# OPTIMAL PERIODIC ORBITS FOR NON RECURRENT RATIONAL FUNCTIONS

MARIUSZ URBAŃSKI

Abstract. We prove that each non-parabolic periodic orbit contained in the $\omega$-limit set of a measure-recurrent optimal orbit for a continuous function defined on the Julia set of a non-recurrent rational function is also optimal. As a by-product, we prove in the next section appropriate versions of shadowing and closing lemmas for non-recurrent rational functions.

## 1. Preliminaries and Introduction

Let $X$ be a compact metric space, $T : X \to X$ be a continuous map, and $\phi : X \to \mathbb{R}$ a continuous function. For every $x \in X$ and $n \geq 1$ put

$$S_n\phi(x) = \frac{1}{n}\sum_{j=0}^{n-1}\phi(T^j(x))$$

and

$$< \phi > (x) = \lim_{n\to\infty} S_n\phi(x)$$

if the limit exists. If $< \phi > (y)$ exists for some $y \in X$ and $< \phi > (y) \geq \limsup_{n\to\infty} S_n\phi(x)$ for each $x \in X$, then the (forward) orbit of $y$ is called an optimal orbit for $T$ and $\phi$. The question about existence of optimal orbits though fundamental has an easy positive answer (see for example [5]). In fact, a slightly stronger result is proven in [5]. In order to describe it fix $x \in T$. If the weak limit

$$\mu_x = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\delta_{T^j(x)}$$

exists, where $\delta_y$ is the Dirac measure concentrated at $y$, then $x$ is said to generate the invariant measure $\mu_x$ and $\mu_x$ is said to be generated by $x$. A point $x \in X$ is said to be measure-recurrent if $x$ generates a $T$-invariant measure $\mu_x$ and $x$ belongs to the topological support of the measure $\mu_x$. The following result has been proved in [5].

**Theorem 1.1.** *If $T : X \to X$ is a continuous map of a compact metric space $X$ and $\phi : X \to \mathbb{R}$ is a continuous function, then there always exists a measure-recurrent optimal orbit for $T$ and $\phi$.*

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping of the Riemann sphere $\overline{\mathbb{C}}$ of degree $\geq 2$. The mapping $f$ is non-recurrent if $c \notin \omega(c)$ for every critical point $c \in J(f)$, where $J(f)$ is the Julia set of

Supported in part by the NSF Grant DMS 0100078.

$f$ and $\omega(c)$ is the $\omega$-limit set of $c$ under $f$. This class contains in particular all expanding, subexpanding and parabolic functions. In [5] the problem of the structure of optimal orbits was dealt with. The authors proved that if $T$ is either an Axiom A diffeomorphism or an expanding map of a smooth compact manifold, then each periodic orbit contained in the $\omega$-limit set of a measure-recurrent optimal orbit for a Lipschitz continuous function $\phi$ is also optimal for $\phi$. We prove a corresponding result (see 3.3) in the context of a non-recurrent rational mapping $f : J(f) \to J(f)$ and all continuous functions $\phi : J(f) \to I\!R$. As a by-product, we prove in the next section appropriate versions of shadowing and closing lemmas for non-recurrent rational mappings. We end this section by recalling the following two notions. Firstly, a point $\omega \in \mathbb{C}$ is called parabolic (rationally indifferent) if it is periodic and there exists $q \geq 1$, a multiple of a period of $\omega$, such that $(f^q)'(\omega) = 1$. The set $\Omega = \Omega(f)$ of all parabolic points of a rational mapping of the Riemann sphere is finite and contained in the Julia set $J(f)$. Secondly, if $A, B$ are two subsets of $\mathbb{C}$, we put

$$\mathrm{dist}(A,B) = \inf\{|b-a| : a \in A, b \in B\} \text{ and } \mathrm{Dist}(A,B) = \sup\{|b-a| : a \in A, b \in B\}.$$

