November 10, 1999

PARABOLIC ITERATED FUNCTION SYSTEMS

by

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Abstract. In this paper we introduce and explore conformal parabolic iterated function systems. We define and study topological pressure, Perron-Frobenius type operators, semi-conformal and conformal measures and the Hausdorff dimension of the limit set. With every parabolic system we associate an infinite hyperbolic conformal iterated function system and we employ it to study geometric and dynamical features (properly defined invariant measures for example) of the limit set.

[1]Research partially supported by NSF Grant DMS 9801583

§**1. Introduction.** In [MU1] we considered conformal infinite iterated function systems and explored geometrical and dynamical properties of its limit set. That paper combined and extended three continuing lines of research. One is the study of an infinite system of similarity maps (*e.g.*, [MW]), the second is the study of a finite system of contracting conformal maps (*e.g.*, [Be], see also [Bo]) and third is the application of the jump transformation (*e.g.* [ADU], [Sch]). We now call the systems considered in [MU1] hyperbolic systems, since the derivatives of the maps in the system were required to be uniformly bounded below one. We continued our investigation of these systems in [MU2] and gave special attention to the limit sets of iterated function systems arising from the standard (real) continued fraction algorithm with restricted entries. In [MU3] our subject of interest was the residual set of the Apollonian packing where we first had to seriously cope with a parabolic system. In the present paper we develop the theory of general parabolic conformal iterated function systems S. Some of our techniques are similar to the methods developed to analyze jump transformations, see for examples, [Sch], [ADU], and [Y]. In section two, we define what it means for S to be parabolic and develop some basic results about its limit set and coding map. In section three, we define the pressure function associated with S and relate this notion to the standard notion of the pressure of a function. Also, in Theorem 3.5 we show how a basic geometric condition of our systems, the cone condition, plays a critical role in our development. This condition allows us to prove a measure theoretic open set condition holds for the image of any shift-invariant Borel measure on the coding space. This in turn allows us to obtain the Billingsley type formula for the Hausdorff dimension of many of these measures in Theorem 3.6. Finally, this allows us to determine some important parameters and features of the pressure function. In section four, we study Perron-Frobenius operators associated with the system S and the corresponding semiconformal measures, *i.e.*, eigen-measures for the dual operators. We also determine h, the Hausdorff dimension of the limit set of a parabolic iterated function system S. In contrast to the hyperbolic systems, there are t-conformal measures for $t > h$ for parabolic systems. In section five, we first describe the structure of these t-conformal measures with $t > h$. Then we associate with the system S an (always infinite) hyperbolic conformal system S^* whose limit set may differ from the limit set of the system S by at most a countable set. The interplay between the parabolic system and this more easily analyzed hyperbolic system is our main tool to study h-conformal measures for the original system S. We prove that if S^* is regular(or, euqivalently possesses an h-conformal measure), then there exists a unique h-conformal measure for S which is atomless. We also study invariant probability measures for S^* and invariant measures for S (which are σ-finite, but which may happen to be infinite) equivalent with the conformal measure for S. In particular we provide necessary and sufficient conditions for these invariant measures ofr the orginal system S to be finite. We also show that the h-dimensional Hausdorff measure of the limit set is always finite and that under the strong open set condition the h-dimensional packing measure is positive. In section six we give several illustrative examples. In particular, we return to the Apollonian packing the subject of [MU3]. Here we study the invariant measure equivalent with h-conformal measure (which is up to a multiplicative constant equal to the h-dimensional Hausdorff measure) showing that this measure is finite. Some of the arguments given in [MU3] which used the general theory presented here are completed. We would like to mention here that although in [MU3] we have considered a slightly different parabolic system and a different hyperbolic system derived from it, the results obtained in the present paper also apply to the setting of [MU3]. We end the paper with a class of one-dimensional examples.

§**2. Preliminaries.** Our setting is this. Let X be a compact connected subset of a Euclidean space $I\!R^d$. Suppose that we have countably many conformal maps $\phi_i : X \to X$, $i \in I$, where I has at least two elements satisfying the following conditions

(1) (Open Set Condition) $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$ for all $i \neq j$.

(2) $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic.

(3) $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then ϕ_ω extends conformally to an open connected set $V \subset I\!R^d$ and maps V into itself.

(4) If i is a parabolic index, then $\bigcap_{n\geq 0} \phi_{i^n}(X) = \{x_i\}$ and the diameters of the sets $\phi_{i^n}(X)$ converge to 0.

(5) (Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ $\forall x, y \in V$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \leq K.$$

(6) $\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.

(7) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.

(8) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi_i'(y)| - |\phi_i'(x)|\big| \leq L||\phi_i'|||y - x|^\alpha,$$

for every $i \in I$ and every pair of points $x, y \in V$.

We call such a system of maps $S = \{\phi_i : i \in I\}$ a subparabolic iterated function system. Let us note that conditions (1),(3),(5)-(7) are modeled on similar conditions which were used to examine hyperbolic conformal systems in [MU1]. Condition (8) also held for many of the systems studied in [MU1] but was not a general requirement. We need this condition in the sequel. If $\Omega \neq \emptyset$, we call the system $\{\phi_i : i \in I\}$ parabolic. As declared in (2) the elements of the set $I \setminus \Omega$ are called hyperbolic. We extend this name to all the words appearing in (5) and (6). By I^* we denote the set of all finite words with alphabet I and by I^∞ all infinite sequences with terms in I. It follows from (3) that for every hyperbolic word ω, $\phi_\omega(V) \subset V$. Note that our conditions insure that $\phi_i'(x) \neq 0$, for all i and $x \in V$. We provide below without proofs all the geometrical consequences of the bounded distortion property (5), abbreviated as (BDP), derived in [MU1] which remain true in our setting. We have for all hyperbolic words $\omega \in I^*$ and all convex subsets C of V

$$\text{(BDP1)} \qquad \mathrm{diam}(\phi_\omega(C)) \le ||\phi'_\omega||\mathrm{diam}(C)$$

and

$$\text{(BDP2)} \qquad \mathrm{diam}(\phi_\omega(V)) \le D||\phi'_\omega||,$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \ge 1$ is a universal constant. Moreover,

$$\text{(BDP3)} \qquad \mathrm{diam}(\phi_\omega(X)) \ge D^{-1}||\phi'_\omega||$$

and

$$\text{(BDP4)} \qquad \phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r),$$

for every $x \in X$, every $0 < r \le \mathrm{dist}(X, \partial V)$, and every hyperbolic word $\omega \in I^*$. Also, there exists $0 < \beta \le \alpha$ such that for all $x \in X$ and for all hyperbolic words $\omega \in I^*$

$$\text{(BDP5)} \qquad \phi_\omega(\mathrm{Int}(X)) \supset \mathrm{Con}\big(\phi_\omega(x), \beta, D^{-1}||\phi'_\omega||\big) \supset \mathrm{Con}\big(\phi_\omega(x), \beta, D^{-2}\mathrm{diam}\phi_\omega(V))\big)$$

where $\mathrm{Con}\big(\phi_\omega(x), \beta, D^{-1}||\phi'_\omega||\big)$ and $\mathrm{Con}\big(\phi_\omega(x), \beta, D^{-2}\mathrm{diam}(\phi_\omega(V))\big)$ denote some cones with vertices at $\phi_\omega(x)$, angles β, and altitudes $D^{-1}||\phi'_\omega||$ and $D^{-2}\mathrm{diam}(\phi_\omega(V))$ respectively. In addition, for every $\omega \in I^*$ (not necessarily hyperbolic) and every $x \in X$, there exists $l(\omega, x) > 0$ such that

$$\text{(BDP6)} \qquad \phi_\omega(\mathrm{Int}(X)) \supset \mathrm{Con}\big(\phi_\omega(x), \beta, l(\omega, x)\big)$$

Frequently, refering to (BDP) we will mean either (BDP) itself or one of the properties (BDP1)-(BDP6). The important point in (BDP6) is that by conformality we can get a cone with vertex x and opening angle β lying in $\phi_\omega(X)$, but we cannot say anything about the height of this cone unless ω is a hyperbolic word in which case we have (BDP5). For each $\omega \in I^* \cup I^\infty$, we define the length of ω by the uniquely determined relation $\omega \in I^{|\omega|}$. If $\omega \in I^* \cup I^\infty$ and $n \le |\omega|$, then by $\omega|_n$ we denote the word $\omega_1\omega_2\ldots\omega_n$. Our first aim in this section is to prove the existence of the limit set. More precisely, we begin with the following lemma.

Lemma 2.1. For all $\omega \in I^\infty$ the intersection $\bigcap_{n\ge0}\phi_{\omega|_n}(X)$ is a singleton.

Proof. Since the sets $\phi_{\omega|_n}(X)$ form a nested sequence of compact sets, the intersection $\bigcap_{n\ge0}\phi_{\omega|_n}(X)$ is not empty. Moreover, it follows from (4) that if ω is of the form τi^∞, $\tau \in I^*, i \in \Omega$, then the diameters of the intersection $\bigcap_{n=0}^k \phi_{\omega|_n}(X)$ tend to 0 and, in the other case, the same conclusion follows immediately from (6). In any case, $\bigcap_{n\ge0}\phi_{\omega|_n}(X)$ is a singleton and we are done. ■

Improving a little bit the argument just given, we have the following.

Lemma 2.2. $\lim_{n\to\infty}\sup_{|\omega|=n}\{\mathrm{diam}(\phi_\omega(X))\} = 0.$

Proof. Let $g(n) = \max_{i\in\Omega}\{\mathrm{diam}(\phi_{i^n}(X)\}$. Since Ω is finite it follows from (4) that $\lim_{n\to\infty} g(n) = 0$. Let $\omega \in I^\infty$. Given $n \geq 0$ consider the word $\omega|_n$. Look at the longest block of the same parabolic element appearing in $\omega|_n$. If the length of this block exceeds $\sqrt{n}$ then, since due to (2) all the maps ϕ_j, $j \in I$, are Lipschitz continuous with a Lipschitz constant ≤ 1, we have $\mathrm{diam}(\phi_{\omega|n}(X)) \leq g(\sqrt{n})$. Otherwise, we can find in $\omega|_n$ at least $\frac{n-\sqrt{n}}{\sqrt{n}} = \sqrt{n} - 1$ distinct hyperbolic indices. It then follows from (6) (and Lipschitz continuity with a Lipschitz constant ≤ 1 of all the maps ϕ_i, $i \in I$) that $\mathrm{diam}(\phi_{\omega|n}(X)) \leq s^{\sqrt{n}-1}$. The proof is finished. ■

We introduce on I^∞ the standard metric $d(\omega,\tau) = \mathrm{e}^{-n}$, where n is the largest number such that $\omega|_n = \tau|_n$. The corollary below is now an immediate consequence of Lemma 2.2.

Corollary 2.3. The map $\pi : I^\infty \to X$, $\pi(\omega) = \bigcap_{n\geq 0} \phi_{\omega|_n}(X)$, is uniformly continuous.

The limit set $J = J_S$ of the system $S = \{\phi_i\}_{i\in I}$ satisfies

$$J = \pi(I^\infty) = \cup_{i\in I}\phi_i(J).$$

We recall the set J is not compact if the index set I is infinite.

Lemma 2.4. If X is a topological disk contained in $\mathbb{C}$, then every parabolic point lies on the boundary of X.

