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FRACTAL MEASURES FOR PARABOLIC IFS

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ABSTRACT. Let h be the Hausdorff dimension of the limit set of a conformal parabolic iterated function system in dimension $d \geq 2$. In case the system of maps is finite, we provide necessary and sufficient conditions for the h-dimensional Hausdorff measure to be positive and finite and also, assuming the strong open set condition holds, characterize when the h-dimensional packing measure of the limit set is positive and finite. We also prove that the upper ball (box)-counting dimension and the Hausdorff dimension of this limit set coincide. As a byproduct we include a compact analysis of the behaviour of parabolic conformal diffeomorphisms in dimension 2 and separately in any dimension greater than or equal to 3.

1. Introduction and preliminaries

Our setting is the following. Let X be a compact subset of a Euclidean space $I\!R^d$ with nonempty interior such that the boundary of X has no isolated points. We consider a countable family of conformal maps $\phi_i : X \to X$, $i \in I$, where I has at least two elements satisfying the following conditions.

(1): (Open Set Condition) $\phi_i(\text{Int}(X)) \cap \phi_j(\text{Int}(X)) = \emptyset$ for all $i \neq j$.

(2): $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i will be called *parabolic* and the set of parabolic indices will be denoted by Ω. All other indices will be called *hyperbolic*.

(3): (extension) There exist an open connected neighbourhood V of X and $s < 1$ such that $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then ϕ_ω extends conformally to V, maps V into itself and $||\phi_\omega'|| \leq s$.

(4): If i is a parabolic index, then $\bigcap_{n\geq 0} \phi_{i^n}(X) = \{x_i\}$ (Thus, the diameters of the sets $\phi_{i^n}(X)$ converge to 0.)

(5): (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$, there exists an open cone $\text{Con}(x, u_x, \alpha, l) \subset \text{Int}(X)$ with vertex x and a central angle of Lebesgue measure α, where $\text{Con}(x, u_x, \alpha, l) = \{y : 0 < (y - x, u_x) \leq \cos\alpha ||y - x|| \leq l\}$ and $||u_x|| = 1$.

(6): $\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.

(7): (Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ $\forall x, y \in V$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{||\phi_\omega'(y)||}{||\phi_\omega'(x)||} \leq K.$$

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(8): There are constants $L \ge 1$ and $\alpha > 0$ such that

$$\Big| ||\phi_i'(y)|| - ||\phi_i'(x)|| \Big| \le L||\phi_i'||(||y-x||)^\alpha,$$

for every $i \in I$ and every pair of points $x, y \in V$.

We call such a system of maps $S = \{\phi_i : i \in I\}$ a conformal iterated function system abbreviated as conformal IFS. If $\Omega = \emptyset$, we call the system S *hyperbolic*; if $\Omega \neq \emptyset$, we call it *parabolic*. Throughout this entire paper we assume that the system S is parabolic.

We would like to emphasize that if $d \ge 2$, then the conditions (7) and (8) are a consequence of condition (3) alone. Indeed, in case $d = 2$, these follow from Koebe's distortion theorem (in its version stated in [Pr]) and the observation that the complex conjugation in $\mathbb{C}$ is an isometry. In case $d \ge 3$, both conditions have been proved in [U2]. Because of an extreme importance of these properties and for the sake of completeness, we include their proof taken from [U2] in the end of Section 2. Finally, since the appropriate results in case $d = 1$ have been proven in [U3], we assume throughout the entire paper that $d \ge 2$.

By I^* we denote the set of all finite words with alphabet I and by I^∞ all infinite sequences with terms in I. It follows from (3) that for every hyperbolic word ω, $\phi_\omega(V) \subset V$. For each $\omega \in I^* \cup I^\infty$, we define the length of ω by the uniquely determined relation $\omega \in I^{|\omega|}$. If $\omega \in I^* \cup I^\infty$ and $n \le |\omega|$, then by $\omega|_n$ we denote the word $\omega_1\omega_2\ldots\omega_n$. In [MU4], we proved that $\lim_{n\to\infty} \sup_{|\omega|=n}\{\mathrm{diam}(\phi_\omega(X))\} = 0$. So, the map $\pi : I^\infty \to X$, $\pi(\omega) = \bigcap_{n\ge 0} \phi_{\omega|_n}(X)$, is uniformly continuous. Its range

$$J = J_S = \pi(I^\infty),$$

the main object of our interest in this paper, is called the *limit set* of the system S. For every integer $q \ge 1$, we denote

$$S^q = \{\phi_\omega : \omega \in I^q\}.$$

Of course, $J_{S^q} = J_S$ and sometimes in the sequel it will be more convenient to consider an appropriate family of iterates S^q of S rather than S itself. The two basic tools we use to study limit sets of parabolic IFS are conformal measures and a hyperbolic system S^* associated with S. The system S^* is given by

$$S^* = \{\phi_{i^nj} : n \ge 1,\ i \in \Omega,\ i \neq j\} \cup \{\phi_k :\ k \in I \setminus \Omega\}.$$

Thus, I_*, the countable set of indices or letters for the system S^* is

$$I_* = \{i^nj : n \ge 1,\ i \in \Omega,\ i \neq j\} \cup \{k :\ k \in I \setminus \Omega\}.$$

This system was described and analyzed in [MU4]. It immediately follows from our assumptions (comp. Theorem 5.2 in [MU4]) that the following is true.

Theorem 1.1. *The system S^* is a hyperbolic conformal iterated function system.*

The limit set generated by the system S^* is denoted by J^*. The following result (see Lemma 5.3 in [MU4]) allows us to reduce our geometric considerations to the limit set S^* and we are able to apply the theory developed for infinite hyperbolic IFS.

Lemma 1.2. *The limit sets J and J^* of the systems S and S^* respectively differ only by a countable set: $J^* \subset J$ and $J \setminus J^*$ is countable.*

Let
$$S^*(\infty) = \bigcap_{F \in \mathcal{F}in} \overline{\bigcup_{a \in I_* \setminus F} \phi_a(X)},$$
where $\mathcal{F}in$ denotes the family all finite subsets of I_*. In [MU1], $S^*(\infty)$ is denoted by $X(\infty)$. The following proposition is an immediate consequence of the condition (4).

Proposition 1.3. *If the alphabet I is finite, then*
$$S^*(\infty) = \{x_i : i \in \Omega\}, \text{ the set of parabolic fixed points.}$$

Following [MU1], given $t \geq 0$, a Borel probability measure m is t-conformal for the system S^* provided $m(J_{S^*}) = 1$ and for every Borel set $A \subset X$ and all $i, j \in I_*$ with $i \neq j$,
$$m(\phi_i(A)) = \int_A |\phi_i'|^t dm \tag{1.1}$$
and
$$m(\phi_i(X) \cap \phi_j(X)) = 0. \tag{1.2}$$
For the system S^*, we define the functions
$$\psi(t) = \sum_{a \in I_*} ||\phi_a'||^t \text{ and } \psi_n(t) = \sum_{a \in I_*^n} ||\phi_a'||^t,$$
and P^*, the *topological pressure function* for the system S^*,
$$P^*(t) = \lim_{n \mapsto \infty} \frac{\log \psi_n(t)}{n}.$$
Finally, the *finiteness parameter* for the system S^* is given by
$$\theta(S^*) = \inf\{t : \psi(t) < \infty\} = \inf\{t : P^*(t) < \infty\}.$$
The system S^* is said to be *hereditarily regular* provided $\psi(\theta(S^*)) = \infty$ and *regular* provided there is some t such that $P^*(t) = 0$. Of course, hreditarily regular systems are regular. Let
$$h = h_S = \dim_{\mathrm{H}}(J_S) = \dim_{\mathrm{H}}(J_{S^*})$$
be the Hausdorff dimension of the limit set J_S. It has been proven in [MU1] that $h = \inf\{t : P^*(t) \leq 0\}$ and if a hyperbolic IFS is regular, then an h-conformal measure exists and is unique. In Section 4 we shall prove the following

Theorem 1.4. *If S is a finite parabolic IFS, then the system S^* is hereditarily regular and, consequently, an h-conformal measure for S^* exists.*

From now on, unless otherwise stated, we will assume that the alphabet I is finite and m will denote the h-conformal measure produced in Theorem 1.4.

Let $\mathcal{H}^t$ denote the t-dimensional Hausdorff measure and $\mathcal{P}^t$, the t-dimensional packing measure. We recall that the system S satisfies the *strong open set condition* if $J_S \cap \mathrm{Int} X \neq \emptyset$.

Noting that in terminology of [MU1] each hereditarily regular IFS is regular, and combining Theorem 1.4, Corollary 4.7 in [MU4] and Corollary 5.10 in [MU4], we get the following.

Theorem 1.5. *If a finite parabolic IFS S satisfies the strong open set condition, then $\mathcal{H}^t(J) < \infty$ and $\mathcal{P}^h(J) > 0$.*

Next we state the main theorem of our paper. It contains a complete description of the h-dimensional Hausdorff and packing measures of the limit set of a finite parabolic IFS.

Theorem 1.6. *Let S be a finite parabolic IFS satisfying the strong open set condition. Then*

(a): *If $h < 1$, then $0 < \mathcal{P}^h(J) < \infty$ and $\mathcal{H}^h(J) = 0$.*
(b): *If $h = 1$, then $0 < \mathcal{H}^h(J) \le \mathcal{P}^h(J) < \infty$.*
(c): *If $h > 1$, then $0 < \mathcal{H}^h(J) < \infty$ and $\mathcal{P}^h(J) = \infty$.*

This sort of theorem has appeared in several contexts, for Kleinian groups in [Su], in the context of parabolic rational functions in [DU], for rational functions with no recurrent critical points in the Julia set (abbreviated as NCP maps), in [U1] and for parabolic Cantor sets (which comprise 1-dimensional parabolic IFS) in [U3]. The idea behind the proofs here are different from those cited. It replies on developing, extending, simplifying and clarifying the approach which originated in [MU3], and employing the necessary and sufficient conditions for the Hausdorff and packing measures to be positive and finite, provided in [MU1] and [MU2]. We note that the inducing procedures proposed in [UZ] indicate that in the case of parabolic rational functions and perhaps even in the case of NCP maps, one can demonstrate appropriate versions of our main theorem as a corollary of Theorem 1.6. We shall also prove in the Section 4 the following.

