MAXIMIZING MEASURES ON METRIZABLE NON-COMPACT SPACES

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Abstract. We prove the existence and uniqueness of maximizing measures for various classes of continuous integrands on metrizable (non-compact) spaces and close subsets of Borel probability measures. We apply these results to various dynamical contexts, especially to hyperbolic mappings of the form $f_\lambda(z) = \lambda e^z$, $\lambda \neq 0$, and associated with them (as in [8] and [9]) canonical maps F_λ of an infinite cylinder. It is then shown that for all hyperbolic maps F_λ and all 0^+-potentials ϕ, the set of (weak) limit points of equilibrium states of of potentials $t\phi$, $t \nearrow +\infty$, is non-empty and consists of dynamically maximizing measures with compact supports.

1. Introduction

In this paper we deal with the general problem of the existence and uniqueness of measures maximizing integrals of certain continuous functions ζ, which we call escaping to $-\infty$, defined on some metrizibale spaces. The significance of such measures in a dynamical context is well explained in [2], [4] and [5]. The setting of the first part of our paper does not require any dynamics. In fact we fix a closed set Ω of Borel probability measures (note that if the referenced metrizable space J is not compact then such set does not have to be compact either) and we look for measures in Ω maximizing integrals of ζ. Obviously, if the space J is compact, then maximizing measures always exist. In a non-compact case this problem becomes critical. We solve it positively in Section 4 for all continuous escaping to $-\infty$ potentials under mild assumptions on the set Ω of considered measures. The next section, Section 5 contains auxiliary, though interesting themselves results from topology and measure theory. In Section 6 we ask the question of uniqueness of maximizing measures of potentials escaping to $-\infty$. This is a delicate problem even in the best understood dynamical context case of subshifts of finite type since one can very easily construct potentials depending only on finitely many coordinates for which this uniqueness fails. Developing the approach from [4] and [2], which in our case meets different type of technical problems, we answer this question positively for G_δ dense subsets in some naturally emerging metric subspaces of the space of all continuous functions. The next section, Section 7 contains straightforward dynamical consequences of the general results proved in the previous sections. In the last section, Section 8, dealing,

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except at its very beginning, exclusively with hyperbolic exponential functions and a natural subclass (1^+-tame) of Hölder continuous potentials, we undertake an approach stemming from thermodynamic formalism. First we show that the Gibbs measures μ_ϕ of such potentials ϕ proven to exist in [10] (comp. [9] for the class of potentials of the form $z \mapsto -t\mathrm{Re}z$ $(t > 1)$ turn out to be equilibrium states for ϕ. We then demonstrate that the family $\{\mu_{t\phi}\}_{t>1}$ is tight when $t \nearrow +\infty$, and that all its limit points are dynamically maximizing measures for ϕ. The estimates obtained in the course of this proof enable us to conclude that all those limit measures have compact supports.

2. Weak Convergence and Tightness

Let X be a metrizable topological space. By $C(X)$ we denote the space of all real-valued continuous functions defined on X, and by $C_b(X)$ its subspace of bounded functions. Denote by $M(X)$ the space of all Borel probability measures on X endowed with the topology of weak convergence. Recall that a sequence $\{\mu_n\}_{n=1}^\infty \subset M(X)$ converges weakly to a measure $\mu \in M(X)$ if and only if

$$\lim_{n\to\infty} \int g d\mu_n = \int g d\mu$$

for every function $g \in C_b(X)$. A family $\mathcal{F} \subset M(X)$ is said to be tight if and only if for every $\varepsilon > 0$ there exists a compact set $F \subset X$ such that $\mu(X \setminus F) \le \varepsilon$ for all $\mu \in \mathcal{F}$. Notice that if X is compact, then every family $\mathcal{F} \subset M(X)$ is obviously tight. We will be however preoccupied mostly with metric spaces which are not necessarily compact, and the concept of tightness is important to us because of the following well-known fact (see for example [1]).

Theorem 2.1. *(Prokhorov) If X is a Polish (complete metrizable and separable) space, then every tight family of measures from $M(X)$ is a pre-compact subset of $M(X)$.*

Since we will deal with several topologies on subsets of $M(X)$, we will call any closed or compact subset of $M(X)$ (endowed with the weak convergence topology) respectively weakly closed or weakly compact.

