REGULARITY PROPERTIES OF HAUSDORFF DIMENSION IN INFINITE CONFORMAL IFS

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Abstract. This paper deals with families of conformal iterated function systems (CIFS). The space of all CIFS, with common seed space X and alphabet I, is successively endowed with the topology of pointwise convergence and a new, weaker topology called λ-topology. It is proved that the pressure and the Hausdorff dimension of the limit set are continuous with respect to the topology of pointwise convergence when I is finite, and are lower semicontinuous, though generally not continuous, when I is infinite. It is then shown that these two functions are, in any case, continuous in the λ-topology. The concepts of analytic, regularly analytic and plane-analytic families of CIFS are also introduced. It is established that if a family of CIFS is regularly analytic, then the Hausdorff dimension function is real-analytic; if a family is plane-analytic, then the Hausdorff dimension function is continuous and subharmonic, though not necessarily real-analytic. These results are then applied to finite parabolic CIFS. Counterexamples highlighting breakdowns of real-analyticity in the Hausdorff dimension among analytic, but not regularly analytic, families are further provided. Such families often exhibit a phenomenon coined phase transitions. Sufficient conditions preventing the occurence of such transitions are supplemented.

1. Introduction

The last 15 years were a period of extensive study of single infinite conformal iterated function systems. Recently, interest in families of such systems has emerged (see [1] and [2] for example). The aim of this paper is to provide a good background for further research in this direction. In section 2, Preliminaries from Iterated Function Systems, we collect the definitions, concepts, and most of the known results concerning single iterated function systems which will be needed in the sequel. In section 3, Preliminaries from Topology, we define and collect some general properties of what we call λ-topologies. In section 4, Continuity in Finite Iterated Function Systems, we topologize (in fact, metricize) the space $\mathrm{CIFS}(X, I)$ of all conformal iterated function systems sharing the same seed space X and the same finite alphabet I. The topological pressure and the Hausdorff dimension functions prove to be continuous with respect to the abovementioned topology. Section 5, Continuity in Infinite Iterated Function Systems, is devoted to systems with a countably infinite alphabet I. We endow the space $\mathrm{CIFS}(X, I)$ with two topologies: the topology of pointwise convergence, which arises from

The research of the first author was supported in part by the Polish KBN Grant 2 PO3A 009 17 and by the FQRNT (Fonds québécois de la recherche sur la nature et les technologies). The first author further wishes to thank the University of North Texas for its kind hospitality and financial support. The research of the second author was supported in part by the NSF Grant DMS 0400481.

a natural metric, and an appropriate, weaker λ-topology. Not only does a λ-converging sequence converge pointwise, but it also satisfies a condition on the difference of the logarithms of the norms of the derivatives of its underlying maps. This condition essentially requires that the maximum distortion of the space created by a sequence of CIFS be comparable to the maximum distortion generated by its limit. We first prove that the pressure and the Hausdorff dimension functions are lower semi-continuous, though not generally continuous, with respect to the topology of pointwise convergence. A class of examples underlining the possible lack of continuity in these two functions is thereafter constructed. Then we demonstrate that both functions are continuous with respect to the λ-topology. The section ends with a complete characterization of the behavior of sequences and their limits in the λ-topology. In section 6, Real-analyticity, Continuity and Subharmonicity in Analytic Families, we define the concepts of analytic, regularly analytic and plane-analytic families of iterated function systems. We note that every plane-analytic family is analytic, and every analytic family on a finite alphabet I is regularly analytic. Regular analyticity forms, up to our knowledge, the most general sufficient condition preserving the real-analyticity of the Hausdorff dimension function. If a family is plane-analytic, then the Hausdorff dimension function proves to be continuous and subharmonic. This result is similar to the one discovered by Ransford [10] for analytic families of hyperbolic rational maps, and our proof is inspired from his work. Like Ransford's, our argumentation relies on the existence of a Bowen-Ruelle-type formula (Theorem 2.1) and on the variational principle for the pressure function. Our result also extends the one obtained by Baribeau and Roy [2] for plane-analytic families of CIFS consisting of similitudes, in which case a completely different proof was brought up. These results are then applied in section 7, Parabolic Iterated Function Systems, to finite parabolic CIFS. Finally, counterexamples showing the possible failure of real-analyticity in analytic, but not regularly analytic, families are provided in section 8, Phase Transitions and Breakdown of Real-analyticity. Such families often display what we call phase transitions. We give sufficient conditions prohibiting these from appearing.

2. Preliminaries from Iterated Function Systems

Let us first describe the setting of conformal iterated function systems introduced in [5]. Let I be a countable (finite or infinite) index set (so-called alphabet) with at least two elements, and let $\Phi = \{\phi_i : X \to X \,|\, i \in I\}$ be a collection of injective contractions from a compact metric space (X, d_X) (sometimes coined seed space) into (X, d_X) for which there exists $0 < s < 1$ such that $d_X(\phi_i(x), \phi_i(y)) \leq s\, d_X(x, y)$ for every $i \in I$ and for every $x, y \in X$. Any such collection Φ is called an iterated function system (abbr. IFS). We define the limit set J_Φ of this system as the image of the coding space I^∞ under a coding map π_Φ as follows. (Remark: We will drop the subscript Φ when a single IFS is considered.) Let I^n denote the space of words of length n with letters in I, $I^* = \bigcup_{n\in I\!\!N} I^n$ be the space of finite words, and I^∞ the space of one-sided infinite words (sequences) of letters in I. For every $\omega \in I^* \cup I^\infty$, we write $|\omega|$ for the length of ω, that is, the unique $n \in I\!\!N \cup \{\infty\}$ such that $\omega \in I^n$. For $\omega \in I^n$,

$n \in I\!N$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \in I\!N$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \dots \omega_n$. Since, given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \in I\!N$, converge to zero and since these sets form a decreasing family, the set

$$\bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton, and we denote its element by $\pi(\omega)$. This defines the coding map $\pi : I^\infty \to X$. Clearly, π is a continuous function when I^∞ is equipped with the topology generated by the cylinders $[i] = \{\omega \in I^\infty : \omega_1 = i\}$, $i \in I$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X).$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i \in I} \phi_i(J)$. Note that if I is finite, then J is compact, and this property usually fails when I is infinite.

An IFS $\Phi = \{\phi_i : X \to X \,|\, i \in I\}$ is said to satisfy the Open Set Condition (OSC) if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. (We do not exclude the possibility that $\overline{\phi_i(U)} \cap \overline{\phi_j(U)} \neq \emptyset$.)

An IFS Φ satisfying the OSC is called conformal (abbr. CIFS) if $X \subset I\!R^d$ for some $d \in I\!N$ and the following conditions are satisfied:

- $U = \mathrm{Int}_{I\!R^d}(X)$;
- There exists an open connected set V, with $X \subset V \subset I\!R^d$, such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V (Notice that for $d = 1$ this just means that the maps ϕ_i, $i \in I$, are C^1 monotone diffeomorphisms; for $d \geq 2$ the words C^1 conformal mean holomorphic or antiholomorphic; and for $d > 2$, this means that the maps ϕ_i, $i \in I$, are Möbius transformations. The proof of the last statement can be found in [3] for instance, where it is called Liouville's theorem.);
- There exist $\gamma, l > 0$ such that for every $x \in X$ there is an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l;
- Bounded Distortion Property (BDP): There exists $K \geq 1$ such that

  $$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

  for every $\omega \in I^*$ and every $x, y \in V$, where $|\phi'_\omega(x)|$ denotes the norm of the derivative.

As demonstrated in [5], infinite CIFS, unlike finite ones, may not possess a conformal measure. There are even continued fraction systems which do not admit a conformal measure (see [6], Example 6.5). Thus, the infinite systems naturally break into two main classes, irregular and regular systems. This dichotomy can be determined from the existence of a conformal measure or, equivalently, the existence of a zero of the topological pressure function. Recall that the topological pressure $\mathrm{P}(t) = \mathrm{P}_\Phi(t)$, $t \geq 0$, is defined as follows. For every $n \in I\!N$, set

$$\mathrm{P}^{(n)}(t) = \sum_{\omega \in I^n} ||\phi'_\omega||^t.$$

Then

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}^{(n)}(t) = \inf_{n \in I\!N} \frac{1}{n} \log \mathrm{P}^{(n)}(t).$$

Recall also that the shift map $\sigma : I^* \cup I^\infty \to I^* \cup I^\infty$ is defined for each $\omega \in I^* \cup I^\infty$ as

$$\sigma\Big(\{\omega_n\}_{n=1}^{|\omega|}\Big) = \{\omega_{n+1}\}_{n=1}^{|\omega|-1}.$$

If the function $\zeta = \zeta_\Phi : I^\infty \to I\!R$ is given by the formula

$$\zeta(\omega) = \log |\phi'_{\omega_1}(\pi(\sigma(\omega)))|,$$

then $\mathrm{P}(t) = \mathrm{P}(t\zeta)$, where $\mathrm{P}(t\zeta)$ is the classical topological pressure of the function $t\zeta$ when I is finite (so the space I^∞ is compact), and is understood in the sense of [4] and [9] when I is infinite. The finiteness parameter, $\theta = \theta_\Phi$, of the system is defined by $\inf\{t \geq 0 : \mathrm{P}^{(1)}(t) < \infty\} = \theta$. In [5], it was shown that the topological pressure function $\mathrm{P}(\cdot)$ is non-increasing on $[0, \infty)$, (strictly) decreasing, continuous and convex on $[\theta, \infty)$, and $\mathrm{P}(d) \leq 0$. Of course, $\mathrm{P}(0) = \infty$ if and only if I is infinite. The following characterization of the Hausdorff dimension $h = h_\Phi$ of the limit set $J = J_\Phi$ was proved in [5], Theorem 3.15. For every $F \subset I$, we write $\Phi|_F$ for the subsystem $\{\phi_i\}_{i \in F}$ of Φ.

Theorem 2.1.

$$h_\Phi = \sup\{h_{\Phi|_F} : F \subset I \textit{ is finite }\} = \inf\{t \geq 0 : \mathrm{P}(t) \leq 0\}.$$

If $\mathrm{P}(t) = 0$, *then* $t = h_\Phi$.

The system Φ was called regular provided there is some $t \geq 0$ such that $\mathrm{P}(t) = 0$. It follows from the strict decrease of P on $[\theta, \infty)$ that such a t is unique. Also, the system is regular if and only if there is a t-conformal measure. A Borel probability measure m is said to be t-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi'_i|^t \, dm,$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0$$

for every pair $i, j \in I$, $i \neq j$.

There are natural subclasses of regular systems. Following [5] still, a system Φ is said to be strongly regular if $0 < \mathrm{P}(t) < \infty$ for some $t \geq 0$. As an immediate application of Theorem 2.1 we get the following:

Theorem 2.2. *A system Φ is strongly regular if and only if $h > \theta$.*

Also, a system $\Phi = \{\phi_i\}_{i\in I}$ was called hereditarily regular or cofinitely regular provided every nonempty cofinite subsystem $\Phi' = \{\phi_i\}_{i\in I'}$ (i.e. I' is a cofinite subset of I) is regular. A finite system is clearly cofinitely regular, and it was shown in [5] that an infinite system is cofinitely regular exactly when the pressure is infinite at the finiteness parameter:

Theorem 2.3. *An infinite system Φ is cofinitely regular if and only if* $\mathrm{P}(\theta) = \infty \Leftrightarrow \mathrm{P}^{(1)}(\theta) = \infty \Leftrightarrow \{t \geq 0 : \mathrm{P}(t) < \infty\} = (\theta, \infty) \Leftrightarrow \{t \geq 0 : \mathrm{P}^{(1)}(t) < \infty\} = (\theta, \infty)$.

