Conformal families of measures for fibred systems

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Abstract

In this paper, we give sufficient conditions for the existence of conformal families of measures for fibred systems. We describe a general construction principle, modelled on the principle developed by Denker and Urbanski. For those systems that are weakly topologically exact along fibres and fibrewise countably critical, we show that each measure in the constructed family is supported on the whole fibre where it is naturally defined. We illustrate our result with a natural example, namely, a family of non fibrewise expanding fibred polynomials which are weakly topologically exact along fibres and fibrewise countably critical.[1]

1 Introduction

Throughout this paper, a fibred system is a collection $\mathcal{Y} = (Y, T, X, S, \pi)$, where Y and X are compact metrizable spaces, $T : Y \to Y$ and $S : X \to X$ are continuous maps and $\pi : Y \to X$ is a continuous surjective map which satisfies $\pi \circ T = S \circ \pi$. Thus, T preserves the fibres $Y_x := \pi^{-1}(\{x\})$, $x \in X$. The restriction of T to the fibre over x, Y_x, will be denoted by T_x; so $T_x : Y_x \to Y_{Sx}$.

In 1999, Denker and Gordin [3] (See also [2, 4]) showed that compact fibred systems, whose fibre maps are Ruelle expanding and topologically exact, admit, for every globally continuous, uniformly fibrewise Hölder continuous potential, a unique Gibbs family of conditional measures on their fibres. This result reduces to the celebrated Ruelle-Perron-Frobenius theorem in the non-fibred case (or, in other words, in the one-fibre case, that is, when X consists of only one point).

Recall that a family $\{m_x\}_{x\in X}$ of probability measures on Y is called a measurable system of conditional probabilities for $\mathcal{Y}$ if $m_x(Y_x) = 1$ for all $x \in X$ and the integral $\int g(y)\, dm_x(y)$ is a Borel-measurable function of x for every bounded Borel-measurable function g on Y.

Denker and Gordin [3] defined the notion of Gibbs family for fibred systems in the following manner. A system $\{m_x\}_{x\in X}$ of conditional probabilities for $\mathcal{Y}$ is called a

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Gibbs family for a measurable function $\varphi : Y \to I\!R$ if there exists a positive Borel-measurable function $A : X \to I\!R$ with the following property:
For all $x \in X$ the Jacobian of m_x with respect to the map T is given by

$$\frac{d(m_{Sx} \circ T_x)}{dm_x} = A(x) \exp(-\varphi) \quad m_x\text{-a.e.}$$

In his Ph.D. thesis, the first author considered whether such a family exists for systems having fibre maps that are expansive rather than Ruelle expanding. He showed that, whenever the base space X is countable, compact fibred systems, whose fibre maps are uniformly open, expansive and topologically exact, and whose fibres are continuous in the Hausdorff metric, possess, for every continuous potential, a Gibbs family, though this family is generally not unique. Moreover, he proved that, without any restriction on the cardinality of X, these systems admit a unique Gibbs family for every globally continuous, uniformly fibrewise locally constant potential (See [7]). The approach he used is based upon Denker and Gordin's results and the existence of a change of metric on Y, compatible with the topology, that transforms the initially fibre expansive system into a fibre expanding one under the reign of the new metric (See [7, 8]).

In this paper, we generalize the first result mentioned in the previous paragraph. We give sufficient conditions for the existence of conformal (or Gibbs) families of measures for fibred systems which are weakly topologically exact along fibres. However, we do not assume that the systems are submitted to any kind of expansion along their fibres. Neither do we assume that the systems have continuous fibres in the Hausdorff metric. Moreover, we do not impose any restriction upon the cardinality of the fibres (cf. [3]) and we do allow the base map S to be countable-to-one (cf. [3]). When the base space is countable, the conformal families obtained satisfy all the requirements imposed by Denker and Gordin. Since we work along individual great orbits, and the partition of a system into its great orbits is usually not measurable, it seems that the measurability of the disintegration obtained is hardly provable, not to say beyond the scope of our method, in the case where X is uncountable.

We describe a general construction principle, inspired by the construction Denker and the second author of this paper did concerning conformal measures for dynamical systems. We provide the reader with a natural example of a non fibrewise expanding family of fibred polynomials that are weakly topologically exact along fibres, some of whom have a countable base. These latter hence admit a conformal family of measures which may not be unique.

2 Construction of Conformal Families of Measures

Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system and φ be a continuous function on Y. Assume that T is bounded-to-one on fibres and fibrewise surjective, that is, there is a $M \in I\!N$ such that T_x is a surjective, at most M-to-1 map for every $x \in X$. Assume also that S is countable-to-one.

In analogy to the construction principle for conformal measures enounced by Denker and Urbanski [1], we will rely on the analytical fact that the transition parameter $c = \limsup_{n\to\infty} a^n/n$ of a sequence of real numbers $(a^n)_{n\in I\!N}$ is uniquely determined by the fact that $\sum_{n\in I\!N} \exp(a^n - ns)$ converges for $s > c$, and diverges for $s < c$. For $s = c$, the series may converge or diverge. However, there is a sequence $(b^n)_{n\in\mathbb{N}}$ of positive real numbers satisfying

$$\sum_{n=1}^{\infty} b^n \exp(a^n - ns) \begin{cases} < \infty, & s > c \\ = \infty, & s \leq c \end{cases}$$

and $\lim_{n\to\infty} b^n/b^{n+1} = 1$ (See [1, Lemma 3.1]).