3. Functions escaping to $-\infty$

Let J be a metrizable topological space. A function $\zeta : J \to \mathbb{R}$ is said to escape to $-\infty$ provided that for every $t \in \mathbb{R}$ there exists a compact set $F \subset J$ such that $\zeta(J \setminus F) \subset (-\infty, t)$. We then say that $\zeta \in C_{-\infty}(J)$. We shall prove the following easy but interesting fact.

Proposition 3.1. *If J is a metrizable space, $\zeta \in C_{-\infty}(J)$, then for every $t \in I\!R$, the set $\zeta^{-1}([t,+\infty))$ is compact.*

Proof. Suppose that $\zeta \in C_{-\infty}(J)$ and fix $t \in I\!R$. Then there exists a compact set $F \subset J$ such that $\zeta(J \setminus F) \subset (-\infty, t)$. So, $\zeta^{-1}([t,+\infty)) \subset F$, and since $\zeta^{-1}([t,+\infty))$ is closed, we conclude that this set is compact. In order to prove the opposite implication, suppose that the right-hand side of our equivalence is true and fix $s \in I\!R$. Then $\zeta(J \setminus \zeta^{-1}([s,+\infty)) = \zeta(\zeta^{-1}((-\infty,s))) \subset (-\infty,s)$, and as $\zeta^{-1}([s,+\infty))$ is compact, we are done. ■

Proposition 3.2. *If J is a metrizable space and $C_{-\infty}(J) \neq \emptyset$, then J is a locally compact σ-compact Polish space.*

Proof. Let $x \in J$. Since $x \in \zeta^{-1}\Big((\zeta(x)-1,\infty)\Big)$, since $\zeta^{-1}\Big((\zeta(x)-1,\infty)\Big) \subset J$ is an open set, and since $\zeta^{-1}\Big((\zeta(x)-1,\infty)\Big) \subset \zeta^{-1}\Big([\zeta(x)-1,\infty)\Big)$, where the latter sat is compact, we see that J is locally compact. Thus, it it is completely metrizable. Since $J = \bigcup_{n=0}^{\infty} \zeta^{-1}([n,+\infty))$, the metrizable space J is σ-compact, and therefore separable. We are done. ■

We also need the following straightforward.

Proposition 3.3. *If $\zeta \in C_{-\infty}(J)$, then the function $\zeta : J \to I\!R$ is bounded above and it takes on its supremum.*

Proof. Indeed, take an arbitrary $t \in \zeta(J)$. Then $\zeta^{-1}([t,+\infty))$ is a compact subset of J and therefore the supremum

$$\sup(\zeta) = \sup\Big(\zeta|_{\zeta^{-1}([t,+\infty))}\Big) < +\infty$$

is attained on $\zeta^{-1}([t,+\infty))$. ■

It follows from this proposition that the integral $\int \zeta d\mu$ (allowed to be $-\infty$) is well-defined for every $\mu \in M(J)$ and is $< +\infty$.

4. Existence of Maximizing Measures

Call a triple (J, ζ, Ω) maximizable if J is a metrizable space, $\zeta \in C_{-\infty}(J)$, and Ω is a non-empty weakly closed subset of $M(J)$ such that $\int \zeta d\mu \in (-\infty,+\infty)$ for some $\mu \in \Omega$. The set Ω is then called ζ-acceptable. Notice that this holds if for instance Ω contains at least one measure with compact support. The name "maximizable" will be wholly justified by the last result of this section. For every $T \in I\!R$ set

$$\Sigma(\zeta,\Omega,T) = \{\mu \in \Omega : \int \zeta d\mu = T\} \ \text{ and } \ \Sigma_+(\zeta,\Omega,T) = \{\mu \in \Omega : \int \zeta d\mu \geq T\}.$$

We shall prove the following.

Lemma 4.1. *For every $T \in I\!R$ the set $\Sigma_+(\zeta,\Omega,T)$ is weakly compact.*

Proof. For every $\mu \in \Sigma_+(\zeta,\Omega,T)$ and every $s>0$, we have

$$\begin{aligned} T \le \int \zeta d\mu &= \int_{\zeta^{-1}((-\infty,-s))} \zeta d\mu + \int_{\zeta^{-1}([-s,+\infty))} \zeta d\mu \\ &\le -s\mu\Big(\zeta^{-1}((-\infty,-s))\Big) + \sup(\zeta)\mu\Big(\zeta^{-1}([-s,+\infty))\Big) \\ &\le -s\mu\Big(\zeta^{-1}((-\infty,-s))\Big) + \max\{0,\sup(\zeta)\}. \end{aligned}$$

