. Then at least one of the following two sets has a positive measure  $\mu_1$ :

$$A = \{x \in X : E_{\mu_1}(\rho|\mathcal{B})(x) < \rho(x)\} \quad \text{and} \quad A^c = \{x \in X : E_{\mu_1}(\rho|\mathcal{B})(x) < \rho(x)\}.$$

Assume without loss of generality that  $\mu_1(A) > 0$ . Then, on the one hand,

$$\begin{aligned} \mu_2(A) &= \int_X \mathbb{1}_A d\mu_2 = \int_X E_{\mu_2}(\mathbb{1}_A|\mathcal{B}) d\mu_2 = \int_X E_{\mu_2}(\mathbb{1}_A|\mathcal{B}) \rho d\mu_1 \\ &= \int_X E_{\mu_1}(\mathbb{1}_A|\mathcal{B}) \rho d\mu_1 = \int_X E_{\mu_1}(E_{\mu_1}(\mathbb{1}_A|\mathcal{B}) \rho) d\mu_1 \\ &= \int_X E_{\mu_1}(\mathbb{1}_A|\mathcal{B}) \rho E_{\mu_1}(\rho|\mathcal{B}) d\mu_1, \end{aligned}$$

and on the other hand,

$$\begin{aligned} \mu_2(A) &= \int_X \rho \mathbb{1}_A d\mu_1 = \int_X E_{\mu_1}(\rho \mathbb{1}_A|\mathcal{B}) d\mu_1 \\ &> \int_X E_{\mu_1}(E_{\mu_1}(\rho|\mathcal{B}) \mathbb{1}_A) d\mu_1 \\ &= \int_X E_{\mu_1}(\rho|\mathcal{B}) E_{\mu_1}(\mathbb{1}_A|\mathcal{B}) d\mu_1. \end{aligned}$$

So,  $\mu_2(A) > \mu_2(A)$ , and this contradiction finishes the proof.  $\square$

Our second auxiliary fact, interesting itself, is this.

**Proposition 6.9.** *The number 1 is the only unitary eigenvalue of the Perron-Frobenius operator  $\hat{\mathcal{L}}_\phi : C(J) \rightarrow C(J)$  and the corresponding eigenspace is equal to  $\mathbb{C}\rho$ .*

*Proof.* Suppose that

$$\hat{\mathcal{L}}_\phi g = \xi g$$

with some  $\xi \in \mathbb{C}$  of modulus one and some non-zero function  $g \in C(J)$ . Having Lemma 6.5, it follows from Theorem 4.9 and Exercise 2 (p. 326/327) in [Sch] that the unitary eigenvalues of the operator  $\hat{\mathcal{L}}_\phi : C(J) \rightarrow C(J)$  form a finite cyclic group. There thus exists  $l \geq 1$  such that  $\xi^l = 1$ . We then have

$$\hat{\mathcal{L}}_\phi^l g = g.$$

Since  $\hat{\mathcal{L}}_\phi$  preserves the class of real-valued functions, the same is true for  $\text{Reg}$  and also for  $\text{Img}$ . Hence, it is sufficient to consider the case when  $g : J(f) \rightarrow \mathbb{R}$ . Denote  $g_+ = \max\{0, g\}$  and  $g_- = \min\{0, g\}$ . Since the operator  $\hat{\mathcal{L}}_\phi$  is positive, we have  $g_1^+ = \hat{\mathcal{L}}_\phi^l g_0^+ \geq 0$ ,  $g_1^- = \hat{\mathcal{L}}_\phi^l g_0^- \leq 0$  and  $g = \hat{\mathcal{L}}_\phi^l g = g_1^+ + g_1^-$ . Clearly there is not a unique decomposition of  $g$  into a positive and negative function. However, the functions  $g_+$  and  $g_-$  are extremal in the sense that they are the smallest functions that have this property. Consequently  $g_1^+ \geq g_+$  and  $g_1^- \leq g_-$ . Since these functions are continuous and since  $\int g_1^+ dm_\phi = \int g_+ dm_\phi$ , we have  $g_1^+ = g_+$  and, for the same reasons,  $g_1^- = g_-$ . Therefore,

$$\hat{\mathcal{L}}_\phi^l g_+ = g_+ \quad \text{and} \quad \hat{\mathcal{L}}_\phi^l g_- = g_-.$$

Suppose that  $g_+$  does not vanish. Let  $\hat{g} = m_\phi(g_+)^{-1}g_+$ . Then

$$\hat{\mathcal{L}}_\phi^l \hat{g} = \hat{g}, \quad \hat{g} \geq 0, \quad \text{and} \quad \int \hat{g} dm_\phi = 1.$$

Since the map  $f^l : J \rightarrow J$  is topologically exact and since  $\hat{g}$  is continuous, we thus conclude that  $\hat{g}$  is strictly positive. Hence, the two invariant measures  $\mu_\phi = \rho_\phi m_\phi$  and  $\hat{\mu} = \hat{g} m_\phi$  are equivalent with uniformly bounded Radon-Nikodym derivatives. Therefore, all versions of expected values  $E_{\hat{\mu}}(u|\mathcal{I})$  and  $E_{\mu_\phi}(u|\mathcal{I})$  are well defined on a set of common measure 1 for  $\hat{\mu}$  and  $\mu_\phi$ , where  $u : J \rightarrow \mathbb{R}$  is any  $\hat{\mu}$  (equivalently  $\mu_\phi$ ) integrable function and  $\mathcal{I}$  is the  $\sigma$ -algebra of all  $f^l$ -invariant subsets of  $J$ . Since at almost every point  $z \in J$  both  $E_{\hat{\mu}}(u|\mathcal{I})$  and  $E_{\mu_\phi}(u|\mathcal{I})$  are equal to the Birkhoff's ergodic averages

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} u \circ f^{lj}(z),$$

we therefore conclude that

$$E_{\hat{\mu}}(u|\mathcal{I}) = E_{\mu_\phi}(u|\mathcal{I}) \quad \text{a.e.}$$

Hence, it follows from Lemma 6.8 that the Radon-Nikodym derivative is measurable with respect to the  $\sigma$ -algebra  $\mathcal{I}$ . This means that the ratio  $\hat{g}/\rho_\phi$  is constant on grand orbits of almost all points in  $J$ . Since by topological exactness of the map  $f^l : J \rightarrow J$ , the grand orbit of every point in  $J$  is dense in  $J$ , and since the function  $\hat{g}/\rho_\phi$  is continuous, we conclude that the function  $\hat{g}/\rho_\phi$  is constant throughout  $J$ . As both measures  $\hat{\mu}$  and  $\mu_\phi$  are probabilistic, this implies that  $\hat{g}/\rho_\phi$  is equal to 1, i.e.  $\hat{g} = \rho_\phi$ .

So,  $g_+ = a\phi$  with some  $a \in \mathbb{C}$ , and likewise,  $g_- = b\phi$  with some  $b \in \mathbb{C}$ . Consequently,  $g = g_+ - g_- = (a - b)\rho_\phi$ . We are done.  $\square$

For every bounded operator  $A : B \rightarrow B$  of a Banach space  $B$ , let  $B_u$  be the closure of the linear span of unitary eigenvectors of  $A$ , and let  $B_0 = \{g \in B : \lim_{n \rightarrow \infty} A^n g = 0\}$ . M. Lyubich proved in [Ly] that if  $A : B \rightarrow B$  is an almost periodic operator, then  $B = B_u \oplus B_0$ , the direct sum of closed vector subspaces. We shall prove the following.