Remark that every cofinitely regular system is strongly regular, and every strongly regular system is regular. Finally, we coin a new class of systems. These are regular systems at the threshold between strongly regular and irregular systems:

Definition 2.4. *A system Φ is named critically regular if $P(\theta) = 0$.*

3. Preliminaries from Topology

Fix an arbitrary set Y among whose elements there is no empty set and suppose that a function $\lambda : Y^{\mathbb{N}} \to Y \cup \{\emptyset\}$ is given with the following properties:

(a) If $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) \in Y$, then for every increasing sequence $\{n_k\}_{k=1}^\infty$, $\lambda\Big(\{x_{n_k}\}_{k=1}^\infty\Big) = \lambda\Big(\{x_n\}_{n=1}^\infty\Big)$;

(b) Consider a sequence $\{x_n\}_{n=1}^\infty \in Y^{\mathbb{N}}$. If there exists $x \in Y$ such that for every increasing sequence $\{n_k\}_{k=1}^\infty$ there exists a subsequence $\{n_{k_j}\}_{j=1}^\infty$ such that $\lambda\left(\{x_{n_{k_j}}\}_{j=1}^\infty\right) = x$, then $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) = x$;

(c) If $x \in Y$, then $\lambda\Big(\{x\}_{n=1}^\infty\Big) = x$.

A sequence $\{x_n\}_{n=1}^\infty \in Y^{\mathbb{N}}$ is called λ-converging if $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) \in Y$, and then $\lambda\Big(\{x_n\}_{n=1}^\infty\Big)$ is called the λ-limit of the sequence $\{x_n\}_{n=1}^\infty$. Otherwise, this sequence is said to be λ-diverging. A set $F \subset Y$ is declared to be closed if the λ-limit of every λ-converging sequence of points from F belongs to F. Clearly, $\emptyset$ and Y are closed sets. We shall next prove the following (actually straightforward) facts:

Lemma 3.1. *If $\{F_t\}_{t\in T}$ is an arbitrary family of closed subsets of Y, then $\bigcap_{t\in T} F_t$ is a closed subset of Y. If F and E are closed subsets of Y, then $F \cup E$ is a closed subset of Y, too.*

Proof. Suppose that $\{x_n\}_{n=1}^\infty$ is a λ-converging sequence of points from $\bigcap_{t\in T} F_t$. Then $\{x_n\}_{n=1}^\infty$ is a λ-converging sequence of points from each F_t. Since each set F_t is closed, $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) \in F_t$ for all $t \in T$. Consequently, $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) \in \bigcap_{t\in T} F_t$, and the first part of our lemma is proven. In order to prove the second part, suppose that $\{y_n\}_{n=1}^\infty$ is a λ-converging sequence of points from $F \cup E$. Then at least one of the sets F or E contains infinitely many

y_n's. Without loss of generality, we may assume that F has this property. This means that there is an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y_{n_k} \in F$ for every $k \in I\!N$. Since F is closed, it follows from (a) that $\lambda\Big(\{y_n\}_{n=1}^\infty\Big) = \lambda\Big(\{y_{n_k}\}_{k=1}^\infty\Big) \in F \subset F \cup E$. The second part of the lemma is thus proved. ■

It follows from this lemma that the family of complements of closed subsets of Y forms a topology on Y, which will be called the λ-topology on Y.

Lemma 3.2. *Consider an arbitrary sequence $\{x_n\}_{n=1}^\infty \in Y^{I\!N}$. If $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) \in Y$, then the sequence $\{x_n\}_{n=1}^\infty$ converges with respect to the λ-topology on Y and its limit with respect to this topology is $\lambda\Big(\{x_n\}_{n=1}^\infty\Big)$. Conversely, if the sequence $\{x_n\}_{n=1}^\infty$ converges with respect to the λ-topology to some point $x \in Y$, then $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) = x$.*

Proof. Take a sequence $\{x_n\}_{n=1}^\infty \in Y^{I\!N}$. Suppose that $x := \lambda\Big(\{x_n\}_{n=1}^\infty\Big) \in Y$ but that the sequence $\{x_n\}_{n=1}^\infty$ does not converge to x with respect to the λ-topology. This means that there are an open set $U \subset Y$ containing x and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $x_{n_k} \notin U$ for all $k \in I\!N$. So, $x_{n_k} \in Y \setminus U$ for all $k \in I\!N$. Since $Y \setminus U$ is closed, using (a), we deduce that $x = \lambda\Big(\{x_{n_k}\}_{k=1}^\infty\Big) \in Y \setminus U$. This contradicts the fact that $x \in U$ and establishes the first part of our lemma. In order to prove the second part, suppose that the sequence $\{x_n\}_{n=1}^\infty$ converges with respect to the λ-topology to some point $x \in Y$. Seeking a contradiction once again, assume first that $\lambda\Big(\{x_{n_k}\}_{k=1}^\infty\Big) = \emptyset$ for every increasing sequence $\{n_k\}_{k=1}^\infty$. Then, in view of (c), $x_n = x$ for finitely many n's only. Denote the largest of these n's by $q-1$ (it is allowed that $q-1=0$). Then, the sequence $\{x_n\}_{n=q}^\infty$ does not contain x and $\lambda\Big(\{x_{n_k}\}_{k=1}^\infty\Big) = \emptyset$ for every increasing sequence $\{n_k\}_{k=1}^\infty$ with $n_1 \geq q$. Consequently, the set $\{x_n\}_{n=q}^\infty$ is trivially closed, as it admits no λ-converging sequence. Therefore, $Y \setminus \{x_n\}_{n=q}^\infty$ is open and contains x. This contradicts the assumption that x is the limit of $\{x_n\}_{n=q}^\infty$ with respect to the λ-topology. So, we may assume that $\{x_n\}_{n=1}^\infty$ contains a λ-converging subsequence $\{x_{n_k}\}_{k=1}^\infty$. It then follows from the first part of this lemma that $\lambda\Big(\{x_{n_k}\}_{k=1}^\infty\Big) = x$. Since every subsequence of $\{x_n\}_{n=1}^\infty$ converges to x with respect to the λ-topology, we may conclude that every subsequence of the sequence $\{x_n\}_{n=1}^\infty$ contains a subsequence λ-converging to the point x. It thus follows from (b) that $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) = x$. This completes the proof. ■

Lemma 3.3. *Let Y and Z be topological spaces, Y being endowed with a λ-topology. Let $f : Y \to Z$ be an arbitrary function. Then the following two conditions are equivalent:*

(i) *$f : Y \to Z$ is continuous;*

(ii) *If a sequence $\{x_n\}_{n=1}^\infty \in Y^{I\!N}$ is λ-converging, then $\lim_{n\to\infty} f(x_n) = f\Big(\lambda\Big(\{x_n\}_{n=1}^\infty\Big)\Big)$.*

Proof. The implication (i)⇒(ii) follows immediately from Lemma 3.2. So, suppose that (ii) is true and, in order to prove (i), consider an arbitrary closed set $F \subset Z$. Let $\{x_n\}_{n=1}^\infty$

be an arbitrary λ-converging sequence of points from $f^{-1}(F)$. Then $f\Big(\lambda\Big(\{x_n\}_{n=1}^\infty\Big)\Big) = \lim_{n\to\infty} f(x_n) \in F$ since $F \subset Z$ is closed and each point $f(x_n)$ is in F. Hence, $\lambda\Big(\{x_n\}_{n=1}^\infty\Big) \in f^{-1}(F)$ and therefore, according to the definition of the λ-topology, $f^{-1}(F)$ is closed. This proves that $f : Y \to Z$ is continuous. ■

4. Continuity in Finite Iterated Function Systems

Given a set $X \subset I\!R^d$ and a countable (finite or infinite) set I, denote by $\mathrm{CIFS}(X, I)$ the family of all conformal iterated function systems with seed set X and alphabet I. Suppose throughout this section that I is finite and define the metric ρ on $\mathrm{CIFS}(X, I)$ by the formula below. Given $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$, set

$$\rho(\Phi, \Psi) = ||\Phi - \Psi|| + ||\Phi' - \Psi'|| = \sum_{i\in I}\Big(||\phi_i - \psi_i|| + ||\phi_i' - \psi_i'||\Big).$$

Our goal in this section is to show that, given $t \geq 0$, the pressure function $P(t) : \mathrm{CIFS}(X, I) \to I\!R \cup \{\infty\}$, $\Phi \mapsto P_\Phi(t)$, is continuous and that so is the Hausdorff dimension function $h : \mathrm{CIFS}(X, I) \to [0, \infty)$, $\Phi \mapsto h_\Phi$. We start with the following intermediate result:

Lemma 4.1. *The coding map $\pi : \mathrm{CIFS}(X, I) \to C(I^\infty, X)$, $\Phi \mapsto \pi_\Phi$, is continuous, where $C(I^\infty, X)$ is the space of continuous maps from I^∞ to X endowed with the supremum norm. Moreover, for all $\Phi, \Psi \in \mathrm{CIFS}(X, I)$,*

$$||\pi_\Psi - \pi_\Phi|| \leq \min\Big\{\frac{1}{1 - ||\Phi'||}, \frac{1}{1 - ||\Psi'||}\Big\}||\Psi - \Phi|| \leq \min\Big\{\frac{1}{1 - ||\Phi'||}, \frac{1}{1 - ||\Psi'||}\Big\}\rho(\Phi, \Psi).$$

Proof. We shall show by induction that for every $x \in X$, every $n \in I\!N$, every $\omega \in I^n$ and all $\Phi, \Psi \in \mathrm{CIFS}(X, I)$

$$|\psi_\omega(x) - \phi_\omega(x)| \leq ||\Psi - \Phi|| \sum_{j=0}^{n-1} ||\Phi'||^j. \tag{4.1}$$

Indeed, this is obvious when $n = 1$. So, suppose that (4.1) is true for some $n \in I\!N$. Take an arbitrary $x \in X$ and $\omega \in I^{n+1}$. By our inductive assumption we have that

$$\begin{aligned}
|\psi_\omega(x) - \phi_\omega(x)| &\leq |\psi_{\omega_1}(\psi_{\sigma\omega}(x)) - \phi_{\omega_1}(\psi_{\sigma\omega}(x))| + |\phi_{\omega_1}(\psi_{\sigma\omega}(x)) - \phi_{\omega_1}(\phi_{\sigma\omega}(x))| \\
&\leq ||\Psi - \Phi|| + ||\phi_{\omega_1}'|| \cdot |\psi_{\sigma\omega}(x) - \phi_{\sigma\omega}(x)| \\
&\leq ||\Psi - \Phi|| + ||\Phi'|| \cdot |\psi_{\sigma\omega}(x) - \phi_{\sigma\omega}(x)| \\
&\leq ||\Psi - \Phi|| + ||\Phi'|| \cdot ||\Psi - \Phi|| \sum_{j=0}^{n-1} ||\Phi'||^j \\
&= ||\Psi - \Phi|| \sum_{j=0}^{n} ||\Phi'||^j.
\end{aligned}$$