Hence, $\mu\Big(\zeta^{-1}((-\infty,-s))\Big) \le s^{-1}\Big(\max\{0,\sup(\zeta)\}-T\Big)$. Therefore, for every $\varepsilon>0$, taking $s_\varepsilon = \varepsilon^{-1}\Big(\max\{0,\sup(\zeta)\}-T\Big)$, we see that $\mu\Big(\zeta^{-1}((-\infty,-s_\varepsilon))\Big) \le \varepsilon$ for all $\mu \in \Sigma_+(\zeta,\Omega,T)$. Since $J\setminus\zeta^{-1}((-\infty,-s_\varepsilon)) = \zeta^{-1}([s_\varepsilon,+\infty))$ is a compact set, we therefore see that $\Sigma_+(\zeta,\Omega,T)$ forms a tight family of measures. Thus, by Prokhorov's Theorem, $\Sigma_+(\zeta,\Omega,T)$ is weakly pre-compact, and we are left to show that $\Sigma_+(\zeta,\Omega,T)$ is weakly closed in $M(J)$. Toward this end consider an arbitrary sequence $\{\mu_n\}_{n=1}^\infty \subset \Sigma_+(\zeta,\Omega,T)$ converging weakly to a measure $\mu \in M(J)$. Since Ω is weakly closed, $\mu \in \Omega$. For every $k\ge 1$ put

$$\zeta_k = \max\{\zeta,-k\}.$$

Then $\zeta_k \in C_b(J(F))$ for every $k\ge 1$ and the non-increasing sequence $\{\zeta_k\}_{k=1}^\infty$ converges pointwise to ζ. Using Lebesgue's monotone convergence theorem we therefore get

$$\begin{aligned} \int \zeta d\mu = \int \lim_{k\to\infty} \zeta_k d\mu = \lim_{k\to\infty}\int \zeta_k d\mu = \lim_{k\to\infty}\Big(\lim_{n\to\infty}\int \zeta_k d\mu_n\Big) \\ \ge \liminf_{k\to\infty}\limsup_{n\to\infty}\int \zeta d\mu_n \ge \limsup_{n\to\infty} T = T. \end{aligned}$$

So, $\mu \in \Sigma_+(\zeta,\Omega,T)$ and we are done. ■

Let

$$s(\zeta) = \sup\{\int \zeta d\mu : \mu \in \Omega\}.$$

Since $\zeta \in C_{-\infty}(J)$, we have $s(\zeta) < +\infty$. Since Ω is ζ-acceptable, $s(\zeta) > -\infty$ and $\Sigma_+(\zeta,\Omega,T) \ne \emptyset$ for all $T < s(\zeta)$. Since in addition $\Sigma(\zeta,\Omega,s(\zeta)) = \bigcap_{n=1}^\infty \Sigma_+\left(\zeta,\Omega,s(\phi)-\frac{1}{n}\right)$, and since the sequence $\{\Sigma_\phi^+\left(s(\phi)-\frac{1}{n}\right)\}_{n\ge 1}$ is descending, as an immediate consequence of Lemma 4.1, we get the following.

Corollary 4.2. *The set $\Sigma(\zeta,\Omega,s(\zeta))$ is non-empty and weakly compact.*

5. Topology and Measure; Auxiliary Results

In this section we fix a metrizable space J and a function $\zeta \in C_{-\infty}(J)$. It is easy to verify that the function $||\cdot||_\zeta : C_b(J) \to [0,\infty)$ given by the formula

$$||\phi||_\zeta = \sup\left\{\frac{|\phi(z)|}{1+|\zeta(z)|} : z \in J\right\}$$

defines a norm on the linear space $C_b(J)$. The metric $\rho_\zeta : C_b(J) \times (C_b(J) \to [0,+\infty)$ canonically associated with the norm $||\cdot||_\zeta$ is given by the formula

$$\rho_\zeta(\phi,\psi) = ||\psi - \phi||_\zeta.$$

The topology induced by the metric ρ_ζ obviously depends on the function ζ. In fact two functions $\zeta_1, \zeta_2 \in C_{-\infty}(J)$ induce the same topology (are equivalent) if and only if

$$\sup\left\{\max\left\{\frac{1+|\zeta_2(z)}{1+|\zeta_1(z)|}, \frac{1+|\zeta_1(z)}{1+|\zeta_2(z)|}\right\} : z \in J\right\} < +\infty.$$

Despite this inconvenience the most transparent advantage of working with the metric ρ_ζ instead of the metric induced by the standard supremum norm is that even in the non-compact case, we have the following.

