POROSITY IN CONFORMAL INFINITE ITERATED FUNCTION SYSTEMS

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Abstract. In this paper we deal with the problem of porosity of limit sets of conformal (infinite) iterated function systems. We provide a necessary and sufficient condition for the limit sets of these systems to be porous. We pay special attention to the systems generated by continued fractions with restricted entries and we give a complete description of the subsets I of positive integers such that the set J_I of all numbers whose continued fraction expansion entries are contained in I, is porous. We then study such porous sets in greater detail examining their Hausdorff dimensions, Hausdorff measures, packing measures and other geometric characteristics. We also show that the limit set generated by the complex continued fraction algorithm is not porous, the limit sets of all plane parabolic iterated function systems are porous, and of all real parabolic iterated function systems are not porous. We provide a very effective necessary and sufficient condition for the limit set of a finite conformal iterated function system to be porous.

1. Introduction, Preliminaries

A bounded subset X of a Euclidean space is said to be *porous* if there exists a positive constant $c > 0$ such that each open ball B centered at a point of X and of an arbitrary radius $0 < r \leq 1$ contains an open ball of radius cr disjoint from X. If only balls B centered at a fixed point $x \in X$ are discussed, X is called porous at x.
Obviously the following, formally weaker, requirement also defines porosity. There exist positive constant $c, \kappa > 0$ such that each open ball B centered at a point of X and of an arbitrary radius $0 < \kappa r \leq 1$ contains an open ball of radius cr disjoint from X. Fixing κ, c is called a porosity constant of X.

It is easy to see that each porous set has the box counting dimension lesser than the dimension of the Euclidean space it is contained in. Further relations between porosity and dimensions can be found for example in [M2] and [Sa]. A much weaker property, called also porosity, was introduced in [De]. For a survey concerning this concept see [Za]. In this paper we will only be interested in the notion of porosity described in the first paragraph of this section.

We deal with the problem of porosity of limit sets of conformal (infinite) iterated function systems. We provide a necessary and sufficient condition for the limit sets of these systems to be porous. We pay special attention to the systems generated by continued fractions with restricted entries. Let us describe the setting of conformal (infinite) iterated function systems introduced in [MU1]. Let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric

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space X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \le s\rho(x,y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let I^n denote the space of words of length n, I^∞ the space of infinite sequences of symbols in I, $I^* = \bigcup_{n\ge 1} I^n$ and for $\omega \in I^n$, $n \ge 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2\ldots\omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \ge 1$, converge to zero and since they form a decreasing family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the Open Set Condition if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \ne j$. (We do not exclude $\overline{\phi_i(U)} \cap \overline{\phi_j(U)} \ne \emptyset$.)

An iterated function system S satisfying the Open Set Condition is said to be conformal if $X \subset I\!R^d$ for some $d \ge 1$ and the following conditions are satisfied.

1a: $U = \mathrm{Int}_{I\!R^d}(X)$.

1b: There exists an open connected set V such that $X \subset V \subset I\!R^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V. (Note that for $d = 1$ this just means that all the maps ϕ_i, $i\imath I$, are C^1 monotone diffeomorphisms, for $d \ge 2$ the words C^1 conformal mean holomorphic or antiholomorphic, and for $d > 2$ the maps ϕ_i, $i\imath I$ are Möbius transformations. The proof of the last statement can be found in [BP] for example, where it is called Liouville's theorem.)

1c: There exist $\gamma, l > 0$ such that for every $x \in X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.

1d: Bounded Distortion Property(BDP). There exists $K \ge 1$ such that

$$|\phi'_\omega(y)| \le K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets C of V

$$\mathrm{diam}(\phi_\omega(C)) \le ||\phi_\omega'||\mathrm{diam}(C) \tag{1.1}$$

and

$$\mathrm{diam}(\phi_\omega(V)) \le D||\phi_\omega'||, \tag{1.2}$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \ge 1$ is a constant depending only on V. Moreover,

$$\mathrm{diam}(\phi_\omega(X)) \ge D^{-1}||\phi_\omega'|| \tag{1.3}$$

and

$$\phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi_\omega'||r) \tag{1.4}$$

for every $x \in X$, every $0 < r \le \mathrm{dist}(X, \partial V)$, and every word $\omega \in I^*$.

We want to end this section with a short description of the content of this paper. In Section 2 we provide a necessary and sufficient condition for the limit sets of conformal systems to be porous and we provide a very effective necessary and sufficient condition for the limit set of a finite conformal system to be porous.

In Section 3 we consider an arbitrary subset I of positive integers $I\!\!N$ and we investigate the set J_I consisting of all those $x \in (0,1)$ that in the continued fraction expansion

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots}}}}$$

each partial denominator x_i, $i \ge 1$, is in I. In Theorem 3.3 an effective necessary and sufficient condition for the set J_I to be porous is provided. This actually enables us to call I porous if the corresponding limit set J_I is porous. We examine some special subsets of $I\!\!N$ as prime numbers, arithmetic progressions, geometric progressions, powers with a fixed exponent from the point of view of porosity.

In Section 4 we prove that the limit set generated by the complex continued fraction algorithm is, not porous.

The Section 5 is devoted to study parabolic iterated function systems. We show that the limit set of a plane parabolic system is porous, including the residual set of Apollonian packing, whereas the limit sets of real parabolic systems are not porous.

2. General Results

We keep the notation and terminology from the previous section and we start this section with the following result whose natural place is in [MU1].

Theorem 2.1. *If $S = \{\phi_i\}_{i\in I}$ is a conformal i.f.s. and $\mathrm{Int}X \setminus \overline{J} \ne \emptyset$, then $J \subset I\!\!R^d$ is a nowhere-dense set.*

Proof. Consider an arbitrary point $x \in J$ and a radius $r > 0$. By the definition of the limit set there exists $\omega \in I^*$ such that $\phi_\omega(X) \subset B(x,r)$. By the Open Set Condition $\phi_\omega(\mathrm{Int}X \setminus \overline{J}) \subset B(x,r)$ is then an open set disjoint from J and we are done. $\square$

The main result of this section is the following.

Theorem 2.2. *Let $S = \{\phi_i\}_{in\in I}$ be a conformal i.f.s.. Then the following three conditions are equivalent*

(a): *The limit set J is porous.*
(b): $\exists(c > 0)\,\exists(\xi > 0)\,\forall(i \in I)\,\forall(0 < r \le \xi)$
if $r \ge \mathrm{diam}(\phi_i(X))$ then there exists $x_i \in B(\phi_i(X), r) \cap X$ such that

$$B(x_i, cr) \cap J = \emptyset.$$

(c): $\exists(\kappa \ge 1)\,\exists(c > 0)\,\exists(\xi > 0)\,\exists(\beta \ge 1)\,\forall(i \in I)\,\forall(0 < r \le \xi)$
if $r \ge \beta\mathrm{diam}(\phi_i(X))$ then there exists $x_i \in B(\phi_i(X), \kappa r) \cap X$ such that

$$B(x_i, cr) \cap J = \emptyset.$$

Proof. It is obvious that $(a) \Rightarrow (b) \Rightarrow (c)$. So, suppose that condition (c) is satisfied. Decreasing $c > 0$ if necessary, we may assume that it holds with $\xi \ge 2KD^3\beta$. Fix an arbitrary $x = \pi(\omega) \in J$, $\omega \in I^\infty$, and a positive radius $r < 2KD^3\beta$. Let $n \ge 1$ be the least integer such that

$$\phi_{\omega|_n}(X) \subset B\left(x, \frac{r}{2KD^2\beta}\right).$$

Suppose first that $n = 1$. Then $r \ge \beta\mathrm{diam}(\phi_{\omega_1}(X))$ and, as $r < 2KD^3\beta$, we conclude from (c) that $B(x_{\omega_1}, cr) \cap J = \emptyset$. Since also $B(x_{\omega_1}, cr) \subset B(x, cr + \kappa r) \subset B(x, (c+\kappa)r)$, we are done in this case with the porosity constant $\le c/2$. So, suppose in turn that $n \ge 2$. Then

$$\mathrm{diam}(\phi_{\omega|_n}(X)) \le \frac{r}{KD^2\beta} \quad \text{and} \quad \mathrm{diam}(\phi_{\omega|_{n-1}}(X)) \ge \frac{r}{2KD^2\beta}. \tag{2.1}$$

Therefore by (1.2) and (1d)

$$\begin{aligned}\mathrm{diam}(\phi_{\omega_n}(X)) &\le D||\phi'_{\omega_n}|| \le DK\frac{||\phi'_{\omega|_n}||}{||\phi'_{\omega|_{n-1}}||} \le DK||\phi'_{\omega|_{n-1}}||^{-1}D\mathrm{diam}(\phi_{\omega|_n}(X))\\ &\le D^2Kr(\beta KD^2)^{-1}||\phi'_{\omega|_{n-1}}||^{-1} = \beta^{-1}r||\phi'_{\omega|_{n-1}}||^{-1}.\end{aligned}$$

Hence,

$$r||\phi'_{\omega|_{n-1}}||^{-1} \ge \beta\mathrm{diam}(\phi_{\omega_n}(X)). \tag{2.2}$$

Also by (2.1)

$$r||\phi'_{\omega|_{n-1}}||^{-1} \le Dr\mathrm{diam}^{-1}(\phi_{\omega|_{n-1}}(X)) \le 2KD^3\beta. \tag{2.3}$$