We define sets $(E_x^n)_{n\in\mathbb{N},x\in X}$ in the following way. For each $x \in X$, choose a finite subset E_x^1 of Y_x. Then, having successively defined the sets $(E_x^k)_{x\in X}$, $1 \leq k \leq n$, for a fixed $n \in \mathbb{N}$, define $E_x^{n+1} = T_x^{-1}(E_{Sx}^n)$ for every $x \in X$. (Since T_x is surjective, we have in particular that $T_x(E_x^{n+1}) = E_{Sx}^n$.) Thereafter, define the sequence $(a_x^n)_{n\in\mathbb{N}}$ for every $x \in X$, where $a_x^n = \log \sum_{y\in E_x^n} \exp[(S_n\varphi)(y)]$ and $S_n\varphi = \sum_{0\leq k<n} \varphi \circ T^k$, and denote by c_x the transition parameter of the sequence $(a_x^n)_{n\in\mathbb{N}}$. We then have the following result.

Lemma 1 *The transition parameter function $c : X \to \mathbb{R}$ is constant over great orbits, that is, if $x' \in [x] = \{x_0 \in X | S^m(x_0) = S^n(x) \text{ for some } m, n \in \mathbb{Z}_+\}$, then $c_{x'} = c_x$.*

Proof. It is sufficient to show that for every $x \in X$, it holds that $c_x = c_{Sx}$. For this, given $x \in X$ and $s \in \mathbb{R}$, we compare the $(n+1)$st term of the series $\sum_{n\in\mathbb{N}} \exp(a_x^n - ns)$ with the nth term of $\sum_{n\in\mathbb{N}} \exp(a_{Sx}^n - ns)$:

$$\begin{aligned} \frac{\exp(a_x^{n+1} - (n+1)s)}{\exp(a_{Sx}^n - ns)} &= \exp(-s)\exp(a_x^{n+1} - a_{Sx}^n) \\ &= \exp(-s)\frac{\sum_{y\in E_x^{n+1}} \exp[(S_{n+1}\varphi)(y)]}{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]} \\ &= \exp(-s)\frac{\sum_{y\in E_x^{n+1}} \exp(\varphi(y))\exp[(S_n\varphi)(Ty)]}{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]} \end{aligned}$$

As T is bounded-to-one on fibres and $T_x(E_x^{n+1}) \subseteq E_{Sx}^n$, we obtain an upper bound:

$$\begin{aligned} \frac{\exp(a_x^{n+1} - (n+1)s)}{\exp(a_{Sx}^n - ns)} &\leq \exp(-s + \|\varphi\|_\infty)M\frac{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]}{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]} \\ &= \exp(-s + \|\varphi\|_\infty)M. \end{aligned}$$

Whereas, as $T_x(E_x^{n+1}) \supseteq E_{Sx}^n$, we get a lower bound:

$$\begin{aligned} \frac{\exp(a_x^{n+1} - (n+1)s)}{\exp(a_{Sx}^n - ns)} &\geq \exp(-s - \|\varphi\|_\infty)\frac{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]}{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]} \\ &= \exp(-s - \|\varphi\|_\infty). \end{aligned}$$

The simultaneous existence of these bounds ensures that both series have the same nature for all $s \in \mathbb{R}$ and therefore $c_x = c_{Sx}$ for every $x \in X$.

The following result is an immediate consequence of the previous lemma.

Corollary 2 *There exists a sequence of positive functions* $(b^n : X \to I\!R_+)_{n \in I\!N}$ *which are constant over great orbits and are such that for every* $x \in X$*, it holds that*

$$\sum_{n=1}^{\infty} b_x^n \exp(a_x^n - ns) \begin{cases} < \infty, & s > c_x \\ = \infty, & s \le c_x \end{cases}$$

and $\lim_{n\to\infty} b_x^n / b_x^{n+1} = 1$.

Now, for each $x \in X$ and $s > c_x$, define the normalized measure

$$m_x^s = \frac{1}{M_x^s} \sum_{n=1}^{\infty} \sum_{y \in E_x^n} b_x^n \exp[(S_n\varphi)(y) - ns]\delta_y,$$

where

$$M_x^s = \sum_{n=1}^{\infty} \sum_{y \in E_x^n} b_x^n \exp[(S_n\varphi)(y) - ns]$$

and δ_y is the Dirac measure supported at y. Observe that, for each $x \in X$, all measures $(m_x^s)_{s > c_x}$ satisfy $m_x^s(Y_x) = 1$, and so do their accumulation points. They also share the same support, supp $m_x^s \subseteq Y_x$, and their common support is uniquely determined by the sets $(E_x^n)_{n \in I\!N}$. Furthermore, note that if there exists $\epsilon > 0$ such that for every $y \in Y$ and $\delta > 0$, there is $n = n(y, \delta) \in Z\!\!\!Z_+$ for which