Proof. Suppose on the contrary that a parabolic point $x_i \in \mathrm{Int}(X)$. Let $D^1 = \{z \in \mathbb{C} : |z| < 1\}$ and let $R : D^1 \to \mathrm{Int}(X)$ be the Riemann map (conformal homeomorphism) such that $R(0) = x_i$. Consider the composition $R^{-1}\circ\phi_i\circ R : D^1 \to D^1$. Then $|(R^{-1}\circ\phi_i\circ R)'(0)| = |R'(0)|^{-1}|R'(0)| = 1$. Thus by Schwarz's lemma $R^{-1}\circ\phi_i\circ R$ is a rotation. Since $\phi_i = R\circ(R^{-1}\circ\phi_i\circ R)\circ R^{-1}$, it follows that $\phi_i(X) = R\circ(R^{-1}\circ\phi_i\circ R)\circ R^{-1}(X) = X$. This contradiction finishes the proof. ■

§3. Topological pressure and associated parameters.

Given a set $F \subset I$ and a function $f : F^\infty \to \mathbb{R}$ we define the topological pressure of f with respect to the shift map $\sigma : F^\infty \to F^\infty$ to be

$$\mathrm{P}_F(\sigma, f) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\omega\in F^n}\exp(\sup_{\tau\in[\omega]}\sum_{j=0}^{n-1} f(\sigma^j(\tau))\right),$$

where $[\omega] = \{\tau \in F^\infty : \tau|_{|\omega|} = \omega\}$. Since the sequence

$$n \mapsto \log\left(\sum_{\omega\in F^n}\exp(\sup_{\tau\in[\omega]}\sum_{j=0}^{n-1} f(\sigma^j(\tau))\right)$$

is subadditive, the limit exists. If $F = I$, we suppress the subscript F and write simply $\mathrm{P}(\sigma, f)$ for $\mathrm{P}_I(\sigma, f)$.

We call $f : I^\infty \to I\!\!R$ acceptable if it is uniformly continuous and $\max_{i\in I}\{\sup(f|_{[i]}) - \inf(f|_{[i]})\} < \infty$. We shall prove the following.

Theorem 3.1. If $f : I^\infty \to I\!\!R$ is acceptable, then

$$\mathrm{P}(\sigma, f) = \sup\{P_F(\sigma, f)\},$$

where the supremum is taken over all finite subsets F of I.

Proof. The inequality $\mathrm{P}(\sigma, f) \geq \sup\{P_F(\sigma, f)\}$ is obvious. To prove the converse suppose first that $\mathrm{P}(\sigma, f) < \infty$. Fix $\varepsilon > 0$. By the acceptability of f, there exists $l \geq 1$ such that $|f(\omega) - f(\tau)| < \varepsilon$, if $\omega|_l = \tau|_l$ and $M = \max_{i\in I}\{\sup(f|_{[i]}) - \inf(f|_{[i]})\} < \infty$. Now, fix $k \geq l$. By subadditivity,

$$\frac{1}{k}\log\left(\sum_{|\omega|=k}\exp(\sup_{\tau\in[\omega]}\sum_{j=0}^{k-1} f(\sigma^j(\tau)))\right) \geq \mathrm{P}(f).$$

For each $F \subset I$ and $m \in N$, set

$$\Gamma_m(F, f) = \sum_{\omega\in F^m}\exp(\sup_{\tau\in[\omega]}\sum_{j=0}^{m-1} f(\sigma^j(\tau)),$$

So, there exists $q \geq 1$ so large that writing $F = \{1, 2 \ldots, q\}$ we have $\frac{1}{k}\log\Gamma_k(F, f) \geq \mathrm{P}(f) - \varepsilon$. Put

$$\overline{f} = \sum_{j=0}^{k-1} f \circ \sigma^j.$$

Then for every $n \geq 1$, we have

$$\begin{aligned}\Gamma_{kn}(F, f) &= \sum_{\omega\in F^{kn}}\exp\left(\sup_{\tau\in[\omega]}\sum_{j=0}^{n-1}\overline{f}\circ\sigma^{kj}(\tau)\right) \geq \sum_{\omega\in F^{kn}}\exp\left(\sum_{j=0}^{n-1}\inf\big(\overline{f}|_{[\sigma^{kj}\omega]}\big)\right)\\ &\geq \sum_{\omega\in F^{kn}}\exp\left(\sum_{j=0}^{n-1}\inf\big(\overline{f}|_{[\sigma^{kj}\omega|_k]}\big)\right).\end{aligned}$$

But, if $j \leq n-1$, then $\inf\big(\overline{f}|_{[\sigma^{kj}\omega|_k]}\big) \geq \sup\big(\overline{f}|_{[\sigma^{jk}\omega|_k]}\big) - \varepsilon(k-l) - Ml$. Hence,

$$\begin{aligned}\Gamma_{kn}(F, f) &\geq \sum_{\omega\in F^{kn}}\exp\left(\sum_{j=0}^{n-1}\sup\big(\overline{f}|_{[\sigma^{kj}\omega|_k]}\big) - (k-l)\varepsilon - Ml\right)\\ &= \mathrm{e}^{-\varepsilon(k-l)n - Mln}\sum_{\omega\in F^{kn}}\exp\left(\sum_{j=0}^{n-1}\sup\big(\overline{f}|_{[\sigma^{kj}\omega|_k]}\big)\right)\\ &= \left(\mathrm{e}^{-\varepsilon(k-l) - Ml}\sum_{\tau\in F^k}\exp\big(\sup\big(\overline{f}|_{[\tau]}\big)\big)\right)^n.\end{aligned}$$

Therefore,

$$\mathrm{P}_F(\sigma, f) \geq \lim_{n\to\infty} \frac{1}{kn} \log \Gamma_{kn}(F, f) \geq \frac{-\varepsilon(k-l)}{k} - \frac{Ml}{k} + \mathrm{P}(f) - \varepsilon \geq \mathrm{P}(f) - 3\varepsilon,$$

provided k is large enough. Thus, letting $\varepsilon \searrow 0$, the theorem follows. The case $\mathrm{P}(\sigma, f) = \infty$ can be treated similarly. ■

Looking at this theorem we should notice that our definition of pressure coincides with a more complicated one given in [Sa] although we will not use this information in our paper. We say a σ-invariant Borel probability measure μ on I^∞ is finitely supported provided there exists a finite set $F \subset I$ such that $\mu(F^\infty) = 1$. The well-known variational principle (see [Wa], *cf.* [PU]) tells us that for every finite set $F \subset I$

$$\mathrm{P}_F(f) = \sup\{\mathrm{h}_\mu(\sigma) + \int f d\mu\},$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures μ with $\mu(F^\infty) = 1$. Applying Theorem 3.1, we therefore get the following.

Theorem 3.2. If $f : I^\infty \to I\!R$ is acceptable, then

$$\mathrm{P}(\sigma, f) = \sup\{\mathrm{h}_\mu(\sigma) + \int f d\mu\},$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures μ which are finitely supported.

Our next goal is to relate the pressure of the "volume potential" function g to the way pressure is defined in [MU1]. We consider the function $g : I^\infty \to I\!R$ given by the formula

$$g(\omega) = \log |\phi'_{\omega_1}(\pi(\sigma(\omega)))|.$$

Using heavily condition (8), we shall prove the following.

Proposition 3.3. The function g defined above is acceptable.

Proof. Fix $n \geq 1$ and $\omega, \tau \in I^\infty$ such that $\omega|_n = \tau|_n$. It then follows from (8) that

$$\begin{aligned}
|g(\omega) - g(\tau)| &= \big|\log |\phi'_{\omega_1}(\pi(\sigma(\omega)))| - \log |\phi'_{\omega_1}(\pi(\sigma(\tau)))|\big| \\
&\leq \frac{\big||\phi'_{\omega_1}(\pi(\sigma(\omega)))| - |\phi'_{\omega_1}(\pi(\sigma(\tau)))|\big|}{\min\{|\phi'_{\omega_1}(\pi(\sigma(\omega)))|, |\phi'_{\omega_1}(\pi(\sigma(\omega)))|\}} \\
&\leq L \frac{||\phi'_{\omega_1}||}{\min\{|\phi'_{\omega_1}(\pi(\sigma(\omega)))|, |\phi'_{\omega_1}(\pi(\sigma(\omega)))|\}} |\pi(\sigma(\omega)) - \pi(\sigma(\tau))|^\alpha.
\end{aligned}$$

If ω_1 is a hyperbolic index, then using the bounded distortion property, we get

$$|g(\omega) - g(\tau)| \leq LK|\pi(\sigma(\omega)) - \pi(\sigma(\tau))|^\alpha.$$

On the other hand, since there are only finitely many parabolic indices, there is a positive constant M such that if ω_1 is parabolic, then

$$|g(\omega) - g(\tau)| \le LM|\pi(\sigma(\omega)) - \pi(\sigma(\tau))|^\alpha.$$

Let $L' = L\max\{K, M\}$. Since X being compact is bounded, taking $n = 1$, it follows from the last inequalities that $\max_{i\in I}\{\sup\big(g|_{[i]}\big) - \inf\big(g|_{[i]}\big)\} \le L'\mathrm{diam}^\alpha(X) < \infty$. The uniform continuity of g follows from inequality $|g(\omega) - g(\tau)| \le L'|\pi(\sigma(\omega)) - \pi(\sigma(\tau))|^\alpha$ and Corollary 2.3. The proof is complete. ■

In [MU1], for each $t \ge 0$ we have defined $\mathrm{P}(t)$ by the formula

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} ||\phi'_\omega||^t,$$

where $||\phi'_\omega|| = \sup\{|\phi'_\omega(x)| : x \in X\}$. This analytic definition was used simply because many dynamic and geometric facts about the limit set can be garnered from its basic properties as a real function. Similarly, we define for each $W \subset X$

$$\mathrm{P}_W(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{|\omega|=n} ||\phi'_\omega||_W^t,$$

where $||\phi'_\omega||_W = \sup\{|\phi'_\omega(x)| : x \in W\}$. Let us note that

$$\mathrm{P}_W(t) = \inf\{s : \sum_{n\ge1}\sum_{|\omega|=n} ||\phi'_\omega||_W^t e^{-sn} < \infty\}.$$

We now introduce some notation. For each $i \in \Omega$, let $I_{gi}^p = \{\omega \in I^p : \omega_p \ne i\}$.

Lemma 3.4. $\mathrm{P}(\sigma, tg) = \mathrm{P}(t)$.