Theorem 1.7. *If S is a finite parabolic IFS, then*

$$\overline{\dim_{\mathrm{B}}(J)} = \dim_{\mathrm{H}}(J),$$

where $\overline{\dim_{\mathrm{B}}(J)}$ denotes the upper ball-counting dimension, also called the box-counting dimension, Minkowski dimension or capacity.

One more note for the reader. The dynamical properties of the parabolic IFS proven in Sections 2 and 3 and needed for the proofs of Theorem 1.6 and Theorem 1.7 are provided in the beginning of Section 4 in a unified fashion. Therefore, the reader only interested in Theorem 1.6 and Theorem 1.7 may actually read Section 4 independently of Section 2 and Section 3.

Section 2 mainly concerns the dynamical properties of a single parabolic conformal diffeomorphism in $I\!\!R^d$, $d \ge 3$ and can be viewed as an introduction to the technically more complicated Section 3 which deals with dynamical properties of a single simple parabolic holomorphic map in $I\!\!R^2$. Both sections provide a compact systematic description of the quantitative behaviour of parabolic maps needed for the proofs in Section 4. The qualitative behaviour of a single parabolic holomorphic map considerd in Section 3 is known as Fatou's flower theorem (see [Al] for additional historical information). Some quantitative results can be also found in these papers. At the end of Sections 2 and 3 some facts about parabolic iterated function systems are proven.

We end this section with two terminologies. Given two sets $A, B \subset I\!\!R^d$, we denote

$$\mathrm{dist}(A,B) = \inf\{||a-b|| : (a,b) \in A \times B\} \text{ and } \mathrm{Dist}(A,B) = \sup\{||a-b|| : (a,b) \in A \times B\}.$$

2. The case $d \geq 3$

As we mentioned in the introduction, it is known (see [BP] and [Ha]) that in every dimension $d \geq 3$ each C^1 conformal homeomorphism A defined on an open connected subset of $I\!\!R^d$ extends to the entire space $I\!\!R^d$ and takes on the form

$$A = \eta D \circ i_{a,r} + b, \tag{2.1}$$

where $0 < \eta \in R$ is a positive scalar, D is a linear isometry of $I\!\!R^d$, $i_{a,r}$ is either the inversion with respect to some sphere centered at a point a and with radius r, or the identity map, and $b \in I\!\!R^d$. If $i_{a,r}$ is an inversion, then for every $z \in I\!\!R^d$

$$||A'(z)|| = \frac{\eta r^2}{||z-a||^2}.$$

Definition 2.1. *We say a conformal map $A : I\!\!R^d \to I\!\!R^d$ is parabolic provided it has a fixed point $\omega \in I\!\!R^d$ such that $||A'(\omega)|| = 1$ and there is a point $\xi \in I\!\!R^d \setminus \{\omega\}$ and $\lim_{n\to\infty} A^n\xi = \omega$.*

If A is a conformal map and fixes ω, then setting

$$\tilde{A} = i_{\omega,1}^{-1} \circ A \circ i_{\omega,1} = i_{\omega,1} \circ A \circ i_{\omega,1},$$

we have $\tilde{A}$ is conformal and $\tilde{A}(\infty) = \infty$. Therefore,

$$\tilde{A} = \lambda D + c,$$

where $\lambda > 0$, D is an orthonormal matrix, and $c \in I\!\!R^d$. From now on, without loss of generality, we will assume that $\omega = 0$, *i.e.*, ω is the origin and we will write i for $i_{0,1}$.

Lemma 2.2. *If $A : I\!\!R^d \to I\!\!R^d$ is a parabolic conformal map and if λ is the scalar involved in the formula for $\tilde{A}$, then $\lambda = 1$.*

Proof. If $\lambda < 1$, then $\tilde{A} : I\!\!R^d \to I\!\!R^d$ is a strict contraction and due to Banach's contraction principle, it has a fixed point $b \in I\!\!R^d$ such that $\lim_{n\to\infty} \tilde{A}^n(z) = b$ for every $z \in I\!\!R^d$. However, this is a contradiction, since $\lim_{n\to\infty} \tilde{A}^n(i(\xi)) = \infty$. Thus, $\lambda \geq 1$. Assume $\lambda > 1$. Then for every $z \in I\!\!R^d \setminus \{0\}$,

$$\begin{aligned}||A'(z)|| &= ||i'(\tilde{A}(i(z))\tilde{A}'(i(z))i'(z)|| = \lambda||\tilde{A}(i(z))||^{-2}||z||^{-2} = \lambda||z||^{-2}||\lambda D(||z||^{-2}(z)) + c||^{-2} \\ &= \lambda||(\lambda||z||^{-1}D(z) + c||z||)||^{-2} = \lambda^{-1}[||D(z/||z||) + (||z||/\lambda)c||]^{-2}.\end{aligned}$$

Since $\lim_{z\to 0} ||z|| = 0$ and since $||D(z)|| = ||z||$, we deduce that $||A'(0)|| = \lim_{z\to 0} ||A'(z)|| = \lambda^{-1} < 1$. This contradiction shows that ≤ 1, and consequently $\lambda = 1$. The proof is complete. ■

Next, we want to estimate the rate at which $\tilde{A}^n(z)$ goes to $+\infty$.

Lemma 2.3. *If $A : \mathbb{R}^d \to \mathbb{R}^d$ is a parabolic conformal map, then there exists a non-zero vector $b \in \mathbb{R}^d$ and a positive constant κ such that for every $z \in \mathbb{R}^d$ and every positive integer n*

$$||\tilde{A}^n z - nb|| \le ||z|| + \kappa.$$

Proof. By a straightforward induction, we get

$$\tilde{A}^n z = D^n z + \sum_{j=0}^{n-1} D^j(c).$$

Write $c = b + a$, where b is a fixed point (*a priori* perhaps 0) of D and a belongs to W, the orthogonal complement of the vector space of the fixed points of D. Since $\lim_{n\to\infty} \tilde{A}^n(i(\xi)) = \infty$, W is not the trivial subspace of $\mathbb{R}^d$. In addition, $D(W) = W$ and $D - \mathrm{Id} : W \to W$ is invertible. Since

$$(D - \mathrm{Id})\left(\sum_{j=0}^{n-1} D^j(a)\right) = D^n a - a$$

and since $||D^n a - a|| \le 2||a||$, we therefore conclude that for every $n \ge 1$

$$\left\|\sum_{j=0}^{n-1} D^j(a)\right\| \le 2||a|| \cdot \left\|(D - \mathrm{Id})|_W^{-1}\right\|.$$

Hence,

$$||\tilde{A}^n z - nb|| = \left\|D^n z + \sum_{j=0}^{n-1} D^j(a)\right\| \le ||z|| + 2\left\|(D - \mathrm{Id})|_W^{-1}\right\| \cdot ||a||.$$

Again, since $\lim_{n\to\infty} \tilde{A}^n(i(\xi)) = \infty$, we finally conclude that $b \ne 0$ and the proof is complete. ■

As an immediate consequence of this lemma we get the following.

Corollary 2.4. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a parabolic conformal map. For every compactum $F \subset \mathbb{R}^d$, there exists a constant $B_F \ge 1$ and integer $M_F \in \mathbb{N}$ such that for every $n \ge M_F$ and every $z \in F$*

$$B_F^{-1} n \le ||\tilde{A}^n z|| \le B_F n.$$

Lemma 2.5. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a parabolic conformal map. For every compactum $L \subset \mathbb{R}^d \setminus \{0\}$, there exist a constant $C_{L,1} \ge 1$ and integer $N_L \in \mathbb{N}$ such that for every $n \ge N_L$ and every $z \in L$*

$$C_{L,1}^{-1} n^{-2} \le ||(A^n)'(z)|| \le C_{L,1} n^{-2} \text{ and } \mathrm{diam}(A^n(L)) \le C_{L,1} n^{-2}$$

Proof. By the Chain Rule, we find for every $z \in \mathbb{R}^d \setminus \{0\}$

$$||(A^n)'(z)|| = ||i'(\tilde{A}^n(i(z)))|| \cdot ||(\tilde{A}^n)'(i(z))|| \cdot ||i'(z)|| = ||\tilde{A}^n(i(z))||^{-2} ||z||^{-2}.$$

For every $z \in L$, $\mathrm{Dist}^{-2}(0, L) \leq ||z||^{-2} \leq \mathrm{dist}^{-2}(0, L)$, and in view of Corollary 2.4, if $n \geq M_{i(L)}$, then $B_{i(L)}^{-1} n \leq ||\tilde{A}^n z|| \leq B_{i(L)} n$. Consequently, if $z \in L$ and $n \geq M_{i(L)}$, we have

$$\Big(B_{i(L)} \mathrm{Dist}(0, L)\Big)^{-2} n^{-2} \leq ||(A^n)'(z)|| \leq B_{i(L)}^2 \mathrm{dist}^{-2}(0, L) n^{-2}$$

and the proof is complete. ■

Lemma 2.6. *Let $A : I\!\!R^d \to I\!\!R^d$ be a parabolic conformal map. For every compactum $L \subset I\!\!R^d \setminus \{0\}$, there exists a constant $C_{L,2} \geq 1$ such that for all integers k, n with $n \geq k \geq 1$,*

$$\mathrm{Dist}(A^k(L), A^n(L)) \leq C_{L,2} \left| k^{-1} - (n+1)^{-1} \right|$$

and

$$\mathrm{Dist}(A^n(L), 0) \leq C_{L,2} n^{-1}.$$

Proof. Let us start with the second inequality. If $n \geq M_{i(L)}$ and $z \in L$, then, by Corollary 2.4, we get $||A^n z|| = ||\tilde{A}^n(i(z))||^{-1} \leq B_{i(L)} n^{-1}$ and the second inequality follows provided $C_{L,2}$ is sufficiently large.