Hence, condition (c) is applicable with $i = \omega_n$ and the radius $r||\phi'_{\omega|_{n-1}n}||^{-1}$. Using (2.2) we get

$$\begin{aligned}\phi_{\omega|_{n-1}}\Big(B\Big(x_{\omega_n}, cr||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) &\subset \phi_{\omega|_{n-1}}\Big(B\Big(\phi_{\omega|_n}(X), cr||\phi'_{\omega|_{n-1}}||^{-1} + \kappa r||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) \subset \\ &\subset \phi_{\omega|_{n-1}}\Big(B\Big(\pi\Big(\sigma^{n-1}(\omega)\Big), \beta^{-1}r||\phi'_{\omega|_{n-1}}||^{-1} + cr||\phi'_{\omega|_{n-1}}||^{-1} + \kappa r||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) \\ &\subset B(x, (2+\kappa)r).\end{aligned}$$

Since $B\Big(x_{\omega_n}, cr||\phi'_{\omega|_{n-1}}||^{-1}\Big)$ may not be contained in X, we need the following reasoning to conclude the proof. In view of (2.3) and the Cone Condition, we get for some $y \in \mathrm{Con}\Big(x_{\omega_n}, \alpha, \min\{c, l(2KD^3\beta)^{-1}\}r||\phi'_{\omega|_{n-1}}||^{-1}\Big)$,

$$\begin{aligned}\phi_{\omega|_{n-1}}\Big(B\Big(x_{\omega_n}, cr||\phi'_{\omega|_{n-1}n}||^{-1}\Big)\Big) &\supset \\ &\supset \phi_{\omega|_{n-1}}\Big(\mathrm{Con}\Big(x_{\omega_n}, \alpha, \min\{cr||\phi'_{\omega|_{n-1}}||^{-1}, l\}\Big)\Big) \\ &\supset \phi_{\omega|_{n-1}}\Big(\mathrm{Con}\Big(x_{\omega_n}, \alpha, \min\{c, l(2KD^3\beta)^{-1}\}r||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) \\ &\supset \phi_{\omega|_{n-1}}\Big(B\Big(y, c'\min\{c, l(2KD^3\beta)^{-1}\}r||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) \\ &\supset B\Big(\phi_{\omega|_{n-1}}(y), K^{-1}c'\min\{c, l(2KD^3\beta)^{-1}\}r\Big),\end{aligned}$$

where $0 < c' \le 1$ is so small that each central cone $\mathrm{Con}(z, \alpha, k)$ contains an open ball of radius $c'k$. Since $\mathrm{Con}\Big(x_{\omega_n}, \alpha, \min\{cr||\phi'_{\omega|_{n-1}}||^{-1}, l\}\Big) \subset \mathrm{Int}(X)$ and since $J \cap B\Big(x_{\omega_n}, cr||\phi'_{\omega|_{n-1}}||^{-1}\Big) = \emptyset$, we conclude that $J \cap B\Big(\phi_{\omega|_{n-1}}(y), K^{-1}c'\min\{c, l(2KD^3\beta)^{-1}\}r\Big) = \emptyset$. The proof is complete. □

Theorem 2.3. *There exists a conformal system $S = \{\phi_i\}_{i\in I}$ whose limit set is not porous but the limit set of each proper subsystem of S is porous.*

Proof. Let $X = [0,1]$, let $I = \{1, 2, \dots\}$ and let $\phi_i : [0,1] \to [0,1]$ be given by the formula

$$\phi_i(x) = \frac{x+1}{2^i},\ \ i \ge 1.$$

Then $S = \{\phi_i\}_{i\in I}$ is a conformal iterated function system and its limit set is equal to $[0,1]$. So, it is not porous (as a subset of $I\!R$). If we now remove at least one element j from I, then each set $\phi_I\left(\bigcup_{k\neq j}\phi_k(X)\right)$ consists of two intervals and the gap between them has the length $2^{-j}|\phi_i(X)|$. Since, in addition $\frac{\phi_{i+1}(X)|}{|\phi_i(X)|} = 1/2$, it is not difficult to check that the condition (b) of Theorem 2.2 is satisfied for the system $I \setminus \{j\}$ (the balls $B(x_i, cr)$ disjoint from $J_I \setminus \{j\}$ must be contained in in gaps of the sets $\phi_n\left(\bigcup_{k\neq j}\phi_k(X)\right)$. The proof is complete. □

This Theorem shows that one cannot replace the set I by any of its cofinite subsystems (i.e. those whose complements in I are finite) in Theorem 2.2. As an immediate consequence of Theorem 2.2 we get however the following.

Theorem 2.4. *If I is infinite and there exists a cofinite subset F of I such that one of the following conditions is satisfied, then the limit set J_I is porous.*

(a): $\exists(c>0)\,\exists(\xi>0)\,\forall(i\in I\setminus F)\,\forall(0<r\le\xi)$
if $r\ge \mathrm{diam}(\phi_i(X))$ *then there exists* $x_i\in B(\phi_i(X),r)\cap X$ *such that*
$$B(x_i,cr)\cap J=\emptyset.$$
(b): $\exists(\kappa\ge 1)\,\exists(c>0)\,\exists(\xi>0)\,\exists(\beta\ge 1)\,\forall(i\in I\setminus F)\,\forall(0<r<\xi)$
if $r\ge \beta\mathrm{diam}(\phi_i(X))$ *then there exists* $x_i\in B(\phi_i(X),\kappa r)\cap X$ *such that*
$$B(x_i,cr)\cap J=\emptyset.$$

(Infinity of I was needed to have some holes (no matter how small for the set F). The following example shows that the limit set of a finite system does not have to be porous. Indeed, take $I=1,2$ and consider two contractions $\{\phi_i : i=1,2\}$ defined on the set $X=[0,1]$ by the formulas
$$\phi_1(x)=\frac{x}{2} \text{ and } \phi_2(x)=\frac{x}{2}+\frac{1}{2}.$$
Then $S=\{\phi_i : i=1,2\}$ is a finite conformal iterated function system and its limit, the interval $[0,1]$ is not porous as a subset of $I\!R$. As an immediate consequence of Theorem 2.2 we get however the following.

Theorem 2.5. *If $S=\{\phi_i\}_{i\in I}$ is a finite conformal i.f.s. and* $\mathrm{Int}X\setminus\overline{J}\ne\emptyset$, *then the limit set J_I is porous.*

Note that this theorem is actually obvious since one can drag the "hole" in $\mathrm{Int}X$ to any scale via the maps ϕ_ω, $\omega\in I^*$, (In an infinite case one does not hav to fill in in this way all scales).

3. Real Continued Fractions

Following [MU2] let us consider an infinite subset I of $I\!N$ and then the iterated function system $\{\phi_i\}_{i\in I}$, where $X=[0,1]$, $V=(-1/2,2)$ and for every $i\in I$,
$$\phi_i(x)=\frac{1}{x+i}$$
The limit set J_I of this iterated function system is the set of those numbers in $[0,1]$ whose all continued fraction expansion entries are in I. Following [MU2] we call a subset $E\subset I$ cluster (or segment, subinterval of I) if $E=[\min(E),\sup(E)]$. By $|E|$, the length of the cluster E, we mean its cardinality, e.i. $\sup(E)-\min(E)+1$. Our main goal in this section is to prove Theorem 3.3 and, as an intermediate step, we prove the following first characterization of the sets I whose limit sets J_I are porous. Already this characterization demonstrates that infinite subsets of $I\!N$ rather reluctantly give rise to porous limit sets.

Theorem 3.1. *Let I be an infinite subset of $I\!N$. Then the following three conditions (a), (b) and (c) are equivalent.*

(a): *The limit set J_I is porous.*

(b): $\exists(0<\sigma<1)\,\forall(i\in I)\,\forall(i\le q\le i(i+1))$
either

(1): *there exist* $k, l \geq 1$ *such that*

$$i \leq k \leq \frac{q(i+1)}{q-(i+1)},\ l \geq \frac{qk}{q-\sigma k}$$

and $[k+1, l-1] \subset I\!N \setminus I$

or

(2): *there exist* $k, l \geq 1$ *such that*

$$\frac{qi}{q+i} \leq l \leq i,\ k \leq \frac{ql}{q+\sigma l}$$

and $[k+1, l-1] \subset I\!N \setminus I$.

(c): $\exists(\lambda \geq 1)\,\exists(0 < \sigma < 1)\,\forall(i \in I)\,\forall(\lambda i \leq q \leq \lambda^{-1} i(i+1))$
either

(1): *there exist* $k, l \geq 1$ *such that*

$$i \leq k \leq \frac{q(i+1)}{q-(i+1)},\ l \geq \frac{qk}{q-\sigma k}$$

and $[k+1, l-1] \subset I\!N \setminus I$

or

(2): *there exist* $k, l \geq 1$ *such that*

$$\frac{qi}{q+i} \leq l \leq i,\ k \leq \frac{ql}{q+\sigma l}$$

and $[k+1, l-1] \subset I\!N \setminus I$.