Proof. First, we show $\mathrm{P}(t) = \mathrm{P}_J(t)$. Clearly, $\mathrm{P}_J(t) \le \mathrm{P}(t)$. To prove the converse inequality, suppose $\mathrm{P}_J(t) < s$. Then using (5)

$$\begin{aligned}
&\sum_{n\ge1}\sum_{|\omega|=n} ||\phi'_\omega||^t e^{-sn} = \\
&\quad = \sum_{n\ge1}\sum_{|\omega|=n,\omega_n\notin\Omega} ||\phi'_\omega||^t e^{-sn} + \sum_{n\ge1}\sum_{i\in\Omega}\sum_{k=1}^{n}\sum_{\omega\in I_{gi}^{n-k}} ||\phi'_{\omega i^k}||^t e^{-sn} \\
&\quad \le K^t \sum_{n\ge1}\sum_{|\omega|=n,\omega_n\notin\Omega} ||\phi'_\omega||_J^t e^{-sn} + K^t \sum_{n\ge1}\sum_{i\in\Omega}\sum_{k=1}^{n}\sum_{\omega\in I_{gi}^{n-k}} ||\phi'_{\omega i}||_J^t e^{-sn} \\
&\quad \le K^t \sum_{n\ge1}\sum_{|\omega|=n,\omega_n\notin\Omega} ||\phi'_\omega||_J^t e^{-sn} + K^t \sum_{n\ge1}\sum_{i\in\Omega}\sum_{k=1}^{n}\sum_{\omega\in I_{gi}^{n-k}} ||\phi'_{\omega i}||_J^t e^{-s(n-k+1)} \\
&\quad \le K^t \sum_{n\ge1}\sum_{|\omega|=n} ||\phi'_\omega||_J^t e^{-sn} < \infty.
\end{aligned}$$

So, $\mathrm{P}(t) \leq s$ and consequently $\mathrm{P}(t) \leq \mathrm{P}_J(t)$. Next, we compute

$$\begin{aligned}
\mathrm{P}(\sigma, tg) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\big(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} tg(\sigma^j(\tau))\big) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\big(\sup_{\tau\in[\omega]} \sum_{j=1}^{n} t\log|\phi'_{\omega_j}(\pi(\sigma^j(\tau)))|\big) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \sup_{\tau\in[\omega]} |\phi'_\omega(\pi(\sigma^n\tau))|^t = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} ||\phi'_\omega||_J = \mathrm{P}_J(t) = \mathrm{P}(t).
\end{aligned}$$

The proof is complete. ■

Our next goal is to prove that the image of all shift-invariant measures satisfies a measure theoretic open set condition. At this point we make essential use of the cone condition.

Theorem 3.5. If μ is a shift-invariant Borel probability measure on I^∞, then

$$\mu \circ \pi^{-1}(\phi_\tau(X) \cap \phi_\rho(X)) = 0$$

for all incomparable words $\tau, \rho \in I^*$.

Proof. We begin by describing a situation which we show to be impossible. Suppose that there exist a point $x \in X$, an integer $q \geq 1$ and an increasing sequence $\{n_k\}_{k\geq 1}$ of positive integers along with pairwise different words $\rho^{(k)}, \tau^{(k)} \in E^{n_k}$ such that

$$x \in \phi_{\tau^{(k)}}(X) \cap \phi_{\rho^{(k)}}(X)$$

and $\rho^{(k)}|_{n_k-q} = \tau^{(k)}|_{n_k-q}$. Passing to a subsequence we may assume that $n_{k+1} - n_k > q$ for every $k \geq 1$.We shall construct by induction with respect to $k \geq 1$ a sequence $\{C_k\}_{k\geq 1}$ such that

(a) For every k, C_k consists of at least $k+1$ incomparable words from $\{\rho^{(j)}, \tau^{(j)} : j \leq k\}$

Set $C_1 = \{\rho^{(1)}, \tau^{(1)}\}$. Suppose now that C_k has been defined. If $\rho^{(k+1)}$ does not extend any word in C_k, then we form C_{k+1} by adding $\rho^{(k+1)}$ to C_k. We can do a similar thing in case $\tau^{(k+1)}$ does not extend any word in C_k. If, on the other hand, $\rho^{(k+1)}$ extends some word κ in C_k and $\tau^{(k+1)}$ extends a word η in C_k, then κ and η are both extended by ρ^{k+1} since for $j \leq k$, $|\rho^{(j)}|, |\tau^{(j)}| \leq n_j \leq n_k$, $n_{k+1} > n_k + q$, and $\rho^{(k+1)}|_{n_{k+1}-q} = \tau^{(k+1)}|_{n_{k+1}-q}$. Since the words in C_k are incomparable, $\kappa = \eta$ and this is the only word in C_k which is extended by both $\rho^{(k+1)}$ and $\tau^{(k+1)}$. In this case we form C_{k+1} by taking away κ and adding both $\rho^{(k+1)}$ and $\tau^{(k+1)}$. Now the sets $\{\phi_\kappa(X) : \kappa \in C_k\}$ are nonoverlapping since the words are incomparable. By (BDP6) we get $k+1$ pairwise disjoint open cones each with vetex x and opening angle β. This is clearly impossible if k is large enough. So it is impossible to have such a point x.

Let μ be a shift-invariant probability measure and suppose that $\mu\circ\pi^{-1}(\phi_\tau(X)\cap\phi_\rho(X))>0$ for some incomparable words $\tau,\rho\in I^*$. Without loss of generality we may assume that $|\tau|=|\rho|$. Put $E=\phi_\tau(X)\cap\phi_\rho(X)$ and set

$$E_\infty=\bigcap_{k=0}^\infty\bigcup_{n=k}^\infty\bigcup_{|\omega|=n}\phi_\omega(E).$$

In view of what we have just proved $E_\infty=\emptyset$. On the other hand, by (BDP6) for every $n\ge 0$, each element of X belongs to at most $1/\beta$ elements of $\phi_\omega(X)$, $\omega\in I^n$, (we assume that $\lambda_{d-1}(S^{d-1})=1$). Using this and σ-invariance of measure μ, we get for every $n\ge 0$

$$\begin{aligned}\mu\circ\pi^{-1}\left(\bigcup_{|\omega|=n}\phi_\omega(E)\right)&\ge\beta^{-1}\sum_{|\omega|=n}\mu\circ\pi^{-1}\big(\phi_\omega(E)\big)\ge\beta^{-1}\sum_{|\omega|=n}\mu\big(\omega\pi^{-1}(E)\big)\\&=\beta^{-1}\mu\circ\pi^{-1}(E),\end{aligned}$$

where $\omega A=\{\omega\kappa:\kappa\in A\}$ for every set $A\subset I^\infty$. Hence, for every $k\ge 0$, $\mu\circ\pi^{-1}\left(\bigcup_{n=k}^\infty\bigcup_{|\omega|=n}\phi_\omega(E)\right)\ge\beta^{-1}\mu\circ\pi^{-1}(E)$, and therefore $\mu\circ\pi^{-1}(E_\infty)\ge\beta^{-1}\mu\circ\pi^{-1}(E)>0$. This contradiction finishes the proof. ■

Let

$$\theta=\theta(S)=\inf\{t\ge 0:\mathrm{P}(t)<\infty\}.$$

Following [MU1] we call θ the finiteness parameter of the system S. If μ is a Borel probability measure supported on X, we denote the Hausdorff dimension of μ by $\mathrm{HD}(\mu)$, the infimum of the Hausdorff dimensions of sets with μ measure 1. Let $\alpha=\{[i]:i\in I\}$ be the partition of I^∞ into initial cylinders of length 1. We let $\mathrm{H}_\mu(\alpha)$ denote the entropy of the partition α with respect to μ. Taking into account Theorem 3.5, the following theorem relating the Hausdorff dimension of a measure and the ratio of the entropy to the Lyuponov exponent was proven for hyperbolic systems in [HMU] and [U2].

Theorem 3.6. If μ is a shift-invariant ergodic Borel probability measure on I^∞ such that $\mathrm{H}_\mu(\alpha)<\infty$, $\chi_\mu(\sigma)=\int -g\,d\mu<\infty$ and either $\chi_\mu(\sigma)>0$ or $\mathrm{h}_\mu(\sigma)>0$ ($\mathrm{h}_\mu(\sigma)>0$ implies $\chi_\mu(\sigma)>0$), then

$$\mathrm{HD}(\mu\circ\pi^{-1})\le\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}.$$

If additionally, $\mu\circ\pi^{-1}(\phi_i(X)\cap\phi_j(X))=0$ for all $i\ne j\in I$ then

$$\mathrm{HD}(\mu\circ\pi^{-1})=\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}.$$

The same proof goes through in our case replacing only the bounded distortion property by the consequence of (8) which says that $\frac{|\phi_i'(y)|}{|\phi_i'(x)|}\le\exp\big(L|y-x|^\alpha\big)$ for all $i\in I$ and all $x,y\in V$. Let

$$\beta=\beta(S)=\sup\{\mathrm{HD}(\mu\circ\pi^{-1})\},$$

where the supremum is taken over all ergodic finitely supported (so shift-invariant) measures of positive entropy. Of course there are many such measures. We shall prove the following.

Proposition 3.7. The pressure function $\mathrm{P}(t)$ has the following properties:

(1) $\mathrm{P}(t) \geq 0$ for all $t \geq 0$

(2) $\mathrm{P}(t) > 0$ for all $0 \leq t < \beta$.

(3) $\mathrm{P}(t) = 0$ for all $t \geq \beta$.

(4) $\mathrm{P}(t)$ is non-increasing.

(5) $\mathrm{P}(t)$ is strictly decreasing on $[\theta, \beta]$.

(6) $\mathrm{P}(t)$ is continuous and convex on (θ, ∞).

Proof. (1). Let i be a parabolic index and let x_i be the corresponding parabolic point. Then $\pi(i^\infty) = x_i$ and let μ be the Dirac measure supported on i^∞. Of course, μ is ergodic, finitely supported, and $\int tg\, d\mu = t \log |\phi_i'(x_i)| = 0$. Hence, by Theorem 3.2 and Proposition 3.3, $P(\sigma, tg) \geq \mathrm{h}_\mu(\sigma) + \int tg d\mu = 0$ and (1) is proved.

(2). Suppose that $0 \leq t < \beta$. Then there exists an ergodic, σ-invariant finitely supported, measure μ of positive entropy such that $\mathrm{HD}(\mu \circ \pi^{-1}) > t$. So, Theorem 3.6 applies to give $t < \mathrm{HD}(\mu \circ \pi^{-1}) \leq \mathrm{h}_\mu(\sigma)/\chi_\mu(\sigma)$ which due to Theorem 3.2 and Proposition 3.3 implies that $\mathrm{P}(\sigma, tg) \geq \mathrm{h}_\mu(\sigma) + \int tg\, d\mu > 0$.

(3). Suppose that $\mathrm{P}(t) > 0$ for some $t \geq 0$. Then in view of Theorem 3.2 and Proposition 3.3 there exists an ergodic σ-invariant finitely supported measure μ such that $\mathrm{h}_\mu(\sigma) - t\chi_\mu(\sigma) > 0$. Therefore $\mathrm{h}_\mu(\sigma) > 0$ and hence, by Theorem 3.5, $t < \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)} = \mathrm{HD}(\mu \circ \pi^{-1}) \leq \beta$. We are done.

(4). Suppose that $t_1 < t_2$. It is clear from the definition of pressure that $\mathrm{P}(t_2) = \infty$ implies $\mathrm{P}(t_1) = \infty$. So, we may assume $\theta \leq t_1 < t_2$. Fix $\varepsilon > 0$. By Theorem 3.2 and Proposition 3.3 there exists an ergodic finitely supported measure μ_2 such that $\mathrm{h}_{\mu_2}(\sigma) + \int t_2 g d\mu_2 \geq \mathrm{P}(\sigma, t_2 g) - \varepsilon$. Then by Theorem 3.2 and Proposition 3.3, $\mathrm{P}(\sigma, t_1 g) \geq \mathrm{h}_{\mu_2}(\sigma) + \int t_1 g d\mu_2 = \mathrm{h}_{\mu_2}(\sigma) + \int t_2 g d\mu_2 + \int (t_1 - t_2) g d\mu_2 \geq \mathrm{h}_{\mu_2}(\sigma) + \int t_2 g d\mu_2 \geq \mathrm{P}(\sigma, t_2 g) - \varepsilon$. Letting $\varepsilon \searrow 0$, we are done.