Towards obtaining the first inequality, for every set $M \subset I\!\!R^d$, let $\mathrm{conv}(M)$ denote the convex hull of M. Obviously, $\mathrm{conv}(M) \subset B(M, \mathrm{diam}(M))$ and $\mathrm{diam}(\mathrm{conv}(M)) = \mathrm{diam}(M)$. By using Lemma 2.3, we have for every $u \in L$ and $n \in I\!\!N$,

$$\begin{aligned} ||\tilde{A}^{n+1}(i(u)) - \tilde{A}^n(i(u))|| &\leq ||\tilde{A}^{n+1}(i(u)) - (n+1)b - (\tilde{A}^n(i(u)) - nb) + b|| \\ &\leq 2(||i(u)|| + \kappa) + ||b|| \leq 2(\mathrm{Dist}(0, i(L)) + \kappa) + ||b|| := M. \end{aligned}$$

Next, choose a positive integer N_0 such that $\mathrm{Dist}(0, \mathrm{conv}(\cup_{t \geq N_0} \tilde{A}^t(i(L)))) = H > 0$ and $N_0 ||b|| > \mathrm{Dist}(0, i(L)) + \kappa + ||b|| := M$. We claim there is a positive constant C such that if $u, v \in L, k \geq N_0$ and $j \geq 0$, then

$$||A^{k+j+1}(v) - A^{k+j}(u)|| \leq C \frac{1}{(k+j+1)^2}.$$

In order to see this, note that

$$\begin{aligned} &||A^{k+j+1}(v) - A^{k+j}(u)|| \leq \\ &\quad \leq ||i(\tilde{A}^{k+j+1}(i(v))) - i(\tilde{A}^{k+j+1}(i(u)))|| + ||i(\tilde{A}^{k+j+1}(i(u))) - i(\tilde{A}^{k+j}(i(u)))|| \\ &\quad \leq \sup\{||i'(w)|| : w \in [\tilde{A}^{k+j+1}(i(v)), \tilde{A}^{k+j+1}(i(u))]\} ||\tilde{A}^{k+j+1} i((v)) - \tilde{A}^{k+j+1} i((u))|| \\ &\qquad + \sup\{||i'(w)|| : w \in [\tilde{A}^{k+j}(i(u)), \tilde{A}^{k+j+1}(i(u))]\} ||\tilde{A}^{k+j+1} i((u)) - \tilde{A}^{k+j} i((u))|| \\ &\quad \leq \mathrm{diam}(i(L)) \sup\{||w||^{-2} : w \in [\tilde{A}^{k+j+1}(i(v)), \tilde{A}^{k+j+1}(i(u))]\} \\ &\qquad + M \sup\{||w||^{-2} : w \in [\tilde{A}^{k+j}(i(u)), \tilde{A}^{k+j+1}(i(u))]\} \end{aligned}$$

Now, if $w \in [\tilde{A}^{k+j+1}(i(v)), \tilde{A}^{k+j+1}(i(u))]$, then by Lemma 2.3, $||w - (k+j+1)b|| \leq \mathrm{Dist}(0, i(L)) + \kappa$ and $||w|| \geq (k+j+1)[||b|| - (\mathrm{Dist}(0, i(L)) + \kappa)/N_0]$. Also, since $||\tilde{A}^{k+j}(i(u)) - (k+j+1)b|| \leq ||i(u)|| + \kappa + ||b||$, if $w \in [\tilde{A}^{k+j}(i(u)), \tilde{A}^{k+j+1}(i(u))]$, then $||w - (k+j+1)b|| \leq \mathrm{Dist}(0, i(L)) + \kappa + ||b||$ and $||w|| \geq (k+j+1)[||b|| - (\mathrm{Dist}(0, i(L)) + \kappa + ||b||)/N_0] \geq (k+j+1)[||b|| - M/N_0]$.

Combining these inequalities establishes our claim.

Therefore, if $N_0 \le k \le n$ we have

$$\begin{aligned}\mathrm{Dist}(A^k(L), A^n(L)) &\le \sum_{j=0}^{n-k-1} \mathrm{Dist}\Big(A^{k+j+1}(L), A^{k+j}(L)\Big) \\ &\le \sum_{j=0}^{n-k} C(k+j)^{-2} \le C_{L,2}(k^{-1} - (n+1)^{-1})\end{aligned}$$

for some constant $C_{L,2} \ge 1$. Clearly, increasing $C_{L,2}$ appropriately, we see that the last inequality is also true for all $1 \le k \le n$. The proof of the first part of our lemma is thus complete. ■

Lemma 2.7. *For every compactum $L \subset \mathbb{R}^d \setminus \{0\}$ there exist a constant $C_{L,3} \ge 1$ and an integer $q \ge 0$ such that for all $k \ge 1$ and all $n \ge k+q$*

$$\mathrm{dist}(A^k(L), A^n(L)) \ge C_{L,3}\Big(k^{-1} - n^{-1}\Big)$$

and

$$\mathrm{dist}(A^n(L), 0) \ge C_{L,3} n^{-1}.$$

Proof. First, notice that it follows from Lemma 2.3 that if $w, z \in i(L)$ and $k, n \in N$, then

$$(n-k)||b|| - 2(\mathrm{Dist}(0, i(L)) + \kappa) \le ||\tilde{A}^n(w) - \tilde{A}^k(z)||.$$

Therefore, there is a positive integer q_0 such that if $n - k \ge q_0$, then $||\tilde{A}^n(w) - \tilde{A}^k(z)|| \ge (1/2)||b||(n-k)$. Let N_0 be as in the proof of Lemma 2.6 and $M_{i(L)}$ be as in Corollary 2.4. Let $k, n \ge N_1 = \max\{N_0, M_{i(L)}\}$. Consider two arbitrary points $z, w \in i(L)$ and parametrize the line segment γ joining $\tilde{A}^k(z)$ and $\tilde{A}^n(w)$ as

$$\gamma(t) = \tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)),\ t \in [0,1].$$

The curve $i(\gamma)$ is a subarc of either a cirlce or a line and let $l(i(\gamma))$ be its length. We have

$$\begin{aligned}l(i(\gamma)) &= \int_0^1 ||(i \circ \gamma)'(t)||dt = \int_0^1 ||i'(\gamma(t))||||\gamma'(t)||dt = ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \int_0^1 ||\gamma(t)||^{-2}dt \\ &= ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \int_0^1 ||\tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)||^{-2}dt \\ &\ge ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \int_0^1 \Big(||\tilde{A}^k(z)|| + t||\tilde{A}^n(w) - \tilde{A}^k(z)||\Big)^{-2}dt \\ &= ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \cdot ||\tilde{A}^n(w) - \tilde{A}^k(z)||^{-1} \int_{||\tilde{A}^k(z)||}^{||\tilde{A}^k(z)||+||\tilde{A}^n(w)-\tilde{A}^k(z)||} u^{-2}du \qquad (2.2) \\ &= ||\tilde{A}^k(z)||^{-1} - \Big(||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)||\Big) \\ &= \frac{||\tilde{A}^n(w) - \tilde{A}^k(z)||}{||\tilde{A}^k(z)|| \cdot \Big(||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)||\Big)}\end{aligned}$$

We have $||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \le B_{i(L)}k + C_{i(L),1}(1/k - 1/(n+1))$. So, there is a constant U such that $||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \le Un$. In view of Corollary 2.4, there is a constant Q_0 such that

$$l(i(\gamma)) \ge Q_0 \frac{||\tilde{A}^n(w) - \tilde{A}^k(z)||}{kn}.$$

Thus, there is a constant Q such that if $k \ge N_1$ and $n \ge k + q_0$, then

$$l(i(\gamma)) \ge Q(k^{-1} - n^{-1}). \tag{2.3}$$

If $i(\gamma)$ is a line segment, then

$$||A^n(i(w)) - A^k(i(z))|| = l(i(\gamma)) \ge Q(k^{-1} - n^{-1}). \tag{2.4}$$

If, however, $i(\gamma)$ is an arc of a circle, then consider the ray

$$g(t) = \tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)),\ t \in (-\infty, 0].$$

Proceeding exactly as in the formula (2.2) and using the estimate $||g(t)|| \le ||\tilde{A}^k(z)|| - t||\tilde{A}^n(w) - \tilde{A}^k(z)||$, we get

$$l(i(g)) \ge \int_{||\tilde{A}^k(z)||}^{\infty} u^{-2}du = ||\tilde{A}^k(z)||^{-1}.$$

And applying Corollary 2.4 we get $l(i(g)) \ge B_{i(L)}^{-1}k^{-1} \ge B_{i(L)}(k^{-1} - n^{-1})$. Therefore, invoking (2.3), we deduce that both arcs joining the points $A^k(i(z))$ and $A^n(i(w))$ on the circle $i(\{\tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)) : t \in I\!R \cup \{\infty\}\})$ have the length $\ge \min\{B_{i(L)}, Q\}(k^{-1} - n^{-1})$. Thus, taking also in account (2.4), we see there is a constant P_0 such that if $k, n \ge N_1$ and $n - k \ge q_0$, then

$$\mathrm{dist}(A^k(L), A^n(L)) \ge P_0(k^{-1} - n^{-1}).$$

Since 0 is not an element of $\cup_{j=1}^{N_1} A^j(L)$, and since it follows from Lemma 2.3 that $A^k(L) \to 0$ as $k \to \infty$, there is a constant $C_{L,3}$ such that the first part of the conclusion of the lemma holds. Applying the proven part of the lemma, we conclude that

$$\mathrm{dist}(A^n(L), 0) = \lim_{k\to\infty} \mathrm{dist}(A^n(L), A^k(L)) \ge \lim_{k\to\infty} C_{L,3}(n^{-1} - k^{-1}) = C_{L,3}n^{-1}.$$

The proof is complete. ■

We end this section we by proving the following two results concerning general parabolic IFS in dimension $d \ge 3$. The first is a straightforward consequence of Lemma 2.3.