**Theorem 6.10.** *We have*

$$C(J)_u = \mathbb{C}\rho_\phi, \quad C(J)_0 = \left\{ g \in C(J) : \int g dm_\phi = 0 \right\}.$$

In addition, if  $g = g_u + g_0$  with  $g_u \in C(J)_u$  and  $g_0 \in C(J)_0$ , then  $g_u = (\int g dm_\phi)\rho_\phi$  and the sequence  $(\hat{\mathcal{L}}_\phi^n g)_0^\infty$  converges to  $(\int g dm_\phi)\rho_\phi$  uniformly on  $J$ .

*Proof.* The fact that  $C(J)_u = \mathbb{C}\rho_\phi$  is the content of Lemma 6.9. If  $g \in C(J)_0$ , then  $\int g dm_\phi = \lim_{n \rightarrow \infty} \int \hat{\mathcal{L}}_\phi^n g dm_\phi = 0$ . so,

$$(6.16) \quad C(J)_0 \subset \left\{ g \in C(J) : \int g dm_\phi = 0 \right\}.$$

If, on the other hand,  $\int g dm_\phi = 0$ , write uniquely  $g = g_u + g_0$  where  $g_u \in C(J)_u$  and  $g_0 \in C(J)_0$ . Then  $\int g_u dm_\phi = \int g_0 dm_\phi - \int g dm_\phi = 0 - 0 = 0$ , whence  $g = (\int g_0 dm_\phi)\rho_\phi = 0\rho_\phi = 0$ . Hence,  $g = g_0 \in C(J)_0$ . The inclusion  $\{g \in C(J) : \int g dm_\phi = 0\} \subset C(J)_0$  is proved, and together with (6.16) it yields

$$C(J)_0 = \left\{ g \in C(J) : \int g dm_\phi = 0 \right\}.$$

The second assertion of our theorem is now obvious and the third assertion follows from the first one too since we know that  $g - (\int g dm_\phi)g \in C(J)_0$ , and therefore, because of Lyubich's Theorem,  $\lim_{n \rightarrow \infty} (g - (\int g dm_\phi)g) = 0$ , the limit considered in the Banach space  $C(J)$ . We are done.  $\square$

We shall now record two mixing properties resulting from this theorem. The proof of the first one is the same as the proof of Corollary 37 in [DU], the second property is also its straightforward consequence.

**Theorem 6.11.** *The dynamical system  $(J, f, \mu_\phi)$  is metrically exact. This means that  $\bigcap_{n=0}^{\infty} f^{-n}(\mathcal{B})$  consists only of sets of measure 0 and 1, where  $\mathcal{B}$  denotes the  $\sigma$ -algebra of Borel sets of  $J$ . In particular, Rokhlin's natural extension of  $(J, f, \mu_\phi)$  is a  $K$ -system and the dynamical system  $(J, f, \mu_\phi)$  is mixing of any order. In particular, it is ergodic.*

**Theorem 6.12.** (*Decay of Correlations*) If  $g \in C(J)$  and  $h \in L^1(m_\phi) = \mathcal{L}^1(\mu_\phi)$ , then

$$\lim_{n \rightarrow \infty} \int h \circ f^n \cdot g d\mu_\phi = \int h d\mu_\phi \int g d\mu_\phi.$$

Given  $H \geq 0$  and  $0 \leq t < \kappa_f$  let  $\mathcal{P}_H^t(f)$  denote the class of all Hölder continuous potentials  $\phi : J \rightarrow \mathbb{R}$  such that  $\|\phi\|_\alpha \leq H$  and  $\sup(\phi) - \inf(\phi) \leq t$ . Call all such potentials  $(H, t)$ -admissible. By Arzela-Ascoli theorem,  $\mathcal{P}_H^t(f)$  is a compact subset of  $C(J)$ . Another crucial technical fact for the uniqueness of equilibrium states is the following refinement of Proposition 6.3.

**Lemma 6.13.** For every  $H \geq 0$ ,  $0 \leq t < \kappa_f$ , and every relatively compact set  $K \subset C(\mathbb{P}^k)$ , the set  $\{\hat{\mathcal{L}}_{\hat{\phi}_q}^n(g) : \phi \in \mathcal{P}_H^t(f), g \in K, n \geq 0\}$  is relatively compact.

*Proof.* Suppose first that  $K$  is a singleton, i.e.  $K = \{g\}$  for some  $g \in C(J)$ . Observe that the number  $\theta > 0$  coming from (4.9) can be taken the same for all  $\phi \in \mathcal{P}_H^t(f)$ . The constructions of  $Q_q^+$  and then of  $\hat{Q}_q$  lead to a number  $\hat{Q} > 0$  such that  $\hat{Q}_q \leq \hat{Q}$  for all  $\phi \in \mathcal{P}_H^t(f)$ . Having this, one verifies that given  $\varepsilon > 0$  the numbers,  $\delta_1$  through  $\delta_5$  and  $\delta$  appearing in the proof of Proposition 6.3 can be taken the same for all  $\phi \in \mathcal{P}_H^t(f)$ . Then the proof of Proposition 6.3 shows that the set  $\{\hat{\mathcal{L}}_{\hat{\phi}_q}^n(g) : \phi \in \mathcal{P}_H^t(f), n \geq 0\}$  is relatively compact. Now consider the general case. Take a sequence  $(\hat{\mathcal{L}}_{(\hat{\phi}_j)_q}^{n_j}(g_j))_{j=1}^\infty$ . Since the set  $K$  is relatively compact and  $\mathcal{P}_H^t(f)$  is compact, passing to a subsequence, we may assume without loss of generality that  $(g_j)_{j=1}^\infty$  converges uniformly to some  $g \in C(J)$  and  $((\hat{\phi}_j)_q)_{j=1}^\infty$  converges uniformly to some  $\phi \in \mathcal{P}_H^t(f)$ . Then

$$\left\| \hat{\mathcal{L}}_{(\hat{\phi}_j)_q}^{n_j}(g_j) - \hat{\mathcal{L}}_{(\hat{\phi}_j)_q}^{n_j}(g) \right\|_\infty = \left\| \hat{\mathcal{L}}_{(\hat{\phi}_j)_q}^{n_j}(g_j - g) \right\|_\infty \leq \hat{Q} \|g_j - g\|_\infty.$$

Since  $\lim_{j \rightarrow \infty} \hat{Q} \|g_j - g\|_\infty = 0$  and since, in view of the already proved singleton part from our lemma, the sequence  $(\hat{\mathcal{L}}_{(\hat{\phi}_j)_q}^{n_j}(g))_{j=1}^\infty$  is relatively compact, we conclude that  $(\hat{\mathcal{L}}_{(\hat{\phi}_j)_q}^{n_j}(g_j))_{j=1}^\infty$  contains a converging subsequence. We are done.  $\square$

+++

As an immediate consequence of this lemma we get the following.

**Lemma 6.14.** For every  $H \geq 0$ ,  $0 \leq t < \log d$ , and every relatively compact set  $K \subset C(J)$ , the set  $\{\hat{\mathcal{L}}_\phi^n(g) : \phi \in \mathcal{P}_H^t(f), g \in K, n \geq 0\}$  is relatively compact.

## 7. PRESSURE VERSUS EIGENVALUE

In this section, developing the idea of Gromov from [Gr], we prove equality of the topological pressure  $P(\phi)$  and the logarithm  $\log \lambda$ . This major part of our argument is done in the following.