Proof. $(a) \Rightarrow (b)$. Since J_I is porous, condition (b) of Theorem 2.2 is satisfied. Fix $i \in I$, and $q \in [i, i(i+1)]$. By condition (b) of Theorem 2.2 there exists a point $x_i \in \left(\frac{1}{i+1} - \frac{(1-c)^{-1}}{q}, \frac{1}{i} + \frac{(1-c)^{-1}}{q}\right)$ such that $B\left(x_i, \frac{c(1-c)^{-1}}{q}\right) \cap \overline{J} = \emptyset$. Since $1/i \in \overline{J}$, either

$$B\left(x_i, \frac{c(1-c)^{-1}}{q}\right) \subset (1/i, \infty) \ \text{ or } \ B\left(x_i, \frac{c(1-c)^{-1}}{q}\right) \subset (-\infty, 1/i).$$

Suppose that the first case holds. Let $k \geq 1$ be the largest integer such that $1/k \geq x_i + \frac{c(1-c)^{-1}}{q}$ (such an integer exists assuming that $c > 0$ is small enough) and let $l \geq 2$ be the least integer such that $1/i \leq x_i - \frac{c(1-c)^{-1}}{q}$. Then $l \leq i$ and $[k+1, l-1] \subset I\!N \setminus I$. Since

$$\frac{1}{l} \leq x_i - \frac{c(1-c)^{-1}}{q} \leq \frac{1}{i} + \frac{(1-c)^{-1}}{q} - \frac{c(1-c)^{-1}}{q} = \frac{1}{i} + \frac{1}{q},$$

we get

$$l \geq \frac{1}{\frac{1}{i} + \frac{1}{q}} = \frac{qi}{q+i}.$$

Since $\frac{1}{k} - \frac{1}{l} \geq \frac{2c(1-c)^{-1}}{q}$, we get

$$k \leq \frac{1}{\frac{1}{i} + \frac{2c(1-c)^{-1}}{q}} = \frac{ql}{q + 2c(1-c)^{-1}l}$$

and we are done in this case. Suppose in turn that

$$B\left(x_i, \frac{c(1-c)^{-1}}{q}\right) \subset (-\infty, 1/i).$$

Let, similarly as above, $k \geq 1$ be the largest integer such that $1/k \geq x_i + \frac{c(1-c)^{-1}}{q}$ and let $l \geq 1$ the least integer such that

$$\frac{1}{l} \leq \max\left\{0, x_i - \frac{c(1-c)^{-1}}{q}\right\}$$

(we allow $l = \infty$). Then $k \geq i$ and $[k+1, l-1] \subset I\!N \setminus I$. Since

$$\begin{aligned}\frac{1}{k} &\geq x_i + \frac{c(1-c)^{-1}}{q} \geq \frac{1}{i+1} - \frac{(1-c)^{-1}}{q} + \frac{c(1-c)^{-1}}{q} \\ &= \frac{1}{i+1} - \frac{1}{q},\end{aligned}$$

we get

$$k \leq \frac{q(i+1)}{q-(i+1)}.$$

If $\max\left\{0, x_i - \frac{c(1-c)^{-1}}{q}\right\} = 0$, then $l = \infty$ and we are done. So, suppose that $x_i > \frac{c(1-c)^{-1}}{q}$. By the definition of l, $1/l \leq x_i - \frac{c(1-c)^{-1}}{q}$ and therefore $\frac{1}{k} - \frac{1}{l} \geq \frac{2c(1-c)^{-1}}{q}$. Hence

$$l \geq \frac{qk}{q - 2c(1-c)^{-1}k}$$

The proof of the implication $(a) \Rightarrow (b)$ is thus complete.

The implication $(b) \Rightarrow (c)$ is obvious. So suppose that condition (c) is satisfied with some $\lambda \geq 1$ and some $0 < \sigma < 1$. Set

$$\beta = \max\{2\lambda, 144/\sigma\}, \;\; \xi = \beta^{-1}(\lambda+1)^{-2}$$

and consider arbitrary $i \in I$ and $\frac{\beta}{i(i+1)} \leq r \leq \xi$. Suppose first that

Case A: $0 < r < \frac{1}{\lambda i}$. Consider the least $q \geq 1$ such that $1/q \leq r$. Then $q \geq r^{-1} > \lambda i$ and, since $q \geq 2$,

$$q = \frac{q}{q-1}(q-1) \leq 2\frac{1}{r} \leq \frac{2}{\beta} i(i+1) \leq \lambda^{-1} i(i+1).$$

Hence, it follows from condition (c) that there exist k and l produced either by case (1) or case (2) of this condition. Suppose first that case (1) holds. Put

$$b = \left[\frac{qk}{q - \sigma k}\right], \;\; x_i = \frac{1}{2}\left(\frac{1}{k+1} + \frac{1}{b}\right) \in [0,1] \;\text{ and }\; r_i = \frac{1}{2}\left(\frac{1}{k+1} - \frac{1}{b}\right).$$

Then $B(x_i, r_i) = \left(\frac{1}{b}, \frac{1}{k+1}\right)$, and since $[k+1, l-1] \subset I\!N \setminus I$,

$$B(x_i, r_i) \cap J = \emptyset \tag{3.1}$$

Also

$$\frac{1}{i+1} - \frac{1}{b} \le \frac{1}{i+1} - \frac{q - \sigma k}{qk} = \frac{1+\sigma}{q} \le 2r.$$

Since in addition $\frac{1}{k+1} \le \frac{1}{i+1}$, we conclude that

$$x_i \in B\Big(\phi_i([0,1]), 2r\Big). \tag{3.2}$$

In order to complete this case we need some auxiliary estimates. First, as $\beta \ge 12/\sigma$,

$$\frac{1}{k(k+1)} \le \frac{1}{i(i+1)} \le \frac{r}{\beta} \le \frac{\sigma}{12} r \tag{3.3}$$

and then, as $b+1 \ge \frac{qk}{q-\sigma k} \ge k$,

$$\frac{1}{b(b+1)} \le \frac{2}{(b+1)^2} \le \frac{2}{k^2} \le \frac{4}{k(k+1)} \le \frac{\sigma}{3} r. \tag{3.4}$$

Using now the definition of q along with (3.3) and (3.4), we get the following last estimate we need in this case

$$\begin{aligned}\frac{1}{k+1} - \frac{1}{b} &= \left(\frac{1}{k} - \frac{1}{b+1}\right) - \frac{1}{k(k+1)} - \frac{1}{b(b+1)} \ge \frac{1}{k} - \frac{q-\sigma k}{qk} - \frac{\sigma}{12} r - \frac{\sigma}{3} r \\ &= \frac{q-1}{q}\frac{\sigma}{q-1} - \frac{\sigma}{12} r - \frac{\sigma}{3} r \ge \frac{\sigma}{2} r - \frac{\sigma}{12} r - \frac{\sigma}{3} r = \frac{\sigma}{12} r.\end{aligned}$$

Combining this, inequalities (3.2) and (3.1), we see that in this case condition (c) of Theorem 2.2 is satisfied with the constant $c = \frac{\sigma}{24}$.

Suppose now that the case (2) holds. Put

$$a = \left[\frac{ql}{q+\sigma l}\right] + 1,\ \ x_i = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{l}\right) \in [0,1] \ \text{ and } r_i = \frac{1}{2}\left(\frac{1}{a} - \frac{1}{l}\right).$$

Then $B(x_i, r_i) = (1/l, 1/a)$ and, since $[k+1, l-1] \subset I\!N \setminus I$,

$$B(x_i, r_i) \cap J = \emptyset. \tag{3.5}$$

Also

$$\frac{1}{a} - \frac{1}{i} \le \frac{q+\sigma l}{ql} - \frac{1}{i} = \frac{qi + \sigma il - ql}{qil} \le \frac{li + lq - \sigma il - ql}{qil} = \frac{(1-\sigma)il}{qil} = \frac{1-\sigma il}{q} < r.$$

Since in addition $1/l \ge 1/i$, we deduce that

$$x_i \in B\Big(\phi_i([0,1]), r\Big). \tag{3.6}$$

In order to complete this case we also need some auxiliary estimate.

$$\frac{1}{a} \leq \frac{q+\sigma l}{ql} = \frac{1}{l} + \frac{\sigma}{q} \leq \frac{i+q}{qi} + \sigma r = \frac{1}{q} + \frac{1}{i} + \sigma r$$
$$\leq (1+\sigma)r + \frac{1}{i} \leq \Big((1+\sigma)\lambda^{-1} + 1\Big)\frac{1}{i} < \frac{3}{i}$$

and therefore, as $\beta \geq \frac{144}{\sigma}$,

$$\frac{1}{a(a-1)} \leq \frac{2}{(a-1)^2} \leq \frac{18}{i^2} \leq \frac{36}{i(i+1)} < \frac{\sigma}{4}r.$$

Hence

$$\frac{1}{a} - \frac{1}{l} = \frac{1}{a-1} - \frac{1}{l} - \frac{1}{a(a-1)} \geq \frac{q+\sigma l}{ql} - \frac{1}{l} - \frac{\sigma}{4}r = \frac{\sigma}{q} - \frac{\sigma}{4}r \geq \frac{\sigma}{2}r - \frac{\sigma}{4}r = \frac{\sigma}{4}r$$

and we are done in this case too.