$$T^n\Big(B(y,\delta) \cap Y_{\pi y}\Big) \supseteq B\Big(T^n(y), \epsilon\Big) \cap Y_{S^n \pi y},$$

and if the finite sets $(E_x^1)_{x \in X}$ are chosen so that they are ϵ-dense in Y_x for every $x \in X$, then supp $m_x^s = Y_x$ for all $s > c_x$ and all $x \in X$. This follows from the fact that, by definition, $E_x^n = (T_{S^{n-2}x} \circ \cdots \circ T_{Sx} \circ T_x)^{-1}(E_{S^{n-1}x}^1)$ for all $x \in X$ when $n \ge 2$. This is evidently the case when $\mathcal{Y}$ is weakly topologically exact along fibres, and, consequently, when $\mathcal{Y}$ is topologically exact along fibres in the sense of Denker and Gordin (cf. [2, 3]).

Definition 3 *A fibred system* $\mathcal{Y} = (Y, T, X, S, \pi)$ *is said to be weakly topologically exact along fibres if for every* $y \in Y$ *and* $\delta > 0$*, there exists* $n = n(y, \delta) \in Z\!\!\!Z_+$ *such that*

$$T^n\Big(B(y,\delta) \cap Y_{\pi y}\Big) = Y_{S^n \pi y}.$$

In summary, we have the following.

Lemma 4 *For each* $x \in X$*, the measures* $(m_x^s)_{s > c_x}$ *satisfy* $m_x^s(Y_x) = 1$*. They further have the same support, and this latter is uniquely determined by the sets* $(E_x^n)_{n \in I\!N}$*. Moreover, if the system is weakly topologically exact along fibres, then their support is the whole fibre where they are defined, namely, supp* $m_x^s = Y_x$ *for all* $s > c_x$ *and all* $x \in X$*.*

Now, fix $x \in X$ and $s > c_x$. Let $B \subseteq Y_x$ be a measurable set such that $T_x B$ is measurable in Y_{Sx} and $T_x|_B$ is one-to-one. As $T_x^{-1}(E_{Sx}^n) = E_x^{n+1}$ for every $n \in I\!N$, we get

$$\begin{aligned} m_{Sx}^s(T_x B) &= \frac{1}{M_{Sx}^s} \sum_{n=1}^{\infty} \sum_{y \in E_{Sx}^n \cap TB} b_{Sx}^n \exp[(S_n\varphi)(y) - ns] \\ &= \frac{1}{M_{Sx}^s} \sum_{n=1}^{\infty} \sum_{y \in B \cap T_x^{-1}(E_{Sx}^n)} b_x^n \exp[(S_n\varphi)(Ty) - ns] \\ &= \frac{1}{M_{Sx}^s} \sum_{n=1}^{\infty} \sum_{y \in B \cap E_x^{n+1}} b_x^n \exp[(S_{n+1}\varphi)(y) - (n+1)s] \cdot \exp[s - \varphi(y)]. \quad (1) \end{aligned}$$

Let

$$\Delta_B(s) = \left| \frac{1}{M^s_{Sx}} \sum_{n=1}^{\infty} \sum_{y\in B\cap E^{n+1}_x} b^n_x \exp[(S_{n+1}\varphi)(y)-(n+1)s]\cdot\exp[s-\varphi(y)] \right.$$
$$\left. -\frac{M^s_x}{M^s_{Sx}} \int_B \exp[c_x-\varphi(y)]dm^s_x(y) \right| \quad (2)$$

and observe that

$$M^s_x \int_B \exp[c_x-\varphi(y)]dm^s_x(y) = \sum_{n=1}^{\infty} \sum_{y\in B\cap E^n_x} b^n_x \exp[(S_n\varphi)(y)-ns]\cdot\exp[c_x-\varphi(y)].$$

Therefore

$$\Delta_B(s) = \frac{1}{M^s_{Sx}} \left| \sum_{n=1}^{\infty} \sum_{y\in B\cap E^{n+1}_x} b^{n+1}_x e^{c_x} \exp[(S_{n+1}\varphi)(y)-(n+1)s]\cdot\exp[-\varphi(y)] \right.$$
$$\left. \cdot\left(\frac{b^n_x}{b^{n+1}_x} e^{s-c_x}-1\right) - b^1_x \sum_{y\in B\cap E^1_x} e^{c_x-s} \right|.$$

The second term in the absolute value is clearly bounded. Henceforth, we study the properties of the ratio M^s_x/M^s_{Sx} when $s\searrow c_x$. Both $M^s_x \nearrow \infty$ and $M^s_{Sx}\nearrow\infty$ as $s\searrow c_x = c_{Sx}$. But what can be said of their ratio? Does $\lim_{s\searrow c_x} M^s_x/M^s_{Sx}$ exist? If so, is it greater than 0? Or does M^s_x/M^s_{Sx} remain at least bounded as $s\searrow c_x$?