(5). Suppose $\theta \leq t_1 < t_2 < \beta$. Since $\mathrm{P}(\sigma, t_2 g) > 0$, in view of Theorem 3.2 and Proposition 3.3 there exists an ergodic σ-invariant finitely supported, measure μ_2 such that

$$\mathrm{h}_{\mu_2}(\sigma) + \int t_2 g d\mu_2 \geq \max\left\{\frac{1}{2}, 1 - \frac{t_2 - t_1}{4\beta}\right\} \mathrm{P}(\sigma, t_2 g) \tag{3.2}$$

Then $\mathrm{h}_{\mu_2}(\sigma) \geq \mathrm{P}(\sigma, t_2 g)/2 > 0$ and therefore by Theorem 3.6, $\frac{\mathrm{h}_{\mu_2}(\sigma)}{\chi_{\mu_2}(\sigma)} = \mathrm{HD}(\mu_2 \circ \pi^{-1}) \leq \beta$. Hence $\int -g d\mu_2 \geq \mathrm{h}_{\mu_2}(\sigma)/\beta \geq \mathrm{P}(\sigma, t_2 g)/2\beta$. Thus, using (3.2), Theorem 3.2 and

Proposition 3.3, we get

$$
\begin{aligned}
\mathrm{P}(\sigma, t_1 g) &\geq \mathrm{h}_{\mu_2}(\sigma) + \int t_1 g d\mu_2 = \mathrm{h}_{\mu_2}(\sigma) + \int t_2 g d\mu_2 + \int (t_1 - t_2) g d\mu_2 \\
&\geq \mathrm{P}(\sigma, t_2 g) - \mathrm{P}(\sigma, t_2 g)\frac{t_2 - t_1}{4\beta} + \mathrm{P}(\sigma, t_2 g)\frac{t_2 - t_1}{2\beta} \\
&= \mathrm{P}(\sigma, t_2 g) + \mathrm{P}(\sigma, t_2 g)\frac{t_2 - t_1}{4\beta} > \mathrm{P}(\sigma, t_2 g).
\end{aligned}
$$

(6). An application of Hölder's inequality shows that each function

$$
t \mapsto \sum_{|\omega|=n} \exp\big(\sup_{\tau \in [\omega]} \sum_{j=0}^{n-1} g(\sigma^j(\tau))\big)
$$

is log convex. Therefore the map $t \mapsto \mathrm{P}(t)$, $t \in (\theta, \infty)$, is convex and consequently continuous. ■

Let us remark that it is possible that $\beta = \theta$. We will call such systems "strange" and deal with them in more detail in sections 5 and 6. Also, although it can happen that $\theta_S = 0$, we always have $P(0) \geq \log 2$ and therefore $h > 0$.

§**4. The Perron-Frobenius operator, semiconformal measures and Hausdorff dimension.** It follows from Proposition 3.7that β is the first zero of the pressure function. We shall provide below more characterizations of this number. Given $t > \theta(S)$ we define the associated Perron-Frobenius operator acting on $C(X)$ as follows

$$
\mathcal{L}_t(f)(x) = \sum_{i \in I} |\phi_i'(x)|^t f(\phi_i(x)).
$$

Notice that the nth composition of $\mathcal{L}$ satisfies:

$$
\mathcal{L}_t^n(f)(x) = \sum_{|\omega|=n} |\phi_\omega'(x)|^t f(\phi_\omega(x)).
$$

Consider the dual operator $\mathcal{L}_t^*$ acting on the space of finite Borel measures on X as follows

$$
\mathcal{L}_t^*(\nu)(f) = \nu(\mathcal{L}_t(f)).
$$

Notice that the map $\nu \mapsto \mathcal{L}_t^*(\nu)/\mathcal{L}_t^*(\nu)(1)$ sending the space of Borel probability measures into itself is continuous and by the Schauder-Tichonov theorem it has a fixed point. In other words $\mathcal{L}_t^*(\nu) = \lambda\nu$, for some probability measure ν, where $\lambda = \mathcal{L}_t^*(\nu)(1) > 0$. A probability measure m is said to be (λ, t)-semiconformal provided that $\mathcal{L}_t^*(m) = \lambda m$. If $\lambda = 1$ we simply speak about t-semiconformal measures. Repeating a short argument from the proof of Theorem 3.6 of [MU] we shall first prove the following.

Lemma 4.1. If m is a (λ, t)-semiconformal measure for the system S with $\lambda > 0$, then $m(J) = 1$.

Proof. For each $n \geq 1$ let $X_n = \cup_{|\omega|=n}\phi_\omega(X)$. The sets X_n form a descending family and $\bigcap_{n\geq 1} X_n = J$. Notice that $1\!\!1_{X_{|\omega|}} \circ \phi_\omega = 1\!\!1_X$ for all $\omega \in I^*$ and therefore, using (λ, t)-semiconformality of m, we obtain for every $n \geq 1$.

$$\begin{aligned}\lambda^n m(X_n) &= \int 1\!\!1_{X_n} d\mathcal{L}_t^{*n}(m) = \int \mathcal{L}_t^n(1\!\!1_{X_n})dm = \int \sum_{|\omega|=n} |\phi_\omega'|^t(1\!\!1_{X_n}\circ\phi_\omega)dm \\ &= \int \sum_{|\omega|=n} |\phi_\omega'|^t dm = \int 1\!\!1_X d\mathcal{L}_t^{*n}(m) \\ &= \int \lambda^n 1\!\!1_X dm = \lambda^n.\end{aligned}$$

Thus, $m(X_n) = 1$ and therefore $m(J) = m\big(\bigcap_{n\geq 1} X_n\big) = 1$. The proof is complete. ■

We set

$$\psi_n(t) = \sum_{|\omega|=n} ||\phi_\omega'||^t.$$

We note that $\theta(S) = \inf\{t : \psi(t) = \psi_1(t) < \infty\}$. In order to demonstrate the existence of $(\mathrm{e}^{\mathrm{P}(t)}, t)$-semiconformal measures we shall prove the following.

Lemma 4.2. If $t > \theta(S)$ and $\mathcal{L}_t^*(m) = \lambda m$ for some measure m on X, then $\lambda = \mathrm{e}^{\mathrm{P}(t)}$.

Proof. We first show the easier part that $\lambda \leq \mathrm{e}^{\mathrm{P}(t)}$. Indeed, for all $n \geq 1$

$$\lambda^n = \int \mathcal{L}_t^n(1\!\!1_X)dm = \int \sum_{|\omega|=n} |\phi_\omega'(x)|^t dm(x) \leq \int \sum_{|\omega|=n} ||\phi_\omega'||^t dm = \sum_{|\omega|=n} ||\phi_\omega'||^t$$

and therefore

$$\log\lambda \leq \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} ||\phi_\omega'||^t = \mathrm{P}(t). \tag{4.1}$$

In order to prove the opposite inequality, for each $p \geq 1$, let $T_p = \sum_{\omega\in I_g^p} ||\phi_\omega'||^t$, where I_g^p is the set of those words $\omega \in I^p$ such that ω_{p-1}, ω_p are not the same parabolic element. For each n,

$$\begin{aligned}\psi_n(t) &= \sum_{|\omega|=n} ||\phi_\omega'||^t \\ &\leq \sum_{\omega\in I_g^n} ||\phi_\omega'||^t + \sum_{i\in\Omega}\sum_{\omega\in I_g^{n-1}} ||\phi_\omega'||^t ||\phi_i'||^t + \sum_{i\in\Omega}\sum_{\omega\in I_g^{n-2}} ||\phi_\omega'||^t ||\phi_{ii}'||^t + \ldots + \sum_{i\in\Omega} ||\phi_{i^n}'||^t \\ &\leq \sum_{k=0}^n \#\Omega T_k,\end{aligned}$$

where $T_0 = 1$. Take $0 \le q(n) \le n$ that maximizes T_k. Then $\psi_n \le (n+1)\#\Omega T_{q(n)}$ and therefore

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n}\log\psi_n \le \liminf_{n\to\infty}\left(\frac{\log(n+1)}{n} + \frac{q(n)}{n}\cdot\frac{1}{q(n)}\log T_{q(n)} + \frac{1}{n}\log\#\Omega\right)$$

$$\le \max\left\{0, \limsup_{n\to\infty}\frac{1}{n}\log T_n\right\}. \tag{4.2}$$

Let

$$\tilde{\mathcal{L}}_t^n(1) = \sum_{\omega\in I_g^n} |\phi'_\omega|^t.$$

It follows from condition (5) that for all $n \ge 1, \omega \in I_g^n$ and all $x \in X$

$$||\phi'_\omega||^t \le K^t|\phi'_\omega(x)|^t.$$

Summing we have $T_n \le K^t\tilde{\mathcal{L}}_t^n(1)(x)$ and integrating this inequality with respect to the measure m, we get

$$T_n \le K^t \int \tilde{\mathcal{L}}_t^n(1)(x)dm(x) \le K^t\lambda^n.$$

Thus, by (4.2)

$$\mathrm{P}(t) \le \max\{0, \limsup_{n\to\infty}\frac{1}{n}\log T_n\} \le \max\{0, \log\lambda\}.$$

If now $t < \beta(S)$, then by Proposition 3.7(2), $\mathrm{P}(t) > 0$, and we therefore get $\mathrm{P}(t) \le \log\lambda$. Thus, we are done in this case. So, suppose that $t \ge \beta(S)$. Then by Proposition 3.7(3), $\mathrm{P}(t) = 0$ and in view of (4.1) we are left to show that $\lambda \ge 1$. In order to do it fix an arbitrary $0 < \eta < 1$. It follows from conditions (4) and (2) that for all n large enough, say $n \ge n_0$, $|\phi'_{i^n}(x)| \ge \eta^n$ for all $i \in \Omega$ and all $x \in X$. Fix $j \in \Omega$. We then have for all $n \ge n_0$

$$\lambda^n = \int 1\!\!1 d\mathcal{L}^{*n}(m) = \int \sum_{|\omega|=n} |\phi'_\omega|^t dm \ge \int |\phi'_{j^n}|^t dm \ge \int \eta^{tn} dm = \eta^{tn}.$$

Thus $\lambda \ge \eta^t$ and letting $\eta \nearrow 1$ we get $\lambda \ge 1$. The proof is complete. ■

Lemma 4.3. For every $t > \theta(S)$ a $(\mathrm{P}(t), t)$-semiconformal measure exists.

Proof. In view of Lemma 4.2, it suffices to prove the existence of an eigenmeasure of the conjugate operator $\mathcal{L}_t^*$. But this has been done in the paragraph preceding Lemma 4.1 which completes the proof. ■

Let $e = e(S)$ be the infimum of the exponents for which a t-semiconformal measure exists. We shall shortly see this infimum is a minimum. Also, let $h = h_S$ be the Hausdorff dimension of the limit set J. As an immediate consequence of Proposition 3.7(3) and Lemma 4.3 we get the following.

Lemma 4.4. $e(S) \le \beta(S)$.