## 2. Shadowing and Closing Lemmas

From now on throughout the entire paper $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is assumed to be a non-recurrent rational mapping. We recall that given $\alpha > 0$ a sequence $\{x_i\}_{i=0}^\infty$ is called and $\alpha$-pseudo-orbit if $|x_{i+1} - f(x_i)| \leq \alpha$ for all $i \geq 0$. We call a pseudo-orbit $\{x_n\}_{n=0}^\infty$ $\theta$-well behaving provided that $x_{n+1} = f(x_n)$ if $x_{n+1} \in B(\Omega, \theta)$ and $x_k \notin B(\Omega, \theta)$ for infinitely many $k$'s. It follows from Mane's theorem (see [1], comp. Lemma 2.13 in [2]) that for every $\theta > 0$ and every $\epsilon > 0$ there exists $\tilde{\epsilon} \in (0, \epsilon)$ such that if $x \notin B(\Omega, \theta)$, then for every $n \geq 0$ and the diameters of all connected components of $f^{-n}(B(x, \tilde{\epsilon}))$ do not exceed $\epsilon$. We shall prove the following version of the Anosov-Bowen shadowing lemma appropriate in the context of non-recurrent rational functions.

**Lemma 2.1.** *For every $\theta > 0$ and every $\epsilon > 0$ there exists $\delta(\theta, \epsilon) > 0$ such that if $\{x_n\}_{n=0}^\infty$ is a $\theta$-well behaving $\delta(\theta, \epsilon)$-pseudo-orbit, then $\{x_n\}_{n=0}^\infty$ is $\epsilon$-shadowable.*

*Proof.* Put $\eta = (\widetilde{\epsilon/2})/2$. Therefore, if $f^k(z) \notin B(\Omega, \theta)$, then

$$\mathrm{diam}\Big(C_k(z, \overline{B}(f^k(z), 2\eta))\Big) \leq \epsilon/2. \tag{2.1}$$

In view of Lemma 5.3 from [3] there exists $q \geq 1$ such that if $k \geq q$ and $f^k(z) \notin B(\Omega, \theta)$, then

$$\mathrm{diam}\Big(C_k(z, \overline{B}(f^k(z), 2\eta))\Big) \leq \eta/2. \tag{2.2}$$

Take $\delta > 0$ so small that

$$(\ldots((\delta||f'|| + \delta)||f'|| + \delta)||f'|| + \ldots + \delta)||f'|| \leq \eta, \tag{2.3}$$

where in this inequality $||f'||$ occurs $q-1$ times. We now extend our pseudo-orbit $\{x_n\}_{n=0}^\infty$ to a pseudo-orbit $\{x_n\}_{n=-q}^\infty$ such that $f^q(x_{-q}) = x_0$ and $x_{n-q} = f^n(x_{-q}) \notin B(\Omega, \theta)$ for all

$0 \le n \le q-1$. We call a piece $\{x_i\}_{i=m}^n$ of the pseudo-orbit $\{x_n\}_{n=-q}^\infty$ of category I if $x_m, x_n \notin B(\Omega,\theta)$ and $n-m=q$ and of category II if $x_m, x_n \notin B(\Omega,\theta)$ and $\{x_i\}_{i=m+1}^{n-q} \subset B(\Omega,\delta)$. If now $\{x_i\}_{i=m}^n$ is a block of category I, then by (2.3), $|x_n - f^q(x_m)| \le \eta$. Hence $f^q(x_m) \in \overline{B}(x_n,\eta) \subset B(f^q(x_m),2\eta)$ and therefore $D_m^n$, the connected component of $f^{-q}(\overline{B}(x_n,\eta)$ containing $x_m$ is contained in $C_q(x_m, \overline{B}(f^q(x_m),2\eta))$. In view of (2.2)

$$D_m^n \subset C_q(x_m, \overline{B}(f^q(x_m),2\eta)) \subset \overline{B}(x_m,\eta). \tag{2.4}$$

