METRICAL DIOPHANTINE ANALYSIS FOR TAME PARABOLIC ITERATED FUNCTION SYSTEMS

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Abstract. In this paper we study various aspects of tame finite parabolic iterated function systems which satisfy a certain open set condition. The first goal in our analysis of these systems is a detailed investigation of the conformal measure on the associated limit sets. We derive a formula which describes in a uniform way the scaling of this measure at arbitrary limit points. The second goal is to provide a metrical Diophantine analysis for these parabolic limit sets in the spirit of theorems of Jarník and Khintchine in number theory. Subsequently, we show that this Diophantine analysis gives rise to refinements of the description of the conformal measure in terms of Hausdorff and packing measures with respect to certain gauge functions.

1. Introduction

For a large class of fractal sets the idea of an iterated function system has turned out to be a very convenient and efficient concept. Traditionally, the development of fractal geometry was always very much inspired by various phenomena which appear in conformal analysis and number theory. In this paper we continue this tradition by studying metrical Diophantine aspects of certain tame parabolic iterated function systems. This study generalizes results for geometrically finite Kleinian groups with parabolic elements (obtained in [S1] [S2] [S3] [SV], see also [HV] [Su]) and for parabolic rational rational functions (obtained in [SU1] [SU2]), which represent complex analytic analogues of Jarník's number theoretical theorem on well-approximable numbers ([J] [B]) and Khintchine's on a qualitative description of the 'essential support' of the 1-dimensional Lebesgue measure ([K]).

The paper is organized as follows. In Section 2 we first define the class of tame finite parabolic iterated function systems which satisfy the Super Strong Open Set Condition (SSOSC). We then recall a few immediate geometrical implications of the bounded distortion properties. In Section 3 we study the h-conformal measures arising from these parabolic systems. (Here, h denotes the Hausdorff dimension of the limit set associated to such a system.) We obtain a formula which describes in a uniform way the scaling of this measure at arbitrary elements of the limit set. As a by-product we obtain an estimate on the local behaviour of the h-conformal measure at parabolic points. In Section 4 we analyse the limit sets from a Diophantine point of view. Our general approach here follows roughly the analysis given in [S1] [S2] [SV] [SU1] [SU2]. Nevertheless, the construction of the main tool, namely the measure μ on a Cantor-like subset of the limit set, is different. This construction is simplified and its geometrical and dynamical significance is clarified. Finally, we establish various limit laws leading up to the

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Khintchine Limit Law for tame parabolic iterated function systems. Subsequently, we show that these laws provide some efficient control on the fluctuations of the h-conformal measure, giving rise to refinements of the description of the h-conformal measure in terms of Hausdorff and packing measures with respect to some gauge functions.

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2. **Preliminaries**

We begin by giving a description of our setting. Let X be a compact subset of some Euclidean space $I\!R^d$ such that X has non-empty interior and is contained in a bounded connected open set V. Suppose that there are countably many conformal maps $\phi_i : X \to X$, $i \in I$, with I having at least two elements. Then the system $S = \{\phi_i : i \in I\}$ is called a conformal iterated function system if and only if the following eight conditions are satisfied.

- **(1):** (Open Set Condition) $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$ for all $i \neq j$.
- **(2):** $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic.
- **(3):** For all $n \geq 1, \omega = (\omega_1, ..., \omega_n) \in I^n$ we have that if ω_n is a hyperbolic index or if $\omega_{n-1} \neq \omega_n$, then ϕ_ω admits a conformal extension to $V \subset I\!R^d$ which maps V into itself.
- **(4):** If i is a parabolic index, then $\bigcap_{n\geq 0} \phi_{i^n}(X) = \{x_i\}$ (Hence in particular, the diameter of the set $\phi_{i^n}(X)$ tends to 0 for n tending to infinity.)
- **(5):** (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, u_x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, $||u_x|| = 1$ and central angle α. Here, we have set $\mathrm{Con}(x, u_x, \alpha, l) := \{y : 0 < (y - x, u_x) \leq \cos\alpha ||y - x|| \leq l\}$.
- **(6):** There exists $0 < s < 1$ such that for all $n \geq 1, \omega \in I^n$ we have that if ω_n is a hyperbolic index or if $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.
- **(7):** (Bounded Distortion Property) There exists $K \geq 1$ such that for all $n \geq 1, \omega = (\omega_1, ..., \omega_n) \in I^n$ and $x, y \in V$ we have that if ω_n is a hyperbolic index or if $\omega_{n-1} \neq \omega_n$, then
$$|\phi_\omega'(y)| \leq K\, |\phi_\omega'(x)|.$$
- **(8):** There are constants $L \geq 1, \alpha > 0$ such that
$$\Big||\phi_i'(y)| - |\phi_i'(x)|\Big| \leq L||\phi_i'|||y - x|^\alpha \quad \text{for all } \ i \in I\, \forall x, y \in V.$$

Note that if $\Omega = \emptyset$, then the system S is called hyperbolic, and that if $\Omega \neq \emptyset$, then S is called parabolic. Throughout this paper we shall always assume without further notice that the system S is parabolic and the alphabet I is finite.

We now state a few immediate geometrical consequences of the bounded distortion properties (7), (8) and the cone condition (5). For the proofs of these statements we refer to [MU1].

For all hyperbolic words $\omega \in I^*$ and all convex subsets C of V we have that

$$\operatorname{diam}(\phi_\omega(C)) \leq ||\phi'_\omega|| \operatorname{diam}(C) \tag{2.1}$$

and

$$\operatorname{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||. \tag{2.2}$$

Here, the norm $||\cdot||$ is the supremum norm on V, and $D \geq 1$ denotes a universal constant. Moreover, for every $x \in X$, $0 < r \leq \ Dist(X, \partial V)$, and for every hyperbolic word $\omega \in I^*$ we have that

$$\operatorname{diam}(\phi_\omega(X)) \geq D^{-1}||\phi'_\omega|| \tag{2.3}$$

and

$$\phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r). \tag{2.4}$$

Also, there exists $0 < \beta \leq \alpha$ such that for all $x \in X$ and for all hyperbolic words $\omega \in I^*$

$$\phi_\omega(\operatorname{Int}(X)) \supset \operatorname{Con}\Big(\phi_\omega(x), \beta, D^{-1}||\phi'_\omega||\Big) \supset \operatorname{Con}\Big(\phi_\omega(x), \beta, D^{-2}\operatorname{diam}\phi_\omega(V))\Big), \tag{2.5}$$

where $\operatorname{Con}\Big(\phi_\omega(x), \beta, D^{-1}||\phi'_\omega||\Big)$ and $\operatorname{Con}\Big(\phi_\omega(x), \beta, D^{-2}\operatorname{diam}(\phi_\omega(V))\Big)$ denote some cones with vertices at $\phi_\omega(x)$, angles β, and altitudes $D^{-1}||\phi'_\omega||$ and $D^{-2}\operatorname{diam}(\phi_\omega(V))$ respectively. Finally, for every $\omega \in I^*$ (not necessarily hyperbolic) and every $x \in X$, there exists an altitude $l(\omega, x) > 0$ such that

$$\phi_\omega(\operatorname{Int}(X)) \supset \operatorname{Con}\Big(\phi_\omega(x), \beta, l(\omega, x)\Big). \tag{2.6}$$

We should like to emphasize that for $d \geq 2$ the conditions (7) and (8) with $\alpha = 1$ can be deduced from condition (3). For $d \geq 3$, this has been shown in [U1]. For $d = 2$, conditions (7) and (8) follow from Koebe's distortion theorem combined with the observation that complex conjugation in $\mathbb{C}$ is an isometry.

Let I^* denote the set of all finite words in the alphabet I, and let I^∞ be the set of all infinite sequences with entries in I. By condition (3), we have that $\phi_\omega(V) \subset V$, for every hyperbolic word ω. For each $\omega \in I^* \cup I^\infty$, we define the length of ω by the uniquely determined relation $\omega \in I^{|\omega|}$. If $\omega \in I^* \cup I^\infty$ and $n \leq |\omega|$, then we write $\omega|_n$ to denote the word $\omega_1\omega_2 \ldots \omega_n$. In [MU1] it was shown that $\lim_{n\to\infty} \sup_{|\omega|=n}\{\operatorname{diam}(\phi_\omega(X))\} = 0$. Hence, the map $\pi : I^\infty \to X$, given by $\pi(\omega) = \bigcap_{n\geq 0} \phi_{\omega|_n}(X)$, is uniformly continuous. Now, the limit set $J = J_S$ of the system S can be defined as the range of the map π, that is we define

$$J = \pi(I^\infty).$$

In order to introduce the notion of tameness we define, for every $i \in \Omega$,

$$X_i = \bigcup_{j \in I \setminus \{i\}} \phi_j(X).$$

We call a parabolic conformal iterated function system $S = \{\phi_i : i \in I\}$ tame if $x_i \notin X_i$, for every $i \in \Omega$. Also, we say that S satisfies the Super Strong Open Set Condition (SSOSC) if the following two conditions are satisfied.

$$\bigcup_{i \in I \setminus \Omega} \phi_i(X) \cup \bigcup_{i \in \Omega} \phi_i(X_i) \subset \mathrm{Int} X; \tag{2.7}$$

$$X \cap \bigcup_{i \in I} \phi_i(X) = \{x_i : i \in \Omega\}. \tag{2.8}$$

Unless stated otherwise, for the remaining part of this section we shall assume that S is a tame parabolic finite conformal iterate function system satisfying (SSOSC). The inclusion in (2.7) implies that there exists $0 < \hat{R} < \mathrm{Dist}(X, \partial V)$ such that

$$B\left(\bigcup_{i \in I \setminus \Omega} \phi_i(X) \cup \bigcup_{i \in \Omega} \phi_i(X_i), 2\hat{R} \right) \subset \mathrm{Int} X. \tag{2.9}$$

Also, for each $\omega \in I^*$ and every $A \subset B(x_i, 2\hat{R})$ we have that

$$\phi_\omega(A) \cap J = \phi_\omega(A \cap J). \tag{2.10}$$

Note that in order to derive the latter formula, we have to use the fact that the system S is tame. Furthermore, for all $i \in \Omega, \omega \in I^*$ we have that

$$\pi^{-1}(\pi(\omega i^\infty)) = \omega i^\infty. \tag{2.11}$$

Following [MU1], given $t \geq 0$, a Borel probability measure m is called t-conformal for the system S if $m(J) = 1$ and if for every Borel set $A \subset X$ and for each $i, j \in I$ with $i \neq j$, we have that

$$m(\phi_i(A)) = \int_A |\phi_i'|^t dm \tag{2.12}$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0. \tag{2.13}$$

Recall that a parabolic system S is called regular if and only if there exists a t-conformal measure (cf. [MU 1]). Then $t = h$ is the Hausdorff dimension of the limit set (see [MU1]). Combining Theorem 1.4 in [MU2] and Corollary 5.8 in [MU1], we immediately have the following result.

Theorem 2.1. *A parabolic finite iterated function system is regular.*

Hence, since the systems which we consider in this paper are finite, it follows that they are regular. The associated h-conformal measure will always be denoted by m. We shall require the following distortion properties.

Lemma 2.2. *There exists a positive constant $R^* < \hat{R}$ such that the following holds. For each hyperbolic word $\tau \in I^*$ and for every $\omega \in I^\infty$ we have that ϕ_τ is well-defined on $B(\pi(\omega), R^*)$. Moreover*

$$\frac{|\phi'_\tau(y)|}{|\phi'_\tau(x)|} \le K \quad \text{for all } \; x, y \in B(\pi(\omega), R^*),$$

and

$$K^{-h}|\phi'_\tau(\pi(\omega))|^h m(B(\pi(\omega), r)) \le m\Big(\phi_\tau(B(\pi(\omega), r))\Big) \le K^h|\phi'_\tau(\pi(\omega))|^h m(B(\pi(\omega), r))$$

for every $r \in [0, R^*]$.