Now, suppose that m is t-semiconformal or equivalently,

$$\int \sum_{\omega \in I^n} |\phi'_\omega|^t (f \circ \phi_\omega)\, dm = \int f\, dm, \tag{4.4}$$

for every continuous function $f : X \to I\!R$. Since this equality extends to all bounded measurable functions f, we get

$$m(\phi_\omega(A)) = \sum_{\tau \in I^n} \int |\phi'_\tau|^t (1_{\phi_\omega(A)} \circ \phi_\tau)\, dm \ge \int_A |\phi'_\omega|^t\, dm \tag{4.5}$$

for all $n \ge 1$, $\omega \in I^n$ and all Borel subsets A of X.

Our next task in this section is to note that $h \le e$. But this follows immediately from the following lemma whose proof, using (4.4), is the same as the proof of Lemma 4.3 of [MU1].

Lemma 4.5. If m is a t-semiconformal measure, then $\mathcal{H}^t \llcorner J \ll m$ and the Radon-Nikodym derivative $\frac{d\mathcal{H}^t}{dm}$ is uniformly bounded from above.

Since obviously $\beta \le h$, we have thus proved the following.

Theorem 4.6. $e = \beta = h =$the minimal zero of the pressure function.

As an immediate of Lemma 4.5, Lemma 4.3, Proposition 3.7(3) and Theorem 4.6 we get the following

Corollary 4.7. The h-dimensional Hausdorff measure of the limit set J is finite.

§5. The associated hyperbolic system. Conformal and invariant measures. In this section we describe how to associate to our parabolic system a new system which is hyperbolic and we apply its properties to study the original system, in particular to prove the existence of h-conformal measures. However we begin this section with a result describing the structure of t-semiconformal measures with exponents $t > h$. Let

$$\Omega_* = \{\phi_\omega(x_i) : i \in \Omega, \omega \in I^*\}$$

So, Ω_* is the set of orbits of parabolic points. The following theorem allows us to conclude a t-semiconformal measure is conformal provided the parabolic orbits do not mix.

Theorem 5.1. If $t > h$ and m_t is a t-semiconformal measure, then m_t is supported on Ω_*, that is $m_t(\Omega_*) = 1$. If for every $\omega \in I^*$ and every $i \in \Omega$, $\pi^{-1}(\phi_\omega(x_i)) = \omega i^\infty$, then each t-semiconformal measure ($t > h$) is t-conformal.

Proof. For every $r > h$ let m_r be an r-semiconformal measure. Note that the existence of at least one such measure (for every $r > h$) has been proved in Lemma 4.3, *cf.* also Proposition 3.7(3) and Theorem 4.6. Repeating the reasoning from Proposition 3.6 of [MU1], we see that for every $r > h$ there exists a Borel probability measure $\tilde{m}_r$ on I^∞ such that $\tilde{m}_r \circ \pi^{-1} = m_r$ and $\tilde{m}_r([\omega]) = \int |\phi'_\omega|^r dm_r$, for all $\omega \in I^*$. Now, fix $t > h$ and $h < s < t$. Let $\tilde{\Omega}_* = \{\omega i^\infty : i \in \Omega, \omega \in I^*\}$. If $\omega \notin \tilde{\Omega}_*$, then there exists an increasing infinite sequence $\{n_k\}_{k=1}^\infty$ such that either $\omega_{n_k} \notin \Omega$ or $\omega_{n_k - 1} \neq \omega_{n_k}$. In either case, using condition (5) we get

$$
\begin{aligned}
m_t([\omega|_{n_k}]) &= \int |\phi'_{\omega_{n_k}}|^t dm_t \le ||\phi'_{\omega_{n_k}}||^t = ||\phi'_{\omega_{n_k}}||^{t-s} ||\phi'_{\omega_{n_k}}||^s \\
&\le ||\phi'_{\omega_{n_k}}||^{t-s} K^s \int |\phi'_{\omega_{n_k}}|^s dm_s = K^{-s} ||\phi'_{\omega_{n_k}}||^{t-s} m_s([\omega|_{n_k}]).
\end{aligned}
\tag{5.0}
$$

It immediately follows from conditions (6) and (2) that $\lim_{k\to\infty} ||\phi'_{\omega_{n_k}}|| = 0$. Combining this and (5.0) we conclude that $\tilde{m}_t(I^\infty \setminus \tilde{\Omega}_*) = 0$ or equivalently $m_t(\tilde{\Omega}_*) = 1$. Since $\pi^{-1}(\Omega_*) \supset \tilde{\Omega}_*$, we get $m_t(\Omega_*) = \tilde{m}_t \circ \pi^{-1}(\Omega_*) \ge \tilde{m}_t(\tilde{\Omega}_*) = 1$. The proof of the first part of Theorem 5.1 is complete. The proof of the second part is an immediate consequence of (4.4) applied to the indicator functions of the sets of the form $\phi_\omega(A)$, where $\omega \in I^*$ and A is a Borel subset of X. ■

Consider now the system S^* generated by I_*, where

$$I_* = \{\phi_{i^n j} : n \ge 1,\ i \in \Omega,\ i \neq j\} \cup \{\phi_k :\ k \in I \setminus \Omega\}.$$

It immediately follows from our assumptions that the following is true.

Theorem 5.2. The system S^* is a hyperbolic conformal iterated function system.

The limit set generated by the system S^* is denoted by J^*.

Lemma 5.3. The limit sets J and J^* of the systems S and S^* respectively differ only by a countable set: $J^* \subset J$ and $J \setminus J^*$ is countable.

Proof. Indeed, it is obvious that $J^* \subset J$. On the other hand, the only infinite words generated by S but not generated by S^* are of the form ωi^∞, where ω is a finite word and i is a parabolic element of S. ■

Definitions. If S is an iterated function system with limit set J, then a measure ν supported on J is said to be invariant for the system S provided

$$\nu(E) = \nu\left(\bigcup_{i\in I} \phi_i(E)\right)$$

and ν is said to be ergodic for the system S provided $\nu(E) = 0$ or $\nu(J \setminus E) = 0$ whenever $\nu(E \Delta \bigcup_{i\in I} \phi_i(E)) = 0$.

Let us make some notation. Let $J_0 \subset J$ consist of all points with a unique code under S. For each $x = \pi(\omega) \in J_0$ express $\omega = i^n \tau$, where i is a parabolic element, $n \geq 0$, $\tau_1 \neq i$ and define $n(x) = n$. For each $k \geq 0$, put

$$B_k = \{x \in J_0 : n(x) = k\} \text{ and } D_k = \{x \in J_0 : n(x) \geq k\}.$$

Theorem 5.4. Suppose that μ^* on J^* is a probability measure invariant under S^* and $\mu^*(J_0) = 1$. Define the measure μ by setting for each Borel set $E \subset J_0$,

$$\mu(E) = \sum_{k=0} \sum_{|\omega|=k} \mu^*(\phi_\omega(E) \cap D_k) = \sum_{k \geq 1} \sum_{i \in \Omega} \mu^*(\phi_{i^k}(E)) + \mu^*(E) \tag{5.1}$$

Then μ is a σ-finite invariant measure for the system S and μ^* is absolutely continuous with respect to μ. If, for each $i \in I$, the measure $\mu^* \circ \phi_i$ is absolutely continuous with respect to the measure μ^*, then μ and μ^* are equivalent and if μ^* is ergodic for the system S^*, then μ is ergodic for the system S. Moreover, in this last case μ is unique up to a multiplicative constant.

Proof. Let us check first that μ is S-invariant. Indeed,

$$\begin{aligned}
\mu\left(\bigcup_{j \in I} \phi_j(E)\right) &= \sum_{k \geq 1}^{\infty} \sum_{i \in \Omega} \mu^*\left(\phi_{i^k}(\bigcup_{j \in I} \phi_j(E))\right) + \sum_{j \in I} \mu^*(\phi_j(E)) \\
&= \sum_{k=1}^{\infty} \sum_{i \in \Omega} \sum_{j \in I} \mu^*(\phi_{i^k j}(E)) + \sum_{j \in I} \mu^*(\phi_j(E)) \\
&= \sum_{k=1}^{\infty} \sum_{i \in \Omega} \sum_{j \in I} \mu^*(\phi_{i^k j}(E) \cap D_{k+1}) + \sum_{k=1}^{\infty} \sum_{i \in \Omega} \sum_{j \in I} \mu^*(\phi_{i^k j}(E) \cap B_k) \\
&\quad + \sum_{j \in I} \mu^*(\phi_j(E)) \\
&= \sum_{k=2}^{\infty} \sum_{i \in \Omega} \mu^*(\phi_{i^k}(E)) + \sum_{k=1}^{\infty} \sum_{i \in \Omega} \sum_{j \neq i} \mu^*(\phi_{i^k j}(E)) + \sum_{j \in I} \mu^*(\phi_j(E)) \\
&= \sum_{k=2}^{\infty} \sum_{i \in \Omega} \mu^*(\phi_{i^k}(E)) + \mu^*(E) + \sum_{j \in \Omega} \mu^*(\phi_j(E)) = \mu(E),
\end{aligned}$$

Where the last equality holds due to invariance of μ^* under S^*. The invariance of μ has been proved. Since $J_0 = \bigcup_{n \geq 0} B_n$, in order to show that μ is σ-finite it suffices to demonstrate that $\mu(B_n) < \infty$ for every $n \geq 0$. And indeed, given $n \geq 0$ we have

$$\mu(B_n) = \sum_{k \geq 1} \sum_{i \in \Omega} \mu^*(\phi_{i^k}(B_n)) + \mu^*(B_n). \tag{5.2}$$

Now, for every $i \in \Omega$,

$$\phi_{i^k}(B_n) \subset B_k \cup B_{n+k}.$$

Hence, $\mu(B_n) \le 2\#\Omega \sum_{k=0}^{\infty} \mu^*(B_k) = 2\#\Omega\mu^*\left(\bigcup_{k=0}^{\infty} B_k\right) = 2\#\Omega\mu^*(X) = 2\#\Omega$. Thus, μ is σ-finite. It follows in turn from (5.1) that $\mu(E) = 0$ implies $\mu^*(E) = \mu^*(E \cap D_0) = 0$. So, μ^* is absolutely continuous with respect to μ.
Now suppose that for each $i \in I$, the measure $\mu^* \circ \phi_i$ is absolutely continuous with respect to the measure μ^*. If $\mu^*(E) = 0$, then $\mu^*(\phi_\omega(E)) = 0$ for all $\omega \in I^*$. Thus, it follows from (5.1) that $\mu(E) = 0$ and the equivalence of μ and μ^* is shown. Suppose now that E is S-invariant, implying that $\bigcup_{i \in I} \phi_i(E) \subset E$. Then $\bigcup_{\omega \in I_*} \phi_\omega(E) \subset E$ and since μ^* is ergodic, either $\mu^*(E) = 0$ or $\mu^*(E^c) = 0$. Since μ is absolutely continuous with respect to μ^*, this implies that either $\mu(E) = 0$ or $\mu(E^c) = 0$. Hence μ is ergodic and the proof is complete. ■

Theorem 5.5. If the assumptions of Theorem 5.4 are satisfied, then the σ-finite measure μ produced there is finite if and only if

$$\sum_{n \ge 1} n\mu^*(B_n) < \infty.$$

Proof. Let us set $B_n^i = \{x \in J_0 : x = \pi(j^n\tau), j \in \Omega \setminus \{i\}, \tau \in I^\infty, \tau_1 \ne j\}$ and $D^i = \bigcup_{n \ge 0} B_n^i$. By (5.2), we can write

$$\begin{aligned}
\mu(J) = \sum_{n \ge 0} \mu(B_n) &= \sum_{n \ge 0} \sum_{k \ge 0} \sum_{i \in \Omega} \mu^*(\phi_{i^k}(B_n)) \\
&= \sum_{k \ge 0} \sum_{n \ge 0} \mu^*(B_{k+n}) + \sum_{k \ge 0} \sum_{n \ge 0} \sum_{i \in I} \mu^*(\phi_{i^k}(B_n^i)) \\
&= \sum_{k \ge 0} \sum_{n \ge 0} \mu^*(B_{k+n}) + \sum_{k \ge 0} \sum_{i \in \Omega} \mu^*(\phi_{i^k}(D^i)) \\
&= \sum_{n \ge 0} (n+1)\mu^*(B_n) + \sum_{n \ge 0} \mu^*(B_n) = \sum_{n \ge 0} (n+2)\mu^*(B_n).
\end{aligned}$$

The proof is therefore complete. ■

We recall from [MU1] that a probability measure m is said to be t-conformal for the system S provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i, j \in I$ with $i \ne j$,

$$m(\phi_i(A)) = \int_A |\phi_i'|^t dm \tag{5.3}$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0. \tag{5.4}$$

A straightforward computation shows (*e.g.* [MU1], p. 118) that any t-conformal measure is t-semiconformal. We also recall from [MU1] that a conformal hyperbolic system is regular if $\mathrm{P}(h)=0$ or equivalently an h-conformal measure exists. We shall now prove a little but useful lemma concerning general hyperbolic systems.