Since for every $\tau \in I^\infty$, $\pi_\Psi(\tau) = \lim_{n\to\infty} \psi_{\tau|_n}(x)$ and $\pi_\Phi(\tau) = \lim_{n\to\infty} \phi_{\tau|_n}(x)$, letting $n \to \infty$ in (4.1), we obtain

$$|\pi_\Psi(\tau) - \pi_\Phi(\tau)| \le ||\Psi - \Phi|| \sum_{j=0}^{\infty} ||\Phi'||^j = \frac{1}{1 - ||\Phi'||} ||\Psi - \Phi|| \le \frac{1}{1 - ||\Phi'||} \rho(\Psi, \Phi).$$

The proof is complete. ■

We are now in a position to study the pressure function:

Lemma 4.2. *For every $t \ge 0$, the function $\Phi \mapsto \mathrm{P}(t\zeta_\Phi)$, $\Phi \in \mathrm{CIFS}(X, I)$, is continuous.*

Proof. Fix $\Phi \in \mathrm{CIFS}(X, I)$, $\gamma \in (0, \inf |\Phi'|)$, and $\delta > 0$ so small that $\inf |\Psi'| \ge \gamma$ for all $\Psi \in B(\Phi, \delta)$. Then for every $i \in I$ and every $x \in X$, we have that

$$\Big| \log |\psi_i'(x)| - \log |\phi_i'(x)| \Big| \le \gamma^{-1} |\psi_i'(x) - \phi_i'(x)| \le \gamma^{-1} ||\Psi' - \Phi'|| \le \gamma^{-1} \rho(\Psi, \Phi).$$

Thus, for every $\omega \in I^\infty$, we get that

$$\begin{aligned}
&\Big| t \log |\psi_{\omega_1}'(\pi_\Psi(\sigma\omega))| - t \log |\phi_{\omega_1}'(\pi_\Phi(\sigma\omega))| \Big| \\
&\quad = |t| \Big| \log |\psi_{\omega_1}'(\pi_\Psi(\sigma\omega))| - \log |\phi_{\omega_1}'(\pi_\Phi(\sigma\omega))| \Big| \\
&\quad \le |t| \Big(\Big| \log |\psi_{\omega_1}'(\pi_\Psi(\sigma\omega))| - \log |\phi_{\omega_1}'(\pi_\Psi(\sigma\omega))| \Big| + \Big| \log |\phi_{\omega_1}'(\pi_\Psi(\sigma\omega))| - \log |\phi_{\omega_1}'(\pi_\Phi(\sigma\omega))| \Big| \Big) \\
&\quad \le |t| \Big(\gamma^{-1} \rho(\Psi, \Phi) + \gamma^{-1} |\pi_\Psi(\sigma\omega) - \pi_\Phi(\sigma\omega)| \Big) \\
&\quad \le \gamma^{-1} |t| \Big(1 + (1 - ||\Phi'||)^{-1} \Big) \rho(\Psi, \Phi).
\end{aligned}$$

Therefore $||t\zeta_\Psi - t\zeta_\Phi|| \le \gamma^{-1} |t| \Big(1 + (1 - ||\Phi'||)^{-1} \Big) \rho(\Psi, \Phi)$ and, consequently, the function $\Phi \mapsto t\zeta_\Phi \in C(I^\infty)$ is continuous. Since the pressure function $\mathrm{P} : C(I^\infty) \to I\!R$ is Lipschitz continuous (with Lipschitz constant 1), the proof is complete. ■

We shall next investigate the Hausdorff dimension function:

Theorem 4.3. *The Hausdorff dimension function $h : \mathrm{CIFS}(X, I) \to (0, \infty)$, $\Phi \mapsto h_\Phi$, is continuous.*

Proof. Fix $\Phi \in \mathrm{CIFS}(X, I)$ and $\varepsilon > 0$. In view of the previous lemma, there exists $\delta > 0$ such that if $\Psi \in B(\Phi, \delta)$, then

$$\begin{aligned}
\max\Big\{ \Big| \mathrm{P}\Big((h_\Phi - \varepsilon)\zeta_\Psi\Big) - \mathrm{P}\Big((h_\Phi - \varepsilon)\zeta_\Phi\Big) \Big|, \Big| \mathrm{P}\Big((h_\Phi + \varepsilon)\zeta_\Psi\Big) - \mathrm{P}\Big((h_\Phi + \varepsilon)\zeta_\Phi\Big) \Big| \Big\} \\
\le \frac{1}{2} \min\Big\{ \mathrm{P}\Big((h_\Phi - \varepsilon)\zeta_\Phi\Big), -\mathrm{P}\Big((h_\Phi + \varepsilon)\zeta_\Phi\Big) \Big\}.
\end{aligned}$$

Therefore

$$\mathrm{P}\Big((h_\Phi-\varepsilon)\zeta_\Psi\Big) \geq \mathrm{P}\Big((h_\Phi-\varepsilon)\zeta_\Phi\Big) - \frac{1}{2}\mathrm{P}\Big((h_\Phi-\varepsilon)\zeta_\Phi\Big) = \frac{1}{2}\mathrm{P}\Big((h_\Phi-\varepsilon)\zeta_\Phi\Big) > 0$$

and

$$\mathrm{P}\Big((h_\Phi+\varepsilon)\zeta_\Psi\Big) \leq \mathrm{P}\Big((h_\Phi+\varepsilon)\zeta_\Phi\Big) + \frac{1}{2}\Big[-\mathrm{P}\Big((h_\Phi+\varepsilon)\zeta_\Phi\Big)\Big] = \frac{1}{2}\mathrm{P}\Big((h_\Phi+\varepsilon)\zeta_\Phi\Big) < 0.$$

It follows from the first inequality that $h_\Psi > h_\Phi - \varepsilon$, and from the second inequality that $h_\Psi < h_\Phi + \varepsilon$. So $|h_\Psi - h_\Phi| < \varepsilon$, and we are done. ■

5. Continuity in Infinite Iterated Function Systems

Throughout the entire section the alphabet I is assumed to be countably infinite. Whenever convenient, we assume that $I = I\!N$. We endow CIFS(X, I) with two topologies. First, given $\Phi, \Psi \in$ CIFS(X, I), set

$$\rho_\infty(\Phi,\Psi) = \sum_{i=1}^{\infty} 2^{-i} \min\Big\{1, ||\phi_i - \psi_i||, ||\phi_i' - \psi_i'||\Big\}.$$

It is easy to verify that this formula defines a metric on the space CIFS(X, I) and that a sequence $\{\Phi^n\}_{n=1}^\infty$ converges to Φ with respect to this metric if and only if for every $i \in I$,

$$\lim_{n\to\infty} \max\Big\{||\phi_i^n - \phi_i||, ||(\phi_i^n)' - \phi_i'||\Big\} = 0.$$

This implies in particular that for every $i \in I$,

$$\lim_{n\to\infty} ||\phi_i^n|| = ||\phi_i|| \text{ and } \lim_{n\to\infty} ||(\phi_i^n)'|| = ||\phi_i'||.$$

The topology induced by the metric ρ_∞ on CIFS(X, I) is called the pointwise convergence topology. It is now important to make the following observation:

Lemma 5.1. *A sequence $\{\Phi^n\}_{n=1}^\infty$ converges to Φ pointwise if and only if for every $\omega \in I^*$,*

$$\lim_{n\to\infty} \max\Big\{||\phi_\omega^n - \phi_\omega||, ||(\phi_\omega^n)' - \phi_\omega'||\Big\} = 0.$$

Proof. The "if" part is trivial. Since $I^* = \cup_{k=1}^\infty I^k$, we will prove the "only if" part by induction on k. When $k = 1$, we have $\omega \in I$, and the result is immediate. So, let $k > 1$ and $\omega \in I^k$, and suppose that for every $\tau \in I^{k-1}$,

$$\lim_{n\to\infty} \max\Big\{||\phi_\tau^n - \phi_\tau||, ||(\phi_\tau^n)' - \phi_\tau'||\Big\} = 0.$$

Then, on the one hand, for every $x \in X$,

$$\begin{aligned}\Big|\phi_\omega^n(x) - \phi_\omega(x)\Big| &= \Big|\phi_{\omega_1}^n(\phi_{\sigma\omega}^n(x)) - \phi_{\omega_1}(\phi_{\sigma\omega}(x))\Big| \\ &\le \Big|\phi_{\omega_1}^n(\phi_{\sigma\omega}^n(x)) - \phi_{\omega_1}^n(\phi_{\sigma\omega}(x))\Big| + \Big|\phi_{\omega_1}^n(\phi_{\sigma\omega}(x)) - \phi_{\omega_1}(\phi_{\sigma\omega}(x))\Big| \\ &\le ||(\phi_{\omega_1}^n)'|| \cdot ||\phi_{\sigma\omega}^n - \phi_{\sigma\omega}|| + ||\phi_{\omega_1}^n - \phi_{\omega_1}||.\end{aligned}$$

Thus,

$$||\phi_\omega^n - \phi_\omega|| \le ||(\phi_{\omega_1}^n)'|| \cdot ||\phi_{\sigma\omega}^n - \phi_{\sigma\omega}|| + ||\phi_{\omega_1}^n - \phi_{\omega_1}||.$$

Both terms on the right-hand side tend to 0 since $\lim_{n\to\infty} ||\phi_{\sigma\omega}^n - \phi_{\sigma\omega}|| = 0$ by our inductive hypothesis, and $\lim_{n\to\infty} ||(\phi_{\omega_1}^n)'|| = ||\phi_{\omega_1}'||$ and $\lim_{n\to\infty} ||\phi_{\omega_1}^n - \phi_{\omega_1}|| = 0$ by the basis step.