Since for every $i \in \{0,1,\dots,n-m\}$, $f^i(D_m^n) \subset C_{q-i}\Big(f^i(x_m), B(f^q(x_m),2\eta)\Big)$, it follows from (2.3) and (2.1) that

$$\begin{aligned}\mathrm{Dist}\Big(x_{m+i}, f^i(D_m^n)\Big) &\le |x_{m+i} - f^i(x_m)| + \mathrm{diam}\Big(C_{q-i}\Big(f^i(x_m), B(f^q(x_m),2\eta)\Big)\Big)\\ &\le \eta + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned} \tag{2.5}$$

If $\{x_i\}_{i=m}^n$ is a block of category II, then

$$|x_{m+i} - f^i(x_m)| \le \eta \tag{2.6}$$

for all $i \in \{0,1,\dots,n-m\}$. And indeed, if $i=0$, then $|x_{m+i} - f^i(x_m)| = |x_m - x_m| = 0 < \eta$, if $i \ge 1$ and $i+m \le n-q$, then $x_{i+m} = f^i(x_m)$ since our pseudo-orbit $\{x_i\}_{i=0}^\infty$ is $\theta$-well behaving and $\{x_i\}_{i=m}^{n-q+1} \subset B(\Omega,\theta)$. If $n+m \ge n-q+1$, then (2.6) follows from (2.3). Hence

$$f^{n-m}(x_m) \in \overline{B}(x_n,\eta) \subset \overline{B}(f^{n-m}(x_m),2\eta)$$

and therefore $D_m^n$, the component of $f^{-(n-m)}(\overline{B}(x_n,\eta))$ containing $x_m$ is contained in the set $C_{n-m}\Big(x_m, \overline{B}(f^{n-m}(x_m),2\eta)\Big)$. In view of (2.2)

$$D_m^n \subset C_{n-m}\Big(x_m, \overline{B}(f^{n-m}(x_m),2\eta)\Big) \subset B(x_m,\eta). \tag{2.7}$$

Since for every $i \in \{0,1,\dots,n-m\}$, $f^i(D_m^n) \subset C_{n-m-i}(f^i(x_m), B(f^{n-m}(x_m),2\eta)$, it follows from (2.6) and (2.1) that

$$\begin{aligned}\mathrm{Dist}\Big(x_{m+i}, f^i(D_m^n)\Big) &\le |x_{m+i} - f^i(x_m)| + \mathrm{diam}\Big(C_{n-m-i}(f^i(x_m), B(f^{n-m}(x_m),2\eta))\Big)\\ &\le \eta + \frac{\epsilon}{2} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned} \tag{2.8}$$

Now, since our pseudo-orbit is $\theta$-well behaving, there exists an increasing to infinity sequence $\{n_k\}_{k=1}^\infty$ such that $x_{n_k} \notin B(\Omega,\theta)$ for all $k \ge 1$. For each $k \ge 1$ one can decompose the pseudo-orbit $\{x_j\}_{j=-q}^{n_k}$ into blocks of category I and II:

$$[x_{j_{s-1}^{(k)}}, x_{j_{s-1}^{(k)}+1}, \dots, x_{j_s^{(k)}}],\ [x_{j_{s-2}^{(k)}}, x_{j_{s-2}^{(k)}+1}, \dots, x_{j_{s-1}^{(k)}}],\ [x_{j_{s-3}^{(k)}}, x_{j_{s-3}^{(k)}+1}, \dots, x_{j_{s-2}^{(k)}}], \dots,$$
$$[x_{j_2^{(k)}}, x_{j_2^{(k)}+1}, \dots, x_{j_3^{(k)}}]$$

where $j_s^{(k)} = n_k$, and the last block $[x_{j_1^{(k)}}, x_{j_2^{(k)}+1}, \dots, x_{j_2^{(k)}}]$, where $j_1^{(k)} = -q$ and

$$\limsup_{k\to\infty} j_2^{(k)} \le 0. \tag{2.9}$$

Fixing $k \geq 1$ define now in the backward direction the sequence $\{F_i\}_{i=s}^2$ as follows. $F_s = D_{j_{s-1}^{(k)}}^{j_s^{(k)}}$ and suppose that $F_i \subset D_{j_i^{(k)}}^{j_{i+1}^{(k)}}$ has been defined for some $3 \leq i \leq s$. It follows from the definition of the sets $D_m^n$ and (2.4) that the intersection $D_{j_{i-1}^{(k)}}^{j_i^{(k)}} \cap f^{-\left(j_i^{(k)} - j_{i-1}^{(k)}\right)}(F_i)$ is not empty. Denote this intersection by $F_{i-1}$. Fix a point $z_k \in F_2$. It follows from (2.5) and (2.8) that