First, let us note that Lemma 2.6 shows that a conformal parabolic map in $I\!R^d, d \ge 3$ has a unique fixed point.

Proposition 2.8. *If $\{\phi_i : X \to X\}_{i\in I}$ is an at least 3-dimensional parabolic conformal IFS (I is allowed to be infinite), then x_i, the only fixed point of a parabolic map ϕ_i, belongs to ∂X.*

Proof. In view of Lemma 2.3, for every $R > 0$ large enough and every $n \geq 1$, the set $\tilde{\phi}_i(\{z : ||z|| > R\})$ is not contained in $\{z : ||z|| > R\}$. Consequently, for every neighbourhood U of x_i, the set $\phi_i^n(U)$ does not converge to x_i. Since however $\lim_{n\to\infty} \phi_i^n(X) = x_i$, the point x_i cannot belong to Int$X$. The proof is complete. ■

In [U2] we have demonstrated that in the case $d \geq 3$ the Bounded Distortion Property (1d) and the property (1e) are satisfied automatically. Because of the importance of these properties for our geometric considerations in Section 4 and for the sake of completeness, we present below their proof taken from [U2].

Theorem 2.9. *If $\{\phi_i\}_{i\in I}$ is a collection of maps satisfying condition (3), then the conditions (7) and (8) are also satisfied, perhaps with a smaller set V and a sufficiently high iterate S^q of S. The property (8) takes on the following stronger form*

$$\Big| ||\phi'_\omega(y) - ||\phi'_\omega(x)|| \Big| \leq K||\phi'_\omega|| ||y - x|| \tag{2.5}$$

for all hyperbolic words $\omega \in I^*$, all $x, y \in V$ and some sufficiently large K.

Proof. Let U be an open neighbourhood of X such that $\text{dist}(U, \partial V) > 0$. Fix a hyperbolic word $\omega \in I^*$. In view of (2.1) there exist $\lambda_\omega > 0$, a linear isometry A_ω, an inversion (or the identity map) $i_\omega = i_{a_\omega, r_\omega}$ and a vector $b_\omega \in I\!R^d$ such that $\phi_\omega = \lambda_\omega A_\omega \circ i_\omega + b_\omega$. In case when i_ω is the identity map the statement of our theorem is obvious. So, we may assume that i_ω is an inversion. Then for every $z \in I\!R^d$

$$||\phi'_\omega(z)|| = \frac{\lambda_\omega r_\omega^2}{||z - a_\omega||^2}.$$

Hence, for all $x, y \in I\!R^d$

$$\frac{||\phi'_\omega(y)||}{||\phi'_\omega(x)||} = \frac{||x - a_\omega||^2}{||y - a_\omega||^2}. \tag{2.6}$$

Since $\phi_\omega(V) \subset V$, $a_\omega \notin V$. Therefore, for all $x, y \in U$

$$\frac{||x - a_\omega||}{||y - a_\omega||} \leq \frac{||x - y|| + ||y - a_\omega||}{||y - a_\omega||} = 1 + \frac{||x - y||}{||y - a_\omega||} \leq 1 + \frac{\text{diam}(U)}{\text{dist}(U, \partial V)} \tag{2.7}$$

Thus,

$$\frac{||\phi'_\omega(y)||}{||\phi'_\omega(x)||} \leq \left(1 + \frac{\text{diam}(U)}{\text{dist}(U, \partial V)}\right)^2.$$

and condition (7) holds. In order to prove the second part we may assume without loosing generality that $||\phi'_\omega(x)|| \le ||\phi'_\omega(y)||$. Using (2.6) and (2.7), we then get

$$\begin{aligned}
\Big| ||\phi'_\omega(y)|| - ||\phi'_\omega(x)|| \Big| &\le ||\phi'_\omega|| \left(\frac{||\phi'_\omega(y)||}{||\phi'_\omega(x)||} - 1 \right) = ||\phi'_\omega|| \left(\frac{||x-a_\omega||^2}{||y-a_\omega||^2} - 1 \right) \\
&= ||\phi'_\omega|| \left(\frac{||x-a_\omega||}{||y-a_\omega||} - 1 \right) \left(\frac{||x-a_\omega||}{||y-a_\omega||} + 1 \right) \\
&\le ||\phi'_\omega|| \left(2 + \frac{\mathrm{diam}(U)}{\mathrm{dist}(U,\partial V)} \right) \frac{||x-y||}{||y-a_\omega||} \\
&\le ||\phi'_\omega|| \left(2 + \frac{\mathrm{diam}(U)}{\mathrm{dist}(U,\partial V)} \right) \frac{||x-y||}{\min\{||y-a_\omega||, ||x-a_\omega||\}} \\
&\le \left(2 + \frac{\mathrm{diam}(U)}{\mathrm{dist}(U,\partial V)} \right) \frac{1}{\mathrm{dist}(U,\partial V)} ||\phi'_\omega|| ||y-x||.
\end{aligned}$$

Now cover X by finitely many balls with a positive distance to ∂V. Join them by smooth compact arcs contained in V to obtain a connected set M whose closure is contained in V. Form the new set U, an open connected neighbourhood of X with a positive distance to the boundary of V, by adding to M sufficiently small open neighbourhoods of these compact arcs. We may require these neighbourhoods to be topological closed balls (in $I\!R^d$) with smooth boundaries. Finally, the boundary of U itself can be taken to be smooth and combining (3) and (4) along with proven distortion property we can easily deduce that $\phi_\omega(U) \subset U$ if only $|\omega|$ is large enough. The proof is complete. ■

3. **The plane case,** $d = 2$

We call a holomorphic map ϕ, defined around a point $\omega \in \mathbb{C}$, simple parabolic if $\phi(\omega) = \omega$, $\phi'(\omega) = 1$ and ϕ is not the identity map. Then on a sufficiently small neighbourhood of ω, the map ϕ has the following Taylor series expansion:

$$\phi(z) = z + a(z-\omega)^{p+1} + b(z-\omega)^{p+2} + \dots$$

with some integer $p \ge 1$ and $a \in \mathbb{C} \setminus \{0\}$. Being in the circle of ideas related to Fatou's flower theorem (see [Al] for extended historical information), we now want to analyze qualitatively and especially quantitatively the behaviour of ϕ in a sufficiently small neighbourhood of the parabolic point ω. Let us recall that the rays coming out from ω and forming the set

$$\{z : (a(z-\omega)^p < 0\}$$

are called attracting directions and the rays forming the set

$$\{z : (a(z-\omega)^p > 0\}$$

are called repelling directions. Fix an attractive direction, say $A = \omega + \sqrt[p]{-a^{-1}(0,\infty)}$, where $\sqrt[p]{\cdot}$ is a holomorphic branch of the pth radical defined on $\mathbb{C} \setminus a^{-1}(0,\infty)$. In order to simplify our analysis let us change the system of coordinates with the help of the affine map $\rho(z) = \sqrt[p]{-a^{-1}} + \omega$. We then get

$$\phi_0(z) = \rho^{-1} \circ \phi \circ \rho(z) = z - z^{p+1} + b\sqrt[p]{-a^{-1}}z^{p+2} + \dots$$

and $\rho^{-1}(A) = (0, \infty)$ is an attractive direction for ϕ_0. We want to analyze the behaviour of ϕ_0 on an appropriate neighbourhoods of $(0, \epsilon)$, for $\epsilon > 0$ sufficiently small. In order to do it, similarly as in the previous section, we conjugate ϕ_0 on $\mathbb{C} \setminus (-\infty, 0]$ to a map defined "near" infinity. Precisely, we consider $\sqrt[p]{\cdot}$, the holomorphic branch of the pth radical defined on $\mathbb{C} \setminus (-\infty, 0]$ and leaving the point 1 fixed. Then we define the map

$$H(z) = \frac{1}{\sqrt[p]{z}}$$

and consider the conjugate map

$$\tilde{\phi} = H^{-1} \circ \phi_0 \circ H.$$

Straighforward calculations show that

$$\tilde{\phi}(z) = z + 1 + O(|z|^{-\frac{1}{p}}) \tag{3.1}$$

and

$$\tilde{\phi}'(z) = 1 + O(|z|^{-\frac{p+1}{p}}). \tag{3.2}$$

Given now a point $x \in (0, \infty)$ and $\alpha \in (0, \pi)$, let

$$S(x, \alpha) = \{z : -\alpha < \arg(z - x) < \alpha\}.$$

The formula (3.1) shows that for every $\alpha \in (0, \pi)$ there exists $x(\alpha) \in (0, \infty)$ such that for every $x \geq x(\alpha)$

$$\overline{\tilde{\phi}(S(x, \alpha))} \subset S\left(x + \frac{1}{2}, \alpha\right), \tag{3.3}$$

$$|z| \geq B^p \tag{3.4}$$

and

$$\mathrm{Re}(\tilde{\phi}(z)) \geq \mathrm{Re}(z) + \frac{1}{2} \tag{3.5}$$

for all $z \in (S(x, \alpha)$, where B is the constant responsible for $O(|z|^{-\frac{1}{p}})$ in (3.1). The following lemma immediately follows from (3.4), (3.1) and (3.5) by a straightforward induction.

Lemma 3.1. *For every compactum $F \subset S(x(\alpha), \alpha)$ there exists a constant $C_F \geq 1$ such that for every $z \in F$ and every $n \geq 1$*

$$C_F^{-1} n \leq |\tilde{\phi}^n(z)| \leq C_F n.$$

Using a straightforward induction, one gets from (3.1) and Lemma 3.1 that

$$\tilde{\phi}^n(z) = z + n + O\Big(\max\{n^{1-\frac{1}{p}}, \log n\}\Big) \tag{3.6}$$

and

$$\tilde{\phi}^n(z) = \tilde{\phi}^k(z) + (n-k) + O\Big(|n^{1-\frac{1}{p}} - k^{1-\frac{1}{p}}|\Big), \tag{3.7}$$

where the constant involved in $''O''$ depends only on F and ϕ_0. Using Lemma 3.1 and (3.2) we shall prove the following.