Suppose finally that we have
Case B: $r \geq \frac{1}{\lambda i}$. Consider then the least $j \in I$ such that $\frac{1}{j} \leq \frac{1}{i} + r$. Then $j \leq i$ and suppose that $j < 4/r = \lambda^{-1}\left(\frac{1}{(4\lambda)^{-1}r}\right)$ Since also

$$\frac{\beta}{j(j+1)} \leq \frac{\beta}{j^2} \leq \beta\left(\frac{1}{i} + r\right)^2 \leq \beta((\lambda+1)r)^2 \leq r,$$

the Case A is applicable with j, the element of I, and the radius $\frac{r}{4\lambda}$. Then

$$\left|x_j - \frac{1}{i}\right| \leq \left|x_j - \frac{1}{j}\right| + \left|\frac{1}{j} - \frac{1}{i}\right| \leq \frac{2r}{4\lambda} + \frac{1}{j(j+1)} + r \leq \frac{r}{2} + \frac{r}{\beta} + r \leq 3r$$

which implies that

$$x_j \in B\Big(\phi_i([0,1]), 3r\Big).$$

Also

$$x_j \in [0,1] \ \text{ and } \ B\left(x_j, \frac{\sigma}{96\lambda}r\right) \cap J = \emptyset.$$

So, we are done the constant $c \leq \frac{\sigma}{96\lambda}$. Thus we can assume that $j \geq 4/r$. Let $m \geq 1$ the largest integer such that $\frac{1}{m} \geq \frac{1}{i} + r$. Then, by the definition of j and m

$$\left[\frac{1}{j}, \frac{1}{m+1}\right] \cap J = \emptyset \tag{3.7}$$

and

$$\frac{1}{m+1} - \frac{1}{j} \geq \frac{1}{2m} - \frac{1}{j} \geq \frac{1}{2i} + \frac{r}{2} - \frac{1}{j} \geq \frac{r}{2} - \frac{1}{j} \geq \frac{r}{2} - \frac{r}{4} = \frac{r}{4}.$$

Set

$$x_i = \frac{1}{2}\left(\frac{1}{j} + \frac{1}{m+1}\right) \in [0,1].$$

Then $B(x_i, r/8) \subset [1/j, \frac{1}{m+1}]$ and therefore by (3.7)

$$B\Big(x_i, r/8) \cap J = \emptyset. \tag{3.8}$$

Moreover $x_i \geq 1/j \geq 1/i$ and $x_i \leq \frac{1}{m+1} < \frac{1}{i}+r$. In particular $x_i \in B\Big(\phi_i([0,1]), r\Big)$. Combining this and (3.8), we conclude the Case B, and consequently the entire proof of Theorem 3.1. □

Our main aim in this section is to prove Theorem 3.3 which provides a very effective condition for an infinite subset I of $I\!N$ to generate a porous limit set J_I. In order to make the proof more readible we demonstrate first the following lemma which in fact is a part of the proof of Theorem 3.3.

Lemma 3.2. *Suppose that $I \subset I\!N$ is an infinite subset of $I\!N$ such that the limit set is porous. Then there exist $0 < \theta \leq 1$ and $x > 0$ such that for every $i \in I$ and every $x \leq p \leq i/5$ either the interval $[i-p, i]$ or $[i, i+p]$ contains a cluster of $I\!N \setminus I$ of length $\geq \theta p$.*

Proof. We have for every $i \geq 1$,

$$\frac{4i(i+1)}{4i-(i+1)} \geq \frac{4i(i+1)}{4i-(i+1)} = \frac{4}{3}(i+1) \geq \frac{4}{3}i \tag{3.9}$$

and for every $i \geq 24$,

$$\frac{4^{-1}i(i+1)(i+1)}{4^{-1}i(i+1)-(i+1)} = \frac{5i}{i-4} \leq 6 \tag{3.10}$$

In addition, for every $j \geq 1$ and every $q \geq 4(j-1)$

$$\left|\frac{(q+1)j}{q+1-j} - \frac{qj}{q-j}\right| = \frac{j^2}{(q+1-j)(q-j)} \leq \frac{j^2}{3j \cdot 3j} = 1/9 < 1 \tag{3.11}$$

Since J is a porous set, condition (b) of Theorem 3.1 is satisfied. Let $0 < \sigma < 1$ be the constant produced there. Set

$$x = \frac{\sigma}{160}.$$

Consider an arbitrary $i \in I$. Take then $x \leq p \leq i/5$. Then $i \geq 5x \geq 5 \cdot 160 \geq 24$. Consider among the numbers

$$\frac{u(i+1)}{u-(i+1)},\ 4i \leq u \leq 4^{-1}i(i+1)$$

the largest element $\leq i + \frac{p}{2}$. Such an element exists by (3.9) and (3.10) since $p/2 \geq x/2 \geq 6$. Denote the corresponding value of u by q. Let us now distinguish two cases according two Theorem 3.1(b).

Case A: The Case (1) of Theorem 3.1(b) holds for our q. Since $i+p-(k+1) \geq (i+p)-\Big(i+\frac{p}{2}\Big) = \frac{p}{2}-1 \geq \frac{p}{4}$, if $l-1 \geq i+p$, we then have $[k+1, l-1] \supset [k+1, i+p]$ and we conclude from part (1) of Theorem 3.1(b) that the interval $[i, i+p]$ contains a cluster of $I\!N \setminus I$ of length $p/4$. Thus, it is so far enough to take $\theta = 1/4$. So suppose that

$$l-1 \leq i+p-1 \tag{3.12}$$

According to part (1) of Theorem 3.1(b) we have

$$l-k \geq \frac{qk}{q-\sigma k} - k = k\left(\frac{q}{q-\sigma k} - 1\right) \geq i\left(\frac{q}{q-\sigma k} - 1\right) = \frac{qki}{q-\sigma k}. \tag{3.13}$$

Since $i+\frac{p}{2} \leq \frac{11}{10}i \leq \frac{4}{3}i$, it follows from the definition of q and (3.11) that

$$\frac{p}{4} \leq \frac{p}{2} - 1 \leq \frac{q(i+1)}{q-(i+1)} - i. \tag{3.14}$$

Our aim now is to find a universal constant $\eta > 0$ such that

$$\frac{\sigma ki}{q-\sigma k} \geq \eta\left(\frac{q(i+1)}{q-(i+1)} - i\right) \tag{3.15}$$

This inequality can be rewritten in equivalent forms as follows.

$$\sigma kqi - \sigma ki(i+1) \geq \eta\Big(q^2(i+1) - \sigma kq(i+1) - iq^2 + \sigma qki + i(i+1)q - \sigma ki(i+1)\Big)$$

or

$$\sigma kqi \geq \eta\Big(q^2 - \sigma qk(i+1) + \sigma qki + i(i+1)q - \sigma ki(i+1)\Big) + \sigma ki(i+1).$$

Thus, (3.15) will be satisfied if $\sigma kqi \geq \eta\Big(q^2 + \sigma qki + i(i+1)q\Big) + \sigma ki(i+1)$. But

$$\begin{aligned}\eta\Big(q^2 + \sigma qki + i(i+1)q\Big) + \sigma ki(i+1) &\leq \eta\Big(q4^{-1}i(i+1) + i(i+1)q\Big)\\ &= \sigma ki2^{-1}q \leq \eta(2^{-1}+2)qik + 2^{-1}\sigma qki.\end{aligned}$$

So, (3.15) will be satisfied if $\sigma qki \geq 3\eta qki + 2^{-1}\sigma qki$ or equivalently if $2^{-1}\sigma \geq 3\eta$. Therefore (3.15) is satisfied provided that $\eta = \sigma/6$. Combining now (3.13)-(3.15), we obtain

$$l-1-(k+1) \geq \frac{\sigma}{6}\cdot\frac{p}{4} - 2 \geq \frac{\sigma}{48}p,$$

where the last equality we wrote since $p \geq x \geq \frac{96}{\sigma}$. Since by (3.12), $[k+1, l-1] \subset [i, i+p]$ and by part (1) of Theorem 3.1(b), $[k+1, l-1] \subset I\!\!N \setminus I$, we see that $[i, i+p]$ contains a cluster of $I\!\!N \setminus I$ of length $\frac{\sigma}{48}p$. Thus, up to now, it is enough to take $\theta \leq \min\{1/4, \sigma/48\} = \sigma/48$.