**Theorem 7.1.** *Let  $\psi$  be a continuous function  $\psi : \mathbb{P}^k \rightarrow \mathbb{R}$ . Assume that there exists  $\lambda > 0$  and  $Q > 0$  such that*

$$\sup \psi + (k - 1) \log d \leq \log \lambda$$

*and, for all  $n > 0$  the inequality*

$$\mathcal{L}_{\psi}^n(\mathbb{1})(x) < Q\lambda^n$$

*holds. Then  $P(\psi) \leq \log \lambda$ .*

*Proof.* We shall follow the idea of the proof of the inequality

$$h_{top}(f) \leq \log(\deg_{top} f) = k \log d$$

which is due to M. Gromov, ([Gr]) Thus, Gromov's inequality corresponds to the case  $\phi = 0$ . Let us consider the following integral

$$\int_{\mathbb{P}^k} (\exp S_n \psi)(\omega + f^* \omega + \dots + (f^{n-1})^* \omega)^k.$$

As in [Gr] we consider the embedding

$$f_n : \mathbb{P}^k \rightarrow X_n = (\mathbb{P}^k)^n$$

given by the formula

$$f_n(x) = (x, f(x), \dots, f^{n-1}(x))$$

(a generalized graph). In the space  $X_n$  we introduce a Kähler form by putting  $\pi_i : X_n \rightarrow X^{(i)} = \mathbb{P}^k$  (projection to the  $i$ -th coordinate),  $\omega_i = \pi_i^* \omega$ , and, finally,  $\eta = \omega_1 + \dots + \omega_n$ .

Now, let  $E$  be an  $(n, 2\varepsilon)$ -separated set in  $\mathbb{P}^k$ , i.e.  $d_n(x, y) > 2\varepsilon$  for  $x, y \in E, x \neq y$ , where  $d_n$  is a metric in  $\mathbb{P}^k$  given by  $d_n(x, y) = \max_{0 \leq i < n} \{d(f^i x, f^i y)\}$ . Then

$$(7.1) \quad d_{\eta}(f_n(x), f_n(y)) = \left( \sum_{i=0}^{n-1} d(f^i x, f^i y)^2 \right)^{\frac{1}{2}} \geq \max_{0 \leq i < n} \{d(f^i x, f^i y)\} > 2\varepsilon$$

Thus, the balls  $B(f_n(x), \varepsilon), x \in E$  (with respect to the metric  $d_{\eta}$ ) are disjoint. Now, we use Lelong's Lemma ([McM]) for the form  $\eta$  and the embedded complex analytic variety  $f_n(\mathbb{P}^k) \subset X_n$  and conclude that the  $\eta$ -volume of  $f_n(\mathbb{P}^k) \cap B(p, \varepsilon)$ , i.e

$$\int_{f_n(\mathbb{P}^k) \cap B(p, \varepsilon)} \eta^k$$

is bounded below by a constant  $c_{\varepsilon}$ , depending on  $\varepsilon$  only. Now, let us fix  $\delta > 0$  small. Since the function  $\psi$  is uniformly continuous, there exists  $\varepsilon > 0$  such that, if  $d(x, y) < \varepsilon$  then  $|\psi(x) - \psi(y)| < \delta$  and, consequently,

$$e^{-\delta} < \frac{e^{\psi(x)}}{e^{\psi(y)}} < e^{\delta}.$$

Let  $E$  be an  $(n, 2\varepsilon)$ -separated set. Then we can write

$$\begin{aligned}
\sum_{x \in E} \exp S_n \psi(x) &\leq e^{\delta n} \sum_{x \in E} \inf_{y \in B_{d_n}(x, \varepsilon)} \exp S_n \psi(y) \\
&\leq e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x \in E} \left( \inf_{y \in B_{d_n}(x, \varepsilon)} \exp S_n \psi \right) \cdot \eta^k(B(f_n(x), \varepsilon)) \\
(7.2) \quad &= e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x \in E} \inf_{y \in B_{d_n}(x, \varepsilon)} \exp S_n \psi(y) (f^* \eta^k)(f_n^{-1} B_{d_n}(f_n(x), \varepsilon)) \\
&\leq e^{\delta n} \frac{1}{c_\varepsilon} \sum_{x \in E} \inf_{y \in B_{d_n}(x, \varepsilon)} \exp S_n \psi(y) (f^* \eta^k)(B_{d_n}(x, \varepsilon)) \\
&\leq e^{\delta n} \frac{1}{c_\varepsilon} \int_{\mathbb{P}^k} \exp S_n \psi(f_n^* \eta^k)
\end{aligned}$$

where we have used (7.1) in the second inequality. Since  $f_n^*(\omega_i) = (f^i)^* \omega$ , the last integral takes on the form

$$(7.3) \quad \int_{\mathbb{P}^k} \exp S_n \psi(\omega + f^* \omega + \dots + (f^{(n-1)})^* \omega)^k$$

We shall estimate this integral from above. First, notice that, if, instead of the above integral, we had

$$\int_{\mathbb{P}^k} \exp S_n \psi(f^{(n-1)*} \omega)^k$$

then the integral would transform immediately to

$$\int_{\mathbb{P}^k} \mathcal{L}_0^n(\mathbb{1}) d\omega^k$$

since the operator  $f^*$  acts on measures on  $\mathbb{P}^k$  as a conjugate to the operator

$$(7.4) \quad f_* g(x) = \sum_{y \in f^{-1}(x)} g(y).$$

In our case, we have to write the integral (7.3) as a sum of integrals, and then use the above observation.

$$\begin{aligned}
&\int_{\mathbb{P}^k} \exp S_n \psi \left( \omega + f^* \omega + \dots + (f^{(n-1)})^* \omega \right)^k = \\
(7.5) \quad &\int_{\mathbb{P}^k} \exp S_n \psi \left( \sum_{0 \leq i_1, \dots, i_n \leq n-1} (f^{i_1})^* \omega \wedge (f^{i_2})^* \omega \wedge \dots \wedge (f^{i_n})^* \omega \right)
\end{aligned}$$

Since all forms  $(f^{i_1})^*\omega \wedge (f^{i_2})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega$  are positive, we can treat them as measures and estimate

$$\begin{aligned}
(7.6) \quad & \int_{\mathbb{P}^k} \exp S_n \psi \left( \omega + f^*\omega + \cdots + (f^{(n-1)})^*\omega \right)^k \leq \\
& k! \int_{\mathbb{P}^k} \exp S_n \psi \sum_{i_1 \leq i_2 \leq \cdots \leq i_k \leq n-1} (f^{i_1})^*\omega \wedge \cdots \wedge (f^{i_k})^*\omega \\
& = k! \sum_{i=0}^{n-k} \int_{\mathbb{P}^k} \exp S_n \psi \sum_{j_2 \leq \cdots \leq j_k \leq n-i} (f^i)^* (\omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega)
\end{aligned}$$

Using the observation 7.4 again, one can rewrite the above sum as

$$(7.7) \quad k! \sum_{i=0}^{n-k} \int_{\mathbb{P}^k} \mathcal{L}_\psi^i(\mathbb{1})(x) \exp S_{n-i} \psi(x) \left[ \sum_{j_2 \leq \cdots \leq j_k \leq n-i} \omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega \right]$$