Lemma 5.6. If $S=\{\phi_i : X\to X, i\in I\}$ is a regular hyperbolic conformal iterated function system, then its h_S-conformal measure is atomless.

Proof. Suppose to the contrary that $m(z)>0$ for some $z\in J$. Then, by Corollary 3.11 of [MU1], $\tilde m(\omega)>0$ for some $\omega\in\pi^{-1}(z)$, where $\tilde m$ is the measure produced in Lemma 3.6 of [MU1]. Let μ^* be the σ-invariant probability measure produced in Theorem 3.8 of [MU1]. Since for every n, $\mu^*(\sigma^n(\omega))\ge\mu^*(\omega)>0$ and μ^* is a probability measure, ω is eventually periodic meaning that there exist $k\ge 0$ and $q\ge 1$ such that $\sigma^q(\sigma^k(\omega))=\sigma^k(\omega)$. Therefore, we can write $\sigma^k(\omega)=\tau^\infty$, for some $\tau\in I^*$. Since $m(\pi(\omega))>0$, $m(\pi(\tau^\infty))>0$ and by the conformality of m we have $m(\pi(\tau^\infty))=m(\phi_\tau(\pi(\tau^\infty)))=\int_{\pi(\tau^\infty)}|\phi_\tau'|^{h_S}dm<m(\pi(\tau^\infty))$ which is a contradiction finishing the proof. ■

Theorem 5.7. Suppose that S is a parabolic conformal iterated function system and the associated hyperbolic system S^* is regular. Then m, the h-conformal measure for S^* is also h-conformal for S and m is the only h-semiconformal measure for S.

Proof. Let m be the h-conformal measure for the system S^*. We will first show that m is h-conformal for the system S over the limit set J. We will then associate with S one more hyperbolic system S^{**} and use some properties of this system to verify that m is h-conformal for S. Since $m(J^*)=1$, the probability measure m clearly satisfies the first condition for conformality: $m(J)=1$. Next, we will show that m satisfies equation (5.3) for all Borel subsets A of J. Since $J\setminus J^*$ is countable and m is atomless, it suffices to show that (5.3) holds for Borel subsets of J^*. Also, since (5.3) holds whenever i is a hyperbolic index even for Borel subsets of X, we only need to verify (5.3) for parabolic indices. Let

$$\mathcal{G}=\{A : A \text{ is a Borel subset of } J^* \text{ and (5.3) holds } \forall i\in\Omega\}.$$

Since $\mathcal{G}$ is closed under monotone limits, it suffices to show that (5.3) holds for every subset U of J^* which is relatively open. Let

$$\Gamma=\{\omega\in I_*^\infty : \omega=(a_1b_1),(a_2b_2),(a_3b_3),\ldots;\forall n\ a_n,b_n\in\Omega,\ b_n\ne a_n,a_{n+1}\}.$$

Let $W=\pi(\Gamma)$. Using Theorem 3.8 from [MU1] and the Birkhoff ergodic theorem, we see that $m(W)=0$ and $m(\phi_i(W))=0,\forall i\in\Omega$. Let us demonstrate that if $i\in\Omega$ and $\omega=(\omega_1,\omega_2,\omega_3,...)\in I_*^\infty\setminus\Gamma$, then there is some l such that for every $k\ge l$, $(\omega_1,...,\omega_k)\in I_*^*$ and the concatenation $i*\omega_1*...*\omega_k$ can be parsed (or regrouped) so that it represents an element of I_*^*. To see this, first suppose that $\omega_1\in I\setminus\Omega$. Then $l=1$ since $i*\omega_1*...*\omega_k$ can be parsed as $i\omega_1,\omega_2,...\omega_k$ which is an element of I_*^*. Now, suppose $\omega_1=p^nq$ where $p\in\Omega$ and $p\ne q$. If $p=i$, then again $l=1$, since $i*\omega_1*...*\omega_k$ can be parsed as $i^{n+1}q,\omega_2,...\omega_k$ which is an element of I_*^*. If $i\ne p$ and $n>1$, then $i*\omega_1*...*\omega_k$ can be parsed as $(ip,p^{n-1}q,\omega_2,\omega_3,\ldots,\omega_k)\in I_*^*$ and also in this case $l=1$. If, on the other hand, $n=1$ and $p=i$, then $\omega_1=a_1b_1$, where $a_1\in\Omega$ and $b_1\ne a_1$. If $b_1\in I\setminus\Omega$, then $i*\omega_1*...*\omega_k$ can be parsed as $(ia_1,b_1,\omega_2,\omega_3,\ldots,\omega_k)\in I_*^*$ and $l=1$. So, suppose that

$b_1 \in \Omega$. Now, consider ω_2. If $\omega_2 \in I\setminus\Omega$, then the concatenation $i*\omega_1*...*\omega_k$ can be parsed as $(ia_1, b_1\omega_2, \omega_3, \ldots, \omega_k) \in I_*^*$ and $l=2$. Otherwise $\omega_2 = p^n q$, where $p \in \Omega$, $q \neq p$ and $n \geq 1$. If $p = b_1$, then $i*\omega_1*...*\omega_k$ can be parsed as $(ia_1, b_1^{n+1}q, \omega_3, \ldots, \omega_k) \in I_*^*$ and $l=2$. If $p \neq b_1$ and $n > 1$, then $i*\omega_1*...*\omega_k$ can be parsed as $(ia_1, b_1p, p^{n-1}q, \omega_3, \ldots, \omega_k) \in I_*^*$ and $l=2$. If, on the other hand, $n=1$, then $\omega_2 = a_2b_2$, where $a_2 \neq b_1, b_2$. If $b_2 \in I\setminus\Omega$, then $i*\omega_1*...*\omega_k$ can be parsed as $(ia_1, b_1a_2, b_2, \omega_3, \ldots, \omega_k) \in I_*^*$ and $l=2$. So, we may assume that $b_2 \in \Omega$. Now, excluding inductively in this manner the cases when $i*\omega_1*...*\omega_k$ can be parsed in a fashion that it would belong to I_*^*, we would end up with the conclusion that $\omega \in \Gamma$ contrary to our assumption. Now, let $U \subset J^*$ be relatively open. Then there is a set $M \subset I_*^*$, consisting of incomparable words such that $U\setminus W \subset \cup_{\tau\in M}\phi_\tau(J^*) \subset U$, and if $\tau \in M$ then $i*\tau \in I_*^*$. Thus,

$$m(\phi_i(U)) = m(\phi_i(\cup\phi_\tau(J^*))) \cup (U\setminus\cup\phi_\tau(J^*)) = \sum_\tau m(\phi_i(\phi_\tau(J^*)))$$
$$= \sum_\tau \int_{J^*} |(\phi_i\circ\phi_\tau)'|^h dm = \sum_\tau \int_{\phi_\tau(J^*)} |\phi_i'|^h dm = \int_U |\phi_i'|^h dm,$$

where the third equality follows since m is h-conformal for the system S^* and in the fourth equality we additionally employed the change of variables formula. Now, we want to show

$$m(\phi_i(J)\cap\phi_j(J)) = 0$$

whenever $i \neq j$. Again, it suffices to verify this when J is replaced by J^* and at least one of the indices i and j is parabolic. As before there is a set $M_i \subset I_*^*$ of incomparable words such that $J^*\setminus W \subset \cup_{\tau\in M_i}\phi_\tau(J^*) \subset J^*$, and if $\tau \in M_i$ then $i*\tau \in I_*^*$. Also, let $M_j \subset I_*^*$ have similar properties with respect to the index j. Then

$$m(\phi_i(J)\cap\phi_j(J)) = m(\cup_{\tau,\rho\in M_i\times M_j}\phi_{i\tau}(J^*)\cap\phi_{j\rho}(J^*)) \leq \sum_{M_i\times M_j} m(\phi_{i\tau}(J^*)\cap\phi_{j\rho}(J^*)) = 0.$$

Finally, to show that m is conformal, we must demonstrate that (5.3) and (5.4) hold whenever A is a Borel subset of X. Note that it suffices to show that $m(A) = 0$ implies $m(\phi_i(A)) = 0$, for all Borel subsets A of X and all parabolic indices i. In order to prove this, we introduce a new hyperbolic system. The index set for this system is $I_{**} = I^3\setminus\{(i,i,i) : i \in \Omega\}\cup\{p^nq : p \in \Omega, q \neq p, n \geq 2\}$. Let us prove that the system S^{**} satisfies the bounded distortion property. To see this read a word $\omega \in I_{**}^*$ as a word in $I^* : \omega = (\omega_1, \omega_2, ..., \omega_n)$. If $\omega_n \in I\setminus\Omega$, then we have bounded distortion by property (5) of the system S. If $\omega_n \in \Omega$ and $\omega_{n-1} \neq \omega_n$, then again by property (5) we have bounded distortion with constant K. If $\omega_{n-1} = \omega_n$, then $\omega_{n-2} \neq \omega_{n-1}$, by the definition of I_{**}^*. Then the word $\omega|_{n-1}$ satisfies the hypothesis of condition (5) and so

$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} = \frac{|\phi_{\omega|_{n-1}}'(\phi_{\omega_n}(y))||\phi_{\omega_n}'(y)|}{|\phi_{\omega|_{n-1}}'(\phi_{\omega_n}(y))||\phi_{\omega_n}'(y)|} \leq K\max\left\{\frac{||\phi_i'||}{\min\{\phi_i'(x) : x \in X\}} : i \in \Omega\right\},$$

where the last number is finite since Ω is. To see that S^{**} satisfies the open set condition, notice that $\phi_{ijk}(\mathrm{Int}(X))\cap\phi_{pqr}(\mathrm{Int}(X)) = \emptyset$ for all $ijk \neq pqr$. Next consider $\phi_{i^nj}(\mathrm{Int}(X))\cap$