Proposition 5.1. *If $\zeta \in C_{-\infty}(J)$, then the metric space $(C_b(J), \rho_\zeta)$ is separable.*

Proof. Since for every integer n, the inverse-image $\zeta^{-1}([n,+\infty))$ is a compact set, the Banach space $\left(C\left(\zeta^{-1}([n,+\infty))\right), ||\cdot||_\infty\right)$ is separable. Let $S_n \subset C\left(\zeta^{-1}([n,+\infty))\right)$ be a corresponding countable dense subset. Using Tietze's Theorem extend each function $\phi \in S_n$ to a function $\tilde{\phi} \in C_b(J)$ such that $\sup(\tilde{\phi}) = \sup(\phi)$ and $\inf(\tilde{\phi}) = \inf(\phi)$. The set

$$S = \bigcup_{n \in \boldsymbol{Z}} \{\tilde{\phi} : \phi \in S_n\}$$

is obviously countable. We shall show that S is a dense subset of $(C_b(J), \rho_\zeta)$. Indeed, fix $g \in C_b(J)$ and then $\varepsilon > 0$. Fix $n \geq 0$ so large that

$$\frac{2||g||_\infty + 1}{1+n} \leq \varepsilon. \tag{5.1}$$

By the definition of S_n and compactness of the set $\zeta^{-1}([n,+\infty))$, there exists $\phi \in S_n$ such that

$$|g(z) - \phi(z)| \leq \min\left\{1, \varepsilon\Big(1 + \inf\{|\zeta(w)| : w \in \zeta^{-1}([n,+\infty))\}\Big)\right\} \tag{5.2}$$

for all $z \in \zeta^{-1}([n,\infty))$. Then, for every $z \in \zeta^{-1}([n,+\infty))$, we have

$$\frac{|g(z) - \tilde{\phi}(z)|}{1+|\zeta(z)|} = \frac{|g(z) - \phi(z)|}{1+|\zeta(z)|} \leq \varepsilon.$$

It also follows from (5.2) that $|\phi(z)| \le 1 + ||g||_\infty$ for all $z \in \zeta^{-1}([n, +\infty))$, and therefore $|\tilde\phi(w)| \le 1 + ||g||_\infty$ for all $w \in J$. Hence, using (5.1), we get that if $z \in \zeta^{-1}((-\infty, -n))$, then

$$\frac{|g(z) - \tilde\phi(z)|}{1 + |\zeta(z)|} \le \frac{|g(z)| + |\tilde\phi(z)|}{1+n} \le \frac{2||g||_\infty + 1}{1+n} \le \varepsilon.$$

Thus $\rho_\zeta(g\tilde\phi) \le \varepsilon$. So, S is a dense subset of $C_b(J)$, and we are done. ■.

Now let (J, ζ, Ω) be a maximizing triple. Set

$$\Sigma_>(\zeta, \Omega, -\infty) = \{\mu \in \Omega : \int \zeta d\mu > -\infty\} = \bigcup_{T \in I\!R} \Sigma_+(\zeta, \Omega, T).$$

In view of Proposition 5.1 we can fix a dense countable set $\{\phi_n\}_{n=1}^\infty$ in the metric space $(C_b(J), \rho_\zeta)$. Define the function $d_\zeta : M(J) \times M(J) \to [0, +\infty)$ by the formula

$$d_\zeta(\mu, \nu) = \sum_{n=1}^\infty 2^{-n} \min\{1, |\int \phi_n d\nu - \int \phi_n d\mu|\}. \tag{5.3}$$

We shall prove the following.

Lemma 5.2. *The function d_ζ restricted to the Cartesian product $\Sigma_>(\zeta, \Omega, -\infty) \times \Sigma_>(\zeta, \Omega, -\infty)$ defines a metric on $\Sigma_>(\zeta, \Omega, -\infty)$.*

Proof. Obviously the only now non-trivial task is to check that if $d_\zeta(\mu, \nu) = 0$, then $\nu = \mu$. Indeed, if $d_\zeta(\mu, \nu) = 0$, then

$$\int \phi_n d\nu = \int \phi_n d\mu$$

for all $n \ge 1$. Since both measures μ and ν are in $\Sigma_>(\zeta, \Omega, -\infty)$, both integrals $\int(1 + |\zeta|)d\nu$ and $\int(1 + |\zeta|)d\mu$ are finite. Put