Recall that, according to the assumption we have

$$\mathcal{L}_\psi^i(\mathbb{1})(x) \leq Q\lambda^n$$

for all  $x$ . Denote  $\alpha = \sup \psi$ . The second assumption on  $\psi$  says that

$$\alpha + (k-1) \log d < \log \lambda.$$

We can therefore estimate the above sum by

$$(7.8) \quad k!Q \sum_{i=0}^{n-k} \lambda^i \exp((n-i)\alpha) \int_{\mathbb{P}^k} \sum_{j_2 \leq \cdots \leq j_k \leq n-i} \omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega$$

It remains to calculate the total mass of each measure  $\omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega$ . Recall that  $(f^j)^*\omega = d^j \omega$  in  $H^2(\mathbb{P}^k)$ . It is then straightforward to check that

$$\int_{\mathbb{P}^k} \omega \wedge (f^{j_2})^*\omega \wedge \cdots \wedge (f^{j_k})^*\omega = \int_{\mathbb{P}^k} d^{j_2+\cdots+j_k} \omega^k = d^{j_2+\cdots+j_k} \leq dd^{(k-1)(n-i)}$$

since  $j_m \leq n-i$  for all  $2 \leq m \leq k$ . Now, the number of possible choices of  $j_2, \dots, j_k$  can be estimated from above by  $n^k$ . Finally, we can estimate (7.8) by

$$\begin{aligned}
(7.9) \quad & c_k n^k \sum_{i=0}^{n-k} \lambda^i \exp((n-i)\alpha) d^{(k-1)(n-i)} = \\
& = c_k n^k \lambda^n \sum_{i=0}^{n-k} [\exp(-\log \lambda + \alpha + (k-1) \log d)]^{n-i} \\
& = c_k n^k \lambda^n \sum_{i=0}^{n-k} [\exp(-\log \lambda + \alpha + (k-1) \log d)]^{(n-i)}
\end{aligned}$$

The last sum is bounded by a constant depending on  $k$  but independent of  $n$  since  $-\log \lambda + \alpha + (k-1) \log d < 0$ . Therefore, we obtain the following. For every  $\delta > 0$

there exists  $\varepsilon_0 = \varepsilon_0(\delta)$  such that for every  $\varepsilon < \varepsilon_0$  and every  $(n, 2\varepsilon)$ -separated set  $E$  we have

$$\sum_{x \in E} \exp S_n \psi(x) \leq e^{\delta n} C(\varepsilon, k) n^k \lambda^n,$$

where  $C(\varepsilon, k)$  is a constant depending on  $\varepsilon$  and  $k$ . This gives immediately

$$P(\psi) \leq \log \lambda + \delta$$

and, as  $\delta$  was arbitrarily small,

$$P(\psi) \leq \log \lambda.$$

□

## 8. EXISTENCE AND UNIQUENESS OF EQUILIBRIUM STATES

As we show in the following proposition the measure  $\mu_\phi$  turns out to be an equilibrium state for the potential  $\phi : J \rightarrow \mathbb{R}$ .

**Proposition 8.1.** *The invariant measure  $\mu_\phi = \rho_\phi m_\phi$  is an equilibrium state for the potential  $\phi : J \rightarrow \mathbb{R}$ . In addition,  $P(\phi) = \log \lambda$ .*

*Proof.* Let  $\hat{\phi}_q : \mathbb{P}^k \rightarrow \mathbb{R}$  be the extension of  $\phi : J \rightarrow \mathbb{R}$  produced in Lemma 6.1. It follows from this lemma that the function  $\psi = \hat{\phi}_q$  satisfies the assumptions of Theorem 7.1 for the dynamical system  $f^q : \mathbb{P}^k \rightarrow \mathbb{P}^k$ . Applying this theorem we get that  $P(\phi) = \frac{1}{q} P(\hat{\phi}_q) \leq \frac{1}{q} P(\hat{\phi}_q) \leq \log \lambda$ . Therefore,

$$\begin{aligned} h\mu_\phi + \int \phi d\mu_\phi &\geq \int \log J_{\mu_\phi} + \int \phi d\mu_\phi \\ &= \int \log \rho_\phi d\mu_\phi - \int \log \rho_\phi \circ f d\mu_\phi + \log \lambda + \int \phi d\mu_\phi - \int \phi d\mu_\phi \\ &= \log \lambda \geq P(\phi) \end{aligned}$$

Invoking the Variational Principle, i. e. formula (1.1), finishes the proof. □

In order to demonstrate the uniqueness of equilibrium states we shall prove an appropriate version of differentiability of topological pressure. The proof is based on Lemma 6.14 and follows the general scheme presented in [PU].

**Lemma 8.2.** *Fix  $g \in C(J)$ . Fix also  $H \geq 0$  and  $0 \leq t < \log d$ . Then*

$$\lim_{n \rightarrow \infty} \sup_{\phi \in \mathcal{P}_H^t(f)} \left\{ \left\| \frac{\hat{\mathcal{L}}_{\phi_q}^n(S_n g)}{\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1})} - \left( \int g d\mu_\phi \right) \mathbb{1} \right\|_\infty \right\} = 0.$$

*Proof.* We have

$$(8.1) \quad \frac{1}{n} \frac{\hat{\mathcal{L}}_{\phi_q}^n(S_n g)}{\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1})} = \frac{\frac{1}{n} \sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi_q}^n(g \circ f^j)}{\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1})} = \frac{\frac{1}{n} \sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi_q}^{n-j}(g \mathcal{L}_{\phi_q}^j(\mathbb{1}))}{\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1})}.$$

It follows from Lemma 6.14 that

$$(8.2) \quad \text{the set } \{g \mathcal{L}_{\phi_q}^j(\mathbb{1}) : \phi \in \mathcal{P}_H^t(f), j \geq 0\} \text{ is relatively compact.}$$

Applying Lemma 6.14 again we thus conclude that

$$(8.3) \quad \text{the set } \{\hat{\mathcal{L}}_{\phi_q}^{n-j}(g \mathcal{L}_{\phi_q}^j(\mathbb{1})) : \phi \in \mathcal{P}_H^t(f), n \geq 0, 0 \leq j \leq n-1\} \text{ is relatively compact.}$$

Therefore

$$(8.4) \quad \text{the set } \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi_q}^{n-j}(g \mathcal{L}_{\phi_q}^j(\mathbb{1})) : \phi \in \mathcal{P}_H^t(f), n \geq 1 \right\} \text{ is relatively compact.}$$

Fixing  $\phi \in \mathcal{P}_H^t(f)$ , Theorem 6.10 yields that  $\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1})$  converges to  $\rho_\phi$ . Combining this with (8.2) taken with  $g = \mathbb{1}$ , we get that

$$(8.5) \quad \lim_{n \rightarrow \infty} \sup_{\phi \in \mathcal{P}_H^t(f)} \|\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1}) - \rho_\phi\|_\infty = 0.$$

By the definition of  $\hat{Q}_q$  we have

$$(8.6) \quad \begin{aligned} & \left\| \hat{\mathcal{L}}_{\phi_q}^{n-j}(g \mathcal{L}_{\phi_q}^j(\mathbb{1})) - \left( \int g d\mu_\phi \right) \rho_\phi \right\|_\infty = \\ & = \left\| \hat{\mathcal{L}}_{\phi_q}^{n-j}(g \mathcal{L}_{\phi_q}^j(\mathbb{1}) - g \rho_\phi) + \hat{\mathcal{L}}_{\phi_q}^{n-j}(g \rho_\phi - \left( \int g d\mu_\phi \right) \rho_\phi) \right\|_\infty \\ & \leq \|\hat{\mathcal{L}}_{\phi_q}^{n-j}\|_\infty \|g(\mathcal{L}_{\phi_q}^j(\mathbb{1}) - \rho_\phi)\|_\infty + \|\hat{\mathcal{L}}_{\phi_q}^{n-j}(g \rho_\phi) - \left( \int g d\mu_\phi \right) \rho_\phi\|_\infty \\ & \leq \|\hat{Q}\| \|g\|_\infty \|\mathcal{L}_{\phi_q}^j(\mathbb{1}) - \rho_\phi\|_\infty + \|\hat{\mathcal{L}}_{\phi_q}^{n-j}(g \rho_\phi) - \left( \int g d\mu_\phi \right) \rho_\phi\|_\infty. \end{aligned}$$