To answer this latter question, we compare the $(n+1)$st term of M^s_x with the nth term of M^s_{Sx}:

$$\frac{\sum_{y\in E^{n+1}_x} b^{n+1}_x \exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]}$$
$$= \frac{\sum_{y\in E^{n+1}_x} b^{n+1}_x \exp(-s+\varphi(y))\exp[(S_n\varphi)(Ty)-ns]}{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]}$$

As T is bounded-to-one on fibres and $T_x(E^{n+1}_x)\subseteq E^n_{Sx}$, we obtain an upper bound:

$$\begin{aligned}
&\frac{\sum_{y\in E^{n+1}_x} b^{n+1}_x \exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\qquad \leq \exp(-s+\|\varphi\|_\infty)M\frac{\sum_{y\in E^n_{Sx}} b^{n+1}_{Sx}\exp[(S_n\varphi)(y)-ns]}{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\qquad \leq \exp(-s+\|\varphi\|_\infty)MH_{Sx}\frac{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]}{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\qquad = \exp(-s+\|\varphi\|_\infty)MH_{Sx}, \qquad (3)
\end{aligned}$$

where $H_{Sx} = \sup\{b^{n+1}_{Sx}/b^n_{Sx} \mid n\in I\!N\} > 0$. Since $T_x(E^{n+1}_x)\supseteq E^n_{Sx}$, we get a lower bound:

$$\frac{\sum_{y\in E^{n+1}_x} b^{n+1}_x \exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]}$$

$$
\begin{aligned}
&\geq \exp(-s-\|\varphi\|_\infty)\frac{\sum_{y\in E^n_{Sx}} b^{n+1}_{Sx}\exp[(S_n\varphi)(y)-ns]}{\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\geq \exp(-s-\|\varphi\|_\infty)L_{Sx}, \qquad (4)
\end{aligned}
$$

where $L_{Sx}=\inf\{b^{n+1}_{Sx}/b^n_{Sx}\,|\,n\in I\!N\}>0$. It follows immediately from (3) that

$$
\begin{aligned}
\frac{M^s_x}{M^s_{Sx}} &= \frac{\sum_{n=1}^\infty\sum_{y\in E^n_x} b^n_x\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&= \frac{b^1_x e^{-s}\sum_{y\in E^1_x}\exp[\varphi(y)]}{M^s_{Sx}} \\
&\quad +\frac{\sum_{n=1}^\infty\sum_{y\in E^{n+1}_x} b^{n+1}_x\exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{n=1}^\infty\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\leq \frac{b^1_x e^{-s}\sum_{y\in E^1_x}\exp[\varphi(y)]}{M^s_{Sx}} \\
&\quad +\frac{\sum_{n=1}^\infty\exp(-s+\|\varphi\|_\infty)MH_{Sx}\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\leq \exp(-s+\|\varphi\|_\infty)\left\{\frac{b^1_x|E^1_x|}{M^s_{Sx}}+MH_{Sx}\right\}
\end{aligned}
$$

and, from (4), that

$$
\begin{aligned}
\frac{M^s_x}{M^s_{Sx}} &= \frac{\sum_{n=1}^\infty\sum_{y\in E^n_x} b^n_x\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\geq \frac{\sum_{n=1}^\infty\sum_{y\in E^{n+1}_x} b^{n+1}_x\exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{n=1}^\infty\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&\geq \frac{\sum_{n=1}^\infty\exp(-s-\|\varphi\|_\infty)L_{Sx}\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E^n_{Sx}} b^n_{Sx}\exp[(S_n\varphi)(y)-ns]} \\
&= \exp(-s-\|\varphi\|_\infty)L_{Sx}.
\end{aligned}
$$

Hence, under the assumptions that T is bounded-to-one on fibres and $T_x(E^{n+1}_x)=E^n_{Sx}$ for all $n\in I\!N$, the ratio M^s_x/M^s_{Sx} remains bounded as $s\searrow c_x$. More precisely, we have shown that

$$
\exp(-s-\|\varphi\|_\infty)L_{Sx}\leq\frac{M^s_x}{M^s_{Sx}}\leq\exp(-s+\|\varphi\|_\infty)\left\{\frac{b^1_x|E^1_x|}{M^s_{Sx}}+MH_{Sx}\right\}.
$$

The fact that this ratio is bounded and that $M^s_x\nearrow\infty$ as $s\searrow c_x$ suffice to ensure that

$$
\lim_{s\searrow c_x}\Delta_B(s)=0.
$$

Now, for each $[x]\in[X]$, where $[X]$ denotes the set of great orbits of the base system (X,S), choose a sequence $(s_k)_{k\in I\!N}$, with $s_k\searrow c_x$ when $k\nearrow\infty$, such that the sequences $(M^{s_k}_{\tilde{x}}/M^{s_k}_{S\tilde{x}})_{k\in I\!N}$ and $(m^{s_k}_{\tilde{x}})_{k\in I\!N}$ converge for all $\tilde{x}\in[x]$ (the measures are assumed to converge in the weak*-topology). This is possible since the great orbits of (X,S) are all countable, the map S being countable-to-one by assumption. Thereafter, define

$$A'(\tilde{x}) = \lim_{k\to\infty} M_{\tilde{x}}^{s_k}/M_{S\tilde{x}}^{s_k}, \tag{5}$$

and

$$m_{\tilde{x}} = \lim_{k\to\infty} m_{\tilde{x}}^{s_k} \tag{6}$$

for every $\tilde{x} \in [x]$ and each $[x] \in [X]$. Then, given any $x \in X$ and any measurable set $B \subseteq Y_x$ such that $T_x B$ is measurable in Y_{Sx} and $T_x|_B$ is one-to-one, we have

$$\lim_{k\to\infty} \Delta_B(s_k) = 0. \tag{7}$$

Moreover,

$$\lim_{k\to\infty} \int_B \exp[c_x - \varphi(y)] dm_x^{s_k}(y) = \int_B \exp[c_x - \varphi(y)] dm_x(y). \tag{8}$$