$$|x_{j_2^{(k)}+j} - f^j(z_k)| \leq \epsilon \tag{2.10}$$

for all $0 \leq j \leq n_k$. In view of (2.10), passing to a subsequence, we may assume that $j_2^{(k)}$ is the same for all $k \geq 1$, say $j_2^{(k)} = j_2$. Let $y$ be an accumulation point of the sequence $\{z_k\}_{k=1}^\infty$. Fixing $n \geq 1$, it immediately follows from (2.10) and continuity of $f$ that $|x_{j_2^{(k)}+j} - f^j(y)| \leq \epsilon$ for all $0 \leq j \leq n$. Since $n$ is an arbitrary number, we get

$$|x_{j_2^{(k)}+j} - f^j(y)| \leq \epsilon \tag{2.11}$$

for all $j \geq 0$. Since $j_2 \leq 0$, in view of (2.11) the point $z = f^{-j_2}(y)$ shadows the pseudo-orbit $\{x_n\}_{n=0}^\infty$. The proof is finished. ■

Since, because of presence of critical points, the shadowing point constructed in the previous lemma is usually not unique, the standard way of deducing Anosov's closing lemma from Anosov - Bowen shadowing lemma fails. We therefore provide below its direct proof.

**Lemma 2.2.** *For every $\theta > 0$ and every $\epsilon > 0$ there exists $m = m(\theta, \epsilon)$ such that if $q \geq m(\theta, \epsilon)$, $f^q(x) \in J(f) \setminus B(\Omega, \theta)$ and $|f^q(x) - x| \leq \tilde{\epsilon}$, then there exists a point $y \in J(f)$ such that $f^q(y) = y$ and $|f^i(x) - f^i(y)| \leq \epsilon$ for all $i = 0, 1, \ldots, q$.*

*Proof.* Fix $m(\theta, \epsilon)$ so large that $B_\epsilon m^{-\xi \frac{p+1}{p}} < \tilde{\epsilon}/2$, where $\xi, p$ and $B_\epsilon$ come from Lemma 5.3 in [4]. We shall define by induction a sequence $\{C_n\}_{n=0}^\infty$ of compact connected subsets of $\overline{\mathbb{C}}$ as follows.

$$C_0 = \overline{B}(f^q(x), \tilde{\epsilon})$$

and $C_1$ is the connected component of $f^{-q}(C_0)$ containing $x$. In view of Lemma 5.3 from [4], the definition of $m$ and since $q \geq m(\theta, \epsilon)$, we have $\operatorname{diam}(C_1) \leq \tilde{\epsilon}/2$. Therefore

$$C_1 \subset \overline{B}(x, \tilde{\epsilon}/2) \subset \overline{B}\left(f^q(x), \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2}\right) = \overline{B}(f^q(x), \tilde{\epsilon}) = C_0.$$

Suppose now that $C_n$, $n \geq 1$, has been defined and $C_n$ is a connected component of $f^{-q}(C_{n-1})$ contained in $C_{n-1}$. Then also $f^{-q}(C_n) \subset f^{-q}(C_{n-1})$ and we conclude that there exists $C_{n+1}$, a connected component of $f^{-q}(C_n)$ contained in $C_n$. Hence $C_{n+1} \subset C_n$ and $f^q(C_{n+1}) = C_n$. Since by our construction $C_n$ is a connected component of $f^{-qn}(\overline{B}(f^q(x), \tilde{\epsilon}))$, it follows from Lemma 5.3 in [4] that $\lim_{n\to\infty} \operatorname{diam}(C_n) = 0$. Therefore the intersection $\bigcap_{n\geq 1} C_n$ is a singleton and we denote its unique element by $y$. Then

$$f^q\left(\bigcap_{n\geq 1} C_n\right) \subset \bigcap_{n\geq 1} f^q(C_n) = \bigcap_{n\geq 1} C_{n-1} = \bigcap_{n\geq 0} C_n \subset \bigcap_{n\geq 1} C_n = \{y\}.$$