Lemma 3.2. *For every compactum $F \subset S(x(\alpha), \alpha)$ there exists a constant $D_F \geq 1$ such that for every $z \in F$ and every $n \geq 1$*

$$D_F^{-1} \leq |(\tilde{\phi}^n)'(z)| \leq D_F.$$

Proof. For every $z \in S(x(\alpha), \alpha)$ let $g(z) = \tilde{\phi}'(z) - 1$. By the Chain Rule, we have for every $z \in S(x(\alpha), \alpha)$ and every $n \geq 1$

$$(\tilde{\phi}^n)'(z) = \prod_{j=0}^{n-1} \phi'(\tilde{\phi}^j(z)) = \tilde{\phi}'(z) \prod_{j=1}^{n-1} \Big(1 + g(\tilde{\phi}^j(z))\Big).$$

Using (3.2) and and the right-hand side of of Lemma 3.1, we get for every $z \in F$ and every $j \geq 1$ that

$$|g(\tilde{\phi}^j(z))| = O\Big(|\tilde{\phi}^j(z)|^{-\frac{p+1}{p}}\Big) \leq C_F^{-\frac{p+1}{p}} O\Big(j^{-\frac{p+1}{p}}\Big).$$

Since the series $\sum_{j=1}^{\infty} j^{-\frac{p+1}{p}}$ converges, the proof is complete. ■

For every $x \in (0, \infty)$ and $\alpha \in (0, \pi)$ let

$$S_0(x, \alpha) = H(S(x, \alpha))$$

and

$$S_\phi^A(x, \alpha) = \rho \circ H(S(x, \alpha)) = \rho(S_0(x, \alpha)).$$

The regions $S_0(x, \alpha)$ and $S_\phi^A(x, \alpha)$ look like flower petals containning symmetrically a part of the ray $(0, \infty)$ and the ray $A = \omega + \sqrt[p]{-a^{-1}(0, \infty)}$ respectively and form with these rays two "angles" of measures α/π at the point 0 and ω respectively. We recall from the previous section that $\mathrm{conv}(M)$ denotes the convex hull of the set M. Combining Lemma 3.1 and Lemma 3.2 we deduce the following.

Lemma 3.3. *For every $\alpha \in (0, \pi/2)$ and for every compactum $F \subset S(x(\alpha), \alpha)$ there exists a constant $C_F \geq 1$ such that for every $n \geq 1$*

$$C_F^{-1} n \leq \mathrm{dist}(0, \mathrm{conv}(\tilde{\phi}^n(F))) \leq \mathrm{Dist}(0, \mathrm{conv}(\tilde{\phi}^n(F))) \leq C_F n.$$

Let us now use the properties of the map $\tilde{\phi}$ and establish useful facts about the map ϕ.

Lemma 3.4. *For every compactum $L \subset S_\phi^A(x,\alpha)$ there exists a constant $C_L \geq 1$ such that for every $z \in L$ and every $n \geq 1$*

$$C_L^{-1} n^{-\frac{p+1}{p}} \leq |(\phi^n)'(z)|, \ \operatorname{diam}(\phi_n(L)) \leq C_L n^{-\frac{p+1}{p}}.$$

Proof. It of course suffices to prove this lemma for ϕ replaced by ϕ_0. Since $H^{-1}(L)$ is a compact subset of $S(x(\alpha),\alpha)$ and since $H'(z) = -\frac{1}{p} z^{-\frac{p+1}{p}}$, using the Chain Rule along with Lemma 3.1, Lemma 3.2, and (3.4), we deduce that for every $z \in L$ and every $n \geq 1$

$$\begin{aligned}
|(\phi_0^n)'(z)| &= |(H \circ \tilde{\phi}^n \circ H^{-1})'(z)| = |H'(\tilde{\phi}^n(H^{-1}(z))| \cdot |(\tilde{\phi}^n)'(H^{-1}(z)| \cdot |(H^{-1})'(z)| \\
&= \frac{1}{p} |\tilde{\phi}^n(H^{-1}(z))|^{-\frac{p+1}{p}} |(\tilde{\phi}^n)'(H^{-1}(z))| p|z|^{-(p+1)} \\
&\leq D_{H^{-1}(L)}^{\frac{p+1}{p}} C_{H^{-1}(L)} (\operatorname{dist}(0, H^{-1}(L)))^{-(p+1)} n^{-\frac{p+1}{p}}
\end{aligned}$$

and

$$|(\phi_0^n)'(z)| \leq D_{H^{-1}(L)}^{-\frac{p+1}{p}} C_{H^{-1}(L)}^{-1} \operatorname{Dist}(0, H^{-1}(L))^{-(p+1)} n^{-\frac{p+1}{p}}.$$

The proof is complete. ■

Lemma 3.5. *For every compactum $L \subset S_\phi^A(x,\alpha)$ there exists a constant $C_{L,1} \geq 1$ such that for all $k, n \geq 1$*

$$\operatorname{Dist}(\phi^k(L), \phi^n(L)) \leq C_{L,1} \left| \min(k,n)^{-\frac{1}{p}} - (\max(k,n)+1)^{-\frac{1}{p}} \right|$$

and

$$\operatorname{Dist}(\phi^n(L), \omega) \leq C_{L,1} n^{-\frac{1}{p}}.$$

Proof. It suffices again to prove this lemma for ϕ replaced by ϕ_0. Let us prove the first inequality. Without loss of generality we may assume that $n \geq k$. Since $H^{-1}(L)$ and $\operatorname{conv}(H^{-1}(L))$ are compact subsets of $S(x(\alpha),\alpha)$, using (3.1), Lemma 3.3, Lemma 3.1, and

Lemma 3.2, we can estimate for every $j \geq 0$ and all $z, \xi \in L$ as follows

$$
\begin{aligned}
|\phi_0^{k+j+1}(\xi) - \phi_0^{k+j}(z)| &\leq |\phi_0^{k+j+1}(\xi) - |\phi_0^{k+j+1}(z)| + |\phi_0^{k+j+1}(z) - \phi_0^{k+j}(z)| \leq \\
&\leq \sup\{|H'(w)| : w \in \operatorname{conv}\Big(\tilde{\phi}^{k+j+1}(H^{-1}(L))\Big)\}\operatorname{diam}\Big(\operatorname{conv}\Big(\tilde{\phi}^{k+j+1}(H^{-1}(L))\Big)\Big) + \\
&+ \Big(1 + B|\tilde{\phi}^{k+j}(H^{-1}(z))|^{-\frac{1}{p}}\Big) \sup\{|H'(w)| : w \in [\tilde{\phi}^{k+j}(H^{-1}(z)), (\tilde{\phi}^{k+j+1}(H^{-1}(z))]\} \\
&\leq \frac{1}{p}\sup\{|w|^{-\frac{p+1}{p}} : w \in \operatorname{conv}\Big(\tilde{\phi}^{k+j+1}(H^{-1}(L))\Big)\}\operatorname{diam}(\tilde{\phi}^{k+j+1}(H^{-1}(L)) + \\
&+ \frac{2}{p}\sup\{|w|^{-\frac{p+1}{p}} : w \in [\tilde{\phi}^{k+j}(H^{-1}(z)), (\tilde{\phi}^{k+j+1}(H^{-1}(z))]\} \\
&\leq \frac{1}{p}D_{H^{-1}(L)}C_{H^{-1}(L)}\operatorname{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \\
&+ \frac{2}{p}\Big(|\tilde{\phi}^{k+j+1}(H^{-1}(z))| - |\tilde{\phi}^{k+j}(H^{-1}(z))|\Big)^{-\frac{p+1}{p}} \\
&\leq \frac{1}{p}D_{H^{-1}(L)}C_{H^{-1}(L)}\operatorname{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \\
&+ \frac{2}{p}\left(C_{H^{-1}(L)}(k+j+1) - B\Big(|\tilde{\phi}^{k+j}(H^{-1}(z))|^{-\frac{1}{p}} + 1\Big)\right)^{-\frac{p+1}{p}} \\
&\leq \frac{1}{p}D_{H^{-1}(L)}C_{H^{-1}(L)}\operatorname{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \\
&+ \frac{2}{p}\left(C_{H^{-1}(L)}(k+j+1) - B\Big(C_{H^{-1}(L)}^{\frac{1}{p}}(k+j)^{-\frac{1}{p}} + 1\Big)\right) \\
&\leq \frac{1}{p}D_{H^{-1}(L)}C_{H^{-1}(L)}\operatorname{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \frac{4}{p}C_{H^{-1}(L)}^{\frac{p+1}{p}}(k+j+1)^{-\frac{p+1}{p}} \\
&= \frac{1}{p}\Big(D_{H^{-1}(L)}C_{H^{-1}(L)}\operatorname{diam}(H^{-1}(L)) + 4C_{H^{-1}(L)}^{\frac{p+1}{p}}\Big)(k+j+1)^{-\frac{p+1}{p}}
\end{aligned}
$$

where the last inequality has been written assuming that $k \geq 1$ is large enough, say $k \geq q$ and B is the constant coming from (3.1). Denote the constant appearing in the last row of the above formula by C'_L. Using also Lemma 3.4 we then get

$$
\begin{aligned}
\operatorname{Dist}(\phi_0^k(L), \phi_0^n(L)) &\leq \sum_{j=0}^{n-k-1} \operatorname{Dist}(\phi_0^{k+j}(L), \phi_0^{k+j+1}(L)) + \sum_{j=0}^{n-k} \operatorname{diam}(\phi_0^{k+j}(L)) \\
&\leq \sum_{j=0}^{n-k} C'_L(k+j)^{-\frac{p+1}{p}} = C_{L,1}(k^{-\frac{1}{p}} - (n+1)^{-\frac{1}{p}})
\end{aligned}
$$

for some constant $C_{L,1} \geq 1$. Clearly, increasing the constant $C_{L,1}$ appropriately, we see that the last inequality is also true for all $1 \leq k \leq q$. The proof of the first part of Lemma 3.6 is thus complete. The second part is a straightforward consequence of the first one. Indeed, it follows from (3.3) that $\phi^k(L)$ converges to ω if $k \to \infty$. Hence, applying the first part of the lemma, we get

$$\mathrm{Dist}(\phi^n(L),\omega)=\lim_{k\to\infty}\mathrm{Dist}(\phi^n(L),\phi^k(L))\le\lim_{k\to\infty}C_{L,1}(n^{-\frac{1}{p}}-(k+1)^{-\frac{1}{p}})=C_{L,1}n^{-\frac{1}{p}}.$$