We deduce from (5) and (8) that

$$\lim_{k\to\infty} \frac{M_x^{s_k}}{M_{Sx}^{s_k}} \int_B \exp[c_x - \varphi(y)] dm_x^{s_k}(y) = A'(x) \int_B \exp[c_x - \varphi(y)] dm_x(y). \tag{9}$$

It then follows from (7), (2) and (9) that

$$\lim_{k\to\infty} \frac{1}{M_{Sx}^{s_k}} \sum_{n=1}^{\infty} \sum_{y\in B\cap E_x^{n+1}} b_x^n \exp[(S_{n+1}\varphi)(y) - (n+1)s_k] \cdot \exp[s_k - \varphi(y)]$$

$$= A'(x) \int_B \exp[c_x - \varphi(y)] dm_x(y). \tag{10}$$

Finally, using (1) and (10), we obtain that

$$\begin{aligned}
m_{Sx}(T_x B) &= \lim_{k\to\infty} m_{Sx}^{s_k}(T_x B) \\
&= \lim_{k\to\infty} \frac{1}{M_{Sx}^{s_k}} \sum_{n=1}^{\infty} \sum_{y\in B\cap E_x^{n+1}} b_x^n \exp[(S_{n+1}\varphi)(y) - (n+1)s_k] \cdot \exp[s_k - \varphi(y)] \\
&= A'(x) \int_B \exp[c_x - \varphi(y)] dm_x(y) \\
&= A'(x) e^{c_x} \int_B \exp[-\varphi(y)] dm_x(y) \\
&= A(x) \int_B \exp[-\varphi(y)] dm_x(y),
\end{aligned}$$

where $A(x) = A'(x)e^{c_x}$. This proves that $\{m_x\}_{x\in X}$ constitutes a conformal family of measures.

Furthermore, given any $x \in X$ and any measurable set $B \subseteq Y_x$ such that $T_x^n B$ is measurable in $Y_{S^n x}$ and $T_x^n|_B$ is one-to-one, we obtain by induction

$$m_{S^n x}(T_x^n B) = A_n(x) \int_B \exp[-(S_n\varphi)(y)] dm_x(y)$$

where $A_n(x) = A(S^{n-1}x) \cdot \cdots \cdot A(Sx) \cdot A(x)$.

2.1 Sufficient conditions for the conformal family to satisfy supp $m_x = Y_x$ for all $x \in X$

For each $x \in X$, let $Crit(T_x)$ be the set of critical points of T_x, that is, the set of points $y \in Y_x$ that do not have any neighbourhood on which T_x is injective. Furthermore, denote by $Crit(\mathcal{Y})$ the union of the $Crit(T_x)$'s. Note that $Crit(T_x)$ is a compact subset of Y_x for every $x \in X$. Indeed, fix $x \in X$. If $(y_n)_{n\in I\!N}$ is a sequence in $Crit(T_x)$ which converges to y, then, given $\epsilon > 0$, there exists $N = N(\epsilon) \in I\!N$ such that $y_n \in B(y, \epsilon/2)$ for every $n \geq N$. Therefore $B(y_n, \epsilon/2) \subseteq B(y, \epsilon)$ for all $n \geq N$. Since $y_n \in Crit(T_x)$ for every $n \in I\!N$, the map T_x is not injective on $B(y_n, \epsilon/2)$ for each $n \in I\!N$. Hence T_x is not injective on $B(y, \epsilon)$ for every $n \geq N$. This shows that $y \in Crit(T_x)$ and $Crit(T_x)$ is thereby closed, and compact, in Y_x.

The following definition proves to be "critical".

Definition 5 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is said to be fibrewise countably critical if for every $x \in X$ the set $Crit(T_x)$ is countable.*

We now prove the following.

Lemma 6 *If a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is weakly topologically exact along fibres and fibrewise countably critical, then the support of the measures belonging to the previously constructed conformal family is the whole fibre where they are defined, namely, supp $m_x = Y_x$ for all $x \in X$.*