Thus $f^q(y) = y$. Now, in view of the definition of $\tilde{\epsilon}$, for every $i = 0, 1, \ldots, q$, $f^i(C_1)$ contains both $f^i(y)$, $f^i(x)$ and has diameter bounded above by $\epsilon$. We are done. ■

## 3. Optimal orbits for continuous functions

We start with the following.

**Lemma 3.1.** *If $x \in J(f)$ generates a recurrent optimal orbit for a continuous function $\phi$ : $J(f) \to I\!R$, then for every $\epsilon > 0$ and every $\theta > 0$ there exists $0 < \delta \leq \tilde{\epsilon}$ and $q(\epsilon) \geq 1$ such that if $|f^{k+q}(x) - f^k(x)| \leq \delta$ and $f^k(x) \notin B(\Omega, 3\theta)$ for some integer's $k \geq 0$ and $q \geq q(\epsilon)$, then*

$$\left| \frac{1}{q} \sum_{i=k}^{k+q-1} \phi(f^i(x)) - < \phi > (x) \right| \leq \epsilon.$$

*Proof.* Fix $\eta > 0$ so small that $|z - w| \leq \eta$ implies that $|\phi(z) - \phi(w)| \leq \epsilon/2$. Take $\delta = \min\{\tilde{\eta}, \theta, \delta(\theta, \eta)\}/4$, $q(\epsilon) = m(\theta, \epsilon)$ and let $y$ be the periodic point of period $q$ produced in Lemma 2.2 with the point $x$ replaced by $f^k(x)$ and $\epsilon$ replaced by $\eta$. Then $|f^{k+i}(x) - f^i(x)| \leq \eta$ for all $i = 0, 1, \ldots, q$ and therefore

$$\left| \frac{1}{q} \sum_{i=k}^{k+q-1} \phi(f^i(x)) - < \phi > (y) \right| \leq \frac{1}{q} \sum_{i=0}^{q-1} |\phi(f^{i+k}(x)) - \phi(f^i(y))| \leq \frac{\epsilon}{2}. \tag{3.1}$$

Hence

$$\frac{1}{q} \sum_{i=k}^{k+q-1} \phi(f^i(x)) \leq < \phi > (y) + \frac{\epsilon}{2} \leq < \phi > (x) + \frac{\epsilon}{2}.$$

So, it remains to prove that

$$\frac{1}{q} \sum_{i=k}^{k+q-1} \phi(f^i(x)) \geq < \phi > (x) - \epsilon.$$

In order to do it define inductively the increasing sequence $\{k_j\}_{j=1}^{\infty}$ as follows. Since $x \in \omega(x)$, there exists the least integer $k_1 \geq 0$ such that $|f^{k_1}(x) - f^k(x)| \leq \delta/||f'||^q$. Given $k_j$, again since $x \in \omega(x)$, there exists the least integer $k_{j+1} \geq k_j + q$ such that $|f^{k_{j+1}}(x) - f^k(x)| < \delta/||f'||^q$. Remove all the pieces of the form $\{f^{k_j}(x), f^{k_j+1}(x), \ldots, f^{k_j+q-1}(x)\}$ from the orbit $\{f^i(x)\}_{i=0}^{\infty}$ of $x$. Write the remaining sequence as $\{x'_i\}_{i=0}^{\infty}$. Since $|f^{k+q}(x) - f^k(x)|\epsilon\delta$, we get for every

$j \geq 1$ that

$$\begin{aligned}|f^{k_j+q}(x) - f(f^{k_j-1}(x))| &= |f^{k_j+q}(x) - f^{k_j}(x)| \\ &\leq |f^{k_j+q}(x) - f^{k+q}(x)| + |f^{k+q}(x) - f^k(x)| + |f^k(x) - f^{k_j}(x)| \\ &\leq ||f'||^q \frac{\delta}{||f'||^q} + \delta + \frac{\delta}{||f'||^q} \leq 3\delta.\end{aligned}$$