On the other hand, for every $x \in X$,

$$\begin{aligned}\Big|(\phi_\omega^n)'(x) - \phi_\omega'(x)\Big| &= \Big|(\phi_{\omega_1}^n)'(\phi_{\sigma\omega}^n(x)) \cdot (\phi_{\sigma\omega}^n)'(x) - \phi_{\omega_1}'(\phi_{\sigma\omega}(x)) \cdot \phi_{\sigma\omega}'(x)\Big| \\ &\le \Big|(\phi_{\omega_1}^n)'(\phi_{\sigma\omega}^n(x)) \cdot (\phi_{\sigma\omega}^n)'(x) - \phi_{\omega_1}'(\phi_{\sigma\omega}^n(x)) \cdot (\phi_{\sigma\omega}^n)'(x)\Big| \\ &\quad + \Big|\phi_{\omega_1}'(\phi_{\sigma\omega}^n(x)) \cdot (\phi_{\sigma\omega}^n)'(x) - \phi_{\omega_1}'(\phi_{\sigma\omega}^n(x)) \cdot \phi_{\sigma\omega}'(x)\Big| \\ &\quad\quad + \Big|\phi_{\omega_1}'(\phi_{\sigma\omega}^n(x)) \cdot \phi_{\sigma\omega}'(x) - \phi_{\omega_1}'(\phi_{\sigma\omega}(x)) \cdot \phi_{\sigma\omega}'(x)\Big| \\ &\le ||(\phi_{\omega_1}^n)' - \phi_{\omega_1}'|| \cdot ||(\phi_{\sigma\omega}^n)'|| + ||\phi_{\omega_1}'|| \cdot ||(\phi_{\sigma\omega}^n)' - \phi_{\sigma\omega}'|| \\ &\quad + \Big|\phi_{\omega_1}'(\phi_{\sigma\omega}^n(x)) - \phi_{\omega_1}'(\phi_{\sigma\omega}(x))\Big| \cdot ||\phi_{\sigma\omega}'||.\end{aligned}$$

Thus,

$$\begin{aligned}||(\phi_\omega^n)' - \phi_\omega'|| &\le ||(\phi_{\omega_1}^n)' - \phi_{\omega_1}'|| \cdot ||(\phi_{\sigma\omega}^n)'|| + ||\phi_{\omega_1}'|| \cdot ||(\phi_{\sigma\omega}^n)' - \phi_{\sigma\omega}'|| \\ &\quad + \sup_{x\in X}\Big|\phi_{\omega_1}'(\phi_{\sigma\omega}^n(x)) - \phi_{\omega_1}'(\phi_{\sigma\omega}(x))\Big| \cdot ||\phi_{\sigma\omega}'||.\end{aligned}$$

The first term on the right-hand side tends to 0 since $\lim_{n\to\infty} ||(\phi_{\omega_1}^n)' - \phi_{\omega_1}'|| = 0$, and $\lim_{n\to\infty} ||(\phi_{\sigma\omega}^n)'|| = ||\phi_{\sigma\omega}'||$ by our inductive hypothesis (in fact, $\lim_{n\to\infty} ||(\phi_{\sigma\omega}^n)' - \phi_{\sigma\omega}'|| = 0$). So does the second term for this latter reason. Finally, the third term tends to 0 since $\lim_{n\to\infty} ||\phi_{\sigma\omega}^n - \phi_{\sigma\omega}|| = 0$ by our inductive hypothesis and ϕ_{ω_1}' is uniformly continuous on X. ■

We now describe the properties of the pressure and Hausdorff dimension functions in the topology of pointwise convergence.

Theorem 5.2. *The Hausdorff dimension function* $h : \mathrm{CIFS}(X, I) \to (0, \infty)$ *is lower semi-continuous when* $\mathrm{CIFS}(X, I)$ *is equipped with the pointwise convergence topology.*

Proof. Fix $\Phi \in \mathrm{CIFS}(X, I)$ and a sequence $\{\Phi^n\}_{n=1}^\infty$ converging to Φ in the pointwise convergence topology. Take $\varepsilon > 0$. In view of Theorem 2.1, there exists a finite set $F \subset I$

such that $h_{\Phi|_F} > h_\Phi - \varepsilon$. Since the sequence $\{\Phi^n|_F\}_{n=1}^\infty$ converges to $\Phi|_F$ with respect to the metric ρ introduced in section 4, it follows from Theorem 4.3 that

$$\liminf_{n\to\infty} h_{\Phi^n} \geq \liminf_{n\to\infty} h_{\Phi^n|_F} = h_{\Phi|_F} > h_\Phi - \varepsilon.$$

Since ε was chosen arbitrarily, we conclude that $\liminf_{n\to\infty} h_{\Phi^n} \geq h_\Phi$. ■

Based on Theorem 2.1.5 from [9] and on Lemma 4.2, the argument used in Theorem 5.2 gives the following:

Lemma 5.3. *For every $t \geq 0$, the function $\Phi \mapsto \mathrm{P}_\Phi(t)$, $\Phi \in \mathrm{CIFS}(X, I)$, is lower semi-continuous with respect to the pointwise convergence topology on* $\mathrm{CIFS}(X, I)$.

It is not difficult to see that the pressure and Hausdorff dimension functions are not upper semi-continuous (and thus not continuous) in the topology of pointwise convergence. Furthermore, the finiteness parameter function θ is generally neither upper nor lower semi-continuous.

Example. Let X be the unit square in the complex plane with vertices $0, 1, 1+i, i$. Let $X_0 \subset X$ be the square determined by the vertices $0, \frac{1}{2}, \frac{1}{2}+\frac{i}{2}, \frac{i}{2}$, and let $X_1 \subset X$ be the square whose vertices are $\frac{1}{2}+\frac{i}{2}, 1+\frac{i}{2}, 1+i, \frac{1}{2}+i$. One can easily find maps $\phi_n : X \to X_0$, $n \in I\!N$, such that

$$\phi_n(z) = 2^{-(n+1)}z + a_n$$

and $\Phi = \{\phi_n\}_{n=1}^\infty$ is a member of $\mathrm{CIFS}(X, I\!N)$ (one can even require that $\phi_n(X) \cap \phi_m(X) = \emptyset$ if $n \neq m$). Notice that $\theta_\Phi = 0$, which means that Φ is absolutely regular in the sense of [6]. So Φ is cofinitely regular. A short calculation further shows that $h_\Phi = \frac{\log(1+\sqrt{5})}{\log 2} - 1 < 1$. It is also easy to build an irregular system $\Psi = \{\psi_n\}_{n=1}^\infty$ consisting of maps $\psi_n : X \to X_1$, $n \in I\!N$, of the form

$$\psi_n(z) = r_n z + b_n$$

for which $h_\Psi = \theta_\Psi = 1$. Now, for every $n \in I\!N$ construct the system

$$\Phi^n = \{\phi_i\}_{i=1}^n \cup \{\psi_j\}_{j=n+1}^\infty.$$

Since $\Phi, \Psi \in \mathrm{CIFS}(X, I\!N)$ and $X_0 \cap X_1 = \{\frac{1}{2} + \frac{i}{2}\}$, all the systems Φ^n, $n \in I\!N$, satisfy the OSC and, consequently, belong to $\mathrm{CIFS}(X, I\!N)$. Obviously, the sequence $\{\Phi^n\}_{n=1}^\infty$ converges pointwise to Φ. However, it is clear that $\theta_{\Phi_n} = \theta_\Psi$, which shows that the function θ is not upper semi-continuous. It is also easy to see that Φ_n is irregular for each $n \in I\!N$ large enough. Moreover, since the system Ψ is irregular, all its cofinite subsystems are irregular and share the same finiteness parameter. Hence

$$h_{\{\psi_j\}_{j=n+1}^\infty} = \theta_{\{\psi_j\}_{j=n+1}^\infty} = \theta_\Psi = 1.$$

Therefore

$$h_{\Phi^n} \geq h_{\{\psi_j\}_{j=n+1}^\infty} = 1,$$

and thus $\limsup_{n\to\infty} h_{\Phi^n} \geq 1 > h_\Phi$, which establishes that the Hausdorff dimension function is not upper semi-continuous. Finally, this example shows that the pressure function $\Xi \mapsto \mathrm{P}_\Xi(t)$ is not upper semi-continuous at Φ (take any $0 < t < \theta_\Psi$). ■

It is also interesting to observe from this example that there are sequences of irregular CIFS that converge in the pointwise topology to cofinitely regular CIFS. We will later show (see Lemma 5.5) that this cannot happen in the λ-topology we will soon introduce. Below is another example.

Example. In the same setting as before, for every $n \in I\!N$ form the system

$$\Psi^n = \{\psi_i\}_{i=1}^n \cup \{\phi_j\}_{j=n+1}^\infty.$$

Since $\Phi, \Psi \in \mathrm{CIFS}(X, I\!N)$ and $X_0 \cap X_1 = \{\frac{1}{2} + \frac{i}{2}\}$, all the systems Ψ^n, $n \in I\!N$, satisfy the OSC and, consequently, belong to $\mathrm{CIFS}(X, I\!N)$. Of course, the sequence $\{\Psi^n\}_{n=1}^\infty$ converges pointwise to Ψ. However, $\theta_{\Psi_n} = \theta_\Phi = 0$ and Ψ_n is absolutely regular, and thereby cofinitely regular, for all $n \in I\!N$. This shows that the function θ is not lower semi-continuous. ■

It is also worth noticing from this example that there are sequences of cofinitely regular CIFS that converge pointwise to irregular CIFS. We will later show (see Lemma 5.5) that this is impossible in the λ-topology. In fact, it is possible to construct similar examples showing that every class of systems can converge to any class in the pointwise topology.

We now turn our attention to the more delicate problem of identifying a topology with respect to which the pressure and Hausdorff dimension functions are continuous. The first example above shows that these functions generally fail to be upper semi-continuous in the topology of pointwise convergence. In order to remedy this flaw, we introduce a weaker, but rich enough, topology on the space $\mathrm{CIFS}(X, I)$ as follows. Given a sequence $\{\Phi^n\}_{n=1}^\infty$ in $\mathrm{CIFS}(X, I)$ and $\Phi \in \mathrm{CIFS}(X, I)$, we say that $\lambda\Big(\{\Phi^n\}_{n=1}^\infty\Big) = \Phi$ provided that $\{\Phi^n\}_{n=1}^\infty$ converges to Φ in the topology of pointwise convergence and that there exist $C > 0$, $M \in I\!N$ and a finite set $F \subset I$ such that

$$\Big|\log ||\phi_i'|| - \log ||(\phi_i^n)'||\Big| \leq C \tag{5.1}$$

for all $i \in I \setminus F$ and all $n \geq M$. Notice that due to pointwise convergence the set F can always be chosen to be empty and that Φ (if it exists) is unique. If a sequence $\{\Phi^n\}_{n=1}^\infty$ in $\mathrm{CIFS}(X, I)$ does not admit any $\Phi \in \mathrm{CIFS}(X, I)$ for which the above conditions are fulfilled, we declare that $\lambda\Big(\{\Phi^n\}_{n=1}^\infty\Big) = \emptyset$.

We shall now check that the function λ hence defined satisfies the three conditions (a), (b) and (c) specified at the beginning of section 3, and therefore induces a λ-topology. Conditions (a) and (c) are obviously fulfilled. In order to verify condition (b), take a sequence $\{\Phi^n\}_{n=1}^\infty$ in $\mathrm{CIFS}(X, I)$ and $\Phi \in \mathrm{CIFS}(X, I)$ such that $\lambda\Big(\{\Phi^n\}_{n=1}^\infty\Big) \neq \Phi$. If the sequence $\{\Phi^n\}_{n=1}^\infty$ does not converge to Φ pointwise, then there exists an increasing sequence $\{n_k\}_{k=1}^\infty$ such that for every subsequence $\{n_{k_j}\}_{j=1}^{\infty}$, the sequence $\{\Phi^{n_{k_j}}\}_{j=1}^{\infty}$ does not converge to Φ pointwise, and in particular $\lambda\left(\{\Phi^{n_{k_j}}\}_{j=1}^{\infty}\right) \neq \Phi$. We may therefore assume that $\{\Phi^n\}_{n=1}^{\infty}$ converges pointwise to Φ, and that there exist a sequence $\{i_k\}_{k=1}^{\infty}$ in I and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that $\left|\log||\phi'_{i_k}|| - \log||(\phi_{i_k}^{n_k})'||\right| > k$. It is now clear that one cannot choose from the sequence $\{\Phi^{n_k}\}_{k=1}^{\infty}$ any subsequence whose λ-value is Φ. So, condition (b) has been checked. Hence the function λ induces the λ-topology on CIFS(X,I) with respect to which a set $G \subset$ CIFS(X,I) is closed if and only if the λ-limit of every λ-converging sequence in G belongs to G. From now on, unless otherwise specified, we consider CIFS(X,I) as the space topologized by the function λ.