Proof. Fix $x \in X$. For every $n \in I\!N$, it holds that

$$Crit(T_x^n) = \bigcup_{k=0}^{n-1} (T_x^k)^{-1}(Crit(T_{S^k x})).$$

Since T is bounded-to-one on fibres, the set $Crit(T_x^n)$ is countable. We can thereby write $Crit(T_x^n) = \{c_1, c_2, \ldots\}$. On the other hand, for each $y \in Y_x \backslash Crit(T_x^n)$, let $r_y > 0$ be such that $T_x^n|_{B(y,r_y)}$ is injective. Since $\{B(y, r_y)\}_{y\in Y_x\backslash Crit(T_x^n)}$ form an open cover of the secound-countable set $Y_x\backslash Crit(T_x^n)$, one can extract, according to Lindelöf's Theorem, a countable subcover $\{B(y_i, r_i)\}_{i=1}^{\infty}$. Then

$$Y_x = \bigcup_{i=1}^{\infty} B(y_i, r_i) \cup Crit(T_x^n)$$

Since $\mathcal{Y}$ is weakly topologically exact along fibres, given any relative open set $U_x \subseteq Y_x$, there is a $n = n(U_x) \in Z\!\!\!Z_+$ such that $T^n(U_x) = Y_{S^n x}$, and therefore

$$\begin{aligned}
1 &= m_{S^n x}(Y_{S^n x}) \\
&= m_{S^n x}(T_x^n U_x) \\
&= m_{S^n x}\left(\bigcup_{i=1}^{\infty} T_x^n(B(y_i, r_i) \cap U_x) \cup \bigcup_{i:c_i\in U_x} \{T_x^n c_i\}\right) \\
&\leq \sum_{i=1}^{\infty} m_{S^n x}(T_x^n(B(y_i, r_i) \cap U_x)) + \sum_{i:c_i\in U_x} m_{S^n x}(\{T_x^n c_i\}) \\
&= \sum_{i=1}^{\infty} m_{S^n x}(T_x^n(B(y_i, r_i) \cap U_x)) + \sum_{i:c_i\in U_x} m_{S^n x}(\{T_x^n c_i\})
\end{aligned}$$

$$
\begin{aligned}
&= A_n(x)\left[\sum_{i=1}^{\infty}\int_{B(y_i,r_i)\cap U_x}\exp[-(S_n\varphi)(y)]dm_x(y)\right.\\
&\qquad\qquad\left.+\sum_{i:c_i\in U_x}\exp[-(S_n\varphi)(c_i)]m_x(\{c_i\})\right]\\
&\le A_n(x)\exp(n\|\varphi\|_\infty)\left[\sum_{i=1}^{\infty}m_x(B(y_i,r_i)\cap U_x)+\sum_{i:c_i\in U_x}m_x(\{c_i\})\right]
\end{aligned}
$$

This implies in particular that, either there is some y_j with $m_x(B(y_j,r_j)\cap U_x)>0$, or some $c_k\in U_x$ such that $m_x(\{c_i\})>0$. In either case, it follows that $m_x(U_x)>0$. This completes the proof.

2.2 Case of a Countable Base Space X

The functions A, c and $x\mapsto\int\varphi(y)\,dm_x(y)$ are measurable functions for every measurable function φ on Y, for they are defined on a countable space X, and the Borel σ-algebra of a countable metrizable space coincides with the trivial σ-algebra of all subsets of this space.

We thus obtain a conformal family of measures. Note that the construction of this family suggests that it may not be unique, as expected.

3 A Family of Non Fibrewise Expanding Fibred Polynomials Admitting Conformal Measures

Fibred polynomials constitute an interesting family of fibred systems. They have been introduced by Sester [9], following earlier work done in the same spirit by Heinemann [5] and Jonsson [6] on Julia sets for maps of several complex variables. Some work in this direction has also been more recently accomplished by Sumi [11] and the first author of this paper [8].

We first recall some basic definitions (cf. [10]).

Definition 7 *Let X be a compact metric space. A continuous map $P: X\times\mathbb{C}\to X\times\mathbb{C}$ is called fibred polynomial if*

$$P(x,z)=(f(x),P_x(z)),$$

where $f:X\to X$ is a continuous map and $P_x:\mathbb{C}\to\mathbb{C}$ is a polynomial of degree at least two for each $x\in X$.

Let $P: X\times\mathbb{C}\to X\times\mathbb{C}$ be a fibred polynomial. For each $n\in\mathbb{N}$ and $x\in X$, set $P_x^n=P_{f^{n-1}(x)}\circ\ldots\circ P_x$ and $P_x^0=Id_{\mathbb{C}}$. The set

$$F_x=\left\{z\in\mathbb{C}\;\middle|\;\{P_x^n\}_{n\in\mathbb{N}}\text{ is normal at }z\right\}$$

is called Fatou set for P in x, and $J_x=\mathbb{C}\setminus F_x$ Julia set for P in x. Moreover, the Julia set for P is defined as

$$J(P)=\overline{\bigcup_{x\in X}\{x\}\times J_x},$$

whereas

$$F(P) = (X \times \mathbb{C}) \backslash J(P)$$

is called Fatou set for P. The filled-in Julia set for P in x is defined by

$$K_x = \left\{ z \in \mathbb{C} \,\middle|\, \{P_x^n(z)\}_{n \in \mathbb{N}} \text{ is a bounded subset of } \mathbb{C} \right\}$$

and the filled-in Julia set for P is

$$K(P) = \overline{\bigcup_{x \in X} K_x}.$$

Finally, the set

$$Crit(P) = \left\{ (x,z) \in X \times \mathbb{C} \,\middle|\, P_x'(z) = 0 \right\}$$

is called critical set for P and

$$Post(P) = \overline{\bigcup_{n \in \mathbb{N}} P^n(Crit(P))}$$

postcritical set for P.