Thus $\{x'_i\}_{i=0}^\infty$ is a $3\delta$-pseudo-orbit. Since $|f^{k_j}(x) - f^k(x)| \leq \delta$, $f^k(x) \notin B(\Omega, 3\theta)$ and $\delta \leq \theta$, we get $f^{k_j}(x) \notin B(\Omega, 2\theta)$. This implies that $f^{k_j-1}(x) \notin B(\Omega, \theta)$. Also, since $|f^{k_j+q}(x) - f^{k+q}(x)| \leq \delta$, $|f^{k+q}(x) - f^k(x)| \leq \delta$ and $f^k(x) \notin B(\Omega, 3\theta)$, we see that $f^{k_j+q}(x) \notin B(\Omega, \theta)$. Thus the $3\delta$-pseudo-orbit $\{x'_i\}_{i=0}^\infty$ is $\theta$-well behaving and, in view of Lemma 2.1 this pseudo-orbit is $\eta$-shadowed by a true orbit $\{f^i(z)\}_{i=0}^\infty$. Let

$$\alpha = \sup_{1 \leq j < \infty} \frac{1}{q} \sum_{i=k_j}^{k_j+q-1} \phi(f^i(x)).$$

Since $|f^{k_j}(x) - f^k(x)| \leq \delta/||f'||^q$, we get $|f^{k_j+i}(x) - f^{k+i}(x)| \leq \delta$ for all $0 \leq i \leq q-1$ and therefore

$$\begin{aligned}\alpha &\leq \sup_{1 \leq j < \infty} \frac{1}{q} \left( \sum_{i=0}^{q-1} \phi(f^{k+i}(x)) + \sum_{i=0}^{q-1} \Big( \phi(f^{k_j+i}(x)) - \phi(f^{k+i}(x)) \Big) \right) \\ &\leq \frac{1}{q} \sum_{i=0}^{q-1} \phi(f^{k+i}(x)) + \frac{1}{q} \sup_{1 \leq j < \infty} \sum_{i=0}^{q-1} |\phi(f^{k_j+i}(x)) - \phi(f^{k+i}(x))| \qquad (3.2) \\ &\leq \frac{1}{q} \sum_{i=0}^{q-1} \phi(f^{k+i}(x)) + \frac{\epsilon}{2}.\end{aligned}$$

Now, for every $n \geq 0$ there exists $q_n \geq 0$ such that

$$\{x'_i\}_{i=0}^n \cup \{f^i(x) : j \in \{1, 2, \dots, q_n\}, i \in \{k_j, k_j+1, \dots, k_j+q-1\}\}$$

forms an initial segment of the orbit $\{f^l(x)\}_{l=0}^\infty$ of the point $x$. Hence

$$\begin{aligned}< \phi > (x) &= \lim_{n \to \infty} \frac{1}{q_n q + n + 1} \left( \sum_{i=0}^n \phi(x'_i) + \sum_{j=1}^{q_n} \sum_{i=k_j}^{k_j+q-1} \phi(f^j(x)) \right) \\ &\leq \liminf_{n \to \infty} \frac{1}{q_n q + n + 1} \left( \sum_{i=0}^n \phi(x'_i) + q_n q \alpha \right)\end{aligned}$$

or equivalently

$$\liminf_{n \to \infty} \frac{1}{q_n q + n + 1} \left( \sum_{i=0}^n \phi(x'_i) - (n+1) < \phi > (x) + q_n q (\alpha - < \phi > (x)) \right) \geq 0.$$

Again, this equivalently means that

$$\liminf_{n \to \infty} \frac{q_n q}{q_n q + n + 1} \left( \frac{1}{q_n q} \left( \sum_{i=0}^n \phi(x'_i) - (n+1) < \phi > (x) \right) + (\alpha - < \phi > (x) \right) \geq 0.$$