$$R = \max\{\int(1 + |\zeta|)d\mu, \int(1 + |\zeta|)d\nu\} \in (0, +\infty).$$

Fix now an arbitrary function $\phi \in C_b(J)$. Fix $\varepsilon > 0$. By the choice of the sequence $\{\phi_n\}_{n=1}^\infty$, there exists $n \ge 1$ such that $\rho_\zeta(\phi, \phi_n)) < \varepsilon(2R)^{-1}$, which means that $|\phi(z) - \phi_n(z)| \le \varepsilon(1 + |\zeta(z)|)(2R)^{-1}$ for all $z \in J$. Hence

$$\begin{aligned}
|\int \phi d\mu - \int \phi d\nu| &= |\int \phi d\mu - \int \phi_n d\mu + \int \phi_n d\mu - \int \phi_n d\nu + \int \phi_n d\nu - \int \phi d\nu| \\
&= |\int \phi d\mu - \int \phi_n d\mu + \int \phi_n d\nu - \int \phi d\nu| \\
&\le \int |\phi - \phi_n| d\mu + \int |\phi - \phi_n| d\nu \\
&\le \varepsilon(2R)^{-1} \int (1 + |\zeta(z)|) d\mu(z) + \varepsilon(2R)^{-1} \int (1 + |\zeta(z)|) d\nu(z) \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Letting $\varepsilon \searrow 0$, we thus get that $\int \phi d\mu = \int \phi d\nu$. Hence $\mu = \nu$ and we are done. ■

We will also need the following.

Lemma 5.3. *Suppose that (J, ζ, Ω) is a maximizing triple. If $\mu \in \Sigma_>(\zeta, \Omega, -\infty)$, $\mu_k \in \Sigma_>(\zeta, \Omega, -\infty)$ for all $k \geq 1$ and the sequence $\{\mu_k\}_{k=1}^\infty$ converges weakly to μ, then*

$$\lim_{k\to\infty} d_\zeta(\mu_k, \mu) = 0.$$

Proof. Fix $\varepsilon > 0$. There then exists $q \geq 1$ so large that $\sum_{n=q+1}^\infty 2^{-n} \leq \varepsilon/2$. Since the sequence $\{\mu_k\}_{k=1}^\infty$ converges weakly to μ, there exists $l \geq 1$ such that $|\int \phi_n d\mu_k - \int \phi_n d\mu| \leq \varepsilon/2$ for all $n = 1, 2, \dots, q$ and all $k \geq l$. Hence, for all $k \geq l$ we have that

$$d_\zeta(\mu_k, \mu) \leq \sum_{n=1}^q 2^{-n} \Big| \int \phi_n d\mu_k - \int \phi_n d\mu \Big| + \sum_{n=q+1}^\infty 2^{-n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We are done. ■

Two functions $\phi, \psi \in C_{-\infty}(J)$ are said to be boundedly equivalent if and only if $\sup\{|\psi(z) - \phi(z)| : z \in J\} < \infty$ (notice that if $\phi \in C_{-\infty}(J)$, $\psi \in C(J)$ and $\sup\{|\psi(z) - \phi(z)| : z \in J\} < \infty$, then $\psi \in C_{-\infty}(J)$ and ϕ and ψ are boundedly equivalent). We then write $\psi \sim \phi$. Obviously bounded equivalence is an equivalence relation on $C_{-\infty}(J)$. The corresponding equivalnce class of $\phi \in C_{-\infty}(J)$ is denoted by $[\phi]$. Let us record the following obvious fact.

Proposition 5.4. *If $\phi, \psi \in C_{-\infty}(J)$ and $\psi \sim \phi$, then the norms $||\cdot||_\psi$ and $||\cdot||_\phi$ are equivalent and $\Sigma_>(\psi, \Omega, -\infty) = \Sigma_>(\phi, \Omega, -\infty)$.*

As an immediate consequence of the previous results we get the following.

Corollary 5.5. *Suppose that (J, ζ, Ω) is a maximizing triple. The the following hold.*

(a) *The identity map* Id $: \Sigma_>(\zeta, \Omega, -\infty) \to \Sigma_>(\zeta, \Omega, -\infty)$ *from the space $\Sigma_>(\zeta, \Omega, -\infty)$ endowed with the topology of weak convergence to the space $\Sigma_>(\zeta, \Omega, -\infty)$ endowed with the metric d_ζ, is continuous.*
(b) *Each weakly compact subset of $\Sigma_>(\zeta, \Omega, -\infty)$ is a compact set in the metric space $\big(\Sigma_>(\zeta, \Omega, -\infty), d_\zeta\big)$.*
(c) *If $\phi \in [\zeta]$, then for every $T \in \mathbb{R}$, the set $\Sigma_+(\phi, \Omega, T)$ is compact in the metric space $\big(\Sigma_>(\zeta, \Omega, -\infty), d_\zeta\big)$.*
(d) *If $\phi \in [\zeta]$, then the set $\Sigma(\phi, \Omega, s(\phi))$ is compact in the metric space $\big(\Sigma_>(\zeta, \Omega, -\infty), d_\zeta\big)$.*