$\phi_{p^m q}(\mathrm{Int}(X))$, where $n, m \geq 2$. If $i \neq p$, this intersection is empty. Also if $i = p$ and $n \neq m$, the intersection is empty. Otherwise, $q \neq j$ and the intersection is empty. Finally, consider $\phi_{i^n j}(\mathrm{Int}(X)) \cap \phi_{pqr}(\mathrm{Int}(X))$, where $n \geq 2$. If $i \neq p$ or if $i = p$ and $q \neq i$, the intersection is empty. Otherwise, $i = p = q$ and in this case $r \neq i$ since the word (i, i, i) is not allowed in I_{**}. Finally, the hyperbolicity of the system S^{**} is an immediate consequence of property (6). So, S^{**} is a hyperbolic conformal iterated function system. Also, since each element of I_*^∞ can be parsed into an element of I_{**}^∞, we have $J^{**} = J^* = J \setminus \{\text{eventually parabolic points}\}$. Also notice that if the system S^* is regular, then the system S^{**} is regular. To see this note that we have already shown that if m is conformal for S^*, then m is conformal for S over J. Thus m is conformal for S^{**} over J. So, for each n, $1 = \int_J dm = \int_J \sum_{\omega \in I_{**}^n} |\phi_\omega'(x)| dm$. But, for each $x \in J$, we have

$$\sum_{\omega \in I_{**}^n} ||\phi_\omega'||^h \geq \sum_{\omega \in I_{**}^n} |\phi_\omega'(x)|^h \geq (K^{**})^{-h} \sum_{\omega \in I_{**}^n} ||\phi_\omega'||^h,$$

where K^{**} is the distortion constant for the system S^{**} over X. Integrating this formula against the measure m we get

$$\sum_{\omega \in I_{**}^n} ||\phi_\omega'||^h \geq 1 \geq (K^{**})^{-h} \sum_{\omega \in I_{**}^n} ||\phi_\omega'||^h,$$

From this it immediately follows that $\mathrm{P}^{**}(h) = 0$. But, this is equivalent to saying that there is an h-conformal measure m^{**} for the system S^{**} over X. We only need to prove that $m^{**} = m$. Let G be open relative to J^*. Let W be a collection of incomparable words in I_{**} such that $G = \bigcup_{\omega \in W} \phi_\omega(J^*)$. Since m is conformal for S^{**} over J,

$$m(G) = \sum_{\omega \in W} \int_J |\phi_\omega'| dm \leq \sum_{\omega \in W} K^h ||\phi_\omega'|| \leq \sum_{\omega \in W} K^h K^{**h} \int_J |\phi_\omega'| dm^{**} = K^h {K^{**}}^h m^{**}(G).$$

Interchanging m and m^{**} in the above estimate, we get

$$(K^h {K^{**}}^h)^{-1} m{**}(G) \leq m(G) \leq K^h {K^{**}}^h m^{**}(G).$$

From this it follows that m and m^{**} are equivalent. To show that $m = m^{**}$, let A be a Borel subset of X. Then $m(\phi_\omega(A)) = m(\phi_\omega(A \cap J)) + m(\phi_\omega(A \setminus J))$. But, since m^{**} is conformal over $X, m^{**}(A \setminus J) = \int_{A \setminus J} |\phi_\omega'|^h dm^{**} = 0$. So, since m is conformal for S over J, we have $m(\phi_\omega(A)) = \int_{A \cap J} |\phi_\omega'|^h dm = \int_A |\phi_\omega'|^h dm$. Also, one can show that (5.4) holds using the same procedure. Thus, m is conformal for S^{**} over X.

Finally, to see that m is conformal for the entire system S over X, let $i \in \Omega$ and choose an arbitrary $q \neq i$, $q \in I$. Then $iq \in I_*$ and $iqi \in I_{**}$. Thus,

$$\int_{\phi_i(A)} |\phi_{iq}'|^h dm = m(\phi_{iq}(\phi_i(A)) = m(\phi_{iqi}(A)) = \int_A |\phi_{iqi}'|^h dm.$$

So, if $m(A) = 0$, then since $|\phi_{iq}'|^h$ is positive on $\phi_i(A)$, we have $m(\phi_i(A)) = 0$.

In order to prove the second part of our theorem suppose that ν is an arbitrary measure supported on J and satisfying

$$\nu(\phi_\omega(A)) \geq \int_A |\phi'_\omega|^h d\nu \tag{5.8}$$

for all Borel sets $A \subset X$ and all $\omega \in I^*$. We show that m is absolutely continuous with respect to ν. Indeed, for every $\omega \in I_*^*$ we have

$$\nu(\phi_\omega(X)) \geq \int_X |\phi'_\omega|^h d\nu \geq K^{-h}||\phi'_\omega||^h \geq K^{-h} \int_X |\phi'_\omega|^h dm = K^{-h} m(\phi_\omega(X)).$$

Next, consider an arbitrary Borel set $A \subset X$ such that $\nu(A) = 0$. Fix $\varepsilon > 0$. Since ν is regular there exists an open subset G of X such that $A \cap J^* \subset G$ and $\nu(G) \leq \varepsilon$. There now exists a family $\mathcal{F} \subset I_*^*$ of mutually incomparable words such that $A \cap J^* \subset \bigcup_{\omega \in \mathcal{F}} \phi_\omega(X) \subset G$. Lemma 2.6 of [MU1] states that there exists a universal upper bound M on the multiplicity of the family $\{\phi_\omega(X) : \omega \in \mathcal{F}\}$. Hence, using the fact that m is supported on J^*, we obtain

$$\begin{aligned} m(A) = m(A \cap J^*) &\leq m\left(\bigcup_{\omega \in \mathcal{F}} \phi_\omega(X)\right) \leq \sum_{\omega \in \mathcal{F}} m(\phi_\omega(X)) \\ &\leq K^h \sum_{\omega \in \mathcal{F}} \nu(\phi_\omega(X)) \leq K^h M \nu\left(\bigcup_{\omega \in \mathcal{F}} \phi_\omega(X)\right) \leq K^h M \nu(G) \\ &\leq K^h M \varepsilon. \end{aligned}$$

Thus, letting $\varepsilon \searrow 0$, we get $m(A) = 0$ which finishes the proof of the absolute continuity of m with respect to ν. Our next aim is to show that $\nu(J \setminus J^*) = 0$. Suppose on the contrary that $\nu(J \setminus J^*) > 0$. Set $P = \{\phi_\omega(x_i) : i \in \Omega, \omega \in I^*\}$. Since $J \setminus J^* \subset P$, $\nu(P) > 0$. Write $\nu = \nu_0 + \nu_1$, where $\nu_0|_{X \setminus P} = 0$ and $\nu_1|_P = 0$. Thus $\nu_0(P) = \nu(P) > 0$. Since $\phi_\omega(P) \subset P$ for all $\omega \in I^*$, we get for every Borel set $A \subset X$ and every $\omega \in I^*$

$$\nu_0(\phi_\omega(A)) \geq \nu_0(\phi_\omega(A \cap P)) = \nu(\phi_\omega(A \cap P)) \geq \int_{A \cap P} |\phi'_\omega|^h d\nu = \int_A |\phi'_\omega|^h d\nu_0.$$

Hence multiplying ν_0 by $1/\nu_0(X)$, we conclude from what has been proved that m is absolutely continuous with respect to ν_0. Since $\nu_0(X \setminus P) = 0$, this implies that $m(X \setminus P) = 0$, and consequently $m(P) = 1$. Since P is countable we arrive at a contradition with Lemma 5.6. Thus $\nu(J^*) = 1$. Since, by Lemma 4.1, any h-semiconformal measure ν is supported on J and, by (4.5), satisfies (5.8), we conclude that any h-semiconformal measure is supported on J^* and satisfies (5.8). Since, additionally, by regularity of the system S^*, $\mathrm{P}^*(h) = 0$, it follows from Lemma 3.10 of [MU1] that ν is h-conformal for S^*. An application of Theorem 3.9 of [MU1] implying the uniqueness of h-conformal (even h-semiconformal) measures for the hyperbolic system S^*, shows that $\nu = m$. The proof is complete. ■

Following the case of hyperbolic systems (see [MU1]) we call a parabolic system regular if there exists an h-conformal measure for S supported on J^*. Since such a measure is h-conformal for S^*, as an immediate consequence of Theorem 5.7 we get the following.

Corollary 5.8. The parabolic system is regular if and only if the associated system S^* is regular.

Trying to say something about parabolic systems which are not regular we are led to introduce the class of strange systems which by definition are those systems for which there is no t with $0 < \mathrm{P}(t) < \infty$. In the hyperbolic case the strange systems coincide (see [MU2]) with systems which are not strongly regular or equivalently with those with $\theta = h$. This last characterization continues to be true also for parabolic systems and this class may also be characterized by the requirement of the existence of a number α (which then turns out to be $\theta = h$) such that $\mathrm{P}(t) = \infty$ for all $t < \alpha$ and $\mathrm{P}(t) = 0$ for all $t \geq \alpha$. Let us remark that we do not want to call the strange systems "irregular" since the irregular hyperbolic systems are precisely those for which no conformal measure exists whereas for a strange parabolic system the following questions remain open.

Questions. Can there exist a strange parabolic system such that the associated hyperbolic system is regular? Can there exist a strange parabolic system with a purely atomic h-conformal measure?

We shall prove the following.

Proposition 5.9. If the system S is strange, then so is S^*.

Proof. Since $h_{S^*} = h_S$, $\mathrm{P}^*(t) \leq 0$ for all $t \geq h_S$. So, we are only left to show that $\mathrm{P}^*(t) = \infty$ for all $t < h_S$. And indeed, fix $t < h_S$. Since S is strange, $\mathrm{P}(t) = \infty$ and therefore $\psi(t) = \infty$. Since Ω is finite, this implies that $\sum_{i \in I \setminus \Omega} ||\phi_i'||^t = \infty$. But then $\psi^*(t) \geq \sum_{i \in I \setminus \Omega} ||\phi_i'||^t = \infty$. Hence $\mathrm{P}^*(t) = \infty$ and we are done. ■

Let us briefly touch on the packing measure of J. Since J^* is dense in J, as an immediate consequence of Theorem 5.7 and Lemma 4.3 of [MU1] we get the following.

Corollary 5.10. Suppose that S is a parabolic iterated function system and the associated hyperbolic system S^* is regular. If $J \cap \mathrm{Int}(X) \neq \emptyset$ (that is, if the strong open set condition is satisfied), then the h-dimensional packing measure of J is positive.

Let us remark here that in Corollary 4.7 we have proved that the h-dimensional Hausdorff measure of J is finite.
Finally, let us give some results about equivalent ergodic invariant measures for regular systems. As a consequence of Theorem 5.7 we have the following.

Corollary 5.11. Suppose that S is a parabolic iterated function system, the associated hyperbolic system S^* is regular and let m be the corresponding h-conformal measure. Then there exists a unique probability measure μ^* equivalent with m, which is ergodic and invariant under S^* and, up to a multiplicative constant, there exists a unique σ-finite measure μ equivalent with m and ergodic invariant under S.