Thus

$$\begin{aligned}
\alpha- < \phi > (x) &\geq -\liminf_{n\to\infty} \frac{q_n q}{q_n q + n + 1} \cdot \frac{1}{q_n q} \left( \sum_{i=0}^{n} \phi(x_i') - (n+1) < \phi > (x) \right) \\
&= \limsup_{n\to\infty} \frac{1}{q_n q + n + 1} \Big( (n+1) < \phi > (x) - \sum_{i=0}^{n} \phi(x_i') \Big) \\
&= \limsup_{n\to\infty} \frac{1}{q_n q + n + 1} \left( (n+1) < \phi > (x) - \sum_{i=0}^{n} \phi(f^i(z)) + \sum_{i=0}^{n} (\phi(f^i(z)) - \phi(x_i')) \right) \qquad (3.3) \\
&\geq \limsup_{n\to\infty} \frac{n+1}{q_n q + n + 1} \left( < \phi > (x) - \frac{1}{n+1} \sum_{i=0}^{n} \phi(f^i(z)) \right) + \\
&\qquad + \liminf_{n\to\infty} \frac{1}{q_n q + n + 1} \sum_{i=0}^{n} (\phi(f^i(z)) - \phi(x_i')).
\end{aligned}$$

Now, since the orbit $\{f^i(z)\}_{i=0}^{\infty}$ $\eta$-shadows the pseudo-orbit $\{x_i'\}_{i=0}^{\infty}$, we get

$$\begin{aligned}
\left| \frac{1}{q_n q + n + 1} \sum_{i=0}^{n} (\phi(f^i(z)) - \phi(x_i')) \right| &\leq \frac{1}{q_n q + n + 1} \sum_{i=0}^{n} |\phi(f^i(z)) - \phi(x_i')| \\
&\leq \frac{1}{n+1} \sum_{i=0}^{n} |\phi(f^i(z)) - \phi(x_i')| \leq \frac{\epsilon}{2}
\end{aligned}$$

and consequently

$$\liminf_{n\to\infty} \frac{1}{q_n q + n + 1} \sum_{i=0}^{n} (\phi(f^i(z)) - \phi(x_i')) \geq -\frac{\epsilon}{2}. \qquad (3.4)$$

Since $0 \leq \frac{n+1}{q_n q + n + 1} \leq 1$ and since

$$\liminf_{n\to\infty} \left( < \phi > (x) - \frac{1}{n+1} \sum_{i=0}^{n} \phi(f^i(z)) \right) \geq 0,$$

we conclude that

$$\liminf_{n\to\infty} \frac{n+1}{q_n q + n + 1} \left( < \phi > (x) - \frac{1}{n+1} \sum_{i=0}^{n} \phi(f^i(z)) \right) \geq 0.$$

Combining this (3.4) and (3.3), we get $\alpha- < \phi > (x) \geq -\epsilon/2$. Combining in turn this and (3.2), we obtain

$$\frac{1}{q} \sum_{i=0}^{k-1} \phi(f^{k+i}(x)) \geq \alpha - \frac{\epsilon}{2} \geq < \phi > (x) - \epsilon.$$

We are done. ■

Given $\delta, \theta > 0$ an orbit $\{f^i(x)\}_{i=0}^{\infty}$ comes within $(\delta, \theta)$ of a periodic orbit $\{f^i(y)\}_{i=0}^{q-1}$ of period $q$ if $y \notin B(\Omega, 2\theta)$ and if there exists $k \geq 0$ such that $|f^{k+i}(x) - f^i(y)| \leq \delta$ for all $i = 0, 1, \dots, q\}$. We are now in position to prove the following.