Before even considering the pressure and Hausdorff dimension functions, let us have a look at the finiteness parameter function θ:

Lemma 5.4. *The function* $\theta : \mathrm{CIFS}(X,I) \to [0,\infty)$, $\Phi \mapsto \theta_\Phi$, *is locally constant, and thereby continuous, when* CIFS(X,I) *is endowed with the* λ*-topology.*

Proof. We will equivalently prove that if $\{\Phi^n\}_{n=1}^{\infty}$ is a λ-converging sequence, then the sequence $\{\theta_{\Phi^n}\}_{n=1}^{\infty}$ is eventually constant. To do this, let $\Phi = \lambda\Big(\{\Phi^n\}_{n=1}^{\infty}\Big)$. It follows from condition (5.1) that there exist $C > 0$ and $M \in I\!N$ such that

$$e^{-Ct} \leq \frac{||(\phi_i^n)'||^t}{||\phi_i'||^t} \leq e^{Ct}$$

for all $i \in I$, all $n \geq M$ and $t \geq 0$. This readily implies that for every $t \geq 0$ and all $n \geq M$, $\mathrm{P}_{\Phi^n}^{(1)}(t) < \infty$ if and only if $\mathrm{P}_{\Phi}^{(1)}(t) < \infty$. But this means that $\theta_{\Phi^n} = \theta_\Phi$ for all $n \geq M$. ■

The proof of Lemma 5.4 also gives the following:

Lemma 5.5. *If* $\lambda\Big(\{\Phi^n\}_{n=1}^{\infty}\Big) = \Phi \in$ CIFS(X,I), *then the following two conditions are equivalent:*

(i) *The system* Φ *is cofinitely regular;*
(ii) *The systems* Φ^n *are cofinitely regular for all* $n \in I\!N$ *large enough.*

In particular, this shows that the set of all cofinitely regular systems is open and closed in CIFS(X,I) when this space is endowed with the λ-topology. However, this set is neither closed nor open in the pointwise convergence topology, as Examples 4 and 6 from section 8 show.

We shall now concentrate on the pressure function. Here is an intermediate result:

Theorem 5.6. *Let* $k \in I\!N$. *For every* $t \geq 0$, *the function* $\mathrm{P}^{(k)}(t) : \mathrm{CIFS}(X,I) \to (-\infty,\infty]$, $\Phi \mapsto \mathrm{P}_\Phi^{(k)}(t)$, *is continuous when* CIFS(X,I) *is endowed with the* λ*-topology.*

Proof. Fix $\Phi \in \text{CIFS}(X,I)$ and a sequence $\{\Phi_n\}_{n=1}^\infty$ in $\text{CIFS}(X,I)$ such that $\lambda\Big(\{\Phi^n\}_{n=1}^\infty\Big) = \Phi$. Recall that for every $t \geq 0$, $\mathrm{P}_\Phi^{(1)}(t) < \infty$ if and only if $\mathrm{P}_{\Phi^n}^{(1)}(t) < \infty$ for all $n \in I\!\!N$ large enough. Thus, for every $t \geq 0$, $\mathrm{P}_\Phi^{(k)}(t) < \infty$ if and only if $\mathrm{P}_{\Phi^n}^{(k)}(t) < \infty$ for all $n \in I\!\!N$ large enough, since $(K_\Psi^{-1}\mathrm{P}_\Psi^{(1)}(t))^k \leq \mathrm{P}_\Psi^{(k)}(t) \leq (\mathrm{P}_\Psi^{(1)}(t))^k$ for any $\Psi \in \text{CIFS}(X,I)$, where K_Ψ is any constant of bounded distortion for the CIFS Ψ. The result follows immediately from this observation for those $t \in \{t' \geq 0 : \mathrm{P}_\Phi^{(1)}(t') = \infty\}$. So, fix $t \in \{t' \geq 0 : \mathrm{P}_\Phi^{(1)}(t') < \infty\}$. Let $\varepsilon > 0$. It follows from condition (5.1) that there exist $C > 0$ and $M \in I\!\!N$ such that

$$e^{-Ct} \leq \frac{||(\phi_i^n)'||^t}{||\phi_i'||^t} \leq e^{Ct}$$

for all $i \in I$ and all $n \geq M$. Since $\mathrm{P}_\Phi^{(k)}(t) < \infty$, there exists a finite set $G \subset I^k$ such that

$$\sum_{\omega \in I^k \setminus G} ||\phi_\omega'||^t < e^{-kCt}K^{-kt}\frac{\varepsilon}{2},$$

where $K = K_\Phi$ is a constant of bounded distortion for the CIFS Φ. Moreover, as $\{\Phi^n\}_{n=1}^\infty$ converges pointwise to Φ, Lemma 5.1 ensures that $\lim_{n\to\infty} ||(\phi_\omega^n)' - \phi_\omega'|| = 0$ for every $\omega \in I^k$. Hence $\lim_{n\to\infty} ||(\phi_\omega^n)'|| = ||\phi_\omega'||$ for every $\omega \in I^k$, and thus there is $N \in I\!\!N$ such that for every $n \geq N$,

$$\left|\sum_{\omega \in G} ||(\phi_\omega^n)'||^t - \sum_{\omega \in G} ||\phi_\omega'||^t\right| < \frac{\varepsilon}{2}.$$

On the one hand, it follows that for every $n \geq \max\{N, M\}$

$$\begin{aligned}
\mathrm{P}_{\Phi^n}^{(k)}(t) = \sum_{\omega \in I^k} ||(\phi_\omega^n)'||^t &= \sum_{\omega \in G} ||(\phi_\omega^n)'||^t + \sum_{\omega \in I^k \setminus G} ||(\phi_\omega^n)'||^t \\
&\leq \sum_{\omega \in G} ||(\phi_\omega^n)'||^t + \sum_{\omega \in I^k \setminus G} \prod_{j=1}^k ||(\phi_{\omega_j}^n)'||^t \\
&\leq \sum_{\omega \in G} ||(\phi_\omega^n)'||^t + \sum_{\omega \in I^k \setminus G} \prod_{j=1}^k e^{Ct}||\phi_{\omega_j}'||^t \\
&\leq \sum_{\omega \in G} ||(\phi_\omega^n)'||^t + e^{kCt} \sum_{\omega \in I^k \setminus G} \prod_{j=1}^k K^t|\phi_{\omega_j}'(\phi_{\sigma^j\omega}(x))|^t \\
&= \sum_{\omega \in G} ||(\phi_\omega^n)'||^t + e^{kCt}K^{kt} \sum_{\omega \in I^k \setminus G} |\phi_\omega'(x)|^t \\
&\leq \sum_{\omega \in G} ||(\phi_\omega^n)'||^t + e^{kCt}K^{kt} \sum_{\omega \in I^k \setminus G} ||\phi_\omega'||^t \\
&< \sum_{\omega \in G} ||\phi_\omega'||^t + \frac{\varepsilon}{2} + e^{kCt}K^{kt}e^{-kCt}K^{-kt}\frac{\varepsilon}{2} \\
&= \sum_{\omega \in G} ||\phi_\omega'||^t + \varepsilon < \mathrm{P}_\Phi^{(k)}(t) + \varepsilon.
\end{aligned}$$

On the other hand, for every $n \geq \max\{N, M\}$

$$\begin{aligned}
\mathrm{P}^{(k)}_{\Phi^n}(t) = \sum_{\omega\in I^k} ||(\phi^n_\omega)'||^t &> \sum_{\omega\in G} ||(\phi^n_\omega)'||^t \\
&> \sum_{\omega\in G} ||\phi'_\omega||^t - \frac{\varepsilon}{2} \\
&= \sum_{\omega\in I^k} ||\phi'_\omega||^t - \sum_{\omega\in I^k\setminus G} ||\phi'_\omega||^t - \frac{\varepsilon}{2} \\
&> \mathrm{P}^{(k)}_{\Phi}(t) - e^{-kCt}K^{-kt}\frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\
&\geq \mathrm{P}^{(k)}_{\Phi}(t) - \varepsilon.
\end{aligned}$$

Consequently, for every $n \geq \max\{N, M\}$

$$\left|\mathrm{P}^{(k)}_{\Phi^n}(t) - \mathrm{P}^{(k)}_{\Phi}(t)\right| < \varepsilon.$$

Since ε was chosen arbitrarily, we have thus shown that $\lim_{n\to\infty} \mathrm{P}^{(k)}_{\Phi^n}(t) = \mathrm{P}^{(k)}_{\Phi}(t)$. According to Lemma 3.3, the function $\mathrm{P}^{(k)}(t)$ is thereafter continuous for each $t \geq 0$. ■

We will now reach our ultimate objective: the continuity of the pressure function.

Theorem 5.7. *For every $t \geq 0$, the function $\mathrm{P}(t) : \mathrm{CIFS}(X, I) \to (-\infty, \infty]$, $\Phi \mapsto \mathrm{P}_\Phi(t)$, is continuous when $\mathrm{CIFS}(X, I)$ is endowed with the λ-topology.*

Proof. Fix $\Phi \in \mathrm{CIFS}(X, I)$ and a sequence $\{\Phi_n\}_{n=1}^\infty$ in $\mathrm{CIFS}(X, I)$ such that $\lambda\Big(\{\Phi^n\}_{n=1}^\infty\Big) = \Phi$. Recall that for every $t \geq 0$, $\mathrm{P}^{(k)}_{\Phi}(t) < \infty$ if and only if $\mathrm{P}^{(k)}_{\Phi^n}(t) < \infty$ for every $k \in I\!N$ and all $n \in I\!N$ large enough. Thus, for every $t \geq 0$, $\mathrm{P}_\Phi(t) < \infty$ if and only if $\mathrm{P}_{\Phi^n}(t) < \infty$ for all $n \in I\!N$ large enough, and the statement holds immediately for every $t \in \{t' \geq 0 : \mathrm{P}^{(1)}_{\Phi}(t') = \infty\}$. So, fix $t \in \{t' \geq 0 : \mathrm{P}^{(1)}_{\Phi}(t') < \infty\}$. Lemma 5.3 asserts that $\liminf_{n\to\infty} \mathrm{P}_{\Phi^n}(t) \geq \mathrm{P}_\Phi(t)$. Let $\varepsilon > 0$. Choose $k \in I\!N$ such that

$$\frac{1}{k}\log \mathrm{P}^{(k)}_{\Phi}(t) < \mathrm{P}_\Phi(t) + \frac{\varepsilon}{2}.$$

Using Theorem 5.6, pick $N \in I\!N$ such that

$$\left|\frac{1}{k}\log \mathrm{P}^{(k)}_{\Phi^n}(t) - \frac{1}{k}\log \mathrm{P}^{(k)}_{\Phi}(t)\right| < \frac{\varepsilon}{2}$$

for every $n \geq N$. It follows that for every $n \geq N$,

$$\begin{aligned}
\mathrm{P}_{\Phi^n}(t) &\leq \frac{1}{k}\log \mathrm{P}^{(k)}_{\Phi^n}(t) \\
&< \frac{1}{k}\log \mathrm{P}^{(k)}_{\Phi}(t) + \frac{\varepsilon}{2} \\
&< \mathrm{P}_\Phi(t) + \varepsilon.
\end{aligned}$$