Case B: The Case (2) of Theorem 3.1(b) holds for our q. Put

$$s = \frac{q(i+1)}{q-(i+1)} - i.$$

Multiplying this inequality by $q-(i+1)$, we get $qi + q - qi + i^2 + i = qs - si - s$ and, since by (3.14) $s \geq p/4 \geq x/4 \geq 1$, we therefore get

$$i^2 = s(q+i) - 2si - i - s \geq s(q+i) - 4si. \tag{3.16}$$

Hence, using (3.14) again,

$$\frac{i^2}{q+i} \geq s - \frac{4si}{q+i} \geq s - \frac{4si}{5i}\frac{s}{5} \geq \frac{p}{20}.$$

Therefore

$$i - \frac{qi}{i+q} = \frac{i^2}{i+q} \geq \frac{p}{20}. \tag{3.17}$$

On the other hand, it follows from the first part of (3.16) and the definition of q that

$$i-(l-1) \leq i - \frac{qi}{i+q} + 1 = \frac{i^2}{i+q} + 1 \leq s+1 \leq \frac{p}{2}+1 \leq \frac{3}{4}p.$$

Hence, if $k+1 \leq i-p$ (so, we have $[k+1, l-1] \supset [i-p, l-1]$), we deduce from part (2) of Theorem 3.1(b) that

$$\#\Big((I\!N \setminus I) \cap [i-p, i+p]\Big) \geq \#\Big([i-p, l-1]\Big) \leq p - \frac{3}{4}p = \frac{1}{4}p$$

and we are done in this case. So, suppose that

$$k+1 > i-p. \tag{3.18}$$

In view of part (2) of Theorem 3.1(b) we have

$$l-k \geq l - \frac{ql}{q+\sigma l} = l\left(1 - \frac{q}{q+\sigma l}\right) \geq \frac{qi}{i+q}\left(1 - \frac{q}{q+\sigma l}\right). \tag{3.19}$$

Our aim now is to find a universal constant $\alpha > 0$ such that

$$\frac{qi}{i+q}\left(1 - \frac{q}{q+\sigma l}\right) \geq \alpha \frac{i^2}{i+q} \tag{3.20}$$

or equivalently $q\left(1 - \frac{q}{q+\sigma l}\right) \geq \alpha i$. This inequality can be in turn equivalently rewritten in the form $q^2 + \sigma q l - q^2 \geq \alpha q i + \alpha\sigma i l$, $\sigma q l \geq \alpha(qi + \sigma i l)$ and $\sigma \geq \alpha\left(\frac{i}{l} + \sigma\frac{i}{q}\right)$. But

$$\frac{i}{l} + \sigma\frac{i}{q} \leq \frac{i+q}{q} + \sigma\frac{i}{q} = 1 + (1+\sigma)\frac{i}{q} \leq 1 + (1+\sigma)\frac{1}{4} \leq 2.$$

Thus, (3.20) will be satisfied with $\alpha = \sigma/3$. Combining now (3.19), (3.20) and (3.17), we get

$$l-1-(k+1) = l-k-2 \geq \frac{\sigma}{2}\cdot\frac{p}{20} - 2 \geq \frac{\sigma}{80}p,$$

where the last inequality we could write since $p \geq x = \frac{160}{\sigma}$. Since, by (3.18), $[k+1, l-1] \subset [i-p, i]$ and, by part (2) of Theorem 3.1(b), $[k+1, l-1] \subset I\!N \setminus I$, we conclude that $[i-p, i]$ contains a cluster of $I\!N \setminus I$ of length $\frac{\sigma}{80}p$. Thus the proof is complete by setting $\theta = \sigma/80$. $\square$

Theorem 3.3. *Let I be at least two point subset of the set of positive integers $I\!N$. Then the following two conditions are equivalent*

- **(a):** *The corresponding limit set J_I (i.e. all the real numbers whose all continued fraction expansion entries belong to I) is porous.*
- **(b):** *There exist $0 < \theta \leq 1$ and $x > 0$ such that for every $i \in I$ and every $x \leq p \leq i$, the interval $[i-p, i+p]$ contains a cluster of $I\!N \setminus I$ of length θp.*

Proof. The implication (a)⇒(b) follows immediately from Lemma 3.2, perhaps with a smaller constant θ since we now require only $p \le i$ and not $p \le i/3$. In order to prove the opposite implication suppose that condition (b) of our lemma is satisfied. We shall then prove condition (c) of Theorem 3.1 holds with $\lambda = \max\{4, 2x\}$. So, consider an arbitrary integer q such that $4i \le q \le 4^{-1}i(i+1)$. Set $\eta = \theta/2$. Since $\frac{qi}{q+i} \le i$ and since

$$i - \frac{qi}{q+i} = \frac{i^2}{q+i} \ge \frac{\lambda i^2}{i(i+1)}\lambda\frac{i}{i+1} \ge \frac{\lambda}{2} \ge x,$$

either the interval $\left[\frac{qi}{q+i}, i\right]$ contains a cluster of $I\!N \setminus I$ of length $\ge \eta\frac{i^2}{q+i}$ or the interval $\left[i, i+i-\frac{qi}{q+i}\right]$ contains a cluster of $I\!N \setminus I$ of length $\ge \eta\frac{i^2}{q+i}$. Suppose first that we have

Case A: The interval $\left[\frac{qi}{q+i}, i\right]$ contains a cluster $[k,l] \subset I\!N \setminus I$ of length $\ge \eta\frac{i^2}{q+i}$. Then $k \le i$, $l \ge k \ge \frac{qi}{q+i}$ and $l - k \ge \eta\frac{i^2}{q+i}$. This inequality means that $k \le l - \eta\frac{i^2}{q+i}$ and in order to finish the proof in this case, it suffices to find a universal constant $\sigma > 0$ such that

$$i - \eta\frac{i^2}{q+i} \le \frac{ql}{q+\sigma l}$$

This equivalently means that $lq^2 + \sigma q l^2 + qil + \sigma ill^2 - \eta q i^2 - \eta\sigma l i^2 \le q^2 l + qil$ and $\sigma l^2(q+i) \le \eta i^2(q+\sigma l)$. But $\sigma l^2(q+i) \le 2\sigma l^2 q \le 2\sigma i^2 q \le 2\sigma i^2(q+\sigma l)$. So, we are done in this case if only $\sigma \le \eta/2$. Consider now

Case B: The interval $\left[i, 2i - \frac{qi}{q+i}\right]$ contains a cluster of $I\!N \setminus I$ of length $\ge \eta\frac{i^2}{q+i}$. First notice that

$$2i - \frac{qi}{q+i} - \frac{q(i+1)}{q-(i+1)} = \frac{2i^2q - 2i(i+1)q - 2i^2(i+1) - q^2}{(q+i)(q-(i+1))} \le -\frac{2i^2(i+1)+q^2}{(q+i)(q-(i+1))} \le 0.$$

Hence, $2i - \frac{qi}{q+i} \le \frac{q(i+1)}{q-(i+1)}$ and consequently

$$i \le k \le l \le \frac{q(i+1)}{q-(i+1)}.$$

We also know that $l - k \ge \eta\frac{i^2}{q+i}$. Thus $l \ge k + \eta\frac{i^2}{q+i}$, and in order to finish the proof in this case, it suffices to find a universal $\sigma > 0$ so small that

$$k + \eta\frac{i^2}{q+i} \ge \frac{qk}{q-\sigma k}.$$

This equivalently means that $kq^2 + kiq - \sigma q k^2 + \eta q i^2 - \eta\sigma k i^2 \ge kq^2 + qki$ and $\eta q i^2 \ge \sigma(qk^2 + ik^2 + \eta k i^2)$. But $k \le l \le 2i - \frac{qi}{q+i} \le 2i$, and therefore $\sigma(qk^2 + ik^2 + \eta k i^2) \le \sigma(4qi^2 + 4i^3 + 2i^3) \le \sigma(4qi^2 + 4qi^2 + 2qi^2) = 10\sigma q i^2$. So, we are done in this case if only $\sigma \le \eta/10$. The proof is complete. $\square$

We call two subsets of $I\!N$ strongly equivalent if their symmetric difference is finite. We call a subset H of I cofinite if the difference $I \setminus H$ is finite. As an immediate consequence of the characterization of porosity of limit sets of continued fractions provided by Theorem 3.3, we get the following.

Theorem 3.4. *Let I be an infinite subset of positive integers. Then the following conditions are equivalent.*

(a): *The limit set J_F is porous for every subset F of I.*
(b): *The limit set J_I is porous.*
(c): *There exists a cofinite subsystem F of I such that the limit set J_F is porous.*
(d): *The limit set J_F is porous for every set $F \subset I\!N$ strongly equivalent with I.*

Given $I \subset I\!N$, we have defined in [MU2] $\overline{\rho}D(I)$, the upper density dimension of I, as follows. For each $t \geq 0, n \in I\!N$, set

$$\overline{\rho}_t(I) = \sup\left\{\frac{\#(I \cap [k,l])}{(l-k)^t} : k < l, k, l \in I\right\} = \sup\left\{\frac{\#(I \cap [k,l])}{(l-k)^t} : k < l\right\}.$$

Notice that

$$\inf\{t : \overline{\rho}_t(I) < \infty\} = \sup\{t : \overline{\rho}_t(I) > 0.\}$$

This common value was called the upper density dimension of I and will be denoted by $\overline{\rho}D(I)$.