Proof. The statement that $\phi_\tau : B(\pi(\omega), R^*) \to I\!R^d$ is well-define and the first distortion property of the lemma are immediate consequences of the fact that $R^* < \hat{R} < \mathrm{Dist}(X, \partial V)$ and property (7) at the beginning of this section. In order to derive the second distortion property of the lemma, choose $0 < R^* < \hat{R}$ sufficiently small such that, for each $i \in \Omega$,

$$B\Big(\phi_i(X) \cap (I\!R^d \setminus B(x_i, \hat{R})), 2R^*\Big) \subset \mathrm{Int}X. \tag{2.14}$$

If $\pi(\omega) \in \phi_i(X)$ for some $i \in \Omega$, and if $||\pi(\omega) - x_i|| \ge \hat{R}$, then $B(\pi(\omega), 2R^*) \subset \mathrm{Int}X$. The proof in this case then follows immediately from a combination of the conformality of the measure m and distortion property (7). In case that $\pi(\omega) \in \phi_i(X) \cap B(x_i, \hat{R})$, it follows that $B(\pi(\omega), R^*) \subset B(x_i, 2\hat{R})$. Using (2.10) and the conformality of m, we obtain that

$$\begin{aligned} m\Big(\phi_\tau((B(\pi(\omega), r))\Big) &= m\Big(\phi_\tau(B(\pi(\omega), r)) \cap J\Big) = m\Big(\phi_\tau(B(\pi(\omega), r) \cap J)\Big) \\ &= \int_{B(\pi(\omega), r) \cap J} |\phi'_\tau|^h dm = \int_{B(\pi(\omega), r)} |\phi'_\tau|^h dm, \end{aligned}$$

and hence the first distortion property of the lemma gives the proof in this case. Finally, if $\pi(\omega) \notin \bigcup_{i \in \Omega} \phi_i(X)$, then $\pi(\omega) \in \phi_j(X)$ for some $j \in I \setminus \Omega$. In this case (2.9) implies that $B(\pi(\omega), 2R^*) \subset \mathrm{Int}X$, and hence the statement of the lemma follows immediately from (7) and the conformality of m. This proves the lemma. ■

The following fact easily follows from the local analysis of parabolic polints done in [MU2].

Lemma 2.3. *Assuming that $R^* > 0$ is sufficiently small, there exists a constant $C_1 > 0$ such that for every $i \in \Omega$ and every $r \in (0, R^*]$, the intersection $J \cap B(x_i, r) \setminus \{x_i\}$ is contained in a central cone contained in* Int$X$ *with vertex x_i and an angle $\le C_1 r^{p_i}$.*

As an immediate consequence of this lemma we get the following.

Lemma 2.4. *There exists a constant $C_1 > 0$ such that for every $i \in \Omega$, every $r \in (0, R^*]$, and every hyperbolic word ω the intersection $J \cap \overline{B}(\phi_\omega(x_i), r|\phi'_\omega(x_i)|)$ is contained in a central cone with vertex x_i and an angle $\leq C_2 r^{p_i}$.*

We are now in the position to prove the following distortion property.

Lemma 2.5. *There exist constants $\rho, R_* > 0$ such that for every $i \in \Omega$, $x \in J \cap B(x_i, R_*)$, and for each $\omega \in I^*$ the map ϕ_ω is well-defined on $B(x, \rho||x - x_i||)$ and*

$$\frac{|\phi'_\omega(z)|}{|\phi'_\omega(y)|} \leq K \quad \textit{for all } \; y, z \in B(x, \rho||x - x_i||),$$

and furthermore, for every positive $r \leq \rho||x - x_i||$ we have that

$$K^{-h}|\phi'_\omega(x)|^h m(B(x,r)) \leq m\Big(\phi_\omega(B(x,r))\Big) \leq K^h|\phi'_\omega(x)|^h m(B(x,r)).$$

Proof. In view of Lemma 2.3 there exists $R_* > 0$ and $\rho \in (0, 1/2)$ such that $B(x, 2\rho||x - x_i||) \subset \mathrm{Int}X$ for all $x \in J \cap B(x_i, R_*)$. Therefore, all the maps $\phi_\omega : B(x, 2\rho_1||x - x_i||) \to \mathrm{Int}X$ are well-defined, and the second part of our lemma follows from the first part. The first part in turn in the case when $d = 1$ is contained in Lemma 2.6 of [U2]. In the case $d = 2$ it follows immediately from Koebe's distortion theorem and the observation that the complex conjugation is an isometry. In the case $d \geq 3$ it follows from the inequality following formula (4.9) in the proof of Theorem 4.13 in [MU2] that, with $Y = \overline{B(x, \rho||x - x_i||)}$, $W = B(x, 2\rho||x - x_i||)$, one gets $|\phi'_\omega(z)|/|\phi'_\omega(y)| \leq 4$. We are done. ■

The constants R_* and R^* of Lemma 2.2 and Lemma 2.4 will be crucial in the sequel. For later use we define

$$R := \min\{R_*, R^*\}.$$

3. **The geometry of conformal measures**

The main result in this section is the derivation of a 'global formula' for the conformal measure associated with a tame parabolic finite iterated function system. This formula describes in a uniform way the scaling of this measure at arbitrary points in the associated limit set. Our ellaboration of this formula follows closely the discussion in [SV] and [SU], where we obtained this type of formula for geometrically finite Kleinian groups with parabolic elements and for parabolic rational maps.

The section is split into two subsections. In the first we give an estimate for the conformal measure around parabolic points. In the second we then derive the global formula. Subsequently, as a first application of this formula, we obtain a first rough description of how the conformal measure relates to the geometric concepts Hausdorff measure and packing measure.

3.1. **The conformal measure around parabolic points.** We begin this subsection by recalling the following estimates for tame parabolic systems. For $d \geq 2$ a proof can be found in [MU2] (Section 4). For $d = 1$ the estimates are obtained immediately from the considerations in [U2].

Proposition 3.1. *Let S be a tame parabolic system. Then there exists a constant $Q \geq 1$ and an integer $q \geq 0$ such that for every parabolic index $i \in I$ there exists an integer $p_i \geq 1$ such that for every $j \in I \setminus \{i\}$ and for all $n, k \geq 1$ we have that*

$$Q^{-1} n^{-\frac{p_i+1}{p_i}} \leq \inf_X \{|\phi'_{i^n j}|\}, ||\phi'_{i^n j}||, \operatorname{diam}(\phi_{i^n j}(X)) \leq Q n^{-\frac{p_i+1}{p_i}}, \tag{3.1}$$

$$Q^{-1} n^{-\frac{1}{p_i}} \leq \operatorname{Dist}(x_i, \phi_{i^n}(X_i)) \leq \operatorname{Dist}(x_i, \phi_{i^n}(X_i)) \leq Q n^{-\frac{1}{p_i}}, \tag{3.2}$$

$$\operatorname{Dist}(\phi_{i^n}(X_i), \phi_{i^k}(X_i)) \leq Q |n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}}|. \tag{3.3}$$

Furthermore, for $|n - k| \geq q$ we have that

$$\operatorname{Dist}(\phi_{i^n}(X_i), \phi_{i^k}(X_i)) \geq Q |n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}}|. \tag{3.4}$$

The following lemma gives the main result of this section.

Lemma 3.2. *Let m denote the h-conformal measure of the finite parabolic system S. For each $\kappa > 0$ there exists $C_\kappa > 0$ such that for every parabolic index i and for every $x \in \overline{J}$ we have that*

$$C_\kappa^{-1} \, ||x - x_i||^{h+(h-1)p_i} \leq m(B(x, \kappa ||x - x_i||)) \leq C_\kappa \, ||x - x_i||^{h+(h-1)p_i}.$$

In particular, the constant C_κ depends continuously on κ.

Proof. Since the support of m is equal to $\overline{J}$, we may assume without loss of generality that $||x - x_i|| \leq \Delta$ for some fixed $0 < \Delta \leq R$. Let $x = \pi(\omega)$ and $\omega \in I^\infty$ be given. Then $\omega = i^n j \tau$, where $j \neq i$, $n \geq 1$, and $\tau \in I^\infty$. Assuming Δ to be chosen sufficiently small, (3.1) implies that

$$n \geq 2Q^2 \kappa^{-1}. \tag{3.5}$$

For the proof of the first inequality in the measure estimate of the lemma, let

$$T := \{k : \operatorname{Dist}(\phi_{i^k j}(X), \phi_{i^n j}(X)) \leq \kappa ||x - x_i|| - \operatorname{diam}(\phi_{i^n j}(X))\}.$$

Using (3.1), we deduce that

$$m(B(x, \kappa ||x - x_i||)) \geq \sum_{k \in T} m(\phi_{i^k j}(X)) \geq \sum_{k \in T} |\phi'_{i^k j}|_*^h \geq \sum_{k \in T} Q^{-h} k^{-\frac{p_i+1}{p_i} h}.$$

Using (3.2) and (3.1), we have that if

$$Q \left| n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}} \right| \leq \kappa Q^{-1} n^{-\frac{1}{p_i}} - Q n^{-\frac{p_i+1}{p_i}},$$

then it follows that $k \in T$. Hence in particular, if $k \geq n$ and if

$$Q\left(n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}}\right) \leq \kappa Q^{-1} n^{-\frac{1}{p_i}} - Q n^{-\frac{p_i+1}{p_i}}, \tag{3.6}$$

then we have that $k \in T$. Clearly, the statement in (3.6) is equivalent to

$$Qk^{-\frac{1}{p_i}} \geq (Q - \kappa Q^{-1}) n^{-\frac{1}{p_i}} + Q n^{-\frac{p_i+1}{p_i}}.$$

Also, (3.5) implies that $Qn^{-\frac{p_i+1}{p_i}} \leq \kappa(2Q)^{-1} n^{-\frac{1}{p_i}}$. Therefore, if $k \geq n$ and $Qk^{-\frac{1}{p_i}} \geq (Q - \kappa Q^{-1}) n^{-\frac{1}{p_i}} + \kappa(2Q)^{-1} n^{-\frac{1}{p_i}}$, or equivalently if $k \geq n$ and $k \leq \left(1 - \frac{\kappa}{2Q}\right)^{-\frac{1}{p_i}}$, then we have that $k \in T$. It now follows that there exists a constant $\tilde{C}_\kappa > 0$ (which depends continuously on κ) such that

$$\begin{aligned} m(B(x, \kappa||x - x_i||)) &\geq Q^{-h} \sum_{k=n}^{\left(1-\frac{\kappa}{2Q}\right)^{-\frac{1}{p_i}}} k^{-\frac{p_i+1}{p_i}h} \\ &\geq Q^{-h}\left(1 - \frac{p_i+1}{p_i}h\right)\left(\left(1 - \frac{\kappa}{2Q}\right)^{-\frac{1}{p_i}\left(1-\frac{p_i+1}{p_i}h\right)} - 1\right) n^{-\frac{p_i+1}{p_i}h} \\ &\geq \tilde{C}_\kappa n^{-\left(\frac{h+(h-1)p_i}{p_i}\right)}. \end{aligned}$$

Hence, since by (3.6) we have that $||x - x_i|| \leq Qn^{-\frac{1}{p_i}}$, it follows that

$$m(B(x, \kappa||x - x_i||)) \geq \tilde{C}_\kappa Q^{h+(h-1)p_i} ||x - x_i||^{h+(h-1)p_i}. \tag{3.7}$$

In order to prove the second inequality in the measure estimate of the lemma, note that $Qk^{-\frac{1}{p_i}} \leq (1+\kappa)||x - x_i||$ if and only if $k \geq \left(Q^{-1}(1+\kappa)||x - x_i||\right)^{-p_i}$. Using this observation, (3.2) and (3.1), we obtain that

$$\begin{aligned} m(B(x, \kappa||x - x_i||)) &\leq m(B(x_i, (1+\kappa)||x - x_i||) \\ &\leq \sum_{j \neq i} \sum_{k=\left(Q^{-1}(1+\kappa)||x-x_i||\right)^{-p_i}} m(\phi_{i^k j}(X)) \\ &\leq \sum_{j \neq i} \sum_{k=\left(Q^{-1}(1+\kappa)||x-x_i||\right)^{-p_i}} ||\phi'_{i^k j}||^h \\ &\leq \sum_{j \neq i} \sum_{k=\left(Q^{-1}(1+\kappa)||x-x_i||\right)^{-p_i}} Qk^{-\frac{p_i+1}{p_i}h} \\ &\leq 2Q\left(k^{h\frac{p_i+1}{p_i}-1}\right)^{-1} \\ &\leq \left(Q^{-1}(1+\kappa)||x - x_i||\right)^{-p_i\left(1-\frac{p_i+1}{p_i}h\right)} = \hat{C}_\kappa ||x - x_i||^{h+(h-1)p_i}, \end{aligned}$$

where $\hat{C}_\kappa < \infty$ denotes some positive constant (which depends continuously on κ). ■

Corollary 3.3. *There exists a constant $C \geq 1$ such that for each $i \in \Omega$ and for all $0 < r \leq 2\mathrm{diam}(X)$ we have that*

$$C^{-1}\, r^{h+(h-1)p_i} \leq m(B(x_i, r)) \leq C\, r^{h+(h-1)p_i}.$$

Proof. Let $j \neq i$, and choose $n \geq 1$ to be the least integer such that $Q^{-1} n^{-\frac{1}{p_i}} \leq r$. Let $x \in \phi_{i^{n-1}j}(X)$ be fixed. By (3.2) and Lemma 3.2, we have that

$$\begin{aligned} m(B(x_i, r)) &\leq m(x, 2||x - x_i||) \leq C_2\, ||x - x_i||^{h+(h-1)p_i} \leq C_2\, Q(n-1)^{-\frac{h+(h-1)p_i}{p_i}} \\ &\ll n^{-\frac{h+(h-1)p_i}{p_i}} \ll \left(\frac{Q}{2}\right)^{h+(h-1)p_i} r^{h+(h-1)p_i}. \end{aligned}$$