Proof. The first part of this corollary is an immediate consequence of Theorem 3.8 and Corollary 3.11 from [MU1]. That m is h-conformal for S follows from Theorem 5.7. The last part is a consequence of this conformality (the measures $\mu^* \circ \phi_i$ are therefore absolutely continuous with respect to μ^*) and Theorem 5.4. ■

Corollary 5.12. If the assumptions of Corollary 5.11 are satisfied, the σ-invariant measure μ produced there is finite if and only if

$$\sum_{i\in\Omega}\sum_{n=1}^{\infty} n \int_{X_i} |\phi'_{i^n}|^h dm < \infty,$$

where $X_i = \bigcup_{j\neq i} \phi_j(X)$.

Proof. Since by Theorem 3.8 and Corollary 3.11 from [MU1] m and μ^* are equivalent with Radon-Nikodym derivatives are bounded away from 0 and infinity, it therefore follows from Theorem 5.5 that μ is finite if and only if the series $\sum_{n\geq 1} nm(B_n)$ converges. Since $m(B_n) = \sum_{i\in\Omega}\int_{X_i} |\phi'_{i^n}|^h dm$, the proof is complete. ■

Corollary 5.13. If for every $i \in \Omega$ there exists some β_i and a constant $C_i \geq 1$ such that for all $n \geq 1$ and for all $z \in X_i$

$$C_i^{-1} n^{-\frac{\beta_i+1}{\beta_i}} \leq |\phi'_{i^n}(z)| \leq C_i n^{-\frac{\beta_i+1}{\beta_i}},$$

then the σ-finite invariant measure μ produced in Corollary 5.11 is finite if and only if

$$h > 2\max\left\{\frac{\beta_i}{\beta_i+1} : i \in \Omega\right\}.$$

Proof. The proof is an immediate consequence of Corollary 5.12. ■

§**6. Examples.** This section contains examples illustrating some of the ideas developed in this paper. We begin with the following.

Example 6.1. (Apollonian packing) Consider on the complex plane the three points $z_j = \mathrm{e}^{2\pi ij/3}$, $j = 0, 1, 2$ and the following additional three points $a_0 = \sqrt{3}-2$, $a_1 = (2-\sqrt{3})\mathrm{e}^{\pi ij/6}$ and $a_2 = (2-\sqrt{3})\mathrm{e}^{-\pi ij/6}$. Let f_0, f_1, and f_2 be the Möbius transformations determined by the following requirements: $f_0(z_0) = z_0$, $f_0(z_1) = a_2$, $f_0(z_2) = a_1$, $f_1(z_0) = a_2$, $f_1(z_1) = z_1$, $f_1(z_2) = a_0$, $f_2(z_0) = a_1$, $f_2(z_1) = a_0$, and $f_2(z_2) = z_2$. Set $X = \overline{B}(0, 1)$, the closed ball centered at the origin of radius 1. It is straightforward that the images $f_0(X)$, $f_1(X)$

and $f_2(X)$ are mutually tangent (at the points a_0, a_1 and a_2, respectively) disks whose boundaries pass through the triples (z_0, a_1, a_2), (z_1, a_0, a_2) and (z_2, a_0, a_1) respectively. Of course all the three disks $f_0(X)$, $f_1(X)$ and $f_2(X)$ are contained in X and are tangent to X at the points z_0, z_1 and z_2 respectively. Let $S = \{f_0, f_1, f_2\}$ be the iterated function system on X generated by f_0, f_1 and f_2. Notice that all the maps f_0, f_1 and f_2 are parabolic with parabolic fixed points z_0, z_1 and z_2 respectively. It is not difficult to check that all the requirements of a parabolic system are satisfied. Observe that the limit set J of the parabolic system S coincides with the residual set of the Apollonian packing generated by the curvilinear triangle with vertices z_0, z_1, z_2. In [MU3], using a slightly different iterated function system, we have dealt with geometrical properties of J proving that $1 < h = \mathrm{HD}(J) < 2$, $0 < \mathcal{H}^h < \infty$ and $\mathcal{P}^h(J) = \infty$. In this paper we want to study its dynamical properties. Let us first notice that the system S^* is regular. Indeed, we proved in [MU3] that

$$f_0^n(z) = \frac{(\sqrt{3} - n)z + n}{-nz + n + \sqrt{3}}$$

and

$$(f_0^n)'(z) = \frac{3}{(-nz + n + \sqrt{3})^2}.$$

By the symmetry of the situation this implies that

$$|(f_i^n \circ f_j)'(z)| \asymp \frac{1}{n^2}$$

for all $i \neq j$. Hence $\psi^*(t) \asymp \sum_{n \geq 1} \frac{1}{n^{2t}}$, where $\psi^*(t)$ is the psi function of the system S^* introduced just before Lemma 4.2. Thus $\theta(S^*) = 1/2$ and $\psi^*(1/2) = \infty$. Hence, it follows from Theorem 3.20 of [MU1] that the system S^* is regular, even more it is hereditarily regular. Thus, the assumptions of Theorem 5.7 and Corollary 5.11 are satisfied in our case. Let m be the h-conformal measure for S and let μ be an S-invariant σ-finite measure equivalent with m. We shall prove the following.

Theorem 6.2. The invariant measure μ of the Apollonian system $\{f_0, f_1, f_2\}$ is finite.

Proof. In the proof of regularity of S^*, we have observed that $|(f_i^n)'| \asymp 1/n^2$ on X_i, $i = 0, 1, 2$. Since $h > 1 = 2\frac{1}{1+1}$, it therefore follows from Corollary 5.13 that μ is finite. The proof is complete. ■

Example 6.3. A large class of examples appears already in the case when X is a compact subinterval of the real line $\mathbb{R}$. We call such systems one-dimensional. If the parabolic elements ϕ_i of a one-dimensional system S have, around parabolic fixed points x_i, a representation of the form

$$\phi_i(x) = x + a(x - x_i)^{1+\beta_i} + o((x - x_i)^{1+\beta_i}) \tag{6.1}$$

then (see *e.g.*, [U1])

$$|\phi_{i^n}'(x)| \asymp n^{-\frac{\beta_i + 1}{\beta_i}} \tag{6.2}$$

outside every open neighbourhood of x_i. Hence the following theorem is a consequence of Theorem 5.7. and Corollary 5.13.

Theorem 6.4. If S is a one-dimensional parabolic system with finite alphabet and satisfying (6.1), then S is regular and any S-invariant invariant measure μ equivalent with the h_S-conformal measure is finite if and only if $h > 2\max\{\frac{\beta_i}{\beta_i+1} : i \in \Omega\}$.

Proof. The regularity of S^* is checked in exactly the same way as in Example 5.1. So, the system S is regular by Corollary 5.8. Since the other assumptions of Corollary 5.13 are satisfied by 6.2, the proof of this theorem is an immediate consequence of Corollary 5.13. $\blacksquare$

Corollary 6.5. If S is a one-dimensional parabolic system with finite alphabet, and if for all $i \in \Omega$, $\beta_i \geq 1$ (or equivalently if all ϕ_i's are twice differentiable at x_i), then S is regular and the corresponding invariant measure μ equivalent with h_S-conformal measure is infinite.

Proof. The proof is an immediate consequence of Theorem 6.4 and the fact that $h \leq 1$. $\blacksquare$

We would like to close this section with some examples of strange systems.

Example 6.6. Our aim here is to describe a class of one-dimensional systems which are strange. Towards this end consider an arbitrary hyperbolic system $S = \{\phi_i : i \in I\}$ on the interval $X = [0,1]$ such that $\psi(\theta(S)) < \infty$ or equivalently $\mathrm{P}(\theta(S)) < \infty$ (examples of such systems may be found in the section Examples of [MU1]); we may assume that there is an interval $G = [0, \gamma)$ with $G \subset X \setminus \bigcup_{i\in I} \phi_i(X)$. Consider also a parabolic map $\phi : X \to G$ such that 0 is its parabolic point and ϕ has the following representation around 0:

$$\phi(x) = x - ax^{\beta+1} + o(x^{\beta+1}),$$

where $\theta(S)\frac{\beta+1}{\beta} > 1$ and $a > 0$. We shall prove the following.

Theorem 6.7. If $F \subset I$ is a sufficiently large finite set, then the system $S_F = \{\phi\} \cup \{\phi_i : i \in I \setminus F\}$ is strange.

Proof. In view of (6.2) and the relation between $\theta(S)$ and β there exists a constant $C \geq 1$ such that for each $i \in I$, $\sum_{n\geq 1} ||(\phi^n \circ \phi_i)'||^{\theta(S)} \leq C||\phi_i'||^{\theta(S)}$. Since $\psi_S(\theta(S)) < \infty$, for every sufficiently large finite set $F \subset I$ we have $(C+1)\sum_{i\in I\setminus F} ||\phi_i'||^{\theta(S)} < 1$. Hence

$$\begin{aligned}
\psi^*_{S_F}(\theta(S)) &= \sum_{i\in I\setminus F} ||\phi_i'||^{\theta(S)} + \sum_{i\in I\setminus F}\sum_{n\geq 1} ||(\phi^n \circ \phi_i)'||^{\theta(S)} \\
&\leq \sum_{i\in I\setminus F} ||\phi_i'||^{\theta(S)} + C \sum_{i\in I\setminus F} ||\phi_i'||^{\theta(S)} \\
&= (C+1) \sum_{i\in I\setminus F} ||\phi_i'||^{\theta(S)} < 1.
\end{aligned}$$

Hence $P^*_{S_F}(\theta(S)) < 0$ and therefore, as $h_{S^*_F} = h_{S_F}$, $P_{S_F}(t) = 0$ for all $t \geq \theta(S)$. On the other hand, since for every $t < \theta(S)$, $\psi_S(t) = \infty$ and since F is finite, $\psi_{S_F}(t) = ||\phi'||^t + \sum_{i \in I \setminus F} ||\phi_i'||^t = \infty$. Hence $P_{S_F}(t) = \infty$ and the proof is complete. ■

Example 6.8. We would like to construct here an example of parabolic one-dimensional system which is regular but strange. We start of with a hyperbolic regular but strange system $S = \{\phi_i\}_{i \in I}$ on the interval $[0,1]$ such that $\phi_1(0) = 0$ and

$$\bigcup_{i \in I\!\!N} \phi_i([0,1]) = [0,1].$$

A way of constructing such systems is described in Example 5.4 of [MU1]. Since Lebesgue measure is a 1-conformal measure for S (so S is regular) and since S is strange, $\psi(t) = \infty$ for all $0 \leq t < 1$ and $P(1) = 0$. Replace now the contraction ϕ_1 by a parabolic element $\tilde{\phi}_1$ such that 0 is its parabolic point and $\tilde{\phi}_i([0,1]) = \phi_i([0,1])$. Denote the new system by $\tilde{S}$. Then obviously $\psi_{\tilde{S}}(t) = \infty$ for all $0 \leq t < 1$ and consequently, $P_{\tilde{S}}(t) = \infty$ for all $0 \leq t < 1$. Since $\mathrm{HD}(J_{\tilde{S}}) \leq 1$, $P_{\tilde{S}}(t) = 0$ for all $t \geq 1$. Hence, $\tilde{S}$ is strange and $\mathrm{HD}(J_{\tilde{S}}) = 1$. Since

$$\tilde{\phi}_i([0,1]) \bigcup \bigcup_{i \geq 2} \phi_i([0,1]) = [0,1],$$

the Lebesgue measure λ on the interval $[0,1]$ is 1-conformal for the system $\tilde{S}$. Since obviously $\lambda(\{\phi_\omega(0) : \omega \in I\!\!N^*\}) = 0$, the system $\tilde{S}$ is regular. ■

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