Theorem 3.5. *If $I \subset I\!N$ and the limit set J_I is porous, then $\overline{\rho}D(I) < 1$.*

Proof. Let x and θ be taken according to Theorem 3.3. Fix $1 \leq k < l \leq 3k$, $l > k + x$. We shall construct by induction for every $0 \leq n \leq q$ (q will be determined in the end of the inductive procedure) a sequence R_n of at most 2^n mutually disjoint segments of length $< x$ contained in the interval $[k,l]$ and a nested sequence S_n of at most 2^{n-1} mutually disjoint segments of length $\geq x$ contained in $[k,l]$ such that the union of all segments from the families R_n and S_n covers $I \cap [k,l]$ and each element of S_n has the length bounded above by $\left(1 - \frac{\theta}{9}\right)^{n-1}$ multiplied by the length of the only element of S_{n-1} containing it and bounded from below by $(2/9)^{n-1}2(l-k)$. And indeed, as the initial step of our induction we declare $R_1 = \emptyset$ and $S_1 = [k,l]$. Now suppose that $n \geq 0$ and that the sequences R_n and S_n have been already defined. If $S_n = \emptyset$, we set $q = n$ and the construction terminates. Otherwise, let us look at all the segments $I_1, I_2, \ldots, I_u$, $u \leq 2^{n-1}$, forming the family S_n. Fix $1 \leq j \leq u$. If the segment K_j centered at the same point as I_j but of length $|I_j|/3$ is contained in $I\!N \setminus I$, we look at the two segments forming the difference $I_j \setminus K_j$. If the length of each of them is less than x, we take these to intervals to the family R_{n+1}. If however their length is $\geq x$, we take these two intervals to the new family S_{n+1}. So, suppose that K_j contains an element $y \in I$. Since $|K_j|/3 \leq \frac{1}{3} \cdot (l-k+1) \leq \frac{1}{3}l \leq k \leq y$, by Theorem 3.3 there exists an interval $L_j \subset \left[y - \frac{1}{3}|K_j|, y + \frac{1}{3}|K_j|\right] \subset I_j$ such that $|L_j| \geq \frac{\theta}{3}|K_j| = \frac{\theta}{9}|I_j|$ and $L_j \cap I = \emptyset$. We now look at the two intervals forming the difference $I_j \setminus L_j$. Each of them of length $< x$ is declared to belong to the family R_{n+1} and each of them of length $\geq x$ is declared to belong to the family S_{n+1}. Note that $\#S_{n+1} \leq 2\#S_n \leq 2 \cdot 2^{n-1} = 2^n$, $\#R_{n+1} \leq \#R_n + 2\#S_{n+1} \leq 2^n + 2^n = 2^{n+1}$, and the length of each element of the family S_{n+1} is bounded above by $\left(1 - \frac{\theta}{9}\right)$ multiplied by the length of the element of of the family S_n containing it. The lower bound $(2/9)^{n-1}2(l-k)$

on the length of each element of the family S_n also follows immediately from the construction. The inductive construction is complete.

Consider now a pair (k,l) with the same conditions as above, i.e. $1 \le k < l \le 3k$ and $l > k+x$. Define $p \ge 1$ to be the largest integer such that $(2/9)^{p-1}2(l-k) \ge x$. It follows from the construction of sequences R_n and S_n that $q \ge p$. Since $\#\bigcup S_p \le \left(1-\frac{\theta}{9}\right)^{p-1}(l-k)$ and since $\#\bigcup R_p \le 2^p x$, using the definition of p, we obtain

$$\begin{aligned}
\#(I \cap [k,l]) &\le \#\bigcup S_p + \#\bigcup R_p \le \left(1-\frac{\theta}{9}\right)^{p-1}(l-k) + 2^p x \\
&\le \left(1-\frac{\theta}{9}\right)\left(\frac{2}{9}\right)^{p\frac{\log\left(1-\frac{\theta}{9}\right)}{\log(2/9)}}(l-k) + 2\left(\frac{2}{9}\right)^{(p-1)\frac{\log 2}{\log(2/9)}} x \\
&\le \left(1-\frac{\theta}{9}\right)\left(\frac{x}{2}(l-k)^{-1}\right)^{\frac{\log\left(1-\frac{\theta}{9}\right)}{\log(2/9)}}(l-k) + 2x\left(\frac{x}{2}(l-k)^{-1}\right)^{\frac{\log 2}{\log(2/9)}} \\
&= \left(1-\frac{\theta}{9}\right)\left(\frac{x}{2}\right)^{\frac{\log\left(1-\frac{\theta}{9}\right)}{\log(2/9)}}(l-k)^{1-\frac{\log\left(1-\frac{\theta}{9}\right)}{\log(2/9)}} + 2x\left(\frac{x}{2}\right)^{\frac{\log 2}{\log(2/9)}}(l-k)^{\frac{\log 2}{\log(9/2)}}
\end{aligned}$$

Putting now

$$B_1 = \left(1-\frac{\theta}{9}\right)\left(\frac{x}{2}\right)^{\frac{\log\left(1-\frac{\theta}{9}\right)}{\log(2/9)}} + 2x\left(\frac{x}{2}\right)^{\frac{\log 2}{\log(9/2)}}$$

and

$$t = \max\left\{\frac{\log 2}{\log(9/2)}, 1-\frac{\log\left(1-\frac{\theta}{9}\right)}{\log(2/9)}\right\} < 1,$$

we get

$$\#(I \cap [k,l]) \le B_1(l-k)^t. \tag{3.21}$$

If $1 \le k < l \le k+x$, then

$$\#(I \cap [k,l]) \le l-k = (l-k)^{1-t}(l-k)^t \le x^{1-t}(l-k)^t \tag{3.22}$$

Writing $B_2 = \max\{B_1, x^{1-t}\}$, combining (3.21) and (3.22), for every $0 \le m \le n$, we get the following

$$\#(I \cap [3^m,3^n]) \le \sum_{j=m}^{n-1}\#(I \cap [3^j,3^{j+1}]) \le B_2\sum_{j=m}^{n-1}3^{tj} = \frac{B_2}{3^t-1}\left(3^{tn}-3^{tm}\right) \le \frac{B_2}{3^t-1}(3^n-3^m)^t, \tag{3.23}$$

where we could write the last inequality since $b^s - a^s \le (b-a)^s$ for all $b \ge a \ge 0$ and all $0 \le s \le 1$.

Consider now two arbitrary integers $1 \le k < l$ such that $l \ge 3k$. Let $m \ge 0$ be the largest integer such that $3^m \le k$ and let $n \ge 0$ be the least integer such that $3^n \ge l$. In view of (3.23) we then have

$$\#(I\cap[k,l]) \le \#(I\cap[3^m,3^n]) \le \frac{B_2}{3^t-1}(3^n-3^m)^t \le \frac{B_2}{3^t-1}\left(3l-\frac{1}{3}k\right)^t \le 5^t\frac{B_2}{3^t-1}(l-k)^t$$

Thus $\overline{\rho}_t(I) \le B_2\max\{1,\frac{5^t}{3^t-1}\}$ and consequently $\overline{\rho}D(I) \le t < 1$. The proof is complete. $\square$

Let H^t and P^t denote respectively the t-dimensional Hausdorff and packing measures. See for instance [Ma] for their definitions and further properties. Let $h = h_I = \mathrm{HD}(J_I)$ denote the Hausdorff dimension of the limit set J_I. As an immediate consequence of Theorem 3.5 and Theorem 4.10 form [MU2] we get the following.

Corollary 3.6. *If $I \subset I\!N$ and the limit set J_I is porous, then the strong equivalence class of I contains an element F with $\mathrm{H}^{h_F}(J_F) > 0$. More precisely, there exists a number $q \ge 1$ such that if F is strongly equivalent with I and $F \supset I \cup [1,q]$, then $\mathrm{H}^{h_F}(J_F) > 0$.*

Given $I \subset I\!N$ let θ_I be the number introduced in [MU1]. Its precise definition will not be needed here. It immediately follows from Theorem 3.5 and Lemma 3.4 of [MU2] that if J_I is porous, then $\theta_I < 1/2$. Therefore, applying Theorem 5.4 of [MU2], we immediately get the following.

Theorem 3.7. *If I is an infinite subset of $I\!N$ and the limit set J_I is porous, then there exists a number $q \ge 1$ such that if F is strongly equivalent with I and $F \supset [1,q]$, then $\mathrm{P}^{h_F}(J_F) = \infty$.*

In contrast to Corollary 3.6, Proposition 4.4 from [MU2] says the following

Proposition 3.8. *If $h = \mathrm{HD}(J_I) < 2\theta$, then $\mathrm{H}^h(J_I) = 0$.*

We shall now provide an example of a porous set J_I for which the hypothesis of Proposition prop3.8 is satisfied; in particular $\mathrm{H}^h(J_I) = 0$. Recall from [MU2] that a subset $I \subset I\!N$ has strong density zero if

$$\sum_{n\in I} n^{-t} < \infty$$

for all $t > 0$. For other properties and equivalent definitions of sets with strong density zero see Sections 2 and 3 in [MU2]. We begin with the following.

Theorem 3.9. *There exists an infinite set $I \subset I\!N$ which does not have strong density zero and the corresponding limit set J_I is porous.*

Proof. We shall provide a concrete construction imitating the procedure of building the middle-third Cantor set. We shall construct the set I by describing its intersections with all the sets of the form $[4^n, 4^{n+1}]$. So, fix $n \ge 0$. We shall define by induction the families $C_{n,k}$, $0 \le k \le p$ (p will be determined in the process of inductive construction) of finitely many disjoint intervals of length between 1 and $(1/3)^k 3\cdot 4^n$. The construction goes as follows. We set $C_{n,0} = \{[4^n, 4^{n+1}]\}$. Suppose now that the family $C_{n,k}$ has been already defined and let A_k be an arbitrary element of $C_{n,k}$. If $\#A_k \le 12$, then A_k generates no members of $C_{n,k+1}$. Otherwise, we remove from A_k an interval $B_k = [a_k, b_k]$ of length $E(\#A_k/3)$ such that $[a_k - 1, b_k + 1]$ covers the interval of length $\#A_k/3$ and which is centered at the same point as A_k. We then declare the two intervals forming the difference $A_k \setminus B_k$ as members of the family $C_{n,k+1}$. The inductive construction terminates at the first moment p when all the segments in the family $C_{n,p}$ have length ≤ 12. We now define

$$I \cap [4^n, 4^{n+1}] = \bigcup C_{n,p}.$$

For each $n \ge 0$, each $k \ge 1$ and each set $A_{n,k} \in C_{n,k}$, there exists exactly one element $A_{n,k-1}$ such that $A_{n,k} \subset A_{n,k-1}$. In the sequel $A_{n,k-1}$ will be called the parent of $A_{n,k}$ and $A_{n,k}$ a child of $A_{n,k-1}$.