The proof is complete. ■

Lemma 3.6. *For every compactum $L\subset S_\phi^A(x,\alpha)$ there exist a constant $C_{L,2}\le 1$ and an integer $q\ge 0$ such that for all $k\ge 1$ and $n\ge k+q$,*

$$\mathrm{dist}(\phi^k(L),\phi^n(L))\ge C_{L,2}|n^{-\frac{1}{p}}-k^{-\frac{1}{p}}|$$

and

$$\mathrm{dist}(\phi^n(L),\omega)\ge C_{L,2}n^{-\frac{1}{p}}.$$

Proof. It suffices of course to prove this lemma with ϕ replaced by ϕ_0. Consider two arbitrary points $z,\xi\in H^{-1}(L)$ and the line segment γ joining $\tilde\phi^k(z)$ and $\tilde\phi^n(\xi)$. Parametrize it as

$$\gamma(t)=\tilde\phi^k(z)+t(\tilde\phi^n(\xi)-\tilde\phi^k(z)),\ t\in[0,1].$$

Let $l(H(\gamma))$ be the length of the curve (a subarc of either a circle or a line) $H(\gamma)$. We have

$$\begin{aligned}
l(H(\gamma))&=\int_0^1|(H\circ\gamma)'(t)|dt=\int_0^1|H'(\gamma(t))||\gamma'(t)|dt\\
&=|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1|H'(\gamma(t))|dt=\frac{1}{p}|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1|\gamma(t)|^{-\frac{p+1}{p}}dt\\
&=\frac{1}{p}|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1\Big(\tilde\phi^k(z)+t(\tilde\phi^n(\xi)-\tilde\phi^k(z))\Big)^{-\frac{p+1}{p}}dt\\
&\ge\frac{1}{p}|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1\Big(|\tilde\phi^k(z)|+t|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{-\frac{p+1}{p}}dt\\
&=\frac{1}{p}\int_{|\tilde\phi^k(z)|}^{|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|}u^{-\frac{p+1}{p}}du=\Big(|\tilde\phi^k(z)|^{-\frac{1}{p}}-\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{-\frac{1}{p}}\Big) \qquad (3.8)\\
&=\frac{\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{\frac{1}{p}}-|\tilde\phi^k(z)|^{\frac{1}{p}}}{|\tilde\phi^k(z)|^{\frac{1}{p}}\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{\frac{1}{p}}}\\
&\ge C_{H^{-1}(L)}^{-\frac{1}{p}}(3C_{H^{-1}(L)})^{-\frac{1}{p}}\frac{\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{\frac{1}{p}}-|\tilde\phi^k(z)|^{\frac{1}{p}}}{k^{\frac{1}{p}}n^{\frac{1}{p}}},
\end{aligned}$$

where the last inequality has been written due to Lemma 3.1. By the Mean Value Theorem there exists $\eta \in [|\tilde{\phi}^k(z)|, |\tilde{\phi}^k(z)| + |\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)|]$ such that

$$\begin{aligned}
&\Big(|\tilde{\phi}^k(z)| + |\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)|\Big)^{\frac{1}{p}} - |\tilde{\phi}^k(z)|^{\frac{1}{p}} = \\
&\qquad = \frac{1}{p}|\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)|\eta^{\frac{1-p}{p}} \geq \frac{1}{p}|\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)|\Big(|\tilde{\phi}^k(z)| + |\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)|\Big)^{\frac{1-p}{p}} \\
&\qquad \geq \frac{1}{p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}|\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)|
\end{aligned} \tag{3.9}$$

Now, in view of (3.6), $\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z) = \xi - z + O(\max\{n^{1-\frac{1}{p}}, \log n\})$. Hence

$$|\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)| \geq \mathrm{diam}(H^{-1}(L)) + (n-k) - O\Big(\max\{n^{1-\frac{1}{p}}, \log n\}\Big) \geq \frac{1}{2}(n-k)$$

if only $n-k$ is large enough, say $n-k \geq q$. Using this, (3.8) and (3.9), if $n \geq k+q$, then

$$l(H(\gamma)) \geq \frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}} \frac{(n-k)n^{1-\frac{1}{p}}}{k^{\frac{1}{p}}n^{\frac{1}{p}}} \tag{3.10}$$

Since $t \leq t^{\frac{1}{p}}$ for $t \in [0,1]$, we get $1-t \geq 1-t^{\frac{1}{p}}$ for these t, and consequently $1-\frac{k}{n} \geq 1-\left(\frac{k}{n}\right)^{\frac{1}{p}}$ or $\frac{n-k}{n} \geq 1-\left(\frac{k}{n}\right)^{\frac{1}{p}}$. Multiplying this last inequality by $n^{\frac{1}{p}}$, we get $(n-k)n^{\frac{1-p}{p}} \geq n^{\frac{1}{p}} - k^{\frac{1}{p}}$. Combining this and (3.10) yields

$$l(H(\gamma)) \geq \frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big). \tag{3.11}$$

If $H(\gamma)$ is a segment of the line, then

$$|\phi_0^k(H(z)) - \phi_0^n(H(\xi))| = l(H(\gamma)) \geq \frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big). \tag{3.12}$$

If however $H(\gamma)$ is an arc of a circle, then consider the curve

$$g(t) = \tilde{\phi}^k(z) + t(\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)),\ t \in (-\infty, 0].$$

Proceeding exactly as in the formula (3.8) with the estimate $|g(t)| \leq |\tilde{\phi}^k(z)| - t|(\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)|$, we get

$$l(H(\gamma)) \geq \frac{1}{p}\int_{|\tilde{\phi}^k(z)|}^{\infty} u^{-\frac{p+1}{p}} du = |\tilde{\phi}^k(z)|^{-\frac{1}{p}}.$$

Applying now Lemma 3.1 this gives

$$l(H(\gamma)) \geq (C_{H^{-1}(L)})^{-\frac{1}{p}}k^{-\frac{1}{p}} \geq (C_{H^{-1}(L)})^{-\frac{1}{p}}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big).$$

Therefore, invoking (3.11), we deduce that both arcs joining the points $\phi_0^k(H(z))$ and $\phi_0^n(H(z))$ on the circle $H(\{\tilde{\phi}^k(z) + t(\tilde{\phi}^n(\xi) - \tilde{\phi}^k(z)) : t \in I\!R \cup \{\infty\})$ have the length $\geq C(\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big)$, where $C = \min\left\{\frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}, C_{H^{-1}(L)})^{-\frac{1}{p}}\right\}$. Hence $|\phi_0^k(H(z)) - \phi_0^n(H(\xi))| \ge \frac{C}{\pi}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big)$. This and (3.12) imply that

$$\mathrm{dist}\Big(\phi_0^k(H(z)), \phi_0^n(H(\xi))\Big) \ge \frac{C}{\pi}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big)$$

and the proof of the first part of our lemma is complete. Since it follows from (3.3) that $\phi^k(L)$ converges to ω if $k \to \infty$, applying the proven part of the lemma, we conclude that

$$\mathrm{dist}(\phi^n(L), \omega) = \lim_{k\to\infty} \mathrm{dist}(\phi^n(L), \phi^k(L)) \ge \lim_{k\to\infty} C_{L,2}\Big(n^{-\frac{1}{p}} - k^{-\frac{1}{p}}\Big) = C_{L,2}n^{-\frac{1}{p}}.$$

The proof is complete. ■

Remark 3.7. *We would like to remark that all statements proven in this section about the map ϕ continue to be true if we replace the assumption $L \subset S_\phi^A(x(\alpha), \alpha)$ by the assumption $\phi^j(L) \subset S_\phi^A(x(\alpha), \alpha)$ for some $j \ge 0$.*

Lemma 3.8. *If $L \subset \mathbb{C} \setminus \omega$ is a compactum and $\lim_{n\to\infty} \phi^n(L) = \omega$, then there exists an attracting direction A such that for every $\alpha \in (0, \pi)$, $\phi^n(L) \subset S_\phi^A(x(\alpha), \alpha)$ for every $n \ge 0$ large enough.*

Proof. First notice that due to (3.3), if $\phi^k(L) \subset S_\phi^A(x(\alpha), \alpha)$, then $\phi^n(L) \subset S_\phi^A(x(\alpha), \alpha)$ for all $n \ge k$. Suppose now that the statement converse than that claimed in our lemma is true. Since the set of attracting directions is finite, there thus exist $\beta \in (0, \pi)$ and such that for every $n \ge k$

$$\phi^n(L) \cap \bigcup_{i=1}^p S_\phi^{A_i^+}(x(\beta), \beta) = \emptyset, \tag{3.13}$$

where $\{A_1^+, A_2^+, \ldots, A_p^+\}$ is the set of all attracting directions for ϕ at ω. Taking now $\gamma \in (\pi - \beta, \pi)$ we see that the union

$$\bigcup_{i=1}^p S_\phi^{A_i^+}(x(\beta), \beta) \cup \bigcup_{i=1}^p S_{\phi^{-1}}^{A_i^-}(x(\gamma), \gamma)$$

(A_i^- being attracting directions for ϕ^{-1}) forms a deleted neighbourhood of ω. Along with (3.13) this implies that $\phi^n(L) \subset S_{\phi^{-1}}^{A_i^-}(x(\gamma), \gamma)$ for some $i \in \{1, 2, \ldots p\}$ and all $n \ge k$. But since, by (3.3), $\lim_{n\to\infty} \phi^{-n}(S_{\phi^{-1}}^{A_i^-}(x(\gamma), \gamma)) = \omega$, we conclude that $L = \lim_{n\to\infty} \phi^{-n}(\phi^(L)) = \omega$. This contradiction finishes the proof. ■