Now, let $k \geq 1$ denote the least integer such that $Qk^{-\frac{1}{p_i}} \leq r/2$, and let $y \in \phi_{i^k j}(X)$ be fixed. Similar as above, (3.2) and Lemma 3.2 imply that

$$\begin{aligned} m(B(x_i, r)) &\geq m(B(y, ||y - x_i|) \geq C_1\, ||y - x_i||^{h+(h-1)p_i} \geq C_1\, Q^{-(h+(h-1)p_i)} k^{-\frac{h+(h-1)p_i}{p_i}} \\ &\gg (k-1)^{-\frac{h+(h-1)p_i}{p_i}} \geq 2^{-(h+(h-1)p_i)} r^{h+(h-1)p_i}. \end{aligned}$$

■

Lemma 3.4. *For every $\kappa > 0$ there exists $D_\kappa \geq 1$ such that for each $i \in \Omega$, for every sufficiently small $r > 0$, and for all $x \in J \cap B(x_i, \kappa^{-1} r)$ we have that*

$$D_\kappa^{-1} r^{h+(h-1)p_i} \leq m(B(x, r)) \leq D_\kappa r^{h+(h-1)p_i}.$$

Proof. Since $B(x, r) \subset B(x_i, ||x - x_i|| + r) \subset B(x_i, (1 + \kappa^{-1})r)$, it follows from Corollary 3.3 that

$$m(B(x, r)) \leq C(1 + \kappa^{-1})^{h+(h-1)p_i} r^{h+(h-1)p_i}. \tag{3.8}$$

Now, if $r \leq 2||x - x_i||$, then $r = \alpha||x - x_i||$ for some α such that $\kappa \leq \alpha \leq 2$. By Lemma 3.2, we have that $C_\alpha \leq \overline{C} := \sup\{C_t : t \in [\kappa, 2]\} < \infty$. Hence, using Lemma 3.2 once again, it follows that

$$\begin{aligned} m(B(x, r)) &= m(B(x, \alpha||x - x_i||) \leq C_\alpha ||x - x_i||^{h+(h-1)p_i} \\ &\leq \overline{C} \left(\frac{r}{\alpha}\right)^{h+(h-1)p_i} \leq \overline{C} \kappa^{-(h+(h-1)p_i)} r^{h+(h-1)p_i}. \end{aligned} \tag{3.9}$$

Otherwise if $r \geq 2||x - x_i||$, then Corollary 3.3 implies

$$m(B(x, r)) \geq m(B(x_i, r/2)) \geq C \left(\frac{r}{2}\right)^{h+(h-1)p_i} = C 2^{-(h+(h-1)p_i)} r^{h+(h-1)p_i}.$$

Combining the latter estimate, (3.8) and (3.9), the lemma follows. ■

3.2. **The global formula for the conformal measure.** An element $\omega \in I^\infty$ is called pre-parabolic if and only if $\sigma^k \omega = i^\infty$ for some $k \geq 0$ and some $i \in \Omega$. The set of all pre-parabolic elements will be denoted by I_p^∞. Also, a limit point which is not a pre-parabolic element will be referred to as radial, and we write I_r^∞ to denote the set of all radial points.

For each $\omega \in I^\infty$ we fix an increasing sequence of integers $\{n_j(\omega)\}_{j=1}^{k(\omega)}$ as follows. Assume that $n_j(\omega)$ is defined, then we define $n_{j+1}(\omega)$ to be the smallest index which is greater than $n_j(\omega)$ such that either $\omega_{n_{j+1}(\omega)}$ is hyperbolic or $\omega_{n_{j+1}(\omega)} \neq \omega_{n_{j+1}(\omega)-1}$ (note that $n_1(\omega)$ is well-defined). In case $n_{j+1}(\omega)$ does not exist, then $j = k(\omega)$. Note that if $n_{j+1}(\omega) \geq n_j(\omega)+2$, then there exists a unique parabolic index $i = i(\omega, j)$ such that $\omega_l = i$ for all $n_j(\omega) \leq l \leq n_{j+1}(\omega)$. Furthermore, if $n_{j+1}(\omega) = n_j(\omega)+1$, then $i(\omega, j)$ denotes some arbitrary element of Ω. Observe that $k(\omega) = \infty$ if and only if $\omega \in I_r^\infty$. For each j, we define

$$r_j(\omega) := R\,|\phi'_{\omega|_{n_j(\omega)}}(\pi(\sigma^{n_j(\omega)}\omega))|,$$

and we refer to the sequence $\{r_j(\omega)\}_{j=1}^{k(\omega)}$ as to the hyperbolic zoom of ω. Note that by the chain rule and by property (6) of section 2, we have that $\{r_j(\omega)\}_{j=1}^{k(\omega)}$ is a strictly decreasing sequence. Hence, for each $\omega \in I_r^\infty$ and every $0 < r \leq \tilde{R} = \min\{\inf\{|\phi_i'| : i \notin \Omega\}, \inf\{|\phi_{ij}'| : i \in \Omega, j \neq i\}\}$, there exists a unique $j \geq 1$ such that $r_{j+1}(\omega) < r \leq r_j(\omega)$. For a given ω and r, these so determined neighbours $r_{j+1}(\omega)$ and $r_j(\omega)$ in the hyperbolic zoom of ω will be denoted by $r_*(\omega)$ and $r_*(\omega)$ respectively. Also, in this situation we shall write $i(\omega, r)$ to denote the parabolic element $i(\omega, j)$. Finally we define the function ζ, which is given for $\omega \in I^\infty$ and $r > 0$ by

$$\zeta(\omega, r) := \frac{m(B(x,r))}{r^h}.$$

The following theorem is the main result of this section.

Theorem 3.5. *(Global formula for conformal measures) Let S be a tame parabolic finite iterated function system satisfying the* (SSOSC)*. Then, for each $\omega \in I_r^\infty$ and every $0 < r \leq \tilde{R}$ we have with $i = i(\omega, r)$ that*

$$\zeta(\omega, r) \asymp \begin{cases} \left(\frac{r}{r^*(\omega)}\right)^{(h-1)p_i} & \text{for } r^*(\omega) \geq r \geq r^*(\omega)\left(\frac{r_*(\omega)}{r^*(\omega)}\right)^{\frac{1}{p_i+1}} \\ \left(\frac{r_*(\omega)}{r}\right)^{h-1} & \text{for } r_*(\omega) \leq r \leq r^*(\omega)\left(\frac{r_*(\omega)}{r^*(\omega)}\right)^{\frac{1}{p_i+1}}. \end{cases}$$

Proof. Let $\omega \in I_r^\infty$ and $0 < r \leq \tilde{R}$ be fixed. For ease of notation, throughout the proof we shall suppress the dependence on ω in some of the appearing quantities. Let j be determined by the condition $r_* = r_j$. Hence, we have that $r^* = r_{j+1}$. By (3.2), we have that

$$||\pi(\sigma^{n_j}\omega) - x_i|| = ||\pi(\phi_{i^{n_{j+1}-n_j-1}\omega_{n_{j+1}}}(\pi(\sigma^{n_{j+1}}\omega))) - x_i|| \asymp (n_{j+1} - n_j)^{-\frac{1}{p_i}}.$$

Using the chain rule and (3.1), we obtain that

$$\begin{aligned} 1 &= r_{j+1}|\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|^{-1}|\phi'_{\sigma^{n_j}\omega|_{n_{j+1}-n_j-1}}(\pi(\sigma^{n_{j+1}}\omega))|^{-1} \\ &\asymp \left(\frac{r_{j+1}}{r_j}\right)(n_{j+1} - n_j)^{\frac{p_i+1}{p_i}} = \left(\frac{r_*}{r^*}\right)(n_{j+1} - n_j)^{\frac{p_i+1}{p_i}}. \end{aligned}$$

Hence,

$$\left(\frac{r_*}{r^*}\right)^{\frac{1}{p_i}} \asymp ||\pi(\sigma^{n_j}\omega) - x_i||. \tag{3.10}$$

This implies that if

$$r \geq r^*(\omega)\left(\frac{r_*(\omega)}{r^*(\omega)}\right)^{\frac{1}{p_i+1}},$$

then

$$|\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))| \cdot ||\pi(\sigma^{n_j}\omega) - x_i|| \leq r \leq |\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|,$$

and hence,

$$||\pi(\sigma^{n_j}\omega) - x_i|| \leq \frac{r}{|\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|} \leq 1.$$

Now, using property (7) (note that $\phi_{\omega|_{n_j}}$ is hyperbolic) and Lemma 3.4, it follows that

$$\begin{aligned} m(B(\pi(\omega), r)) &\asymp |\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|^h \, m\Big(B(\pi(\sigma^{n_j}\omega), r|\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))|^{-1})\Big) \\ &\asymp r_j^h(rr_j^{-1})^{h+(h-1)p_i} = r^h\left(\frac{r}{r_j}\right)^{(h-1)p_i} = r^h\left(\frac{r}{r^*}\right)^{(h-1)p_i}. \end{aligned}$$

This proves the first case in the theorem. We are now left to consider the case in which

$$r \leq r^*(\omega)\left(\frac{r_*(\omega)}{r^*(\omega)}\right)^{\frac{1}{p_i+1}}.$$

Because of (3.10) (after arranging for appropriate constants), this means that

$$|\phi'_{\omega|_{n_{j+1}}}(\pi(\sigma^{n_{j+1}}\omega))| \leq r \leq \rho|\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}\omega))| \cdot ||\pi(\sigma^{n_j}\omega) - x_i||,$$

where $0 < \rho < 1$ is the constant obtained in Lemma 2.5. Therefore, there exists $n_j \leq u \leq n_{j+1} - 1$ such that

$$\rho|\phi'_{\omega|_{u+1}}(\pi(\sigma^{u+1}\omega))| \cdot ||\pi(\sigma^{u+1}\omega) - x_i|| \leq r \leq \rho|\phi'_{\omega|_u}(\pi(\sigma^u\omega))| \cdot ||\pi(\sigma^u\omega) - x_i||.$$

In particular, this implies that

$$r \asymp \rho|\phi'_{\omega|_u}(\pi(\sigma^u\omega))| \cdot ||\pi(\sigma^u\omega) - x_i||. \tag{3.11}$$

Thus, by using the conformality of m, Lemma 2.5 and Lemma 3.2, it follows that

$$\begin{aligned} m(B(\pi(\omega), r)) &\asymp |\phi'_{\omega|_u}(\pi(\sigma^u\omega))|^h m(B(\pi(\sigma^u\omega), \rho||\pi(\sigma^u\omega) - x_i||)) \\ &\asymp |\phi'_{\omega|_u}(\pi(\sigma^u\omega))|^h ||\pi(\sigma^u\omega) - x_i||^{h+(h-1)p_i} \\ &\asymp r^h||\pi(\sigma^u\omega) - x_i||^{(h-1)p_i}. \end{aligned} \tag{3.12}$$

On the other hand, the chain rule, (3.1) and (3.2) imply that

$$\begin{aligned} 1 &= r_{j+1}|\phi'_{\omega|_u}(\pi(\sigma^u\omega))|^{-1}|\phi'_{\sigma^u\omega|_{n_{j+1}-u-1}}(\pi(\sigma^{n_{j+1}}\omega))|^{-1} \\ &\asymp r_*|\phi'_{\omega|_u}(\pi(\sigma^u\omega))|^{-1}(n_{j+1} - u)^{\frac{p_i+1}{p_i}}, \end{aligned}$$

as well as

$$||\pi(\sigma^u\omega) - x_i||^{-(p_i+1)} \asymp (n_{j+1} - u)^{\frac{p_i+1}{p_i}}.$$

These two latter comparabilities together with (3.11) show that

$$r \asymp r_*||\pi(\sigma^u\omega) - x_i||^{-(p_i+1)}||\pi(\sigma^u\omega) - x_i|| = r_*||\pi(\sigma^u\omega) - x_i||^{-p_i}.$$

Hence, we have that $||\pi(\sigma^u\omega) - x_i|| \asymp (r_*/r)^{1/p_i}$, which together with (3.12) implies that

$$m(B(\pi(\omega), r)) \asymp r^h \left(\frac{r_*}{r}\right)^{h-1}.$$

This proves the second case in the theorem. ■

The following corollaries are immediate consequences of the previous theorem.