We shall now show that I does not have strong density zero. Let $i \in I \cap [4^n, 4^{n+1}]$ for some $n \ge 0$ and let $q \ge 0$ be the largest number such that $i \in A_i \in C_{n,q}$. Since by construction $(1/4)^q 3 \cdot 4^n \le |A_i| \le 12$, we get $4^q \ge 4^{n-1}$, hence $q \ge n-1$. Thus $\#\Big(C_{n,n-1}\Big) \ge 2^{n-1}$ and since each segment of $C_{n,n-1}$ contains at least one element of $I \cap [4^n, 4^{n+1}]$, we get $\#(I \cap [4^n, 4^{n+1}]) \ge 2^{-1}$. Therefore

$$\sum_{k \in I} \frac{1}{k^{1/2}} \ge \sum_{n \ge 0} \#(I \cap [4^n, 4^{n+1}]) \frac{1}{(4^{n+1})^{1/2}} \ge \frac{1}{2} \sum_{n \ge 0} \frac{2^{n-1}}{2^n} = \infty.$$

Thus I does not have strong density zero and we are only left to demonstrate that J_I is porous. So, consider $i \in I$ and $49 \le p \le i$. There exists $n \ge 0$ such that $i \in [4^n, 4^{n+1}]$. Suppose first that $[i-p, i+p] \subset [4^n, 4^{n+1}]$. Since $p \ge 49 \ge 12$ and $i \in I$, there exists a least $k \ge 0$ such that $i \in A_{i,k} \subset [i-p, i+p]$, where $A_{i,k}$ is the only element of the family $C_{n,k}$ such that $i \in A_{i,k}$. Assume first that $k = 0$, i.e. that $A_{i,k} = [4^n, 4^{n+1}]$. Then $[i-p, i+p] = [4^n, 4^{n+1}]$. So, $p = \frac{3}{2} 4^n$ and, according to our construction, two intervals contained in $[4^n, 4^{n+1}]$, at least 4^n long, containing 4^n and 4^{n+1} respectively, are disjoint from I. Therefore both intervals $[i-p, i]$ and $[i, i+p]$ contain a cluster of $I\!N \setminus I$ at least $4^n = \frac{2}{3} p$ long. Thus we are done in this case with $\theta = 2/3$. Assume now that $k \ge 1$ and let $A_{i,k-1}$ be the parent of $A_{i,k}$. Then either $[i-p, i] \subset A_{i,k-1}$ or $[i, i+p] \subset A_{i,k-1}$ (or both inclusions are true). In either case $|A_{i,k-1}| \ge p$. It also follows from the construction of the set I that $|A_{i,k}| \ge \frac{|A_{i,k-1}|}{3} - 1 \ge \frac{|A_{i,k-1}|}{4}$. Hence $|A_{i,k}| \ge p/4$. If $|A_{i,k}| \le 12$, then $p \le 48$ and this case is ruled out by the assumption that $p \ge 49$. Thus $|A_{i,k}| > 12$ and, according to our construction of the set I, there exist intervals at least $|A_{i,k}|/3 \ge p/12$ long contained in $|A_{i,k}|$ and disjoint from I. Since $|A_{i,k} \subset [i-p, i+p]$ and $i \in I$, each of these intervals is contained either in $[i-p, i]$ or $[i, i+p]$. Thus, we are done so far with $\theta = \frac{1}{12}$. So, suppose that $p \ge \frac{16}{15} 48$ and that $[i-p, i+p]$ is not contained in $[4^n, 4^{n+1}]$. Then either $i + p > 4^{n+1}$ or $i - p < 4^n$. Suppose first that $i + p > 4^{n+1}$. Consider the subcase when $i + p > 4^{n+1} + \frac{p}{16}$. Since $4^{n+1} + \frac{p}{16} \le 4^{n+1} + \frac{i}{16} \le 4^{n+1} + 4^{n+1}$, then $[i, i+p] \supset [4^{n+1}, 4^{n+1} + \frac{i}{16}]$, and by the construction of I, $[4^{n+1}, 4^{n+1} + \frac{i}{16}] \cap I = \emptyset$. So, we are done with $\theta = 1/16$ in this case. We now keep on hold for a moment the case $i + p \le 4^{n+1} + \frac{p}{16}$ and we consider the situation when $i - p < 4^n$. If $i - p < 4^n - \frac{p}{16}$, then $4^n - \frac{p}{16} \ge 4^n - \frac{i}{16} \ge 4^n - \frac{4^{n+1}}{16} = 4^n - 4^{n-1}$. Hence $[i-p, i] \supset [4^n - \frac{p}{16}, 4^n]$ and, by the construction of I, $[4^n - \frac{p}{16}, 4^n] \cap I = \emptyset$. So, we are done with $\theta = 1/16$ in this case too. So, we are left with the case when $i + \frac{15}{16} p \le 4^{n+1}$ and simultaneously $i - \frac{15}{16} p \ge 4^n$ which equivalently means that $\left[i - \frac{15}{16} p, i + \frac{15}{16} p\right] \subset [4^n, 4^{n+1}]$. But since $\frac{15}{16} p \ge \frac{15}{16} \left(\frac{16}{15} \cdot 48\right) = 48$, we may apply what we have already proved to find a cluster of length $\frac{1}{12} \cdot \frac{15}{16} p = \frac{5}{64p}$ contained either in $\left[i - \frac{15}{16}p, i\right] \cap (I\!\!N \setminus I) \subset [i-p,i] \cap (I\!\!N \setminus I)$ or in $\left[i, i+\frac{15}{16}\right] \cap (I\!\!N \setminus I) \subset [i, i+p] \cap (I\!\!N \setminus I)$. So, we have checked that the assumptions of Theorem 3.3(b) are satisfied with $x = \frac{15}{16} \cdot 48$ and $\theta = 1/16$. Thus, applying this theorem finishes the proof. □

Theorem 3.10. *There exists an infinite set $I \subset I\!\!N$ such that J_I is porous and the assumptions of Proposition 3.8 are satisfied. In particular $H_h(J_I) = 0$.*

Proof. Let F be the set constructed in Theorem 3.9. By this theorem, F does not have strong density zero, and therefore, in view of Lemma 3.3 (g) from [MU2], $\theta_F > 0$. It consequently follows from Theorem 3.23 of [MU1] that there exists a cofinite subset I of F such that $h_I < 2\theta_I = 2\theta_F$. Thus, the assumptions of Proposition 3.8 are satisfied, and, in particular, $\mathrm{H}^h(J_I) = 0$. Since J_F is porous by Theorem 3.9, J_F is porous by Theorem 3.4. The proof is finished. □

We shall now examine from the point ov view of porosity some well-known infinite subsets of positive integers. First notice that taking in Theorem 3.3(b), $p = i$, we get the following.

Proposition 3.11. *If $I = \{n_k\}_{k=1}^\infty$ is represented as an increasing to infinity sequence of positive integers and the limit set J_I is porous, then*

$$\limsup_{k \to \infty} \frac{n_{k+1}}{n_k} > 1.$$

Since for every integer $p \geq 1$, $\lim_{k \to \infty} \frac{(k+1)^p}{k^p} = 1$, Proposition 3.11 implies immediately the following.

Theorem 3.12. *If $p \geq 1$ is an integer and $I_p = \{n^p\}_{n=1}^\infty$, then the limit set J_I is not porous.*

As an immediate consequence of Proposition 3.11, we get also the following.

Theorem 3.13. *If $I = \{n_k\}_{k=1}^\infty$ is an infinite subsequence of I with bounded gaps, i.e. $\sup_{k \geq 1}\{n_{k+1} - n_k\} < \infty$, then the limit set J_I is not porous.*

Since, by Tchebyschev's theorem, the upper density dimension of the set of prime numbers is equal to 1, the following result is an immediate consequence of of Theorem 3.5.

Theorem 3.14. *If I is the set of all prime numbers, then the limit set J_I is not porous.*

Ending this section with a positive example, as an immediate consequence of Theorem 3.3 we get the following.

Theorem 3.15. *If $a \geq 2$ is an integer and $I_a = \{a^n\}_{n=1}^\infty$, then the limit set J_{I_a} is porous.*

4. Complex Continued Fractions

In this short section we deal with the iterated function system generated by the complex continued fractions algorithm, the primary example in [MU1]. In order to describe this system let $I = \{m + ni : (m,n) \in I\!\!N \times \boldsymbol{Z}\}$, where $\boldsymbol{Z}$ is the set of integers and $I\!\!N$ is the set of positive integers. Let $X \subset \mathbb{C}$ be the closed disc centered at the point $1/2$ with radius $1/2$ and let V be an open topological disk containing X such that $\phi_b(V) \subset V$ for every $b \in I$, where $\phi_b : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is defined by the formula

$$\phi_b(z) = \frac{1}{b+z}.$$

We call $\{\phi_b\}_{b \in I}$ the iterated function system of continued fractions. Let $K \geq 1$ be the Koebe constant (see [Hi]) corresponding to the ratio of radii equal to $2/3$. Let $g : \mathbb{C} \to \mathbb{C}$ be the map $g(z) = 1/z$. As an immediate consequence of Koebe's' distortion theorem and $\frac{1}{4}$-Koebe's' distortion theorem (see [Hi]), we get the following

Lemma 4.1. *If $b \in I$, then $|\phi_b'(z)| = 1/|z+b|^2$ and*

$$K^{-1}\left|b+\frac{1}{2}\right|^{-2} \leq \mathrm{diam}(\phi_b(X)) \leq K\left|b+\frac{1}{2}\right|^{-2}.$$

If, in addition B is a disk contained in $g(B\left(b+\frac{1}{2}, \frac{1}{2}\Re(b)-\frac{1}{2}\right)$, then $g(B)$ contains a disk of diameter $\geq \frac{1}{4}K^{-1}\left|b+\frac{1}{2}\right|^{-2}\mathrm{diam}(B)$.