We end this section with a result concerning parabolic IFS in dimension $d = 2$

Proposition 3.9. *If $S = \{\phi_i : X \to X\}_{i\in I}$ is a parabolic IFS and $d = 2$, then the fixed point of each parabolic element ϕ_i belongs to the boundary of X. In addition, the derivative of each parabolic element evaualted at the corresponding parabolic fixed point is a root of unity.*

Proof. Suppose that $i \in I$ is a parabolic index and that the corresponding fixed point x_i is in Int$X$. Let C_i be the component of $\mathrm{Int}(X)$ containing x_i. So, C_i is an open connected subset of $\mathbb{C}$ missing at least three points, since X is a compact subset of $\mathbb{C}$. Therefore, due to the uniformization theorem, there exists a holomorphic covering map $R : D \to C_i$ sending 0 to x_i, where $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk in $\mathbb{C}$. Since $\phi_i(x_i) = x_i$, $\phi_i(C_i) \subset C_i$. Considering, if necessary, the second iterate of ϕ_i we may assume that ϕ_i is holomorphic. Hence, all its lifts to D (*i.e.*, satisfying the equality $\phi_i \circ R = R \circ \psi$) are holomorphic. Take $\psi : D \to D$, the lift fixing the point 0. Then $\psi'(0) = \phi_i'(x_i)$, whence $|\psi'(0)| = 1$. Therefore, in view of Schwarz's lemma, $\psi : D \to D$ is a rotation with the center at 0. In particular

$$\phi_i(C_i) = \phi_i \circ R(D) = R \circ \psi(D) = R(D) = C_i.$$

This contradicts condition (4) from Section 1. Finally, suppose i is a parabolic index. If $\phi_i'(x_i)$ were not a root of unity, then the images of finitely many iterates of ϕ_i of an open cone witnessing the cone condition at x_i would cover a punctured neighborhood of X. This contradicts the fact the the boundary of X has no isolated points. ■

4. **Proofs of the main theorems**

In order to be apply the results of sections 2 and 3 we need the following. Recall for each parabolic index i, x_i is the unique fixed point of the map ϕ_i.

Proposition 4.1. *If $\{\phi_i : X \to X\}_{i\in I}$ is a parabolic IFS (I is allowed to be infinite), then for every parabolic index $i \in I$ and every $j \in I \setminus \{i\}$, we have $x_i \notin \phi_j(X)$.*

Proof. Suppose on the contrary that $x_i \in \phi_j(X)$ for some parabolic index $i \in I$ and some $j \in I \setminus \{i\}$. Then by the Cone Condition and conformality of ϕ_j, the set $\phi_j(X)$ contains a central cone with positive measure and vertex x_i. On the other hand, since ϕ_i is conformal, $X \setminus \phi_i(X)$ contains no central cone with positive measure and vertex x_i. This is a contradiction since, by the Open Set Condition, $\mathrm{Int}(\phi_i(X)) \cap \mathrm{Int}(\phi_j(X)) = \emptyset$. The proof is complete. ■

Consider a parabolic IFS, $S = \{\phi_i : X \to X\}_{i\in I}$. If S is 2-dimensional, then dealing with the family of second iterates $S^2 = \{\phi_{ij} : i, j \in I\}$, instead of S, we may assume that all the parabolic maps are holomorphic. Also, from Proposition 3.9 the derivative of each parabolic element evaluated at the corresponding parabolic fixed point, is a root of unity. Therefore, for some appropriate positive integer q, the derivative of each parabolic element of S^q evaluated at the corresponding parabolic fixed point is equal to 1. Thus, without loss of generality, we may assume that in case $d = 2$, all the parabolic elements of S are simple parabolic mappings in the sense of Section 3. Grouping now together the results of sections 2 and 3, we deduce that for any given $d \geq 2$, there exists a constant $Q \geq 1$ and an integer $q \geq 0$ such that for every parabolic index $i \in I$ there exists an integer $p_i \geq 1$ such that for every $j \in I \setminus \{i\}$ and all $n, k \geq 1$ we have

$$Q^{-1} n^{-\frac{p_i+1}{p_i}} \leq \inf_X \{||\phi_{i^n j}'(x)||\}, ||\phi_{i^n j}'||, \mathrm{diam}(\phi_{i^n j}(X)) \leq Q n^{-\frac{p_i+1}{p_i}}, \tag{4.1}$$

$$Q^{-1}n^{-\frac{1}{p_i}} \le \operatorname{dist}(x_i, \phi_{i^n j}(X)) \le \operatorname{Dist}(x_i, \phi_{i^n j}(X)) \le Q n^{-\frac{1}{p_i}}, \tag{4.2}$$

$$\operatorname{Dist}(\phi_{i^n j}(X), \phi_{i^k j}(X)) \le Q \left| \min\{k,n\}^{-\frac{1}{p_i}} - (\max\{k,n\}+1)^{-\frac{1}{p_i}} \right| \tag{4.3}$$

and, furthermore, if $|n-k| \ge q$, then

$$\operatorname{dist}(\phi_{i^n j}(X), \phi_{i^k j}(X)) \ge Q |n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}}|. \tag{4.4}$$

We also need the following.

Theorem 4.2. *If $\{\phi_i : X \to X\}_{i \in I}$ is a parabolic IFS (I is allowed to be infinite), then*

$$dim_H(J_S) > \max\left\{ \frac{p_i}{p_i+1} : i \text{ is parabolic} \right\},$$

where p_i is the integer indicated in (4.4).

Proof. Using (4.1), if we take t slightly larger than $\frac{p_i}{p_i+1}$, then $\psi(t)$ can be made as large as we like. Since $P^*(t) \ge -t\log K + \log\psi(t)$, $P^*(t) > 0$. Therefore, $h = \dim_{\mathrm{H}}(J_{S^*}) > \frac{p_i}{p_i+1}$. It therefore immediately follows from Lemma 1.2 that

$$\dim_{\mathrm{H}}(J_S) = \dim_{\mathrm{H}}(J_{S^*}) > \max\left\{ \frac{p_i}{p_i+1} : i \ \text{ is parabolic} \right\}.$$

The proof is complete. ■

If, in addition S is finite, then we conclude from (4.1) that

$$\theta_{S^*} = \max\left\{ \frac{p_i}{p_i+1} : i \ \text{ is parabolic} \right\}$$

and $\psi(\theta_{S^*}) = \infty$. This means that the system S^* is hereditarily regular and we have proved Theorem 1.4.

Lemma 4.3. *For every parabolic index $i \in I$, there exists an open cone $C_i \subset X$ with vertex x_i and such that $x_i \in \overline{J \cap C_i}$.*

Proof. In case $d \ge 3$ this is an immediate consequence of Lemma 2.3. In case $d \ge 3$ this is an immediate consequence of (3.6) and Lemma 3.8. ■

In view of Theorem 1.5 in order to prove Theorem 1.6 it suffices to demonstrate the following four lemmas assuming the finite parabolic system S satisfies the strong open set condition.

Lemma 4.4. *If $h < 1$, then $\mathcal{H}^h(J) = 0$.*

Lemma 4.5. *If $h \le 1$, then $\mathcal{P}^h(J) < \infty$.*

Lemma 4.6. *If $h > 1$, then $\mathcal{P}^h(J) = \infty$.*

Lemma 4.7. *If $h \geq 1$, then $\mathcal{H}^h(J) > 0$.*

Proof of Lemma 4.4. Let $i \in I$ be a parabolic index. Fix $j \in I \setminus \{i\}$. Since $\phi_{i^nj}(X) \subset B(x_i, r)$ if and only if $\mathrm{Dist}(x_i, \phi_{i^nj}(X)) < r$, it follows from (4.2) that if $Qn^{-\frac{1}{p_i}} < r$, then $\phi_{i^nj}(X) \subset B(x_i, r)$. Hence using (4.1) and the conformality of m, we get

$$\begin{aligned} r^{-h} m(B(x_i, r)) &\geq r^{-h} \sum_{n: Qn^{-\frac{1}{p_i}} < r} m(\phi_{i^nj}(X)) \geq r^{-h} \sum_{n > (Qr^{-1})^{p_i}} Q^{-h} n^{-\frac{p_i+1}{p_i}h} \\ &\geq Q^{-h} (\mathrm{const})\, r^{-h} (Q^{p_i} r^{-p_i})^{1 - \frac{p_i+1}{p_i}h} \geq (\mathrm{const})\, r^{-h} r^{-p_i + (p_i+1)h} \\ &= (\mathrm{const})\, r^{p_i(h-1)}. \end{aligned}$$

Since $h < 1$, this implies that $\lim_{r \to 0} r^{-h} m(B(x_i, r)) = \infty$. By Proposition 1.3, $x_i \in S^*(\infty)$, it therefore follows immediately from Lemma 4.9 in [MU1] that $\mathcal{H}^h(J_S) = \mathcal{H}^h(J_{S^*}) = 0$. The proof is finished. ■

Proof of Lemma 4.5. Fix a parabolic index $i \in I$, $j \in I \setminus \{i\}$, $n \geq 1$ and fix r, $2\mathrm{diam}(\phi_{i^nj}(X)) < r \leq 1$. Take an arbitrary point $x \in \phi_{i^nj}(X)$. It follows from (4.3) and the inequality $r > 2\mathrm{diam}(\phi_{i^nj}(X))$ that if $k \leq n$ and $\overline{Q}\Big(k^{-\frac{1}{p_i}} - n^{-\frac{1}{p_i}}\Big) < r$, where we take an appropriate constant $\overline{Q} \geq Q$, then $B(x, r) \supset \phi_{i^kj}(X)$. Hence, using (4.1),Theorem 4.2 and letting $E(x)$ denote the greatest integer in x, we get

$$\begin{aligned} m(B(x, r)) &\geq \sum_{k = E\left(\left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{-p_i}\right) + 1}^{n} m(\phi_{i^kj}(X)) \geq \sum_{k = E\left(\left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{-p_i}\right) + 1}^{n} \overline{Q}^{-h} k^{-\frac{p_i+1}{p_i}h} \\ &\geq (\mathrm{const}) \left(\left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{-p_i\left(1 - \frac{p_i+1}{p_i}h\right)} - n^{1 - \frac{p_i+1}{p_i}h} \right) \\ &\geq (\mathrm{const}) \left(\left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{(p_i+1)h - p_i} - n^{-\frac{1}{p_i}((p_i+1)h - p_i)} \right). \end{aligned} \tag{4.5}$$