Now  $\lim_{n-j \rightarrow \infty} \|\hat{\mathcal{L}}_{\phi_q}^{n-j}(g \rho_\phi) - \left( \int g d\mu_\phi \right) \rho_\phi\|_\infty = 0$  in view of Theorem 6.10, and, in virtue of (8.4), the convergence is uniform with respect to  $\phi \in \mathcal{P}_H^t(f)$ . Thus, invoking also (8.5) and (8.6), we see that

$$\lim_{\substack{j \rightarrow \infty \\ n-j \rightarrow \infty}} \sup_{\phi \in \mathcal{P}_H^t(f)} \|\hat{\mathcal{L}}_{\phi_q}^{n-j}(g \mathcal{L}_{\phi_q}^j(\mathbb{1})) - \left( \int g d\mu_\phi \right) \rho_\phi\|_\infty.$$

Consequently,

$$(8.7) \quad \lim_{n \rightarrow \infty} \sup_{\phi \in \mathcal{P}_H^t(f)} \left\| \frac{1}{n} \sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi_q}^{n-j}(g \mathcal{L}_{\phi_q}^j(\mathbb{1})) - \left( \int g d\mu_\phi \right) \rho_\phi \right\|_\infty = 0.$$

Since it can be easily seen from the proof of Lemma 5.2 that  $\inf\{\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1})(z) : \phi \in \mathcal{P}_H^t(f), n \geq 0, z \in J\}$ , we thus see from (8.7) and (8.5) that

$$\lim_{n \rightarrow \infty} \sup_{\phi \in \mathcal{P}_H^t(f)} \left\| \frac{\frac{1}{n} \sum_{j=0}^{n-1} \hat{\mathcal{L}}_{\phi_q}^{n-j}(g\mathcal{L}_{\phi_q}^j(\mathbb{1}))}{\hat{\mathcal{L}}_{\phi_q}^n(\mathbb{1})} - \left( \int g d\mu_\phi \right) \mathbb{1} \right\|_\infty = 0.$$

We now are done by invoking (8.1).  $\square$

**Proposition 8.3.** *Suppose  $\phi : J \rightarrow \mathbb{R}$  is an admissible potential and  $g : J \rightarrow \mathbb{R}$  is a Hölder continuous function. Then the function  $\mathbb{R} \ni t \mapsto P(\phi + tg) \in \mathbb{R}$  is differentiable on a sufficiently small open neighborhood of zero and*

$$\frac{d}{dt} P(\phi + tg) = \int g d\mu_{\phi+tg}.$$

*Proof.* Since  $\phi : J \rightarrow \mathbb{R}$  is an admissible potential and  $g : J \rightarrow \mathbb{R}$  is a Hölder continuous function, there exist  $H > 0$ ,  $0 \leq s < \log d$ , and  $\delta > 0$  such that  $\phi + tg \in \mathcal{P}_H^t(f)$  for all  $t \in [-\delta, \delta]$ . Then  $\mu_{\phi+tg}$  is well-defined and it follows from Proposition 5.1, Lemma 5.2, and Proposition 8.1 that for every  $x \in J$ ,

$$(8.8) \quad P(\phi + tg) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_{\phi+tg}^n(\mathbb{1})(x).$$

But

$$\frac{d}{dt} \left( \frac{1}{n} \log \mathcal{L}_{\phi+tg}^n(\mathbb{1})(x) \right) = \frac{\frac{1}{n} \mathcal{L}_{\phi+tg}^n(S_n g)(x)}{\mathcal{L}_{\phi+tg}^n(\mathbb{1})(x)} = \frac{\frac{1}{n} \hat{\mathcal{L}}_{\phi+tg}^n(S_n g)(x)}{\hat{\mathcal{L}}_{\phi+tg}^n(\mathbb{1})(x)}.$$

Thus, by Lemma 8.2,  $\frac{d}{dt} \left( \frac{1}{n} \log \mathcal{L}_{\phi+tg}^n(\mathbb{1})(x) \right)$  converges to  $\int g d\mu_{\phi+tg}$  uniformly with respect to  $t \in [-\delta, \delta]$ . Combining this with (8.8), we conclude that the function  $\mathbb{R} \ni t \mapsto P(\phi + tg) \in \mathbb{R}$  is differentiable and the derivative  $\frac{d}{dt} P(\phi + tg)$  is equal to the limit of derivatives,  $\int g d\mu_{\phi+tg}$ . We are done.  $\square$

Now, after all this preparation the proof of the main result of our paper is ready.

**Theorem 8.4.** *If  $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$  is a holomorphic endomorphism of a complex projective space  $\mathbb{P}^k$ , and  $\phi : J \rightarrow \mathbb{R}$  is a Hölder continuous potential satisfying  $\sup(\phi) - \inf(\phi) < \kappa_f$ , then there exists exactly one equilibrium state for  $\phi$ . This equilibrium state is equal to  $\mu_\phi = \rho_\phi m_\phi$  and it is metrically exact.*

*Proof.* The fact that  $\mu_\phi = \rho_\phi m_\phi$  is an equilibrium state for  $\phi$  was established in Proposition 8.1. Its uniqueness follows directly from Proposition 8.3 and Corollary 2.6.7 from [PU]. Metrical exactness of the dynamical system  $(f, \mu_\phi)$  coincides with Theorem 6.11. We are done.  $\square$

## 9. LOCAL DEGREE

Let us start by looking at the following result of Charles Favre (see [Fa]).

**Proposition 9.1.** *For all  $x \in \mathbb{P}^k$  the limit*

$$d(x) = \lim_{n \rightarrow \infty} (\deg_x f^n)^{\frac{1}{n}}$$

*exists. If  $d(x) > 1$  then there exists an irreducible complex subspace  $V$  and integers  $k, l$  such that  $f^k(x) \in V$  and  $f^l(V) = V$ .*

The proof of this theorem is based on Szemerédi's Theorem. The statement, however, is not sufficient for us; we need a uniform estimate of degrees of  $n$ 'th iterates. So, our main result of this section, Proposition 3.3, may be understood as a strengthening of Favre's result. In addition, as a byproduct, we obtain also an elementary proof of Favre's Proposition. For the convenience of the reader we begin with two short definitions that have been already formulated in Section 2, Preliminaries.

**Definition 9.2.** *Given an integer  $n \geq 1$  the periodic critical set  $A_n$  is the union of orbits of all irreducible varieties, that are contained in the critical set and are periodic under an iterate  $f^l$  with some  $l \leq n$ . In particular, an orbit of a critical periodic point of period  $l \leq n$  is in the critical periodic set  $A_n$ .*

For all positive integers  $n$  and  $p$  such that  $n > p$  let  $E_n^p$  be the set defined as follows.

**Definition 9.3.**  *$E_n^p$  is the set of all points  $x \in \mathbb{P}^k$  for which there exists a non-negative integer  $i \leq n - 1$  such that  $f^i(x) \in A_p$ .*

Our main result in this section is the following stated in Preliminaries as proposition 3.3.