The result of this section is the following.

Theorem 4.2. *The limit set J of the iterated function system generated by the complex continued fractions is not porous.*

Proof. For every $b \in I$ with sufficiently real part we will find a radius $1 \geq r_b \geq \mathrm{diam}(\phi_b(X))$ such that if t_b is the radius of maximal disk contained in $B(\phi_b(1/2), 2r_b)$ and disjoint from J, then

$$\lim_{|\Re(b)| \to \infty} \frac{t_b}{r_b} = 0. \tag{4.1}$$

And indeed, given $b \in I$ set

$$r_b = \frac{1}{16}\left|b+\frac{1}{2}\right|^{-2}\left(\Re(b)-\frac{1}{2}\right).$$

We get immediately from this definition that $r_b \leq 1/8 \leq 1$ and from Lemma 4.1 that

$$\mathrm{diam}(\phi_b(X)) \leq 16K\left(\Re(b)-\frac{1}{2}\right)^{-1} r_b \leq r_b$$

if only $\Re(b)$ is large enough. Suppose now that B is a disk contained in $B(\phi_b(1/2), r_b)$ and disjoint from J. This implies that $B(\phi_b(1/2), r_b) \cap \overline{J} = \emptyset$. Since for every $a \in I$, $\phi_a(0) \in \overline{J}$, we conclude that $g(a) = \phi_a(0) \notin B$ or equivalently,

$$a \notin g(B) \tag{4.2}$$

for all $a \in I$. Since, by $\frac{1}{4}$-Koebe's' distortion theorem

$$g(B(\phi_b(1/2), 2r_b) \subset g\left(B\left(b+\frac{1}{2}, \frac{1}{2}\left(\Re(b)-\frac{1}{2}\right)\right)\right)$$

and since $B \subset B(\phi_b(1/2), 2r_b)$, it follows from Lemma 4.1 that $g(B)$ contains a disk of diameter $\geq \frac{1}{4}K^{-1}\mathrm{diam}(B)\left|b+\frac{1}{2}\right|^2$. Combining this and (4.2) we conclude that

$$K^{-1}\mathrm{diam}(B)\left|b+\frac{1}{2}\right|^2 \leq \sqrt{2}.$$

Hence $\mathrm{diam}(B) \leq 64K\sqrt{2}\left(\Re(b)-\frac{1}{2}\right)^{-1} r_b$ and therefore

$$\frac{t_b}{r_b} \leq 32K\sqrt{2}\left(\Re(b)-\frac{1}{2}\right)^{-1}.$$

Thus formula (4.1) is proved and this shows that J is not porous. $\square$

5. Parabolic Iterated function Systems

In this section we explore the problem of porosity of parabolic iterated function systems introduced in [MU3]. We begin with the following.

Definition 5.1. *Let X be a compact topological disk in $\overline{\mathbb{C}}$ with a piecewise smooth boundary. Suppose that we have finitely many conformal maps $\phi_i : X \to X$, $i \in I$, where I has at least two elements and the following conditions are satisfied.*

- **(5pa):** *(Open Set Condition) $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$ for all $i \neq j$.*
- **(5pb):** *$|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic.*
- **(5pc):** *$\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then ϕ_ω extends conformally to an open topological disk $V \subset \overline{\mathbb{C}}$ with a piecewise smooth boundary and ϕ_ω maps V into itself.*
- **(5pd):** *If i is a parabolic index, then $\bigcap_{n\geq 0} \phi_{i^n}(X) = \{x_i\}$ and the diameters of the sets $\phi_{i^n}(X)$ converge to 0.*
- **(5pe):** *(Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ $\forall x, y \in V$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then*
$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \leq K.$$
- **(5pf):** *$\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.*
- **(5pg):** *(Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$ there exists an open cone $Con(x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.*
- **(5ph):** *There are two constants $L \geq 1$ and $\alpha > 0$ such that*
$$\Big||\phi_i'(y)| - |\phi_i'(x)|\Big| \leq L||\phi_i'|||y-x|^\alpha,$$
for every $i \in I$ and every pair of points $x, y \in V$.
- **(5pi):** *$\phi_i(X) \subset \mathrm{Int}(X)$ for every hyperbolic element $i \in I$.*

Any system S satisfying the above conditions (5pa)-(5pi) will be called a plane parabolic iterated function system.

We shall now recall from [MU3] how to associate with any parabolic iterated function system S a canonical, infinite but hyperbolic, iterated function system S^* which essentially has the same limit set as S.

Definition 5.2. *The system S^* is by definition generated by the set of maps of the form $\phi_{i^n j}$, where $n \geq 1$, $i \in \Omega$, $i \neq j$, and the maps ϕ_k, where $k \in I \setminus \Omega$. The corresponding alphabet $\{i^n j : i \in \Omega, i \neq j, n \geq 1\} \cup (I \setminus \Omega)$ will be denoted by I_*.*

The following fact has been proved in [MU3] as Theorem 5.2.

Theorem 5.3. *The system S^* is a (hyperbolic) conformal iterated function system in the sense of Section 1.*

Note that $J_{S^*} = J_S \setminus \{\phi_\omega(x_i) : i \in \Omega, \omega \in I^*\}$. In view of Lemma 2.4 in [MU3], every parabolic point x_i, $i\imath I$, lies on the boundary of X. It is easy to see that $\phi_i'(x_i) = 1$ and the Taylor's series expansion of ϕ_i at x_i has the form

$$\phi_i(z) = z + a(z - x_i)^{p_i+1} + \ldots$$

for some integer $p_i \geq 1$. Changing the system of coordinates via the map $\frac{1}{z-x_i}$ sending x_i to ∞, one can easily deduce that for every $j \neq i$ and for every $n \geq 1$

$$\operatorname{diam}(\phi_{i^n j}(X)) \asymp \operatorname{dist}\Big(\phi_{i^{n+1} j}(X), \phi_{i^n j}(X)\Big) \asymp ||\phi'_{i^n j}|| \asymp n^{-\frac{p_i+1}{p_i}} \tag{5.1}$$

and

$$\operatorname{dist}\Big(x_i, \phi_{i^n j}(X)\Big) \asymp n^{-\frac{1}{p_i}} \tag{5.2}$$

In addition, changing the system of coordinates via the map $\frac{1}{z-x_i}$ we can easily see that the following is true.

Lemma 5.4. *If S is a parabolic iterated function system and x_i, $i \in I$, is a parabolic point, then there exists a constant $C \geq 1$ such that for every $k \geq 1$ and every $n \geq k$, the sets $\phi_{i^n j}(X)$, $j \neq i$, are all contained in corresponding sectors centered at x_i with angular measures bounded above by Cn^{-1}.*

Once this lemma has been established, the following result becomes actually an immediate consequence of Theorem 2.2.

Theorem 5.5. *The limit set of each plane parabolic iterated function system is porous.*

Proof. Indeed, after observing that by (5.1) and (5.2)

$$\frac{\operatorname{diam}\Big(\phi_{i^n j}(X)\Big)}{\operatorname{dist}\Big(x_i, \phi_{i^n j}(X)\Big)} \asymp n^{-1}$$

the proof follows immediately by combining Theorem 2.2 and Lemma 5.4. ∎

In particular, since the residual set of the Apollonian packing is the limit set of a plane parabolic iterated function system (see [PU3]), this residual set is porous.

The situation however changes if we consider so called real parabolic iterated function systems. The difference is that we assume now X to be a compact interval in the real line $I\!R$ and $V \subset I\!R$ is an open interval containing X. We assume in addition that for every parabolic point x_i, $i \in I$, there exists $\beta_i > 0$ such that

$$\phi_i(x) = x - a(x - x_i)^{\beta_i + 1} + o\Big(|x - x_i|^{\beta + 1}\Big). \tag{5.3}$$

By the same method we then get formula (5.1)with p_i replaced by β_i. In this case that formula almost immediately implies the following.

Theorem 5.6. *If S is a real parabolic iterated function system and (5.3) is satisfied, then the corresponding limit set J_S (considered as a subset of $I\!R$) is not porous.*

Proof. According to (5.1), for every $k \geq 1$ all the gaps between points of J_S in the ball $B\left(x_i, k^{-\frac{1}{\beta_i}}\right)$ are of length not exceeding $const\, k^{-\frac{\beta_i+1}{\beta_i}}$. Since

$$\frac{k^{-\frac{\beta_i+1}{\beta_i}}}{k^{-\frac{1}{\beta_i}}} = k^{-1} \to 0 \text{ when } k \to \infty,$$

the limit set J_S is not porous at any parabolic point $x_i \in \Omega$. The proof is complete. $\square$

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