Since ε was chosen arbitrarily, we have thus shown that $\limsup_{n\to\infty} \mathrm{P}_{\Phi^n}(t) \le \mathrm{P}_\Phi(t)$. Consequently,

$$\limsup_{n\to\infty} \mathrm{P}_{\Phi^n}(t) \le \mathrm{P}_\Phi(t) \le \liminf_{n\to\infty} \mathrm{P}_{\Phi^n}(t),$$

that is, $\lim_{n\to\infty} \mathrm{P}_{\Phi^n}(t) = \mathrm{P}_\Phi(t)$. According to Lemma 3.3, the function $\mathrm{P}(t)$ is thereafter continuous. ■

We thereby have a complete classification of the behavior of sequences in $\mathrm{CIFS}(X, I)$ and of their limits:

Lemma 5.8. *If $\lambda\Big(\{\Phi^n\}_{n=1}^\infty\Big) = \Phi \in \mathrm{CIFS}(X, I)$, then the following hold:*

(i) *If Φ is cofinitely regular, then the systems Φ^n are cofinitely regular for all $n \in I\!N$ large enough;*
(ii) *If the systems Φ^n are cofinitely regular for all $n \in I\!N$ large enough, then Φ is cofinitely regular;*
(iii) *If Φ is strongly regular, though not cofinitely regular, then the systems Φ^n are strongly regular, though not cofinitely regular, for all $n \in I\!N$ large enough;*
(iv) *If the systems Φ^n are strongly regular, though not cofinitely regular, for all $n \in I\!N$ large enough, then Φ is either strongly regular, though not cofinitely regular, or critically regular (see Example 5, section 8);*
(v) *If Φ is critically regular, then the systems Φ^n are strongly regular, though not cofinitely regular, or critically regular or irregular, or any combination of these, for all $n \in I\!N$ large enough (see Example 5, section 8);*
(vi) *If the systems Φ^n are critically regular for all $n \in I\!N$ large enough, then Φ is critically regular;*
(vii) *If Φ is irregular, then the systems Φ^n are irregular for all $n \in I\!N$ large enough;*
(viii) *If the systems Φ^n are irregular for all $n \in I\!N$ large enough, then Φ is either critically regular (see Example 5, section 8) or irregular.*

This classification can also be expressed in the form:

Lemma 5.9. *In the space $\mathrm{CIFS}(X, I)$ endowed with the λ-topology,*

(i) *The set of cofinitely regular systems is both open and closed;*
(ii) *The set of strongly regular, though not cofinitely regular systems, is open but generally not closed;*
(iii) *The set of critically regular systems is closed but usually not open;*
(iv) *The set of irregular systems is open but generally not closed;*
(v) *The set of critically regular or strongly regular, though not cofinitely regular, systems is closed but usually not open;*
(vi) *The set of critically regular or irregular systems is closed but generally not open.*

Finally, we look at the Hausdorff dimension:

Theorem 5.10. *The Hausdorff dimension function* $h : \mathrm{CIFS}(X, I) \to (0, \infty)$, $\Phi \mapsto h_\Phi$, *is continuous when* $\mathrm{CIFS}(X, I)$ *is endowed with the* λ*-topology. Moreover, it is locally constant on the open set of irregular systems.*

Proof. The latter part of the statement simply follows from Lemma 5.4 and Theorem 5.7. Thus, the case where Φ is irregular is settled and solely the regular case needs to be addressed. In this case, follow the proof of Theorem 4.3 with $\mathrm{P}\Big(t\zeta_\Phi\Big)$ replaced by $\mathrm{P}_\Phi(t)$. ∎

6. Real-analyticity, Continuity and Subharmonicity in Analytic Families

Let $\Lambda \subset \mathbb{C}^q$, $q \in \mathbb{N}$, be an open analytic submanifold of $\mathbb{C}^q$. Let $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ be a family of CIFS from $\mathrm{CIFS}(X, I)$, where X is contained in V, an open connected subset of $\mathbb{C}^d$, $d \in \mathbb{N}$, and I is a countable alphabet, either finite or infinite. Fix $\lambda_0 \in \Lambda$ and for every $\omega \in I^\infty$, consider the function

$$\lambda \mapsto \kappa_\omega(\lambda) := \frac{\Big(\phi^\lambda_{\omega_1}\Big)'\Big(\pi_\lambda(\sigma(\omega))\Big)}{\Big(\phi^{\lambda_0}_{\omega_1}\Big)'\Big(\pi_{\lambda_0}(\sigma(\omega))\Big)},$$

where $\pi_\lambda := \pi_{\Phi^\lambda} : I^\infty \to X$ is the coding map induced by the CIFS Φ^λ.

A family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is said to be analytic if

(a) For every $x \in X$ and every $i \in I$, the function $\lambda \mapsto \phi^\lambda_i(x)$, $\lambda \in \Lambda$, is analytic.

An analytic family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is called regularly analytic if in addition the following two conditions are satisfied:

(b) Φ^{λ_0} is strongly regular;

(c) There exists $\eta \in (0, 1)$ such that $|\kappa_\omega(\lambda) - 1| \le \eta$ for every $\omega \in I^\infty$ and all $\lambda \in \Lambda$.

Making a synthesis of arguments from the proof of Theorem 6.3 in [13] and from appropriate parts of [14], we obtain the following result:

Theorem 6.1. *If* $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ *is a regularly analytic family of CIFS from* $\mathrm{CIFS}(X, I)$, *then the function* $\lambda \mapsto h_\lambda := h_{\Phi^\lambda}$, $\lambda \in \Lambda$, *is real-analytic.*

We omit the proof since it repeats, essentially word by word, various fragments from [13] and [14]. When I is finite, every analytic family of CIFS from $\mathrm{CIFS}(X, I)$ is regularly analytic, and therefore as an immediate consequence of Theorem 6.1, we get the following corollary:

Corollary 6.2. *If* I *is finite and* $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ *is an analytic family of CIFS from* $\mathrm{CIFS}(X, I)$, *then the function* $\lambda \mapsto h_\lambda$, $\lambda \in \Lambda$, *is real-analytic.*

Note further the following very useful immediate consequence of the Mean Value Inequality.

Observation. If $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is an analytic family of CIFS from $\mathrm{CIFS}(X, I)$ and if there is an open neighbourhood U of λ_0 contained in Λ such that

$$\sup\{||D_\lambda\kappa_\omega(\lambda)|| : \omega \in I^\infty,\ \lambda \in U\} < \infty,$$

where $D_\lambda\kappa_\omega(\lambda) : \mathbb{C}^q \to \mathbb{C}^q$ denotes the differential of the function $\lambda \mapsto \kappa_\omega(\lambda)$ evaluated at the point λ, then there is $r > 0$ such that condition (c) of the definition of regular analyticity is satisfied whenever $\lambda \in B(\lambda_0, r)$.

Notice also that due to Cauchy's formula for the derivative of a holomorphic function, the above condition can be replaced by the one below:

$$\sup\{||\kappa_\omega(\lambda)|| : \omega \in I^\infty,\ \lambda \in U\} < \infty.$$

A family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is called plane-analytic if it is analytic and both Λ and V are open connected subsets of the complex plane $\mathbb{C}$.

Our next result is analog to the one developed by Ransford [10] for analytic families of hyperbolic rational maps. Like Ransford's proof, our argument relies on the existence of a Bowen-Ruelle-type formula (Theorem 2.1) and on the variational principle for the pressure function. Our result also extends the one obtained by Baribeau and Roy [2] for plane-analytic families of CIFS consisting of similitudes.

Theorem 6.3. *If $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is a plane-analytic family of CIFS from* $\mathrm{CIFS}(X, I)$*, then the reciprocal $1/h_\lambda$ of the Hausdorff dimension function is the infimum of a family of positive harmonic functions of λ. In particular, it is continuous and superharmonic with respect to the variable λ.*

Proof. Suppose first that each Φ^λ is regular. For every $\lambda \in \Lambda$, put $\zeta_\lambda := \zeta_{\Phi^\lambda}$, the function associated to the CIFS Φ^λ defined in section 2. According to [12] and [4], each CIFS Φ^λ satisfies for each t-potential $-t\zeta_\lambda$, $t \geq 0$, the variational principle

$$\mathrm{P}(\lambda, t) := \mathrm{P}(-t\zeta_\lambda) = \sup_{\mu\in M(\lambda)} \left\{ h_\mu(\sigma) - t \int_{I^\infty} \zeta_\lambda(\omega)\,d\mu(\omega) \right\},$$

where $M(\lambda)$ is the family of all σ-invariant Borel probability measures on I^∞ such that $t\int_{I^\infty} \zeta_\lambda(\omega)d\mu(\omega) < \infty$, and $h_\mu(\sigma)$ is the measure-theoretic entropy of σ with respect to μ. Let

μ be a σ-invariant Borel probability measure on I^∞. Then we have for every $\lambda \in \Lambda$

$$\begin{aligned}\int_{I^\infty} \zeta_\lambda(\omega) d\mu(\omega) &= \int_{I^\infty} -\log|(\phi^\lambda_{\omega_1})'(\pi_\lambda(\sigma(\omega)))| d\mu(\omega) \\ &= \sum_{i\in I} \int_{[i]} -\log|(\phi^\lambda_i)'(\pi_\lambda(\sigma(\omega)))| d\mu(\omega) \\ &= \sum_{i\in I} \int_{\sigma([i])} -\log|(\phi^\lambda_i)'(\pi_\lambda(\omega))| d(\mu\circ\sigma^{-1})(\omega) \\ &= \sum_{i\in I} \int_{I^\infty} -\log|(\phi^\lambda_i)'(\pi_\lambda(\omega))| d\mu(\omega).\end{aligned}$$

Now, fix $i \in I$ temporarily. Since $(\lambda, z) \mapsto \phi^\lambda_i(z)$ is an analytic function of two complex variables, the function $(\lambda, z) \mapsto \frac{\partial\phi_i}{\partial z}(\lambda, z) = (\phi^\lambda_i)'(z)$ is analytic, too. Moreover, because $0 < |(\phi^\lambda_i)'(z)| < 1$ for every (λ, z), this function is locally bounded (in modulus) away from 0 and 1. Since $\{\lambda \mapsto \pi_\lambda(\omega)\}_{\omega\in I^\infty}$ is a family of analytic functions, this implies that $\{\lambda \mapsto (\phi^\lambda_i)'(\pi_\lambda(\omega))\}_{\omega\in I^\infty}$ is a family of analytic functions, locally uniformly bounded away from 0 and 1. Consequently, $\{\lambda \mapsto -\log|(\phi^\lambda_i)'(\pi_\lambda(\omega))|\}_{\omega\in I^\infty}$ is a family of positive harmonic functions, locally uniformly bounded away from 0 and ∞. Applying Fubini's theorem, we deduce that $\lambda \mapsto \int_{I^\infty} -\log|(\phi^\lambda_i)'(\pi_\lambda(\omega))| d\mu(\omega)$ is a positive harmonic function. Using Harnack's Theorem (see, for instance, [11], Theorem 1.3.9), we conclude that the function $\lambda \mapsto \int_{I^\infty} \zeta_\lambda(\omega) d\mu(\omega)$ is either identically ∞ or an everywhere finite, positive, harmonic function. In particular, this means that either $\mu \in M(\lambda)$ for every $\lambda \in \Lambda$, or $\mu \notin M(\lambda)$ for every $\lambda \in \Lambda$. So, the function $\lambda \mapsto M(\lambda)$, $\lambda \in \Lambda$, is constant. Fixing any $\lambda_0 \in \Lambda$, we have thus shown that

$$\mathrm{P}(\lambda, t) = \sup_{\mu\in M(\lambda_0)} \left\{ h_\mu(\sigma) - t \int_{I^\infty} \zeta_\lambda(\omega) d\mu(\omega) \right\},$$

and that the function $\lambda \mapsto \int_{I^\infty} \zeta_\lambda(\omega) d\mu(\omega)$ is finite, positive and harmonic for every $\mu \in M(\lambda_0)$. Applying Theorem 2.1, we get

$$0 = \mathrm{P}(\lambda, h_\lambda) = \sup_{\mu\in M(\lambda_0)} \left\{ h_\mu(\sigma) - h_\lambda \int_{I^\infty} \zeta_\lambda(\omega) d\mu(\omega) \right\}.$$