Corollary 3.6. *If $\omega \in I_r^\infty$, then for each $j \geq 1$ we have that*

$$m(B(\pi(\omega), r_j(\omega))) \asymp r_j(\omega)^h.$$

Corollary 3.7. *The conformal measure m is a doubling measure. This means that for every $c > 0$ there exists $B > 0$ such that for each $z \in J$ and every $r > 0$ we have that*

$$m(B(z, cr)) \leq Bm(B(z, r)).$$

Finally, as a first non-trivial application of Theorem 3.5 we derive an alternative proof of the following geometrical fact which was obtained under slightly weaker assumptions in [MU2]. For this let H^t and P^t denote the t-dimensional Hausdorff and packing measure respectively.

Theorem 3.8. *If S is a tame finite parabolic system satisfying the* (SSOSC), *then the following holds.*

- **(a):** *If $h > 1$, then $0 < \mathrm{H}^h(J) < \infty$ and $\mathrm{P}^h(J) = \infty$.*
- **(b):** *If $h = 1$, then $0 < \mathrm{H}^h(J), \mathrm{P}^h(J) < \infty$.*
- **(c):** *If $h < 1$, then $0 < \mathrm{P}^h(J) < \infty$ and $\mathrm{H}^h(J) = 0$.*

Additionally, if either measure H^h or P^h is finite and positive, then its normalized version is equal to the conformal measure m.

Proof. In [MU1] (Lemma 5.6 and Theorem 5.7) it was shown that for a tame finite parabolic system satisfying the (SSOSC) the h-conformal measure m is atomless. This combined with Corollary 3.6 and the inverse Frostmann lemma (see [PU]) implies that we always have that $\mathrm{H}^h(J) < \infty$ and $\mathrm{P}^h(J) > 0$. Now, if $h \geq 1$, then Theorem 3.5 immediately gives that, for every $x \in \pi(I_r^\infty)$,

$$\limsup_{r \to 0} \frac{m(B(x, r))}{r^h} \ll 1,$$

which implies that $\mathrm{H}^h(J) > 0$. If in addition $x = \pi(\omega)$, for $\omega \in I_r^\infty$ containing arbitrarily long blocks of i's for some $i \in \Omega$, then we have that

$$\liminf_{r \to 0} \frac{m(B(x, r))}{r^h} \leq \liminf_{r \to 0} \zeta\left(\omega, \rho r^*(\omega)\left(\frac{r_*(\omega)}{r^*(\omega)}\right)^{\frac{1}{p_i+1}}\right) = \liminf_{r \to 0} \left(\frac{r_*(\omega)}{r^*(\omega)}\right)^{\frac{(h-1)p_i}{p_i+1}} = 0.$$

Now, by ergodicity of the measure m (see [MU2], Corollary 5.11) and since m is positive on open sets, it follows that m-almost everywhere we have that

$$\liminf_{r\to 0} \frac{m(B(x,r))}{r^h} = 0.$$

We conclude that $\mathrm{P}^h(J) = \infty$, which proves case (a) of the theorem. Case (b) is an immediate consequence of Theorem 3.5. The proof of case (c) is analogous to the proof of case (a), and we obmit it. ■

4. **Metrical Diophantine analysis**

In this section we give a metrical Diophantine analysis for tame parabolic finite iterated function systems. In the first subsection we calculate the Hausdorff dimensions of certain subsets of the limit set which are of zero h-conformal measure. These sets are comprised of radial elements which under the system have a rather rapid approach to the parabolic points. In particular, these sets are the natural analogues of the sets of well-approximable numbers. In the second subsection we derive various limit laws which give useful approximations of the 'essential support' of the h-conformal measure associated with a tame finite parabolic iterated function system. Subsequently, we show that these laws lead to good estimates on the growth of the function ζ in the global formula (Theorem 3.5), which in turn give rise to a refined description of the conformal measure in terms of Hausdorff measures and packing measures with respect to some explicit gauge functions.

4.1. **Iterated function systems in the spirit of Jarník.**

We first have to introduce the notion of a canonical ball. For $i \in \Omega$, $\delta > 0$ and a hyperbolic word $\omega \in I^*$, we define

$$B_\omega(i) = B_\omega = \overline{B}(\phi_\omega(x_i), R|\phi'_\omega(x_i)|) \quad \text{and} \quad B^\delta_\omega(i) = B^\delta_\omega = \overline{B}(\phi_\omega(x_i), (R|\phi'_\omega(x_i)|)^{1+\delta}).$$

The closed ball B_ω will be referred to as the canonical ball associated with the hyperbolic word ω.
Our main interest in this section will be focused on the sets

$$J^\delta_i := \bigcap_{q\geq 1} \bigcup_{n\geq q} \bigcup_{|\omega|=n} B^\delta_\omega$$

and

$$J^\delta := \bigcup_{i\in\Omega} J^\delta_i.$$

The main result in this section is stated in the following theorem. The proof of this theorem will occupy the remaining part of this section. It will be given in several steps, some of which are formulated in separate lemmata.

Theorem 4.1. *Let $S = \{\phi_i : i \in I\}$ be a tame parabolic finite iterated function system satisfying* (SSOSC). *Then, for every $i \in \Omega$ the following holds.*

(a): *If $h \leq 1$, then*

$$\mathrm{HD}(J^\delta) = \frac{h}{1+\delta}.$$

(b): *If $h \geq 1$, then*

$$\mathrm{HD}(J_i^\delta) = \begin{cases} \frac{h}{1+\delta} & \text{if } \ \delta \geq h-1 \\ \frac{h+\delta p_i}{1+\delta(1+p_i)} & \text{if } \ \delta \leq h-1. \end{cases}$$

In particular, with $p_{min} := \min\{p_i : i \in \Omega\}$, we have that

$$\mathrm{HD}(J^\delta) = \begin{cases} \frac{h}{1+\delta} & \text{if } \delta \geq h-1 \\ \frac{h+\delta p_{min}}{1+\delta(1+p_{min})} & \text{if } \delta \leq h-1. \end{cases}$$

The first step in the proof of the theorem is to give some upper bound for $\mathrm{HD}(J_i^\delta)$.

Lemma 4.2. *For each $i \in \Omega$ and every $\delta > 0$ we have that*

$$\mathrm{HD}(J_i^\delta) \leq \min\left\{\frac{h}{1+\delta}, \frac{h+\delta p_i}{1+\delta(1+p_i)}\right\}.$$

Proof. For $n \geq 1$, let H_n denote the family of all hyperbolic words of length n. For every $\epsilon > 0$ we have that

$$\begin{aligned} \mathrm{H}^{\frac{h}{1+\delta}+\epsilon}(J_i^\delta) &\leq \liminf_{q\to\infty} \sum_{n\geq q} \sum_{\omega \in H_n} \Big((R|\phi'_\omega(x_i)|)^{1+\delta})\Big)^{\frac{h}{1+\delta}+\epsilon} \\ &\leq R^{h+\epsilon(1+\delta)} \liminf_{q\to\infty} \sum_{n\geq q} \sum_{\omega\in H_n} |\phi'_\omega(x_i)|^{h+\epsilon(1+\delta)}. \end{aligned}$$

From Lemma 4.3 and Theorem 4.6 in [MU1] we deduce that exists a $(h+\epsilon(1+\delta))$-semiconformal measure ν. We then apply Theorem 5.1 in [MU1], which gives that ν is in fact $(h+\epsilon(1+\delta))$-conformal, and that $\nu(x_j) > 0$ for some $j \in \Omega$. From the definition of the limit set J it follows that there exists a hyperbolic word $\tau \in I^*$ such that $\phi_\tau(x_j) \in B(x_i, R)$. Hence by Lemma 2.2, we have for every hyperbolic word $\omega \in I^*$ that

$$|\phi'_\omega(x_i)| \leq K|\phi'_\omega(\phi_\tau(x_j))| = K|\phi'_\tau(x_j)|^{-1}|\phi'_{\omega\tau}(x_j)|.$$

Combining this latter estimate and the conformality of ν, it follows that for each $q \geq 0$ and every $n \geq q$ we have that

$$\begin{aligned} \sum_{n\geq q}\sum_{\omega\in H_n} |\phi'_\omega(x_i)|^{h+\epsilon(1+\delta)} &\leq \Big(K|\phi'_\tau(x_j)|^{-1}\Big)^{h+\epsilon(1+\delta)} \sum_{n\geq q}\sum_{\omega\in H_n} |\phi'_{\omega\tau}(x_j)|^{h+\epsilon(1+\delta)} \\ &\leq \sum_{n\geq q}\sum_{|\omega|=n} \nu(\phi_{\omega\tau}(x_j))\nu(x_j)^{-1} \\ &\leq \nu(x_j)^{-1}\nu(\{\phi_\gamma(x_j) : |\gamma| \geq q+|\tau|\}) \leq \nu(x_j)^{-1}. \end{aligned}$$

Hence, $\mathrm{H}^{\frac{h}{1+\delta}+\epsilon}(J_i^\delta) \leq \nu(x_j)^{-1}$, and consequently $\mathrm{HD}(J_i^\delta) \leq \frac{h}{1+\delta}+\epsilon$. By letting ϵ tend to 0, we derive that $\mathrm{HD}(J_i^\delta) \leq \frac{h}{1+\delta}$.

In order to obtain the second upper bound, note that by Lemma 2.4 the intersection $J \cap B(\phi_\omega(x_i), (R|\phi'_\omega(x_i)|)^{1+\delta})$ is contained in a central cone with vertex $\phi_\omega(x_i)$ and angle $\leq C_2 R^{1+\delta}|\phi'_\omega(x_i)|)^{\delta p_i} \asymp (R|\phi'_\omega(x_i)|)^{\delta p_i}$, or equivalently the radius of the base $\preceq (R|\phi'_\omega(x_i)|)^{1+\delta(p_i+1)}$. Thus $J \cap B(\phi_\omega(x_i), (R|\phi'_\omega(x_i)|)^{1+\delta})$ can be covered by at most const $(R|\phi'_\omega(x_i)|)^{-\delta p_i}$ balls of radii $(R|\phi'_\omega(x_i)|)^{1+\delta(p_i+1)}$. Therefore, for every $\epsilon > 0$ we have

$$\begin{aligned} \mathrm{H}^{\frac{h+\delta p_i}{1+\delta(1+p_i)}+\epsilon}(J_i^\delta) &\leq \liminf_{q\to\infty} \sum_{n\geq q} \sum_{\omega\in H_n} \left((R|\phi'_\omega(x_i)|)^{1+\delta(1+p_i)}\right)^{\frac{h+\delta p_i}{1+\delta(1+p_i)}+\epsilon} (R|\phi'_\omega(x_i)|)^{-\delta p_i} \\ &\asymp \liminf_{q\to\infty} \sum_{n\geq q} \sum_{\omega\in H_n} |\phi'_\omega(x_i)|^{h+\epsilon(1+\delta(1+p_i))}. \end{aligned}$$

Now the proof follows exactly in the same way as in the first part. ■

As a first step towards the proof of the lower bound in Theorem 4.1, we obtain the following lemma.