**Proposition 9.4.** *For every  $\beta > 0$  there exist  $p = p(\beta)$  and  $N = N(\beta)$  such that for every  $n > N$  and for every  $x \notin E_n^p$  we have*

$$\#\{j \leq n : f^j(x) \in C\} \leq n\beta.$$

For its proof we need the following three simple lemmas.

**Lemma 9.5.** *Let  $h : X \rightarrow X$  be an arbitrary map and let  $F \subset X$  be an arbitrary subset of  $X$ . Fix  $\alpha > 0$  and consider the union*

$$F_\alpha = \bigcup_{i \leq [\frac{1}{\alpha}]} F \cap h^{-i}(F).$$

Next, consider a trajectory  $x, h(x), \dots, h^{M-1}(x)$  of length  $M > \lceil \frac{1}{\alpha} \rceil$ . If

$$\#\{s < M : h^s(x) \in F_\alpha\} \leq \alpha M,$$

then

$$\#\{s < M : h^s(x) \in F\} \leq 3\alpha M.$$

*Proof.* Divide the trajectory  $x, h(x), \dots, h^{M-1}(x)$  into blocks, ending at consecutive points in the trajectory, which are in  $F$ , i.e:  $B_1 = [x, h(x), \dots, h^{s_1}(x)]$  where  $s_1$  is the smallest iterate of  $x$  which falls to  $F$ , and, inductively,  $B_{m+1} = [h^{s_m+1}x, \dots, h^{s_{m+1}}(x)]$  while the last block has the form  $[h^{s_r}(x), h^{M-1}(x)]$ . Let us choose all blocks  $B_m, m < r$  of length  $\leq \lceil \frac{1}{\alpha} \rceil - 1$ . Notice that then  $h^{s_m}(x) \in F_\alpha$  since the distance  $s_{m+1} - s_m$  is not larger than  $\lceil \frac{1}{\alpha} \rceil$ . Consequently, by our assumption, the number of such blocks is not larger than  $\alpha M$ . Moreover, in the remaining blocks  $B_m, m < r$  every appearance of an element of  $F$  is followed by at least  $\lceil \frac{1}{\alpha} \rceil$  elements which are not in  $F$ . Consequently, the total number of elements of  $F$  in the trajectory can be bounded from above by  $\alpha M + \alpha M + 1 = 2\alpha M + 1 < 3\alpha M$  (since  $M > \lceil \frac{1}{\alpha} \rceil$ ).  $\square$

**Lemma 9.6.** *Let  $h : \mathbb{P}^k \rightarrow \mathbb{P}^k$  be a holomorphic map and let  $\mathcal{D} = \{D_1, \dots, D_t\}$  be a collection of irreducible varieties of the same codimension  $p$ . If, for some  $i$ ,  $h^{-i}(D_s) \cap D_r$  has an irreducible component of the same codimension  $p$ , then  $h^i(D_r) = D_s$ .*

*Proof.* Let  $V$  be a component of  $h^{-i}D_s \cap D_r$  of the same codimension. Since  $V \subset D_r$  and  $D_r$  is irreducible, we have  $V = D_r$  and, consequently,  $D_r \subset h^{-i}(D_s)$ . Thus,  $h^i(D_r) \subset D_s$ . Next, since  $D_s$  is also irreducible and  $\dim h(D_s) = \dim D_r$ , we get  $h^i(D_r) = D_s$ .  $\square$

**Lemma 9.7.** *Under the assumptions of Lemma 9.6, there exists an integer  $l \geq 1$  such that, for  $H = h^l$ , if  $H^i(D_r) = D_s$  for some  $i \in \mathbb{N}$  and some  $D_r, D_s \in \mathcal{D}$ , then  $H(D_s) = D_s$ .*

*Proof.* We build a natural graph with vertices  $D_1, \dots, D_t$ . We put an arrow from  $D_s$  to  $D_r$  if  $D_s$  is mapped under some iterate  $h^i$  of  $h$  onto  $D_r$  and, for all  $1 \leq j < i$ , the image  $h^j(D_s)$  is not a variety in our family  $\mathcal{D}$ . Associate to this arrow the weight  $i$ . To every maximal path in this graph, which is not eventually a loop, we associate its weight defined to be the product of weights of all arrows forming this path. Let  $l$  be a multiple of the lengths of all simple loops and, which in addition, is larger than the weights of all maximal paths in the graph that do not contain loops. Then every variety  $D_s$  is mapped by  $H = h^l$  either on a variety which is not in our family  $\mathcal{D}$ , or onto a variety  $D_r$  which belongs to a loop; thus fixed by  $H$ .  $\square$

We now pass to the proof of the Proposition 9.4. The proof will be performed in two steps. In the first, preparatory Step I, we construct recursively appropriate families of irreducible varieties contained in the critical set. They are then used in the inductive proof of Proposition 9.4 in Step II.

Step I. Construction of families of irreducible varieties  $\mathcal{D}_m^{(1,2,j_3,\dots,j_m)}$ .

Given  $\beta > 0$ , let  $\beta_0 = \beta$ ,  $\beta_1 = \frac{\beta}{2}$ , and  $\beta_m = \frac{\beta_{m-1}}{2k}$  for all  $2 \leq m \leq k$ . This choice of the sequence  $\beta_m$  will become clear in the second step of the proof. First, let us use lemma 9.7 for the map  $f$  and for the collection  $\mathcal{D}_1^{(1)}$  of irreducible components  $C_1, \dots, C_t$  of the critical set  $C$ . The superscript (1) stands here for the codimension 1 of all varieties in the family. We then replace the original map  $f$  by its iterate  $g_1 = f^{l_1}$ , such that for which the statement of Lemma 9.7 is satisfied for the family of irreducible varieties  $\mathcal{D}_1^{(1)}$ . Next we define the family of varieties  $\mathcal{D}_2^{(1,2)}$ . This family consists of all irreducible components of the intersections  $D_r \cap g_1^{-i} D_s$  of codimension 2, where  $i \leq \lfloor \frac{2}{\beta} \rfloor$ . Notice, that, for  $k = 2$  the varieties in the family  $\mathcal{D}_2^{(1,2)}$  are just points. For  $k = 2$  our construction ends at this step.

For  $k > 2$  we proceed as follows. Using Lemma 9.7 again, we find an iterate  $g_2 = g_1^{l_2}$  such that the statement of this lemma is satisfied, i.e. for each  $D_r, D_s \in \mathcal{D}_2^{(1,2)}$ , if  $g_2^i(D_r) = D_s$  for some  $i \geq 1$ , then  $g_2(D_s) = D_s$ . Now, for every  $3 \leq j_3 \leq k$  let  $\mathcal{D}_3^{(1,2,j_3)}$  be the family of all irreducible components of all intersections of the form  $D_r \cap g_2^{-i} D_s$ ,  $D_r, D_s \in \mathcal{D}_2$ , that have codimension  $3 \leq j_3 \leq k$  and  $1 \leq i \leq \lfloor \frac{2}{\beta_1} \rfloor$ . Let

$$\mathcal{D}_3 = \bigcup_{j_3=3}^k \mathcal{D}_3^{(1,2,j_3)}$$

Again, notice that if  $k = 3$  then only  $j_3 = 3$  is possible and the procedure ends at this point.