The following lemma regroups the most fundamental properties of $J(P)$ (See [10, Proposition 2.9]).

Lemma 8 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial with $P(x,z) = (f(x), P_x(z))$. The following statements hold.*

(i) For $x \in X$, we have $P_x^{-1}(F_{f(x)}) = F_x$ and $P_x^{-1}(J_{f(x)}) = J_x$;

(ii) $J(P)$ is forward invariant, that is, $P(J(P)) \subseteq J(P)$;

(iii) If f is surjective, then P is surjective, too;

(iv) If f is an open surjective map, then $J(P)$ is completely invariant, that is, it holds that $P^{-1}(J(P)) = J(P) = P(J(P))$;

(v) The map $x \mapsto J_x$ is lower semicontinuous with respect to the Hausdorff metric for the space of compact subsets of $\mathbb{C}$.

An important family of fibred polynomials are those whose postcritical sets stay away from their Julia sets.

Definition 9 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial. We say that P is hyperbolic along fibres if $Post(P) \subseteq F(P)$.*

Another important family of fibred polynomials are those that are coined expanding along fibres.

Definition 10 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial. We say that P is expanding along fibres if there exist constants $C > 0$ and $\lambda > 1$ such that for each $n \in \mathbb{N}$*

$$\inf_{(x,z) \in J(P)} |(P_x^n)'(z)| \geq C\lambda^n.$$

In fact, Sester has shown that the previous two families are the same (See [10, Theorem 1.1]).

Theorem 11 *A fibred polynomial $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is hyperbolic along fibres if and only if it is expanding along fibres.*

Sester further proved the following (See [10, Proposition 4.1]).

Proposition 12 *If $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is a fibred polynomial which is hyperbolic along fibres, then P has continuous fibres, that is, the application $x \mapsto J_x$ is continuous with respect to the Hausdorff metric on the space of compact subsets of $\mathbb{C}$.*

Let us finally recall two other results (See [10, Corollary 2.6] and [10, Theorem 5.2]).

Theorem 13 *The set K_x is connected for all $x \in X$ if and only if $Crit(P) \subseteq K(P)$.*

Theorem 14 *If $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is a fibred polynomial which is hyperbolic along fibres and if the interior of K_x is a non-empty connected set for all $x \in X$, then each K_x is a κ-quasidisk (with κ independent of x) and therefore each J_x is a κ-quasicircle.*

In [8], the first author of this paper showed that some fibred polynomials, despite not being expanding along fibres, are fibrewise expansive.

Theorem 15 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial on the unit interval $X = [0, 1] \subseteq \mathbb{C}$. Assume that $P(x, z) = (f(x), P_x(z))$, where f has the following properties:*

1. *f is a homeomorphism of the interval;*
2. *$f(x) < x$ for all $0 < x < 1$;*
3. *$f(x) \leq x/8$ for all $x \leq x_*$ and some $x_* > 0$,*

and that $P_x(z) = z^2 + x/4$. Then $\mathcal{Y} = (J(P), P, X, f, \pi_X)$ is a fibrewise expansive, compact system whose fibres are continuous in the Hausdorff metric. Furthermore, $\mathcal{Y}$ is uniformly fibrewise open, but is neither fibrewise expanding nor expansive with respect to the metric induced on $J(P)$ by the usual euclidean norm $\|(x, z)\| = \sqrt{|x|^2 + |z|^2}$.

In the course of the proof of this result, it was established that:

Lemma 16 *J_0 is the unit circle and J_1 the cauliflower.*

Lemma 17 *For each $x \in X$, we have $J_x \subseteq \overline{A_{\frac{1}{2}(1+\sqrt{1-x}),\frac{1}{2}(1+\sqrt{1+x})}}$, where $\overline{A_{\alpha,\beta}} = \{z \in \mathbb{C} \mid \alpha \leq |z| \leq \beta\}$.*

Lemma 18 *Given $\tilde{x} < 1$, the fibred polynomial $P_{\leq \tilde{x}} := P|_{[0,\tilde{x}]\times\mathbb{C}}$ is expanding along fibres and the fibred system $\mathcal{Y}_{\leq \tilde{x}} := (J(P_{\leq \tilde{x}}), P_{\leq \tilde{x}}, [0, \tilde{x}], f, \pi_X)$ generated by $P_{\leq \tilde{x}}$ is fibrewise expanding. Moreover, $\mathcal{Y}_{\leq \tilde{x}}$ has continuous fibres and is not expansive. Furthermore, each J_x ($x \leq \tilde{x}$) is a quasicircle.*

We now prove that these fibred polynomials are weakly topologically exact along fibres.

Lemma 19 *All fibred polynomials described in Theorem 15 are weakly topologically exact along fibres.*

Proof. Notation: For each $x \in X$ and $n \in \mathbb{Z}_+$, let $x_n = f^n(x)$.