Lemma 4.3. *There exists a universal constant $b(d) \geq 1$ such that the following holds. For every open set $G \subset \mathrm{Int}X$ and each $n \geq 1$, there exists a finite set $I_{G,n} \subset \bigcup_{j\geq n} I^j$ of mutually incomparable hyperbolic words, which has the properties that $m\left(\bigcup_{\omega\in I_{G,n}} B_\omega\right) \geq b(d)^{-1}m(G)$ and that the balls in $\{B_\omega : \omega \in I_{G,n}\}$ are pairwise disjoint subsets of G.*

*Proof.*Fix $i \in \Omega$. We define

$$J_\infty := \pi\Big(\{\omega \in I^\infty \setminus \{\tau i^\infty : \tau \in I^*\} : \omega \text{ contains arbitrarily long blocks of } i\text{'s}\}\Big).$$

Then, since the conformal measure m is positive on non-empty open subsets of J, Corollary 5.11 in [MU1] implies that $m(J_\infty) = 1$. Now, let $q \geq 1$ be sufficiently large such that $\phi_{i^q}(X) \subset B(x_i, K^{-1}R)$. It follows from the definition of J_∞ that if $x \in J_\infty$, then there exists an increasing infinite sequence $\{l_j\}_j$ with $l_j \geq n$ for all $j \geq 1$, a sequence $\{q_j\}_j$ with $q_j \geq q+1$ for all $j \geq 1$, and words $\omega^{(j)} \in I^{l_j+q_j}$ such that for all $j \geq 1$ we have that $x \in \phi_{\omega^{(j)}}(X)$, $\omega^{(j)}_{l_j} \neq i$ and $\sigma^{l_j}\omega|_{q_j} = i^{q_j}$. It now follows that $x \in \phi_{\omega^{(j)}|_{l_j+1}}(B(x, K^{-1}R)) \subset B_{\omega^{(j)}}$, that $\omega^{(j)}|_{l_j+1}$ is a hyperbolic word, and that $\lim_{j\to\infty} \mathrm{diam}(B_{\omega^{(j)}}) = 0$. Hence, the set $G \cap J_\infty$ can be covered by canonical balls B_ω for which $|\omega| \geq n$. Let Γ denote such a cover of $G \cap J_\infty$. By the Besicovič Covering Theorem, there exists a universal constant $b(d) \geq 2$ such that Γ contains $b(d)/2$ subfamilies, each consisting of pairwise disjoint elements, such that G is contained in the union of all balls in these subfamilies. It follows that for at least one of these subfamilies, say Γ_0, we have that $m\left(\bigcup_{B_\omega\in\Gamma_0} B_\omega\right) \geq 2/b(d)m(G \cap J_\infty) = 2/b(d)m(G)$. Since there clearly exists a finite subset Γ_f of Γ_0 which has the property that $m\left(\bigcup_{B_\omega\in\Gamma_f} B_\omega\right) \geq \frac{1}{2}m\left(\bigcup_{B_\omega\in\Gamma_0} B_\omega\right)$, the statement of the lemma follows. ■

Proof of Theorem 4.1.
Our next step in the proof of the theorem is the construction of a Cantor set contained in J_i^δ. Crucial for this will be a certain increasing sequence $\{n_l\}_{l\geq 0}$ of non-negative integers, and

it will become clear during the construction how one has to choose this sequence. We begin with by defining for $l \geq 0$ the sets $I_l \subset I^*$ by induction as follows. Let $B_\emptyset := \overline{B}(x_i, R)$ and $I_0 := \{\emptyset\}$. Suppose that I_l has been defined, and let $\omega \in I_l$ be fixed. By Lemma 4.3 there exists a finite set ω^* consisting of hyperbolic words such that

$$\omega^* \subset \bigcup_{k \geq \max\{|\omega|+l, n_{l+1}\}} I^k$$

and the family $\{B_\tau\}_{\tau \in \omega^*}$ consists of pairwise disjoint balls such that $B_\tau \subset \mathrm{Int} B_\omega^\delta$ for every $\tau \in \omega^*$ (note that $\tau|_{|\omega|} = \omega$). In addition

$$m\left(\bigcup_{\tau \in \omega^*} B_\tau\right) \geq \frac{1}{b(d)} m(\mathrm{Int} B_\omega^\delta) \gg m(B_\omega^\delta). \tag{4.1}$$

Here, the latter inequality is a consequence of the conformality of m, Lemma 2.2 and Corollary 3.3. Put

$$I_{l+1} = \bigcup_{\omega \in I_l} \omega^*.$$

Now, let $\{F_l\}_{l \geq 1}$ denote the family of nested non-empty compact subsets of $B_\emptyset$ which is given by

$$F_l := \bigcap_{\omega \in I_l} B_\omega.$$

Note that we have in particular that

$$F = \bigcap_{l \geq 1} F_l \neq \emptyset.$$

Next, for each $l \geq 1$ we construct a Borel probability measure μ_l supported on the set F_{l-1} as follows. Let $\mu_1 := \frac{1}{m(B_\emptyset)} m|_{B_\emptyset}$, and assume that the measure μ_l has already been defined for some $l \geq 1$. Recall that $\omega^* := \{\tau \in I_{l+1} : \tau|_{n_l} = \omega\}$ for $\omega \in I_l$. Now, for each $\omega \in I_l$ and every Borel set $A \subset B_\omega$ we put

$$\mu_{l+1}(A) := \frac{\sum_{\tau \in \omega^*} m(A \cap B_\tau)}{\sum_{\tau \in \omega^*} m(B_\tau)}. \tag{4.2}$$

This defines a Borel probability measure μ_{l+1} on F_l which has the property that $\mu_{l+1}(B_\omega) = \mu_l(B_\omega)$ for every $\omega \in I_l$. A straighforward inductive argument gives that $\mu_q(B_\omega) = \mu_l(B_\omega)$ for every $q \geq l$. Also, since for each $\omega \in \bigcup_{l \geq 0} I_l$ the set $B_\omega \cap F$ is an open subsets of F , we conclude that the weak-limit $\mu := \lim_{l \to \infty} \mu_l$ exists and is supported on F, and that $\mu(B_\omega) = \mu_l(B_\omega)$ for each $l \geq 1$ and every $\omega \in I_l$. For $\omega \in I_l$ and $j \leq l$, let $k_j = k_j(\omega) \leq |\omega|$ denote the unique integer which is determined by $\omega|_{k_j} \in I_j$. Using (4.1) and (4.2), a straightforward inductive argument gives that for every $l \geq 1$ and every $\omega \in I_l$ we have that

$$
\begin{aligned}
\mu(B_\omega) = \mu_l(B_\omega) &= \prod_{j=1}^{l} \frac{m(B_{\omega|_{k_j}})}{\sum_{\tau \in \omega|_{k_{j-1}}^*} m(B_\tau)} = m(B_\omega) \prod_{j=1}^{l-1} \frac{m(B_{\omega|_{k_j}})}{\sum_{\tau \in \omega|_{k_j}^*} m(B_\tau)} \frac{1}{m(B(x_i, R))} \\
&= m(B_\omega) \prod_{j=1}^{l-1} \frac{m(B_{\omega|_{k_j}})}{m(B^\delta_{\omega|_{k_j}})} \exp(O(l)) \\
&= m(B_\omega) \prod_{j=1}^{l-1} \frac{|\phi'_{\omega|_{k_j}}(x_i)|^h}{|\phi'_{\omega|_{k_j}}(x_i)|^h m\left(B\Big(x_i, \big(R|\phi'_{\omega|_{k_j}}(x_i)|\big)^\delta\Big)\right)} \exp(O(l)) \\
&= m(B_\omega) \prod_{j=1}^{l-1} m\left(B\Big(x_i, \big(R|\phi'_{\omega|_{k_j}}(x_i)|\big)^\delta\Big)\right)^{-1} \exp(O(l)).
\end{aligned}
\tag{4.3}
$$

For every $\eta \in \bigcup_{j \geq l-1} I_j$ define

$$
\prod_{l-1}(\eta) := \prod_{j=1}^{l-1} m\left(B\Big(x_i, \big(R|\phi'_{\eta|_{k_j}}(x_i)|\big)^\delta\Big)\right)^{-1}.
$$

Since

$$
\lim_{n \to \infty} \sup\{|\phi'_\omega(x_i)| : \omega \in I^n\} = 0, \tag{4.4}
$$

it follows that there exists $n_0 \geq 1$ such that for each ω with $|\omega| \geq n_0$ we have that

$$
(R|\phi'_\omega(x_i)|)^{1+\delta} \leq \frac{1}{3} R|\phi'_\omega(x_i)|. \tag{4.5}
$$

Since the set I_l is finite, it follows from (4.4) that there exists a positive number $\tilde{R} \leq R$ such that if $\omega \in I_l$ and if $R|\phi'_\tau(x_i)| \leq \tilde{R}$ for some $\tau \in \omega^*$, then $|\omega| \geq n_0$. For fixed $z \in F$ and $0 < r \leq \tilde{R}/3$, consider the family $\mathcal{F}$ of all words $\omega \in \bigcup_{l \geq 0} I_{l+1}$ for which

$$
B_\omega \cap B(z, r) \cap J \neq \emptyset, \; R\,|\phi'_\omega(x_i)| < 3r, \; R\,|\phi'_{\omega|_{k_l}}(x_i)| \geq 3r. \tag{4.6}
$$

We shall now see that the family $\mathcal{F}_* = \{\omega|_{k_l} : \omega \in \mathcal{F}\}$ is a singleton, and that if this is the case with $\{\gamma\} = \mathcal{F}_*$, then it follows that

$$
B(z, r) \subset B_\gamma. \tag{4.7}
$$

For this, fix some element $\gamma \in \mathcal{F}_*$ and $\omega \in \mathcal{F}$ such that $\gamma = \omega|_{k_l}$ and such that $y \in B_\omega \cap B(z, r) \cap J$. Clearly, by construction of the set J, we have that $y \in B^\delta_\gamma$. From (4.5) and (4.6) we deduce that if $x \in B(z, r)$, then

$$
\begin{aligned}
||x - \phi_\gamma(x_i)|| &\leq ||x - z|| + ||z - y|| + ||y - \phi_\gamma(x_i)|| < r + r + (R|\phi'_\omega(x_i)|)^{1+\delta} \\
&\leq 2r + \frac{1}{3} R|\phi'_\omega(x_i)| \leq \frac{2}{3} R|\phi'_{\omega|_{k_l}}(x_i)| + \frac{1}{3} R|\phi'_{\omega|_{k_l}}(x_i)| = R|\phi'_\omega(x_i)| = R|\phi'_\gamma(x_i)|.
\end{aligned}
$$

Hence, we have now obtained that (4.7) holds and in particular, using (4.6) and the construction of the set F, that $\mathcal{F}_* = \{\gamma\}$.

Let $\epsilon > 0$ be fixed. Since $T_0 := \sup\{\prod(\tau) : \tau \in I_{l-1}\} < \infty$, we obtain for n_l sufficiently large and for all $\eta \in I_l$ that

$$T_0 \exp(O(l)) \leq |\phi'_\eta(x_i)|^{-\epsilon}.$$

Combining this estimate and (4.6), it follows that

$$\prod_{l-1}(\gamma) \exp(O(l)) \leq r^{-\epsilon}. \tag{4.8}$$

In order to complete the proof of Theorem 4.1, it is now sufficient to show that $\mu(B(z,r))$ can essentially be estimated from above by $r^{-2\epsilon} r^\theta$, for θ being either $\frac{h}{1+\delta}$ or $\frac{h+\delta p_i}{1+\delta(1+p_i)}$. We split this estimate into the following three different cases.

Case 1^0**:** $r \geq \left(R\,|\phi'_\gamma(x_i)|\right)^{1+\delta}$.

Using (4.7), (4.8) and the conformality of m, we obtain that

$$\begin{aligned}\mu(B(z,r)) &\leq \mu(B_\gamma) = \mu_l(B_\gamma) \leq m(B_\gamma) \prod_{l-1}(\gamma) \exp(O(l)) = |\phi'_\gamma(x_i)|^h \prod_{l-1}(\gamma) \exp(O(l)) \\ &\ll |\phi'_\gamma(x_i)|^h r^{-\epsilon} \ll r^{\frac{h}{1+\delta}-\epsilon},\end{aligned}$$

which completes the discussion for this case.

Before dealing with the remaining cases, note that, using (4.3), (4.6), (4.8) and Corollary 3.3, we have that

$$\begin{aligned}\mu(B(z,r)) &\leq \sum_{\omega\in\mathcal{F}} \mu(B_\omega) = \sum_{\omega\in\mathcal{F}} m(B_\omega) \prod_l(\omega) \exp(O(l)) \leq m(B(z,7r)) \prod_l(\omega) \exp(O(l)) \\ &= m(B(z,7r)) \prod_{l-1}(\omega) \exp\Big(O(l) \left(m\Big(B(x_i, (R|\phi'_\gamma(x_i)|)^\delta)\Big)\right)^{-1}\Big) \\ &\ll r^{-\epsilon}\,|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)}\, m(B(z,7r)).\end{aligned} \tag{4.9}$$

Case 2^0**:** $r \leq \left(R\,|\phi'_\gamma(x_i)|\right)^{1+\delta}$ and $r \geq K^2 R Q^{p_i+2}\left(R\,|\phi'_\gamma(x_i)|\right)^{1+\delta+\delta p_i}$.