It follows from the Mean Value Theorem that there exists some η with $n^{-\frac{1}{p_i}} \leq \eta \leq \overline{Q}^{-1} r + n^{-\frac{1}{p_i}}$ such that

$$\begin{aligned} \left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{(p_i+1)h - p_i} - n^{-\frac{1}{p_i}((p_i+1)h - p_i)} &= ((p_i+1)h - p_i)(\overline{Q}^{-1} r)\eta^{(p_i+1)h - p_i - 1} \\ &= ((p_i+1)h - p_i)\overline{Q}^{-1} r \eta^{(p_i+1)(h-1)} \\ &\geq ((p_i+1)h - p_i)\overline{Q}^{-1} r \left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{(p_i+1)(h-1)}. \end{aligned} \tag{4.6}$$

But, by our constraints on r and by (4.1), $n^{-\frac{1}{p_i}} \le Q^{\frac{1}{p_i+1}}\mathrm{diam}^{\frac{1}{p_i+1}}(\phi_{i^nj}(X)) \le (1/2)Q^{\frac{3}{p_i+1}}r^{\frac{1}{p_i+1}}$. Thus, combining this, (4.6) and (4.5), we get

$$
\begin{aligned}
m(B(x,r)) &\ge (\mathrm{const})\, r\left(\overline{Q}^{-1}r + n^{-\frac{1}{p_i}}\right)^{(p_i+1)(h-1)} \\
&\ge (\mathrm{const})\, r\left(r^{\frac{1}{p_i+1}}\right)^{(p_i+1)(h-1)} = (\mathrm{const})\, r^h.
\end{aligned}
$$

Therefore, the proof follows by applying Theorem 2.5(2) in [MU2] with $\xi = 1$, $\gamma = 1$ and F consisting of hyperbolic indices. ■

Proof of Lemma 4.6. Fix a parabolic index $i \in I$. Since the system is finite, by applying Proposition 4.1, there is some $R > 0$ such that if $0 < r < R$, then $B(x_i, r)$ does not intersect $\phi_j(X)$, for any $j \ne i$. Fix such a radius r. Using (4.2) and (4.1), we derive

$$
\begin{aligned}
r^{-h}m(B(x_i,r)) &\le r^{-h}\sum_{j\ne i}\sum_{n:Q^{-1}n^{-\frac{1}{p_i}}<r} m(\phi_{i^nj}(X)) \le r^{-h}\sum_{j\ne i}\sum_{n>(Qr)^{-p_i}} Q^h||\phi'_{i^nj}||^h \\
&\le Q^h r^{-h}\sum_{j\ne i}\sum_{n>(Qr)^{-p_i}} n^{-\frac{p_i+1}{p_i}h} \\
&\le (\mathrm{const})\#IQ^h\left(\frac{p_i+1}{p_i}h - 1\right) r^{-h}(Qr)^{(-p_i)\left(1-\frac{p_i+1}{p_i}h\right)} \\
&= (\mathrm{const})\, r^{-h+(p_i+1)h-p_i} = (\mathrm{const})\, r^{p_i(h-1)}.
\end{aligned}
$$

Since $h > 1$, this implies that $\lim_{r\to 0} r^{-h}m(B(x_i,r)) = 0$. Applying Lemma 4.13 in [MU1] along with Lemma 4.3 and Proposition 1.3, we conclude that $\mathcal{P}^h(J) = \infty$. ■

Proof of Lemma 4.7. Fix a parabolic index $i \in I$, $j \in I \setminus \{i\}$, $n \ge \max\{2q, q+1\}$ and $x \in \phi_{i^nj}(X)$. Given $1 \ge r > \mathrm{diam}(\phi_{i^nj}(X))$ and using (4.1) twice we obtain

$$
\begin{aligned}
\Sigma_1 &:= \sum_{a\ne i}\sum_{k=n-q}^{n+q} m(\phi_{i^ka}(X)) \le \sum_{a\ne i}\sum_{k=n-q}^{n+q} ||\phi'_{i^ka}||^h \\
&\le \sum_{a\ne i}\sum_{k=n-q}^{n+q} Q^h k^{-\frac{p_i+1}{p_i}h} \le \#IQ^h 2q(n-q)^{-\frac{p_i+1}{p_i}h} = 2\#IqQ^h\left(\frac{n}{n-q}\right)^{\frac{p_i+1}{p_i}h} n^{-\frac{p_i+1}{p_i}h} \qquad (4.7) \\
&\le 2qQ^h\#I2^{\frac{p_i+1}{p_i}h}Q\mathrm{diam}^h(\phi_{i^nj}(X)) \le 2qQ^{h+1}2^{\frac{p_i+1}{p_i}h}\#Ir^h.
\end{aligned}
$$

Put $l = E\left(\left(n^{-\frac{1}{p_i}} - Qr\right)^{-p_i}\right) + 1$ if $Qr < n^{-\frac{1}{p_i}}$ and $l = \infty$ otherwise. Using (4.1) we get

$$\Sigma_2 := \sum_{a\neq i} \sum_{k:|n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}}|<Qr} m(\phi_{i^k a}(X)) \le \sum_{a\neq i} \sum_{k=E\left(\left(Qr+n^{-\frac{1}{p_i}}\right)^{-p_i}\right)}^{l} Q^h k^{-\frac{p_i+1}{p_i}h}$$

$$\le \#IQ^h \sum_{k=E\left(\left(Qr+n^{-\frac{1}{p_i}}\right)^{-p_i}\right)}^{l} k^{-\frac{p_i+1}{p_i}h}.$$

Suppose first that $Qr < n^{-\frac{1}{p_i}}$. Then

$$\Sigma_2 \le \#IQ^h \left(\frac{p_i+1}{p_i}h - 1\right)\left(\left(Qr + n^{-\frac{1}{p_i}}\right)^{-p_i+(p_i+1)h} - \left(n^{-\frac{1}{p_i}} - Qr\right)^{-p_i+(p_i+1)h}\right).$$

It follows now from the Mean Value Theorem that there exists $\eta \in [n^{-\frac{1}{p_i}} - Qr, n^{-\frac{1}{p_i}} + Qr]$ such that

$$\left(Qr + n^{-\frac{1}{p_i}}\right)^{-p_i+(p_i+1)h} - \left(n^{-\frac{1}{p_i}} - Qr\right)^{-p_i+(p_i+1)h} = ((p_i+1)h - p_i)2Qr\eta^{(p_i+1)(h-1)}.$$

Since by (4.1),

$$n^{-\frac{1}{p_i}} \le Q^{\frac{1}{p_i}} \operatorname{diam}(\phi_{i^n j}(X))^{\frac{1}{p_i}} \le Q^{\frac{1}{p_i}} r^{\frac{1}{p_i}},$$

we therefore find

$$\begin{aligned}\Sigma_2 &\le (\text{const})\, r\eta^{(p_i+1)(h-1)} \le (\text{const})\, r\left(Qr + n^{-\frac{1}{p_i}}\right)^{(p_i+1)(h-1)} \\ &\le (\text{const})\, r\left(Q^{\frac{1}{p_i}} r^{\frac{1}{p_i}} + Qr\right)^{(p_i+1)(h-1)} \le (\text{const})\, r\left(r^{\frac{1}{p_i}}\right)^{(p_i+1)(h-1)} \qquad (4.8)\\ &= (\text{const})\, r^h.\end{aligned}$$

Suppose in turn that $Qr \ge n^{-\frac{1}{p_i}}$. Then

$$\begin{aligned}\Sigma_2 &\le Q^h\#I \sum_{k=E\left(\left(Qr+n^{-\frac{1}{p_i}}\right)^{-p_i}\right)}^{\infty} k^{-\frac{p_i+1}{p_i}h} \le Q^h\#I\left(\frac{p_i+1}{p_i}h - 1\right)\left(Qr + n^{-\frac{1}{p_i}}\right)^{-p_i\left(1-\frac{p_i+1}{p_i}h\right)} \\ &= Q^h\#I\left(\frac{p_i+1}{p_i}h - 1\right)\left(Qr + n^{-\frac{1}{p_i}}\right)^{(p_i+1)h-p_i} \le (\text{const})\, r^{(p_i+1)h-p_i} \qquad (4.9)\\ &= (\text{const})\, r^h r^{p_i(h-1)} \le (\text{const})\, r^h.\end{aligned}$$

Since, by (4.4), $m(B(x,r)) \le \Sigma_1 + \Sigma_2$, it follows from (4.7)-(4.9) that $m(B(x,r) \le (\text{const})\, r^h$. Finally, applying Theorem 2.4(3) in [MU2] completes the proof. ■

Proof of Theorem 1.7. It is a straightforward consequence of formulae (4.2), (4.3) and (4.4) that for every parabolic index $i \in I$, every $j \in I \setminus \{i\}$ and every $x \in X$, $\overline{\mathrm{BD}}\left(\{\phi_{i^n j}(x)\}_{n\geq 1}\right) = \frac{1}{\frac{1}{p_i}+1} = \frac{p_i}{p_i+1}$. Hence, it follows from Theorem 4.1 and Theorem 2.11 in [MU2] along with Theorem 3.1 in [MU1] that $\overline{\mathrm{BD}}(J) = \dim_{\mathrm{H}}(J)$. ■

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