Fix $\tilde{x} < 1$ and choose an arc C in $J_{\tilde{x}}$. Since the fibred polynomial $P_{\leq \tilde{x}} := P|_{[0,\tilde{x}]\times \mathbb{C}}$ is expanding along fibres according to Lemma 18, there exists $a > 0$ and $Q \in \mathbb{Z}_+$ such that $\mathrm{diam}(P^q_{\tilde{x}}(C)) \geq a$ for all $q \geq Q$. Choosing Q so large that $d^H(J_x, J_0) < a/2$ for all $x \leq \tilde{x}_Q$ (the second property of f and the continuity of the fibres in the Hausdorff metric, d^H, ensure the existence of such a Q), there is $z_q \in J_0$ such that $\emptyset \neq B(z_q, a/2) \cap J_{\tilde{x}_q} \subseteq P^q_{\tilde{x}}(C)$ for all $q \geq Q$.

For every open set $U \subseteq \mathbb{C}$ with $U \cap J_0 \neq \emptyset$, there is a smallest $k(U) \in \mathbb{Z}_+$ such that $P_0^{k(U)}(U) \supseteq J_0$. Observe that the family $\{B(z, a/2)\}_{z\in J_0}$ forms an open cover of J_0. Let $0 < \delta < a/4$ be a Lebesgue number for this cover. Then the family $\{B(z, \delta)\}_{z\in J_0}$ constitutes an open cover of J_0. By definition of δ, each element of $\{B(z, \delta)\}_{z\in J_0}$ is contained in at least one element of $\{B(z, a/2)\}_{z\in J_0}$. Choose a finite subcover $\{B(z_j, \delta)\}_{j=1}^{l}$ of $\{B(z, \delta)\}_{z\in J_0}$ and let $k = \max_{1\leq j\leq l} k(B(z_j, \delta))$. Then, for each $z \in J_0$, there exists $1 \leq j(z) \leq l$ such that $B(z_{j(z)}, \delta) \subseteq B(z, a/2)$. It thereby follows that $k(B(z, a/2)) \leq k(B(z_{j(z)}, \delta)) \leq k < \infty$. This implies in particular that $P_0^k(B(z, a/2)) \supseteq J_0$ for every $z \in J_0$.

Moreover, since P is an open map and $\{B(z_j, \delta) \times \mathbb{C}\}_{j=1}^{l}$ are open sets in $X \times \mathbb{C}$, the sets $\{P^k(B(z_j, \delta) \times \mathbb{C})\}_{j=1}^{l}$ are open in $X \times \mathbb{C}$ and $P^k(B(z_j, \delta) \times \mathbb{C}) \supseteq \{0\} \times J_0$ for every $1 \leq j \leq l$. The set $\{0\} \times J_0$ being compact in $X \times \mathbb{C}$, there exists $\epsilon > 0$ and $x'' > 0$ such that $P^k(B(z_j, \delta) \times \mathbb{C}) \supseteq [0, x''] \times \overline{A_{1-\epsilon,1+\epsilon}}$ for each $1 \leq j \leq l$. This implies in particular that $P^k(B(z, a/2) \times \mathbb{C}) \supseteq [0, x''] \times \overline{A_{1-\epsilon,1+\epsilon}}$ for every $z \in J_0$. We deduce from the first two properties of f and Lemma 17 that there is $0 < x' \leq x''$ such that $P^k_x(B(z, a/2)) \supseteq J_{x^k}$ for every $z \in J_0$ and all $x \leq x'$ (by simply choosing $0 < x' \leq x''$ so that $\overline{A_{\frac{1}{2}(1+\sqrt{1-x'}),\frac{1}{2}(1+\sqrt{1+x'})}} \subseteq \overline{A_{1-\epsilon,1+\epsilon}}$). It follows that $P^k_x(B(z, a/2) \cap J_x) = J_{x^k}$ for every $z \in J_0$ and all $x \leq x'$.

Now, if necessary, enlarge Q so that $\tilde{x}_Q \leq x'$. Then

$$P^{Q+K}_{\tilde{x}}(C) = P^K_{\tilde{x}_Q}(P^Q_{\tilde{x}}(C)) \supseteq P^K_{\tilde{x}_Q}(B(z_Q, a/2) \cap J_{\tilde{x}_Q}) = J_{\tilde{x}_{Q+K}}.$$

Finally, it follows from standard properties of Julia sets (in complex dynamics) that P_1 is weakly topologically exact along fibres on the cauliflower.

An example of fibred polynomial with a countable base can easily be found by restricting f to the orbit of a single point (or even countably many points) of X.

Example 20 *Fix $0 < x_0 < 1$. Let $P : [0, 1] \times \mathbb{C} \to [0, 1] \times \mathbb{C}$ be one of the fibred polynomials described in Theorem 15 and $X = \{f^n(x_0)|n \in \mathbb{Z}\}$. Then $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is a fibred polynomial and $\mathcal{Y} = (J(P), P, X, f, \pi_X)$ is a uniformly fibrewise open, fibrewise expansive, though not globally expansive, compact system whose fibres are continuous in the Hausdorff metric. Furthermore, this system is weakly topologically exact along fibres and fibrewise countably critical (in fact, $Crit(\mathcal{Y}) = \emptyset$), and thereby admits a conformal family of measures $\{m_x\}_{x\in X}$ such that supp $m_x = Y_x$ for all $x \in X$.*

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