**Proposition 3.2.** *Fix $\theta > 0$. Suppose that $\{f^i(x)\}_{i=0}^\infty$ is a measure-recurrent optimal orbit for a continuous function $\phi : J(f) \to I\!R$. Then for every $\epsilon > 0$ there exist $\delta > 0$ and $m \geq 1$ such that if $\{f^i(y)\}_{i=0}^{q-1}$ is a periodic orbit of period $q \geq m$ and $\{f^i(x)\}_{i=0}^\infty$ comes within $(\delta/2, \theta)$ of $\{f^i(y)\}_{i=0}^{q-1}$, then*

$$< \phi > (x) - \epsilon \leq < \phi > (y) \leq < \phi > (x).$$

*Proof.* Take $\delta > 0$ and $m = q(\epsilon/2)$ ascribed to $\epsilon/2$ and $\theta/3$ as in Lemma 3.1. Since $\{f^i(x)\}_{i=0}^\infty$ comes within $(\delta, \theta)$ of $\{f^i(y)\}_{i=0}^{q-1}$, there exists $k \geq 0$ such that $|f^{k+i}(x) - f^i(y)| \leq \delta/2$ for all $i = 0, 1, \dots, q\}$. Since $\delta \leq \tilde{\epsilon/2}$, we therefore get

$$\left| \frac{1}{q} \sum_{i=0}^{q-1} \phi(f^{k+i}(x)) - < \phi > (y) \right| \leq \frac{1}{q} \sum_{i=0}^{q-1} |\phi(f^{k+i}(x)) - f^i(y)| \leq \frac{\epsilon}{2}.$$

Since $f^k(x) - y| \leq \delta$ and since $y \notin B(\Omega, 2\theta)$, we get that $f^k(x) \notin B(\Omega, \theta)$. Since also

$$|f^{k+q}(x) - f^k(x)| \leq |f^{k+q}(x) - f^q(y)| + |f^q(y) - f^k(x)| \leq \frac{\delta}{2} + |f^k(x) - y| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

Lemma 3.1 applies and we get $< \phi > (x) - \frac{1}{q} \sum_{i=0}^{q-1} \phi(f^{k+i}(x)) \leq \epsilon/2$. Consequently $< \phi > (x) - \epsilon \leq < \phi > (y) \leq < \phi > (x)$ and we are done. ■

Our main result in this paper is the following.

**Theorem 3.3.** *If $\{f^i(x)\}_{i=0}^\infty$ is a measure recurrent optimal orbit for a continuous function $\phi : J(f) \to I\!R$, then for every periodic point $y \in \omega(x) \setminus \Omega(f)$, $< \phi > (y) = < \phi > (x)$.*

*Proof.* Let $q \geq 1$ be a period of $y$ and let

$$\theta = \frac{1}{2} \text{dist}(\Omega, \{f(y), \dots, f^{q-1}(y)\}) > 0.$$

Fix $\epsilon > 0$ and let $\delta > 0$ and $m \geq 1$ be chosen as in Proposition 3.2. Since $y$ is periodic orbit of any period $ql$, $l \geq 1$, we may assume without loss of generality that $q \geq m$. Since $y \in \omega(x) \setminus \Omega$, the orbit $\{f^i(x)\}_{i=0}^\infty$ comes (infinitely often) within $(\delta/2, \theta)$ of $\{f^i(y)\}_{i=0}^{q-1}$. It therefore follows from Proposition 3.2 that $< \phi > (x) - \epsilon \leq < \phi > (y) \leq < \phi > (x)$. Letting $\epsilon \searrow 0$, we obtain $< \phi > (y) = < \phi > (x)$ which finishes the proof. ■

## References


[1] R. Mané, On a theorem of Fatou, Bol. Soc. Bras. Mat., 24 (1933), 1-11.

[2] M. Urbański, Rational functions with no recurrent critical points, Ergod. Th. and Dynam. Sys. 14 (1994), 391-414.

[3] M. Urbański, Geometry and ergodic theory of conformal nonrecurrent dynamics, Ergod. Th. and Dynam. Sys. 17 (1997), 1449-1476.

[4] M. Urbański, Thermodynamic formalism, topological pressure and escape rates for non-recurrent conformal dynamics, Preprint 2000.

[5] G. Yuan, B. Hunt, Optimal orbits of hyperbolic systems, Nonlinearity 12 (1999), 1207-1224.

Mariusz Urbański; Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA
urbanski@unt.edu, http://www.math.unt.edu/∼urbanski