Hence

$$h_\lambda = \sup_{\mu\in M(\lambda_0)} \frac{h_\mu(\sigma)}{\int_{I^\infty} \zeta_\lambda(\omega) d\mu(\omega)}.$$

So

$$\frac{1}{h_\lambda} = \inf_{\mu\in M(\lambda_0)} \frac{\int_{I^\infty} \zeta_\lambda(\omega) d\mu(\omega)}{h_\mu(\sigma)}.$$

Thus, $1/h$ is the infimum of a family of positive harmonic functions. This concludes the proof in the case in which each Φ^λ is regular.

In general, using Theorem 2.1, we see that for $\lambda \in \Lambda$ it holds that $h_\lambda := h_{\Phi^\lambda} = \sup\{h_{\Phi^\lambda|F} : F \subset I \text{ is finite}\}$. So $1/h_\lambda = \inf\{1/h_{\Phi^\lambda|F} : F \subset I \text{ is finite}\}$. Since every finite CIFS is regular, we deduce that each of the functions $\lambda \mapsto 1/h_{\Phi^\lambda|F}$ is the infimum of a family of positive harmonic functions. Therefore $1/h$ is an infimum of infimums, i.e. an infimum, of positive harmonic functions. It follows from Harnack's inequality (see, for example, [11], Theorem 1.3.1) that $1/h$ is continuous. It is then easy to see that $1/h$ is superharmonic. ■

Corollary 6.4. *The functions h and $\log h$ are continuous and subharmonic. Moreover, h satisfies the functional inequality*

$$h\Delta h \geq 2|\nabla h|^2$$

whenever h is C^2.

Proof. Since h and $\log h$ are decreasing convex functions of the superharmonic function $1/h$, an application of Jensen's inequality shows that these functions are subharmonic. The last part of the corollary follows from the expansion of the inequality $\Delta(1/h) \leq 0$. ■

However, h is generally not harmonic. Indeed, consider a plane-analytic family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ of CIFS consisting of two similitudes ϕ_1^λ and ϕ_2^λ with $(\phi_1^\lambda)'(z) = (\phi_2^\lambda)'(z) = e^\lambda$ for every $z \in \mathbb{C}$, where $\Lambda \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$. Then $h_\lambda = h(\lambda) = -\log 2/\mathrm{Re}(\lambda)$ and $(\Delta h)(\lambda) = -2\log 2/(\mathrm{Re}(\lambda))^3$. Now, observe that $(\Delta h)(\lambda) > 0$ for every $\lambda \in \Lambda$. Thus, h is nowhere harmonic. Note also that $h\Delta h = 2|\nabla h|^2$, so the functional inequality in the corollary is sharp.

7. Parabolic Iterated Function Systems

We follow in this short section the definitions, terminology, and notation from [7] and [8]. Assume I is finite. Given a finite set $I_p \subset I$, a (finite) set $\Omega \subset \partial X \subset \mathbb{R}^d$ with the same cardinality as I_p, and a function p defined on I_p whose co-domain G_d is the set $(0, \infty)$ if $d = 1$, $\mathbb{N}$ if $d = 2$ and the singleton $\{1\}$ if $d \geq 3$, we consider $\mathrm{PIFS}(X, I; I_p, \Omega, p)$, the space of all parabolic CIFS whose parabolic indexes are formed by I_p, parabolic fixed points by Ω, and the valency function on I_p is given by the function p. The space $\mathrm{PIFS}(X, I; I_p, \Omega, p)$ is endowed with the same metric structure ρ as at the beginning of section 4. Recall from [7] that to each parabolic iterated function system Φ is associated a hyperbolic iterated function system Φ^* with the alphabet I_*. Improving substantially the calculations from sections 2 and 3 of [8], one can establish the following fact:

Lemma 7.1. *Assume I is finite. Given two finite sets $I_p \subset I$ and $\Omega \subset \partial X \subset \mathbb{R}^d$ with the same cardinality, and a function $p : I_p \to G_d$, the function $* : \mathrm{PIFS}(X, I; I_p, \Omega, p) \to \mathrm{CIFS}(X, I_*)$ is continuous if $\mathrm{CIFS}(X, I_*)$ is equipped with the λ-topology.*

Since $h_{\Phi^*} = h_\Phi$, combining this lemma with Theorem 5.10, we get the following result:

Theorem 7.2. *If I is finite, then the Hausdorff dimension function $h : \mathrm{PIFS}(X, I; I_p, \Omega, p) \to (0, \infty)$ is continuous.*

Improving even more the calculations from sections 2 and 3 of [8], one can prove the following:

Lemma 7.3. *If I is finite and $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is an analytic family of parabolic iterated function systems from $\mathrm{PIFS}(X, I; I_p, \Omega, p)$, then $\{(\Phi^\lambda)^*\}_{\lambda\in\Lambda}$ is a regularly analytic family.*

Combining this lemma and Theorem 6.1, we obtain:

Theorem 7.4. *If I is finite and $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is an analytic family of parabolic iterated function systems from $\mathrm{PIFS}(X, I; I_p, \Omega, p)$, then the Hausdorff dimension function $\lambda \mapsto h_{\Phi^\lambda}$, $\lambda \in \Lambda$, is real-analytic.*

8. Phase Transitions and Breakdown of Real-analyticity

This section is consecrated to a number of examples highlighting the lack of real-analyticity in the Hausdorff dimension function in analytic, but not regularly analytic, families of iterated function systems. We will look for these examples among plane-analytic families $\{\Phi^\lambda\}_{\lambda\in\Lambda}$, where all the maps ϕ_i^λ, $\lambda \in \Lambda$, are of the form

$$\phi_i^\lambda(z) = c_i(\lambda)z + b_i(\lambda),$$

with $b_i(\lambda)$ and $c_i(\lambda)$ depending holomorphically on λ. Although the essence of this section lies in counterexamples, we start with several positive results. Fix $\lambda_0 \in \Lambda$. For each $\lambda \in \Lambda$, let

$$p(\lambda) = \inf_{i\in I} \left|\frac{c_i(\lambda)}{c_i(\lambda_0)}\right| \qquad \text{and} \qquad q(\lambda) = \sup_{i\in I} \left|\frac{c_i(\lambda)}{c_i(\lambda_0)}\right|.$$

The following theorem is an immediate consequence of Theorem 6.1 and the observation that follows Corollary 6.2.

Theorem 8.1. *Assume that Φ^{λ_0} is strongly regular and that for some $R > 0$,*

$$\sup_{\lambda\in B(\lambda_0,R)} q(\lambda) < \infty.$$

Then there exists a neighbourhood $\Lambda_0 \subset \Lambda$ of λ_0 on which the Hausdorff dimension function $h : \Lambda_0 \to (0, \infty)$ is real-analytic.

Before continuing, note that in order to simplify notation, we will write θ_λ instead of θ_{Φ^λ}, h_λ rather than h_{Φ^λ}, and so on. Now, notice that

$$\big(p(\lambda)\big)^t \mathrm{P}^{(1)}_{\lambda_0}(t) \le \mathrm{P}^{(1)}_{\lambda}(t) \le \big(q(\lambda)\big)^t \mathrm{P}^{(1)}_{\lambda_0}(t). \tag{8.1}$$

Thus, if $q(\lambda) < \infty$, then $\mathrm{P}^{(1)}_{\lambda_0}(t) < \infty$ implies that $\mathrm{P}^{(1)}_{\lambda}(t) < \infty$, and $\theta_\lambda \le \theta_{\lambda_0}$. Similarly, if $p(\lambda) > 0$, then $\mathrm{P}^{(1)}_{\lambda}(t) < \infty$ implies that $\mathrm{P}^{(1)}_{\lambda_0}(t) < \infty$, and $\theta_\lambda \ge \theta_{\lambda_0}$ in this case. In particular, if both conditions are simultaneously satisfied, then $\mathrm{P}^{(1)}_{\lambda}(t) = \infty$ if and only if $\mathrm{P}^{(1)}_{\lambda_0}(t) = \infty$ for every $t \ge 0$, and $\theta_\lambda = \theta_{\lambda_0}$. If this turns out to be the case for every $\lambda \in \Lambda$, then the function θ is constant on Λ. This is the case in the forthcoming Example 5. Although these conditions are sufficient for the constancy of θ, they are not necessary, as the first four examples will show.

Proposition 8.2. *Assume that the function θ is constant. Suppose also that Φ^{λ_0} is irregular for some $\lambda_0 \in \Lambda$. If $q(\lambda) < \big(\mathrm{P}^{(1)}_{\lambda_0}(\theta)\big)^{-1/\theta}$ for some $\lambda \in \Lambda$, then Φ^λ is irregular.*

This follows immediately from the rightmost inequality in (8.1). In particular, if the above condition holds on an open neighbourhood Λ_0 of λ_0, then Φ^λ is irregular on that neighbourhood, and $h \equiv \theta$ on Λ_0. This is the case in Example 5 with $\lambda_0 \in \Lambda_0 = B(0,1)$.

Proposition 8.3. *Assume that θ is continuous and that Φ^{λ_0} is strongly regular for some $\lambda_0 \in \Lambda$. Then there is a neighbourhood Λ_0 of λ_0 on which every Φ^λ is strongly regular.*

Proof. Indeed, as $\mathrm{P}^{(1)}_{\lambda_0}(\theta_{\lambda_0}) > 1$, there is a finite set $F \subset I$ such that $\sum_{i\in F} |c_i(\lambda_0)|^{\theta_{\lambda_0}} > 1$. It follows from the continuity of the c_i's that there exists $\delta > 0$ so that $\mathrm{P}^{(1)}_{\lambda}(\theta_\lambda) \ge \sum_{i\in F} |c_i(\lambda)|^{\theta_\lambda} > 1$ for every $\lambda \in B(\lambda_0, \delta) \subset \Lambda$. ■

This explains the absence of 'phase transitions' from strongly regular systems to non strongly regular systems when θ is continuous. As in the irregular case, we can be a little more specific when θ is constant:

Proposition 8.4. *Suppose that θ is constant. If Φ^{λ_0} is strongly regular and for some λ it holds that $p(\lambda) > \big(\mathrm{P}^{(1)}_{\lambda_0}(\theta)\big)^{-1/\theta}$, then Φ^λ is strongly regular. If Φ^{λ_0} is cofinitely regular and $p(\lambda) > 0$ for some λ, then Φ^λ is cofinitely regular.*

This proposition is a straightforward consequence of the leftmost inequality in (8.1).