From (4.7) and Lemma 2.2 we deduce that $r \leq R\,|\phi'_\gamma(x_i)| \leq KR\,|\phi'_{\tau|_n}(\pi(\sigma^n\tau))|$, where $z = \pi(\tau)$ and $\tau|_n = \gamma$. This implies that

$$r/K \leq R\,|\phi'_{\tau|_n}(\pi(\sigma^n\tau))| = R\,|\phi'_\gamma(\pi(\sigma^n\tau))|. \tag{4.10}$$

Now, since $z \in B_\gamma \cap J$, we have that $z \in B^\delta_\gamma$, and therefore that $\pi(\sigma^n\tau) \in B\Big(x_i, K(R\,|\phi'_\gamma(x_i)|)^\delta\Big)$. Let $\sigma^n\tau = i^q\omega$, with $\omega_1 \neq i$. By Proposition 3.1 (formula (3.2)), we have that $||\pi(\sigma^n\tau) - x_i|| \geq Q^{-1} q^{-\frac{1}{p_i}}$. Hence, using the fact that $Q^{-1} q^{-\frac{1}{p_i}} \leq K(R\,|\phi'_\gamma(x_i)|)^\delta$ and Proposition 3.1 (formula (3.1)), we obtain that

$$
\begin{aligned}
R\,|\phi'_{\tau|_{n+q+1}}(\pi(\sigma^{n+q+1}\tau))| &= R\,|\phi'_{\gamma}(\pi(\sigma^{n}\tau))|\cdot|\phi'_{i^q\omega_1}(\pi(\sigma\omega))| \\
&\leq R\,|\phi'_{\gamma}(\pi(\sigma^{n}\tau))|Qq^{-\frac{1}{p_i}} \\
&\leq KRQ^{p_i+2}\,(R|\phi'_{\gamma}(x_i)|)^{1+\delta(p_i+1)} \\
&\leq r/K.
\end{aligned} \tag{4.11}
$$

Hence, it follows that

$$
(r/K)^{*} = R|\phi'_{\gamma}(\pi(\sigma^{n}\tau))| \text{ and } (r/K)_{*} = R|\phi'_{\tau|_{n+q+1}}(\pi(\sigma^{n+q+1}\tau))|.
$$

Choose $\kappa > 0$ to be sufficiently small, which will be specified in the course of the proof. Without loss of generality we may assume that $z \notin J_i^{\delta+\kappa}$. Thus by choosing $r > 0$ to be sufficiently small, we can assume that $z \notin B_{\gamma}^{\delta+\kappa}$, and hence in particular that $\pi(\sigma^{n}\tau) \notin B\Big(x_i, K^{-1}(R|\phi'_{\gamma}(x_i)|)^{\delta+\kappa}\Big)$. Since by Proposition 3.1 (formula (3.2)), we have that $||\pi(\sigma^{n}\tau) - x_i|| \leq Qq^{-\frac{1}{p_i}}$, it follows that $Qq^{-\frac{1}{p_i}} \geq K^{-1}(R|\phi'_{\gamma}(x_i)|)^{\delta+\kappa}$. Hence, using Proposition 3.1 (formula (3.1)), we obtain that

$$
\begin{aligned}
R\,|\phi'_{\tau|_{n+q+1}}(\pi(\sigma^{n+q+1}\tau))| &= R\,|\phi'_{\gamma}(\pi(\sigma^{n}\tau))|\cdot|\phi'_{i^q\omega_1}(\pi(\sigma\omega))| \\
&\geq K^{-1}R\,|\phi'_{\gamma}(x_i)|Q^{-1}q^{-\frac{1}{p_i}} \\
&\geq (R(KQ)^{p_i+2})^{-1}(R\,|\phi'_{\gamma}(x_i)|)^{1+(\delta+\kappa)(p_i+1)}.
\end{aligned} \tag{4.12}
$$

Write $r = c(R|\phi'_{\gamma}(x_i)|)^{1+\delta+\eta}$, for $0 \leq \eta \leq \delta p_i$ and $1 \leq c \leq K^2RQ^{p_i+2}$. Suppose first that the first part of the global formula (Theorem 3.5) holds for the centre z and radius r/K. Using (4.12), we obtain that

$$
\begin{aligned}
cK^{-1}(R|\phi'_{\gamma}(x_i)|)^{1+\delta+\eta} &\geq R|\phi'_{\gamma}(\pi(\sigma^{n}\tau))|\left(\frac{R|\phi'_{\tau|_{n+q+1}}(\pi(\sigma^{n+q+1}\tau))|}{R|\phi'_{\gamma}(\pi(\sigma^{n}\tau))|}\right)^{\frac{1}{p_i+1}} \\
&\geq RK^{-1}|\phi'_{\gamma}(x_i)|\left((R(KQ)^{p_i+2})^{-1}\frac{(R|\phi'_{\gamma}(x_i)|)^{1+(\delta+\kappa)(p_i+1)}}{R|\phi'_{\gamma}(\pi(\sigma^{n}\tau))|}\right)^{\frac{1}{p_i+1}} \\
&\asymp |\phi'_{\gamma}(x_i)|^{1+\delta+\kappa}.
\end{aligned}
$$

Note that if $r>0$ is chosen to be sufficiently small (and hence the word length of γ is large), we have that $\eta \le 2\kappa$. Then, applying Theorem 3.5, (4.9) and Corollary 3.7, we obtain that

$$\begin{aligned}\mu(B(z,r)) &\ll r^{-\epsilon}|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)}m(B(z,r/K)) \asymp r^{-\epsilon}r^h|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)}\left(\frac{r/K}{(r/K)^*}\right)^{h-1}\\ &\asymp r^{-\epsilon}r^h|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)}\left(\frac{r}{|\phi'_\gamma(x_i)|}\right)^{(h-1)p_i}\\ &= r^{-\epsilon}r^{h+(h-1)p_i}|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)-(h-1)p_i}\\ &\asymp r^{-\epsilon}r^{h+(h-1)p_i}r^{\frac{-\delta(h+(h-1)p_i)-(h-1)p_i}{1+\delta+\eta}} = r^{-\epsilon}r^{\frac{hp_i\eta-p\eta+h+h\eta}{1+\delta+\eta}}.\end{aligned}$$

Note that we have

$$\frac{hp_i\eta-p\eta+h+h\eta}{1+\delta+\eta} \ge \frac{h}{1+\delta}-\epsilon \tag{4.13}$$

if and only if

$$\eta(hp_i-p_i+hp_i\delta-p_i\delta+h\delta) \ge -\epsilon(1+\delta+\eta).$$

Clearly, since $\eta \le 2\kappa$, the latter inequality is satisfied if we choose $\kappa>0$ to be sufficiently small. Hence, we can assume without loss of generality that (4.13) holds. It then follows that

$$\mu(B(z,r)) \le r^{\frac{h}{1+\delta}-2\epsilon},$$

which gives the Case 2^0 assuming the first part of the global formula.

Now suppose that the second part of the global formula (Theorem 3.5) holds for the centre $z=\pi(\tau)$ and radius r/K. Then (4.9), Corollary 3.7 and Theorem 3.5 imply that

$$\begin{aligned}\mu(B(z,r)) &\ll r^{-\epsilon}|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)}m(B(z,r/K)) \le r^{-\epsilon}|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)}r^h\left(\frac{(r/K)_*}{r}\right)^{h-1}\\ &\asymp r^{-\epsilon}r|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)p}|\phi'_{\tau|_{n+q+1}}(\pi(\sigma^{n+q+1}\tau))|^{h-1}.\end{aligned}\tag{4.14}$$

If $h\le 1$, then using (4.12), we can continue the estimate in this case as follows.

$$\begin{aligned}\mu(B(z,r)) &\ll r^{-\epsilon}r|\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)p_i}|\phi'_\gamma(x_i)|^{(h-1)(1+(\delta+\kappa)(p+1))}\\ &= r^{-\epsilon}r|\phi'_\gamma(x_i)|^{h+h\kappa p_i+h\kappa-\kappa p_i-1-\delta-\kappa}\\ &= r^{-\epsilon}r|\phi'_\gamma(x_i)|^{h-1-\delta+a\kappa},\end{aligned}$$

where we have set $a:=hp_i+h-p_i-1\le 0$. Hence, we have that

$$\mu(B(z,r)) \ll r^{-\epsilon}rr^{\frac{h-1-\delta+a\kappa}{1+\delta+\eta}} = r^{-\epsilon}r^{\frac{h+\eta+a\kappa}{1+\delta+\eta}} \le r^{-\epsilon}r^{\frac{h+a\kappa}{1+\delta}},$$

where in the last inequality we used the assumption that $h\le 1$. Now, by choosing $\kappa>0$ to be sufficiently small, it follows that

$$\mu(B(z,r)) \le r^{\frac{h}{1+\delta}-2\epsilon}.$$

This completes Case 2^0 for $h \le 1$.

If $h > 1$, then using (4.11), we can continue the estimate in (4.14) as follows.

$$\begin{aligned}\mu(B(z,r)) &\ll r^{-\epsilon} r |\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)} |\phi'_\gamma(x_i)|^{(h-1)(1+\delta(p+1))} \\ &= r^{-\epsilon} r |\phi'_\gamma(x_i)|^{h-1-\delta} = r^{-\epsilon} r r^{\frac{h-1-\delta}{1+\delta+\eta}} = r^{-\epsilon} r^{\frac{h+\eta}{1+\delta+\eta}} \\ &\le \begin{cases} r^{\frac{h}{1+\delta}-\epsilon} & \text{if } \delta \ge h-1 \\ r^{\frac{h+\delta p_i}{1+\delta(1+p_i)}} & \text{if } \delta \le h-1. \end{cases}\end{aligned}$$

Here, the latter inequality is obtained by using the facts that $\eta \le \delta p_i$ and that for $\delta \le h-1$ it holds that $\frac{h+\eta}{1+\delta+\eta}$ decreases if η increases.

Hence, the proof of Case 2^0 is complete.

Case 3^0: $r \le K^2 R Q^{p_i+2} \Big(R|\phi'_\gamma(x_i)|\Big)^{1+\delta+\delta p_i}$.

From (4.9) and Corollary 3.7 we deduce that

$$\begin{aligned}\mu(B(z,r)) &\ll r^{-\epsilon} |\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)} m(B(z,r/K)) = r^{-\epsilon} |\phi'_\gamma(x_i)|^{-\delta(h+(h-1)p_i)} r^h \zeta(z,r/K) \\ &\ll r^{-\epsilon} r^h r^{-\frac{\delta(h+(h-1)p_i)}{1+\delta+\delta p_i}} \zeta(z,r/K) \\ &= r^{\frac{h+\delta p_i}{1+\delta+\delta p_i}-\epsilon} \zeta(z,r/K).\end{aligned} \tag{4.15}$$

If $h \ge 1$, then we can apply Theorem 3.5, and we obtain that

$$\mu(B(z,r)) \ll r^{\frac{h+\delta p_i}{1+\delta+\delta p_i}-\epsilon}.$$

If $h \le 1$, we can assume that $i = i_{\max}$, which means that $p_i = \max\{p_j : j \in \Omega\}$. Let $k \ge 1$ be the index in the hyperbolic zoom associated with the point z and with the radius r/K. If $n_{k+1} = n_k + 1$, then we can proceed as in the previous case to obtain the desired result. Hence, suppose that $n_{k+1} \ne n_k + 1$. It follows that $\sigma^{n_k}\tau = j^u \tau_{n_k+1}$ for some $j \in \Omega$, $u \ge 1$ and $\tau_{n_k+1} \ne j$. Now, for $t \in [(r/K)_*, (r/K)^*]$ we write $\zeta(z,t) = t^{\alpha(t)}$. Then we have that

$$\alpha(t) = \frac{\log \zeta(z,t)}{\log t} = \begin{cases} p_j(h-1) + \frac{p_j(1-h)\log((r/K)^*)}{\log t} & \text{for } (r/K)^* \ge r \ge (r/K)^* \left(\frac{(r/K)_*}{(r/K)^*}\right)^{\frac{1}{p_j+1}} \\ 1-h+\frac{(h-1)\log((r/K)_*)}{\log t} & \text{for } (r/K)_* \le r \le (r/K)^* \left(\frac{(r/K)_*}{(r/K)^*}\right)^{\frac{1}{p_j+1}}. \end{cases}$$

¿From this we deduce that the function α takes on its minimum at $t = (r/K)^* \left(\frac{(r/K)_*}{(r/K)^*}\right)^{\frac{1}{p_j+1}}$. Therefore, we can assume without loss of generality that

$$(r/K) = (r/K)^* \left(\frac{(r/K)_*}{(r/K)^*}\right)^{\frac{1}{p_j+1}}. \tag{4.16}$$

Also, by choosing $\kappa > 0$ to be sufficiently small, we can assume that $z \notin J^{\delta+\kappa}$. For $r > 0$ small, we then have that $z \notin B^{\delta+\kappa}_{\tau|_k}(p_j)$. Now, by the same arguments as those leading to formula (4.12) in Case 2^0, we have that

$$(r/K)_* \geq (R(KQ)^{p_i+2})^{-1}((r/K)^*)^{1+(\delta+\kappa)(p_j+1)}. \tag{4.17}$$

Hence, Theorem 3.5 and (4.16) imply that

$$\zeta(z, r/K) \leq \left(\frac{(r/K)_*}{(r/K)^*}\right)^{\frac{(h-1)p_j}{p_j+1}}. \tag{4.18}$$