For a general  $k$ , proceeding by induction, fix  $m \geq 3$  and assume that for every  $1 \leq j_m \leq k$  the family  $\mathcal{D}_m^{(1,2,j_3,\dots,j_m)}$  of irreducible varieties of codimension  $j_m$  has been defined. Also, assume that the map  $g_m$  (as an appropriate iterate of  $f$ ) has been defined. We assume by induction that the map  $g_m$  has the following property. If, for some  $i \geq 1$ , and  $D_r, D_s \in \mathcal{D}_m^{(1,2,j_3,\dots,j_m)}$ ,  $g_m^i(D_r) = D_s$ , then  $g_m(D_s) = D_s$ . Fix  $1 \leq j_m \leq k$ . If  $j_m = k$  then the procedure ends at this moment (note that the elements of  $\mathcal{D}_m^{(1,2,j_3,\dots,j_m)}$  are then just points). If  $j_m < k$ , we define the families  $\mathcal{D}_{m+1}^{(1,2,j_3,\dots,j_m,j_{m+1})}$ , where  $j_{m+1} > j_m$  as follows. The family  $\mathcal{D}_{m+1}^{(1,2,j_3,\dots,j_m,j_{m+1})}$  consists of all irreducible components of all intersections  $D_r \cap g_m^{-i} D_s$ ,  $i \leq \lfloor \frac{2}{\beta_{m-1}} \rfloor$  that have codimension equal to  $j_{m+1}$ . Note that the range of admissible  $j_{m+1}$ 's is  $\{j_m + 1, \dots, k\}$  but some families  $\mathcal{D}_{m+1}^{(1,2,j_3,\dots,j_m,j_{m+1})}$  may be empty. Finally, for all families  $\mathcal{D}_{m+1}^{(1,2,j_3,\dots,j_m,j_{m+1})}$ , defined in this way, we find, using Lemma 9.7, a common value  $l_{m+1}$  and an iterate  $g_{m+1} = g_m^{l_{m+1}}$  of  $g_m$  such that if  $D_s = g_{m+1}^i(D_r)$  for some  $D_r, D_s \in \mathcal{D}_{m+1}^{(1,2,j_3,\dots,j_m,j_{m+1})}$ , then  $g_{m+1}(D_s) = D_s$ .

Step II. Fix  $N = N_0 = l_1 \cdot l_2 \cdot \dots \cdot l_k N_k$ , where  $N_k > N_k^0$  and  $N_k^0 > \lfloor \frac{1}{\beta_k} \rfloor$  so that the statement of Lemma 9.5 is satisfied for all  $M \geq N_k^0$  and all  $\alpha \geq \beta_k$ . Let  $N_1 = l_2 \cdot l_3 \cdot \dots \cdot N_k$ ,  $N_m = l_{m+1} \cdot \dots \cdot N_k$ . (so that  $N_{m-1} = l_m N_m$ ). Take now an arbitrary

point  $x \in \mathbb{P}^k$  and assume that the trajectory of  $x$ , up to  $f^N(x)$  visits the critical set with a frequency larger than  $\beta$ :

$$\#\{n < N : f^n x \in C\} > \beta N.$$

Since  $N = l_1 \cdot N_1$ , the whole trajectory  $x, f(x), \dots, f^{N-1}(x)$  can be split in a natural way into  $l_1$  trajectories of the points  $f^i(x), i \leq l_1$ , under  $g_1 : f^i(x), g_1(f^i(x)) \dots g^{N_1-1}(f^i(x))$ , and it is evident that there exists a point  $\tilde{x} = f^j(x), j < l_1$  such that

$$\#\{n < N_1 : g_1^n(\tilde{x}) \in C\} > \beta N_1.$$

Let us consider two cases. Either

(1) there exists  $n < N_1, i \leq [\frac{1}{\beta}]$ , such that  $g_1^n(\tilde{x}) \in C_r, g_1^{n+i}(\tilde{x}) \in C_s$  and the component of  $C_r \cap g_1^{-i}(C_s)$  containing  $\tilde{x}$  has codimension 1,

or else, the following holds:

(2) if  $g_1^n(\tilde{x}) \in C_r$  and  $g_1^{n+i}(\tilde{x}) \in C_s$  for some  $n \leq N_1, i \leq [\frac{3}{\beta}]$  then the component of  $C_r \cap g_1^{-i}(C_s)$  containing  $g_1^n(\tilde{x})$  has codimension 2.

If the case (1) occurs, we conclude that the point  $g_1^{n+i}(\tilde{x})$ , where  $n+i < N_1, i \leq [\frac{1}{\beta}]$ , lands in a critical variety  $V$ , which is fixed by  $g_1$  thus periodic for  $f$  with period  $l_1$ . This implies that  $f^m(x) \in V$ , where  $m \leq N, f^{l_1}(V) = V$ , and  $x \in E_N^p$ . The proof is then finished.

In the case (2) we proceed as follows. Using Lemma 9.5, we conclude that

$$\#\left\{0 \leq n \leq N_1 : g_1^n(\tilde{x}) \in \bigcup_{i \leq [\frac{2}{\beta}]} (C \cap g^{-i}C)\right\} > \beta/2N_1.$$

This means that the trajectory  $g_1(\tilde{x}), g_1^2(\tilde{x}), \dots, g_1^{N_1}(\tilde{x})$  of  $\tilde{x}$ , under  $g_1$  visits the varieties  $D_r$  from the family  $\mathcal{D}_2^{(1,2)}$  with frequency larger than  $\beta/2$ , This property is referred to as  $M_1(\tilde{x})$  and reads as follows.

$$(9.1) \quad \#\left\{0 \leq n \leq N_1 : g_1^n(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}_2^{(1,2)}} D_r\right\} > \beta/2N_1 = \beta_1 N_1$$

It is easy now to conclude the proof if  $k = 2$ . Indeed, then the varieties in  $\mathcal{D}^{(1,2)}$  are just points, and we can proceed as follows. Let  $R = R(\beta) = \#\mathcal{D}^{(1,2)}$ . It is evident that, if  $N_1$  is large enough and

$$\#\left\{0 \leq n < N_1 : g_1^n(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}_2^{1,2}} D_r\right\} \geq \beta/3N_1$$

then there exist  $m, m+i < N_1$  and a point  $z$  in the family  $\mathcal{D}_2^{(1,2)}$  such that  $g_1^m(\tilde{x}) = g_1^{m+i}(\tilde{x}) = z$ . Hence, there are  $n < N$ , and  $j \leq [\frac{3}{\beta}]$  such that  $f^n(x) = f^{n+j}(x) = z$

and  $z$  is a critical periodic point. thus  $x \in E_N^p$ , where  $p = [\frac{3}{\beta}]$ . This concludes the proof if  $k = 2$ .