We now describe precisely what we mean by 'phase transition'.

Definition 8.5. *We say that a family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ undergoes a phase transition at $\lambda_0 \in \Lambda$ if in every neighbourhood of λ_0 there is a parameter λ at which Φ^λ is of a different type than Φ^{λ_0}.*

More precisely, given that Φ^{λ_0} is of type T_0, we say that a family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ experiences a phase transition at λ_0 from a type T_0 system to type T systems ($T \neq T_0$) if there is a continuous path $\gamma : [0,1] \to \Lambda$ originating from $\lambda_0 = \gamma(0)$ such that $\Phi^{\gamma(t)}$ is of type T for every $t > 0$.

It is now time to give some concrete examples of this phenomenon. All of the following examples involve similitudes. Recall that a CIFS Φ consisting of similitudes is regular if and only if $\mathrm{P}^{(1)}(t) = 1$ for some $t \geq 0$. Furthermore, if such a t exists, then it is equal to the Hausdorff dimension h of the limit set J of Φ. Indeed, for such a CIFS, it holds that $\mathrm{P}(t) = \log \mathrm{P}^{(1)}(t)$ for every $t \geq 0$.

Example 1. *(Phase transitions from irregular to cofinitely regular systems; breakdown of the real-analyticity of the Hausdorff dimension function)*

Let $I = \{(n,k) \in \mathbb{N} \times \mathbb{N} \,|\, k \leq 2^{n^2-1}\}$ and $\Phi^\lambda = \{\phi^\lambda_{n,k} : \mathbb{C} \to \mathbb{C} \,|\, (n,k) \in I\}$, where $\phi^\lambda_{n,k}(z) = c_{n,k}(\lambda)z + b_{n,k}(\lambda)$, with $c_{n,k}(\lambda) = \lambda^{n^{3/2}} 2^{-(n^2+n)}$ and the $b_{n,k}$'s to be specified later. Then

$$\mathrm{P}^{(1)}_\lambda(t) = \frac{1}{2}\sum_{n\in\mathbb{N}} \exp\Big(n^2(1-t)\log 2 + n^{3/2} t \log|\lambda| - nt\log 2\Big).$$

One deduces from this that $\theta_\lambda = 1$ for every λ ($\theta \equiv 1$). Moreover, $\mathrm{P}^{(1)}_\lambda(\theta) \leq 1/2$ whenever $|\lambda| \leq 1$, the equality prevailing if and only if $|\lambda| = 1$. Hence Φ^λ is irregular and $h_\lambda \equiv \theta \equiv 1$ for every $|\lambda| \leq 1$. However, $\mathrm{P}^{(1)}_\lambda(\theta) = \infty$ when $|\lambda| > 1$. Thus, Φ^λ is cofinitely regular and $h_\lambda > \theta = 1$ for every $|\lambda| > 1$. We are therefore in the presence of phase transitions from irregular to cofinitely regular systems at every point of the unit circle $C(0,1)$. This is accompanied by a breakdown of the real-analyticity of the Hausdorff dimension function on $C(0,1)$, as this function is constant (equal to 1 on $\overline{B(0,1)}\backslash\{0\}$) but non-constant on every neighbourhood of any point of $C(0,1)$. Nonetheless, observe that the Hausdorff dimension function is continuous as it follows from Corollary 6.4 or can be checked by a direct calculation. Note that it is possible to find analytic functions (in fact, even constant functions) $b_{n,k}$'s such that the OSC is satisfied for every $|\lambda| < \sqrt{2}$ through $U = B(0,1)$. Indeed, the sum of the Jacobians of the $\phi^\lambda_{n,k}$'s is $\mathrm{P}^{(1)}_\lambda(2)$, and it turns out that $\mathrm{P}^{(1)}_\lambda(2) < 1$ whenever $|\lambda| < \sqrt{2}$.

Example 2. *(Simultaneous phase transitions from critically regular to irregular and cofinitely regular systems; breakdown of the real-analyticity of the Hausdorff dimension function)*

To witness such phase transitions and breakdown, substitute the restriction $k \leq 2^{n^2}$ for $k \leq 2^{n^2-1}$ in the previous example. One then deduces that $\mathrm{P}^{(1)}_\lambda(t)$ is simply twice what it was earlier. Hence $\theta \equiv 1$, and Φ^λ is cofinitely regular when $|\lambda| > 1$, as before. This time, however, $\mathrm{P}^{(1)}_\lambda(\theta) \leq 1$ for every $|\lambda| \leq 1$, with equality precisely when $|\lambda| = 1$. This means that Φ^λ is critically regular when $|\lambda| = 1$, and irregular when $|\lambda| < 1$.

Example 3. *(Phase transitions from strongly regular, though not cofinitely regular, systems to cofinitely regular systems)*

To observe such transitions, simply replace in the original example the restriction $k \le 2^{n^2-1}$ by $k \le 2^{n^2+1}$. Then $\mathrm{P}_\lambda^{(1)}(t)$ is just 4 times what it was originally. So $\theta \equiv 1$. Now, $1 < \mathrm{P}_\lambda^{(1)}(\theta) \le 2$ for every $\delta < |\lambda| \le 1$ and some $\delta > 0$. This implies that Φ^λ is strongly regular in the annulus $\delta < |\lambda| \le 1$.

Example 4. *(Phase transitions from cofinitely regular to strongly regular, but not cofinitely regular, systems)*

Replace in the original example the $c_{n,k}$'s with $c_{n,k}(\lambda) = \lambda^n 2^{-n^2}$. Then

$$\mathrm{P}_\lambda^{(1)}(t) = \frac{1}{2}\sum_{n\in\mathbb{N}} \exp\Big(n^2(1-t)\log 2 + nt\log|\lambda|\Big).$$

As earlier, $\theta \equiv 1$. Moreover, $\mathrm{P}_\lambda^{(1)}(\theta) = \infty$ for each $|\lambda| \ge 1$, and thereby Φ^λ is cofinitely regular in this region. However, Φ^λ is strongly regular, though not cofinitely regular, when $2/3 < |\lambda| < 1$, for $\mathrm{P}_\lambda^{(1)}(\theta) = |\lambda|/(2(1-|\lambda|))$.

Example 5. *(Simultaneous phase transitions from critically regular to irregular and strongly regular, though not cofinitely regular, systems)*

Set $I = \mathbb{N}$ and choose a sequence of positive real numbers $\{c_n\}_{n\in\mathbb{N}}$ so that $\sum_{n\in\mathbb{N}} c_n = 1$ and $\sum_{n\in\mathbb{N}} c_n^t = \infty$ for every $0 \le t < 1$. Let $\Phi^\lambda = \{\phi_n^\lambda : \mathbb{C} \to \mathbb{C} | n \in \mathbb{N}\}$ consist of the similarities $\phi_n^\lambda(z) = c_n(\lambda)z + b_n(\lambda)$, where $c_n(\lambda) = \lambda c_n$ and the b_n's will be specified later. Then

$$\mathrm{P}_\lambda^{(1)}(t) = |\lambda|^t \sum_{n\in\mathbb{N}} c_n^t.$$

As in all the examples given so far, $\theta \equiv 1$. Furthermore, $\mathrm{P}_\lambda^{(1)}(\theta) = |\lambda|$. This shows that Φ^λ is critically regular when $|\lambda| = 1$, irregular when $|\lambda| < 1$ and strongly regular, but not cofinitely regular, when $|\lambda| > 1$. The phase transitions that take place on $C(0,1)$ are, for similar reasons as before, accompanied by a breakdown in the real-analyticity of h_λ. Note that it is possible to find analytic functions (in fact, even constant functions) b_n's such that the OSC is satisfied for $|\lambda| < 1/\sqrt{\sum_{n\in\mathbb{N}} c_n^2}$.

We now present an example where the function θ is not constant.

Example 6. *Replace in the original example the expression for the functions $\{c_{n,k}\}$ by $c_{n,k}(\lambda) = \lambda^{\frac{1}{4}n^{3/2}} 2^{-(\lambda n^2 + n)}$, and let $\Lambda = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 1/2\}$. Then*

$$\mathrm{P}_\lambda^{(1)}(t) = 2\sum_{n\in\mathbb{N}} \exp\Big(n^2(1 - t\mathrm{Re}(\lambda))\log 2 + \frac{1}{4}n^{3/2}t\log|\lambda| - nt\log 2\Big).$$

In this case, $\theta_\lambda = 1/\mathrm{Re}(\lambda)$. Moreover, $\mathrm{P}_\lambda^{(1)}(\theta_\lambda) = \infty$ for every $|\lambda| > 1$. This means that Φ^λ is cofinitely regular whenever $|\lambda| > 1$. However, $\mathrm{P}_\lambda^{(1)}(\theta_\lambda) < \infty$ for every $|\lambda| \le 1$. Observe further that $\mathrm{P}_\lambda^{(1)}(\theta_\lambda) = 2\sum_{n\in\mathbb{N}} 2^{-n/\mathrm{Re}(\lambda)}$ on $\Lambda \cap C(0,1)$ and is hence a continuous increasing function *of* $\mathrm{Re}(\lambda)$ *there, with* $2/3 < \mathrm{P}_\lambda^{(1)}(\theta_\lambda) \le 2$ *and equality when* $\lambda = 1$. *It is also easy to see that the two complex conjugate parameters* $\lambda_c, \overline{\lambda_c} \in \Lambda \cap C(0,1)$ *such that* $\mathrm{P}_\lambda^{(1)}(\theta_\lambda) = 1$ *are determined by the relation* $\mathrm{Re}(\lambda_c) = \ln 2/\ln 3$. *(Henceforth, we assume that* λ_c *has a positive imaginary part.) The CIFS* Φ^{λ_c} *and* $\Phi^{\overline{\lambda_c}}$ *are critically regular, and there are phase transitions at* λ_c *and* $\overline{\lambda_c}$ *to cofinitely regular systems as* λ *leaves* $\overline{B(0,1)}$*, to irregular systems as* λ *moves away from these points along* $C(0,1)$ *in the direction of* $1/2 + i\,(\pm\sqrt{3}/2)$*, respectively, and to strongly regular, though not cofinitely regular, systems as* λ *moves along* $C(0,1)$ *in the direction of* 1. *There are also phase transitions at every point of the arc* $(\lambda_c, \overline{\lambda_c})$ *from strongly regular, but not cofinitely regular, to cofinitely regular systems, and transitions at every point of the arcs* $(1/2 + i\sqrt{3}/2, \lambda_c)$ *and* $(1/2 - i\sqrt{3}/2, \overline{\lambda_c})$ *from irregular to cofinitely regular systems as* λ *quits* $\overline{B(0,1)}$. *The restriction* $\mathrm{Re}(\lambda) > 1/2$ *ensures that* $\mathrm{P}_\lambda^{(1)}(2) < 1$ *on* Λ*, and consequently that constant functions* $b_{n,k}$*'s can be found so that* Φ^λ *satisfies the OSC with* $U = B(0,1)$.

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