Write now

$$\left(\frac{(r/K)_*}{(r/K)^*}\right)^{\frac{(h-1)p_j}{p_j+1}} = (r/K)^\alpha = (r/K)^* \left(\left(\frac{(r/K)_*}{(r/K)^*}\right)^{\frac{1}{p_j+1}}\right)^\alpha$$

and for every $t \in (0,1)$ consider the number $\alpha(t)$ determined by the equation

$$\left(\frac{t}{(r/K)^*}\right)^{\frac{(h-1)p_j}{p_j+1}} = (r/K)^* \left(\left(\frac{t}{(r/K)^*}\right)^{\frac{1}{p_j+1}}\right)^{\alpha(t)}. \tag{4.19}$$

We are interested in a sufficiently good lower bound on $\alpha(r/K)$. And indeed, solving equation (4.19) for $\alpha(t)$, one easily deduces that the function $t \mapsto \alpha(t)$ is increasing throughout the entire interval $(0,1)$. Therefore, invoking (4.17), we may assume that

$$(r/K)_* = R(KQ)^{p_i+2})^{-1}((r/K)^*)^{1+(\delta+\kappa)(p_j+1)} \leq R(KQ)^{p_i+2})^{-1}((r/K)^*)^{1+\delta(p_j+1)}.$$

Combining this and (4.16), we obtain

$$(r/K) \ll (r/K)^*\Big((r/K)^*\Big)^\delta = \Big((r/K)^*\Big)^{1+\delta}.$$

Then by combining this, (4.18) and (4.17), we get

$$\zeta(z, r/K) \ll \Big((r/K)^*\Big)^{(\delta+\kappa)p_j(h-1)} \ll (r/K)^{\frac{p_j(h-1)(\delta+\kappa)}{1+\delta}}.$$

Substituting this latter inequality in (4.15), we obtain that

$$\mu(B(z,r)) \ll r^{\frac{h+\delta p_i}{1+\delta+\delta p_i} + \frac{\delta p_j(h-1)}{1+\delta}} r^{-\epsilon + \frac{\kappa p_j(1-h)}{1+\delta}}.$$

A straightforward calculation, using the facts that $p_i \geq p_j$ and $h \geq \frac{p_j}{p_j+1}$, shows that

$$\frac{h+\delta p_i}{1+\delta+\delta p_i} + \frac{\delta p_j(h-1)}{1+\delta} \geq \frac{h}{1+\delta}.$$

Hence, if κ is chosen to be sufficiently small, we finally obtain that

$$\mu(B(z,r)) \ll r^{\frac{h}{1+\delta} - \epsilon}.$$

■

4.2. **Limit laws for iterated function systems.** Let us define the set

$$I_* := \{i^n j : i \in \Omega, j \neq i, n \geq 1\} \cup (I \setminus \Omega).$$

Note that a word $\omega \in I^\infty$ can be written uniquely as an infinite word in elements from I_* if and only if ω is not of the form τi^∞ for any $i \in \Omega$ and $\tau \in I^*$. Let

$$\sigma^* : I_*^\infty \to I_*^\infty$$

denote the shift map on I_*^∞. Also, for $i \in \Omega$ and $\omega \in I_*^\infty$ define

$$Q_i(\omega) := \begin{cases} n & \text{if } \omega_1 = i^n j \text{ forsome} n \geq 1 \text{ and } j \neq i \\ 0 & \text{otherwise.} \end{cases}$$

In [MU1] we obtained that the iterated function system $S^* = \{\phi_\omega : \omega \in I^*\}$ is hyperbolic, and furthermore that S^* is regular if and only if S is regular. The shift map σ^* can be interpreted as the symbolic representation of the system S^*. As in the previous section, in this section we shall always assume that S is a tame parabolic finite iterated function system satisfying (SSOSC), and that m is the associated conformal measure for S. Clearly, m is also conformal for S^*. Hence, there exist Borel probability measures $\tilde{m}$ and μ^* on I_*^∞ which are equivalent to eachother (with uniformly bounded Radon-Nikodym derivatives) such that $m = \tilde{m} \circ \pi^{-1}$ and $\mu^* \circ (\sigma^*)^{-1} = \mu^*$ (see [MU1]). For $\epsilon \in I\!R$, $i \in \Omega$ and $n \geq 1$, we define

$$A_{i,n}(\epsilon) := \left\{ \omega \in I_*^\infty : Q_i(\omega) \geq n^{\frac{p_i}{h+(h-1)p_i} - \epsilon} \right\}$$

and

$$A_{i,\infty}(\epsilon) := \{\omega \in I_*^\infty : \sigma^{*n}(\omega) \in A_{i,n}(\epsilon) \text{ for infinitely many } n\}.$$

Lemma 4.4. *For $i \in \Omega$ and $\epsilon \in I\!R$ we have that $\tilde{m}(A_{i,\infty}(\epsilon)) > 0$ if and only if $\epsilon \geq 0$.*

Proof. Using the definition of $\tilde{m}$ and the conformality of m, we obtain

$$\begin{aligned} \sum_{n \geq 1} \mu^*((\sigma^*)^{-n}(A_{i,n}(\epsilon))) &= \sum_{n \geq 1} \mu^*(A_{i,n}(\epsilon)) = \sum_{n \geq 1} \tilde{m}(A_{i,n}(\epsilon)) \\ &= \sum_{n \geq 1} \sum_{k \geq n^{\frac{p_i}{h+(h-1)p_i} - \epsilon}} k^{-\frac{p_i+1}{p_i} h} \\ &\asymp \sum_{n \geq 1} n^{-1+\epsilon \frac{h+(h-1)p_i}{p_i}}. \end{aligned} \tag{4.20}$$

Since we have that $h + (h-1)p_i > 0$ (see [MU2]), it follows that the series

$$\sum_{n \geq 1} \mu^*((\sigma^*)^{-n}(A_{i,n}(\epsilon)))$$

converges for $\epsilon < 0$. Thus, the "weaker part of the Borel-Canteli lemma" gives that $\mu_*(A_{i,\infty}(\epsilon)) = 0$, which then implies that $\tilde{m}(A_{i,\infty}(\epsilon)) = 0$. This proves one direction of the equivalence in the lemma.

In order to prove the remaining part of the lemma, recall the following well-known result from elementary analysis.

- Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of events in a probability space (X, P). If $\sum_{n\in\mathbb{N}} P(X_n) = \infty$ and if $P(X_n \cap X_k) \ll P(X_n)\, P(X_k)$ for all distinct $n, k \in \mathbb{N}$, then $P\left(\limsup_{n\to\infty} X_n\right) \gg 1$.

By using once more formula (4.20), the 'if-part' of the lemma follows from this general result once we have shown that for all $n, k \in I\!N$ with $n > k$ we have that

$$\tilde{m}\left((\sigma^*)^{-k}(A_{i,k}(\epsilon)) \cap (\sigma^*)^{-n}(A_{i,n}(\epsilon))\right) \ll \tilde{m}\left((\sigma^*)^{-k}(A_{i,k}(\epsilon))\right)\, \tilde{m}\left((\sigma^*)^{-n}(A_{i,n}(\epsilon))\right).$$

Since μ^* and $\tilde{m}$ are equivalent, and since μ^* is σ^*-invariant, it follows that in order to obtain this latter inequality it is sufficient to show that

$$\tilde{m}\left(A_{i,k}(\epsilon) \cap (\sigma^*)^{-(n-k)}(A_{i,n}(\epsilon))\right) \ll \tilde{m}(A_{i,k}(\epsilon))\, \tilde{m}(A_{i,n}(\epsilon)).$$

Since the set $A_{i,k}(\epsilon)$ can be written as a union of S^*-cylinders of length 1, it can be written also as a union of cylinders of length $(n-k)$. If $A_{i,k}(\epsilon) = \bigcup B_k(\epsilon)$ denotes such a representation by cylinders of length $(n-k)$, then by the σ^*-invariance of μ^* and by the Bounded Distortion Property (7) and the conformality of m, we have for each $\omega \in A_{i,n}(\epsilon)$ and $\tau \in B_k(\epsilon)$ that

$$\begin{aligned}\tilde{m}\left((\sigma^*)^{-k}(A_{i,k}(\epsilon)) \cap (\sigma^*)^{-n}(A_{i,n}(\epsilon))\right) &\asymp \tilde{m}\left((\sigma^*)^{-(n-k)}(A_{i,n}(\epsilon)) \cap B_k(\epsilon)\right) \\ &\asymp |\phi'_\tau(\pi(\omega))|^h \tilde{m}(A_{i,n}(\epsilon)) \cap (\sigma^*)^{n-k}(B_k(\epsilon))).\end{aligned}$$

This implies that

$$\frac{\tilde{m}((\sigma^*)^{-(n-k)}(A_{i,n}(\epsilon)) \cap B_k(\epsilon))}{\tilde{m}(B_k(\epsilon))} \asymp \frac{|\phi'_\tau(\pi(\omega))|^h \tilde{m}(A_{i,n}(\epsilon))}{|\phi'_\tau(\pi(\omega))|^h} = \tilde{m}(A_{i,n}(\epsilon)),$$

or equivalently that

$$\tilde{m}((\sigma^*)^{-(n-k)}(A_{i,n}(\epsilon)) \cap B_k(\epsilon)) \asymp \tilde{m}(A_{i,n}(\epsilon))\, \tilde{m}(B_k(\epsilon)).$$

If in this latter inequality we sum up over all sets $B_k(\epsilon)$, then we obtain that

$$\tilde{m}((\sigma^*)^{-(n-k)}(A_{i,n}(\epsilon)) \cap A_{i,k}(\epsilon)) \asymp \tilde{m}(A_{i,n}(\epsilon))\, \tilde{m}(A_{i,k}(\epsilon)),$$

which in particular gives the desired inequality. ∎

Lemma 4.5. *For $i \in \Omega$ and $\epsilon \geq 0$ we have that $\tilde{m}(A_{i,\infty}(\epsilon)) = 1$.*

Proof. Let $i \in \Omega$ and $\epsilon > 0$ be fixed. Clearly, we have that $\sigma^*(A_{i,\infty}(\epsilon)) \subset A_{i,\infty}(\epsilon)$. Hence, using the ergodicity of the map σ^* and the previous lemma, the statement of the lemma follows. ∎

Theorem 4.6. (Limit Law (I)) *For $\tilde{m}$-almost every $\omega \in I_*^\infty$ and for all $i \in \Omega$ we have that*

$$\limsup_{n\to\infty} \frac{\log Q_i((\sigma^*)^n(\omega))}{\log n} = \frac{p_i}{h+(h-1)p_i}.$$

Proof. In order to obtain the lower bound for the 'lim sup' in the lemma, fix some $i \in \Omega$ and note that by Lemma 4.5 we have that $\tilde{m}(A_{i,\infty}(0)) = 1$. If $\omega \in A_{i,\infty}(0)$, then by definition, there exists a sequence $(k_j)_{j\in I\!N}$ of natural numbers k_j, such that $(\sigma^*)^{k_j}(\omega) \in A_{i,k_j}(0)$ for all $j \in \mathbb{N}$. This implies for all j that

$$Q_i((\sigma^*)^{k_j}(\omega)) \geq k_j^{p_i/(h+(h-1)p_i)},$$

and hence that

$$\limsup_{n\to\infty} \frac{\log Q_i((\sigma^*)^n(\omega))}{\log n} \geq \frac{p_i}{h+(h-1)p_i}.$$

In order to obtain the upper bound for the 'lim sup' in the lemma, let $\epsilon < 0$ and $i \in \Omega$. By Lemma 4.4, there exists a set $F_i(\epsilon)$ such that $\tilde{m}(F_i(\epsilon)) = 1$, and such that if $\omega \in F_i(\epsilon)$ then there exists a number $n_0 = n_0(\omega) \in I\!N$ with the property that $(\sigma^*)^n(\omega) \notin A_{i,n}(\epsilon)$ for all $n \geq n_0$. Hence, for $\omega \in F_i(\epsilon)$ we have for all $n \geq n_0$ that

$$\limsup_{n\to\infty} \frac{\log Q_i((\sigma^*)^n(\omega))}{\log n} \leq \frac{p_i}{h+(h-1)p_i} - \epsilon.$$

If we put $F_i = \bigcap_{n\geq 1} F_i(-\frac{1}{n})$, then $\tilde{m}(F_i) = 1$ and for each $\omega \in F_i$ we have that

$$\limsup_{n\to\infty} \frac{\log Q_i((\sigma^*)^n(\omega))}{\log n} \leq \frac{p_i}{h+(h-1)p_i}.$$

Hence, for $\omega \in A_{i,\infty}(0) \cap B_i$ we obtain the equality stated in the theorem. ∎

Note that if $Q_i(\omega) = n$, then it follows from (3.3) that $|x_i - \pi(\omega)| \asymp (n+1)^{-1/p_i}$. This now leads to our second limit law.