For an arbitrary  $k$  we use the following inductive procedure. Put  $p = l_1 l_2 \dots l_k$ . Recall that  $\beta_1 = \beta/2$ . Let  $m \leq k$  and assume that for the point  $x$  the following property  $M_{m-1}(x)$  holds.

$$\# \left\{ 0 \leq n < N_{m-1} : g_{m-1}^n(x) \in \bigcup_{D \in \mathcal{D}_m^{(1,2,j_3 \dots j_m)}} D \right\} > \beta_{m-1} N_{m-1},$$

where  $D \in \mathcal{D}_m^{(1,2,j_3 \dots j_m)}$  for some sequence  $(1, 2, j_3 \dots j_m)$  which depends on  $x$ . Note that for  $m = 2$  this is precisely the formula (9.1). Let  $\mathcal{D}_m$  be the union of all families of the form  $\mathcal{D}_m^{(1,2,j_3 \dots j_m)}$ . Recall that  $g_m = g_{m-1}^{l_m}$  is the iterate of  $f$  such that, (see Step 1), if for some  $D_r, D_s \in \mathcal{D}_m$ ,  $g^i(D_r) = D_s$  then  $g_m(D_s) = D_s$ . Since the trajectory

$$\{x, g_{m-1}(x), g_{m-1}^2(x), \dots, g_{m-1}^{N_{m-1}}(x)\}$$

of the point  $x$  under  $g_{m-1}$  can be split into  $l_m$  trajectories of the points  $g_{m-1}^j(x)$ ,  $0 \leq j \leq l_m - 1$ , under  $g_m$ , it is evident that there exists a point  $\tilde{x} = g_{m-1}^j(x)$ ,  $i < l_m$  such that

$$\# \left\{ n \leq N_m : g_m^n(\tilde{x}) \in \bigcup_{D \in \mathcal{D}_m^{(1,2,j_3 \dots j_m)}} D \right\} \geq \beta_{m-1} N_m.$$

Recall that  $N_{m-1} = l_m N_m$ . Now, as in the case of  $m = 1$ , we consider two possibilities.

- (Ind.1) There exist  $n < N_m$ ,  $i \leq [\frac{2}{\beta_{m-1}}]$  and  $D_r, D_s \in \mathcal{D}_m$  of the same codimension, say  $j_m$ , such that  $g_m^n(\tilde{x}) \in D_r$ ,  $g_m^{n+i}(\tilde{x}) \in D_s$ , and the component of  $D_r \cap g_m^{-i}(D_s)$  containing  $\tilde{x}$  has codimension  $j_m$ .
- (Ind.2) If  $g_m^n(\tilde{x}) \in D_r$  and  $g_m^{n+i}(\tilde{x}) \in D_s$  for some  $n < N_m, i \leq [\frac{2}{\beta_{m-1}}]$ , then the component of  $D_r \cap g_m^{-i}(D_s)$  containing  $g_m^n(\tilde{x})$  has codimension larger than  $j_m$ .

If (Ind. 1) occurs then, by Lemma 9.6 and Lemma 9.7, we get that  $g_m(D_s) = D_s$  and  $g_m^{n+i}(\tilde{x})$  lands in an irreducible variety  $D_s$ , which is fixed by  $g_m$ . In particular, this case always occurs when  $j_m = k$ , i.e the varieties  $D_r, D_s$  are just points. Since  $g_m^{n+i}(\tilde{x})$  lands in the variety  $D_s$ , fixed by  $g_m$ , we see that  $g_{m-1}^{j+l_m(n+i)}(x) = g_m^{n+i}(\tilde{x})$  lands in the component  $D_s$ . Note that  $j + l_m(n+i) < N_{m-1}$  and  $g_{m-1}^{l_m}(D_s) = g_m(D_s) = D_s$ . Thus, we conclude that for some  $r < N_{m-1}$  the point  $g_{m-1}^r(x)$  lands in the variety  $D_s$ , which is contained in the critical set and which is periodic under  $g_{m-1}$  with period  $l_m$ . We are then done.

In the case (Ind. 2) we proceed as follows. Using Lemma 9.5, for  $\alpha = \frac{\beta_{m-1}}{2}$ , we can write

$$\# \left\{ n \leq N_m : g_m^n(\tilde{x}) \in \bigcup_{j \leq \lfloor \frac{2}{\beta_{m-1}} \rfloor} \bigcup_{D_r, D_s \in \mathcal{D}_m^{(1,2,j_3, \dots, j_m)}} (D_r \cap g_m^{-j} D_s) \right\} \geq \frac{\beta_{m-1}}{2} N_m.$$

Recall that the family  $\mathcal{D}_{m+1}^{(1,2,j_3, \dots, j_m, j_{m+1})}$  consists of all intersections  $D_r \cap g_m^{-j} D_s$ ,  $1 \leq j \leq \lfloor \frac{2}{\beta_{m-1}} \rfloor$ ,  $D_r, D_s \in \mathcal{D}_m^{(1,2,j_3, \dots, j_m)}$ , which have the same codimension  $j_{m+1}$ .

$$\{D_r \cap g_m^{-i} D_s : D_r, D_s \in \mathcal{D}_m^{(1,2,j_3, \dots, j_m)}\} = \bigcup_{j_{m+1}} \mathcal{D}_{m+1}^{(1,2,j_3, \dots, j_m, j_{m+1})}$$

and, since the range of all possible  $j_{m+1}$ 's is less than  $k$ , we can choose one value of  $j_{m+1}$  such that the following property  $M_m(\tilde{x})$  holds.

$$\# \left\{ n \leq N_m : g_m^n(\tilde{x}) \in \bigcup_{D_r \in \mathcal{D}_{m+1}^{(1,2,j_3, \dots, j_{m+1})}} D_r \right\} \geq \frac{\beta_{m-1}}{2k} N_m = \beta_m N_m.$$

Therefore, we have checked the following. If the condition  $M_{m-1}(x)$  is satisfied for  $x$  then either there exists  $r < N_{m-1}$  so that  $g_{m-1}^r(x)$  falls into a variety which is contained in  $C$  and periodic under  $g_{m-1}$  with period  $l_m$ , or else, the condition  $M_m(\tilde{x})$  is satisfied for some  $\tilde{x} = f^j(x)$ ,  $j \leq l_m$ , and some family  $\mathcal{D}_m^{1,2,j_3, \dots, j_{m+1}}$ . This ends the inductive step.

Now, take an arbitrary point  $x \in \mathbb{P}^k$  and consider its trajectory  $x, f(x), \dots, f^N(x)$ . where  $N = p \cdot N_k$ ,  $N_k \geq N_k^0$ . Let us turn again to the beginning of the Step II. Since the case (1) ends the proof, we are left to consider case (2). This leads to the condition  $M_1(\tilde{x})$  (see (9.1)). We then apply the above inductive procedure, starting from the point  $\tilde{x}$  and  $m = 1$ . If, at some step the case (Ind.1) occurs then the induction stops. It is evident that the number of inductive steps is at most  $k$ . Assume that the procedure ends for some  $m = m_0 \leq k$ . It then follows that for some  $r < N_{m_0-1}$  the point  $g_{m_0-1}^r(\hat{x})$  lands in a variety  $D_s$ , which is contained in the critical set and which is periodic under  $g_{m_0-1}$  with period  $l_{m_0}$ , thus periodic under  $f$  with period  $l_{m_0} \cdot l_{m_0-1} \cdots l_1 \leq p$ . Observe that  $\hat{x}$  is a point in the trajectory of  $\tilde{x}$  and, in fact it is easy to see that  $\hat{x} = f^t(x)$  for some  $t < l_1 + l_1 l_2 + \cdots + l_1 l_2 \cdots l_{m_0-1}$ . Since  $g_{m_0-1} = f^{l_1 l_2 \cdots l_{m_0-1}}$ , we get  $f^t(x) \in D_s$  for some  $t < N$ . Finally, take an arbitrary  $n > l_1 l_2 \cdots l_k N_k^0 = p N_k^0$ . Then  $n = pM + r$  for some  $M \geq N_k^0$  and  $0 \leq r < p$ . It is evident that, if  $\#\{i \leq n : f^i(x) \in C\} > 2n\beta$  then  $\#\{i \leq pM : f^i(x) \in C\} > n\beta$  if  $M$  is large enough. Thus, the statement follows from the proven part for  $N = pM$ .  $\square$

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