Theorem 4.7. (Limit Law (II)) *For $\tilde{m}$-almost every $\omega \in I_*^\infty$ we have for all $i \in \Omega$ that*

$$\limsup_{n\to\infty} \frac{-\log|(\sigma^*)^n(\omega) - x_i|}{\log n} = \frac{1}{h+(h-1)p_i}.$$

Proof. Fix $\omega \in I_*^\infty$ and $i \in \Omega$. By definition of Q_i and using (3.3), we have for $n \in I\!N$ that

$$|\pi((\sigma^*)^n(\omega)) - x_i| \asymp (Q_i((\sigma^*)^n(\omega)) + 1)^{-1/p_i}.$$

Hence, it follows that

$$\lim_{n\to\infty} \left| \frac{-\log|\pi((\sigma^*)^n(\omega)) - x_i|}{\log n} - \frac{\log Q_i((\sigma^*)^n(\omega))}{p_i\ \log n} \right| = 0.$$

Using Limit Law (I), we obtain for $\tilde{m}$-almost all $\omega \in I_*^\infty$ that

$$\begin{aligned}\limsup_{n\to\infty} \frac{-\log|\pi((\sigma^*)^n(\omega))-x_i|}{\log n} &= \frac{1}{p_i}\limsup_{n\to\infty}\frac{\log Q_i((\sigma^*)^n(\omega))}{\log n}\\ &= \frac{1}{h+(h-1)p_i}.\end{aligned}$$

■

Since $\tilde{m}$ is ergodic and positive on non-empty open sets, we have that $\tilde{m}$-almost every point in I_*^∞ has arbitrarily long blocks with parabolic entries only. Taking this observation into account, we now modify on a set of full measure the definition of the hyperbolic zoom $(r_j(\omega))_j$ as follows. For a given $i \in \Omega$ we include only those elements in the hyperbolic zoom for which $n_j(\omega) \geq n_{j-1}(\omega)+2$ and $i(\omega, j) = i$. With other words, we consider subsequences $(r_{j_k}(\omega))_k$ and $(n_{j_k}(\omega))_k$, such that $n_{j_k}(\omega) \geq n_{j_k-1}(\omega)+2$ and $\omega_{n_{j_k-1}(\omega)} = i$. A subsequence of this type will be referred to as the i-restricted hyperbolic zoom, and the i-restricted optimal sequence respectively.

Theorem 4.8. *(*Limit Law (III)*) For each $i \in \Omega$ the i-restricted optimal sequence at $\tilde{m}$-almost every $\omega \in I_*^\infty$ has the property that*

$$\limsup_{k\to\infty}\frac{\log(n_{j_k+1}(\omega)-n_{j_k}(\omega))}{\log j_k} = \frac{p_i}{h+(h-1)p_i}.$$

Proof. Let $i \in \Omega$ and $\omega \in I_*^\infty$. Define the function $N_n : I_*^\infty \to I\!N$ by $(\sigma^*)^n(\omega) = \sigma^{N_n(\omega)}(\omega)$, for every $n \geq 1$. Then we have by induction that $N_j(\omega) = n_j(\omega)$, for all $j \in I\!N$ (this follows, since $n_1(\omega) = N_1(\omega)$ and, assuming that $n_j(\omega) = N_j(\omega)$, since $n_{j+1}(\omega) = n_j(\omega) + N_1(\omega)(\sigma^{n_j(\omega)}(\omega)) = N_{j+1}(\omega)$).

Using Limit Law (II) and the fact that $|\pi(\sigma^{N_{j_k}(\omega)}(\omega)) - x_i| \asymp (N_{j_k+1}(\omega) - N_{j_k}(\omega))^{-1/p_i}$, it follows that for $\tilde{m}$-almost all ω we have that

$$\begin{aligned}\limsup_{k\to\infty}\frac{\log(n_{j_k+1}(\omega)-n_{j_k}(\omega))}{\log j_k} &= \limsup_{k\to\infty}\frac{\log(N_{j_k+1}(\omega)-N_{j_k}(\omega))}{\log j_k}\\ &= \limsup_{k\to\infty}\frac{-p_i\ \log|\pi(\sigma^{N_{j_k}(\omega)}(\omega))-x_i|}{\log j_k}\\ &= \limsup_{k\to\infty}\frac{-p_i\ \log|\pi((\sigma^*)^{j_k}(\omega))-x_i|}{\log j_k}\\ &= \frac{p_i}{h+(h-1)p_i}.\end{aligned}$$

■

Theorem 4.9. *(*Limit Law (IV)*) For each $i \in \Omega$ the i-restricted hyperbolic zoom at $\tilde{m}$-almost every $\omega \in I_*^\infty$ has the property that*

$$\limsup_{k\to\infty}\frac{\log(r_{j_k}(\omega)\,/\,r_{j_k+1}(\omega))}{\log j_k} = \frac{1+p_i}{h+(h-1)p_i}.$$

Proof. For $i \in \Omega$ and $\omega \in I_*^\infty$ we already saw in the proof of Theorem 3.5 that for $k \in \mathbb{N}$ we have that

$$\frac{r_{j_k}(\omega)}{r_{j_k+1}(\omega)} \asymp (n_{j_k+1}(\omega) - n_{j_k}(\omega))^{(1+p_i)/p_i}.$$

Combining this estimate and Limit Law (III), it follows for $\tilde{m}$-almost all $\omega \in I_*^\infty$ that

$$\begin{aligned}\limsup_{k\to\infty} \frac{\log\left(r_{j_k}(\omega)\,/\,r_{j_k}(\omega)\right)}{\log j_k} &= \limsup_{k\to\infty} \frac{1+p_i}{p_i}\,\frac{\log(n_{j_k+1}(\omega) - n_{j_k}(\omega))}{\log j_k} \\ &= \frac{1+p_i}{h+(h-1)p_i}.\end{aligned}$$

■

The following theorem presents the main results in this section.

Theorem 4.10. (The Khintchine Limit Law for parabolic iterated function systems) *The hyperbolic zoom at $\tilde{m}$-almost every $\omega \in I_*^\infty$ has the property that*

$$\limsup_{j\to\infty} \frac{\log\left(r_j(\omega)\,/\,r_{j+1}(\omega)\right)}{\log\log \frac{1}{r_j(\omega)}} = \frac{1+p_{\max}}{h+(h-1)p_{\max}},$$

where we have set $p_{\max} := \max\{p_i : i \in \Omega\}$.

Proof. Observe that for $\tilde{m}$-almost all $\omega \in I_*^\infty$ we have that

$$\begin{aligned}\lim_{j\to\infty} \frac{\log r_j(\omega)}{j} &= \lim_{j\to\infty} \frac{\log|\phi'_{n_j(\omega)}(\pi(\sigma^{n_j(\omega)}(\omega)))|}{j} \\ &= \lim_{j\to\infty} \frac{\log|\phi'_{N_j(\omega)}(\pi(\sigma^{N_j(\omega)}(\omega)))|}{j} \\ &= \lim_{j\to\infty} \frac{\log|\phi'_{N_j(\omega)}(\pi(\sigma^*)^j(\omega))|}{j} \\ &= \chi;\end{aligned}$$

where the latter equality follows from the Birkhoff Ergodic Theorem, using the facts that $(I_*^\infty, \sigma^*, \mu^*)$ is an ergodic system and that

$$\chi := \int_{I_*^\infty} \log|\phi'_{\omega_1}(\pi(\sigma^*)(\omega))|\, dm^*(\omega) > -\infty.$$

Hence, we have that

$$\lim_{j\to\infty} \frac{\log\log \frac{1}{r_j(\omega)}}{\log j} = 1.$$

Now the theorem follows by combining this equality and Limit Law (IV), and noting that

$$\max_{i\in\Omega} \frac{1+p_i}{h+(h-1)p_i} = \frac{1+p_{\max}}{h+(h-1)p_{\max}}.$$

■

Corollary 4.11. *For the function ζ of the h-conformal measure m (see Theorem 3.5) associated with a tame parabolic finite iterated function system satisfying (SSOSC) the following holds.*

(i): *For $h = 1$, we have for all $\omega \in I_*^\infty$ and $0 < r < \operatorname{diam}(I_*^\infty)$ that*

$$\zeta(\omega, r) \asymp 1.$$

(ii): *For $h < 1$, we have for $\tilde{m}$-almost every $\omega \in I_*^\infty$ that*

$$\limsup_{r\to 0} \frac{\log \zeta(\omega, r)}{\log\log \frac{1}{r}} = \frac{(1-h)p_{\max}}{h + (h-1)p_{\max}}.$$

(iii): *For $h > 1$, we have for $\tilde{m}$-almost every $\omega \in I_*^\infty$ that*

$$\liminf_{r\to 0} \frac{\log \zeta(\omega, r)}{\log\log \frac{1}{r}} = \frac{(1-h)p_{\max}}{h + (h-1)p_{\max}}.$$

Proof. The statement (i) of the corollary is an immediate consequence of Theorem 3.5. In order to prove the statement (ii), let $\omega \in I_*^\infty$ and $r > 0$ sufficiently small be given. Without loss of generality we may assume that $r_{j+1}(\omega) \le r < r_j(\omega)$ and that $\omega_{n_j(\omega)+1} = i$, for some $i \in \Omega$. For r in this range, an elementary calculation shows that the maximal value of $\zeta(\omega, r)$ is achieved if r is comparable to $r_{j,\max}(\omega) := r_j(\omega)\left(\frac{r_{j+1}(\omega)}{r_j(\omega)}\right)^{1/(1+p_i)}$. For this value of r we have that

$$\zeta(\omega, r_{j,\max}(\omega)) \asymp \left(\frac{r_j(\omega)}{r_{j+1}(\omega)}\right)^{(1-h)p_i/(1+p_i)}.$$

As we have seen above in the proof of the Khintchine law, for $\tilde{m}$-almost all $\omega \in I_*^\infty$ it is sufficient to restrict the discussion to those indices j for which $\omega_{n_j(\omega)} = i$, with $p_i = p_{\max}$. It follows that for all $\epsilon > 0$ and for m-almost all $\omega \in I_*^\infty$ we eventually have that

$$\frac{(1-\epsilon)(1+p_i)}{h+(h-1)p_i} \log\log \frac{1}{r_j(\omega)} \le_{i.o.} \log \frac{r_j(\omega)}{r_{j+1}(\omega)} \le \frac{(1+\epsilon)(1+p_i)}{h+(h-1)p_i} \log\log \frac{1}{r_j(\omega)}$$

(where '$\le_{i.o.}$' indicates that the inequality holds 'infinitely often', i.e. for some infinite subsequence $(r_{j_i}(\omega)/r_{j_i+1}(\omega))_i$). Hence, the above estimate implies that

$$\left(\log \frac{1}{r_j(\omega)}\right)^{(1-\epsilon)(1-h)p_{\max}/(h+(h-1)p_{\max})}$$
$$\ll_{i.o.} \zeta(\omega, r_{j,\max}(\omega)) \ll \left(\log \frac{1}{r_j(\omega)}\right)^{(1+\epsilon)(1-h)p_{\max}/(h+(h-1)p_{\max})}.$$

This proves the statement (ii) in the corollary. The statement (iii) follows from a similar argument, and we omit its proof. ■

We are now in the position to derive a refinement of the description of the geometric nature of the h-conformal measure given in Theorem 3.8 . Namely, using the latter corollary, we have the following statements concerning its relationship to the packing measure $\mathrm{P}_{\psi_\lambda}$ and

Hausdorff measure $\mathrm{H}_{\psi_\lambda}$ with respect to the dimension function ψ_λ. Here, the function ψ_λ is given for $\lambda \in I\!R$ and positive r by

$$\psi_\lambda(r) := r^h \left(\log \frac{1}{r}\right)^{(1+\lambda)(1-h)p_{\max}/(h+(h-1)p_{\max})}.$$

Corollary 4.12. *If S is a regular tame parabolic iterated function system satisfying (SSOSC), then we have the following table.*

λ vs. h	$h < 1$	$h > 1$
$\lambda > 0$	$m << \mathcal{H}_{\psi_\lambda}$ *and* $\mathcal{H}_{\psi_\lambda}(J) = \infty$	$\exists E_\lambda, m(E_\lambda) = 1$ *s.t.* $\mathcal{P}_{\psi_\lambda}(E_\lambda) = 0$
$\lambda \leq 0$	$\exists F_\lambda, m(F_\lambda) = 1$ *s.t.* $\mathcal{H}_{\psi_\lambda}(F_\lambda) = 0$	$m << \mathcal{P}_{\psi_\lambda}$ *and* $\mathcal{P}_{\psi_\lambda}(J) = \infty$

The symbol '<<' indicates absolute continuity between two measures.

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