Proof. Item (a) is a reformulation of Lemma 5.3. Item (b) is an immediate consequence of item (a). Item (c) is in turn an immediate consequence of item (b) along with Lemma 4.1

and Proposition 5.4. Eventually item (d) is proved in the same way as item (c), only instead of Lemma 4.1 one is to apply Corollary 4.2. ■

Fix again $\zeta \in C_{-\infty}(J)$. The formula

$$\overline{\rho}_\zeta(\phi,\psi) = \sup\left\{\frac{|\psi(z)-\phi(z)|}{1+|\zeta(z)|} : z \in J\right\}$$

defines a metric on $[\zeta]$. We shall prove the following.

Lemma 5.6. *Let (J,ζ,Ω) be a maximizing triple. Then the function $\phi \to s(\phi)$, $\phi \in [\zeta]$ ($[\zeta]$ endowed with the metric $\overline{\rho}_\zeta$) is lower semi-continuous.*

Proof. Fix $\phi \in [\zeta]$. By Corollary 4.2 there exists $\mu \in \Omega$ such that $s(\phi) = \int \phi d\mu \in I\!R$. Then $\int (1+|\zeta|)d\mu \in I\!R$. Fix $\varepsilon > 0$. Take an arbitrary $\psi \in B_{\overline{\rho}}\Big(\phi, \varepsilon(\int(1+|\zeta|)d\mu)^{-1}\Big)$. Then for every $z \in J$ we have $|\psi(z)-\phi(z)| \le \varepsilon(\int(1+|\zeta|)d\mu)^{-1}(1+|\zeta(z)|)$, and therefore

$$\begin{aligned} s(\phi) = \int \phi d\mu &\le \int \psi d\mu + \int \varepsilon\Big(\int (1+|\zeta|)d\mu)^{-1}\Big)(1+|\zeta(z)|)d\mu(z) \\ &= \int \psi d\mu + \varepsilon \le s(\psi)+\varepsilon. \end{aligned}$$

So,

$$\liminf_{\overline{\rho}_\zeta(\psi,\phi)\to 0} s(\psi) \ge s(\phi),$$

and we are done. ■

We end this section with the following.

Lemma 5.7. *Let (J,ζ,Ω) be a maximizing triple. Suppose that $g, g_n \in [\zeta]$, $n \ge 1$, and that $\lim_{n\to\infty} g_n = g$ with respect to the metric $\overline{\rho}_\zeta$ on $[\zeta]$. If $\mu_n \in \Sigma(g_n,\Omega,s(g_n))$ for all $n \ge 1$ and if the sequence $\{\mu_n\}_{n=1}^\infty$ converges weakly to a measure $\mu \in M(J)$, then $\mu \in \Sigma(g,\Omega,s(g))$.*

Proof. Since Ω is weakly compact, $\mu \in \Omega$. Since $g \sim \zeta$,

$$A := \sup\{g(z)-\zeta(z)| : z \in J\} < +\infty.$$

Fix an arbitrary $s > A+1$. Take an arbitrary $n \ge 1$ so large, say $n \ge q$, that $\overline{\rho}_\zeta(g_n,g) < 1/2$. Then for every $z \in g^{-1}((-\infty,-s))$ we get that

$$\begin{aligned} g_n(z) < g(z) + \frac{1}{2}(1+|\zeta|) &\le g(z) + \frac{1}{2}(1+|g(z)|+A) = g(z) + \frac{1}{2}(1+A-g(z)) \\ &= \frac{1}{2}g(z) + \frac{1}{2}(1+A) < -\frac{1}{2}s + \frac{1}{2}(1+A) = \frac{1}{2}(1+A-s) < 0. \end{aligned} \tag{5.4}$$

Hence, for all $n \ge q$, we obtain

$$\int_{g^{-1}([-s,+\infty))} g_n d\mu_n \ge \int_J g_n d\mu_n = s(g_n).$$

Since the set $g^{-1}([-s,+\infty))$ is compact (consequently the sequence $\{g_n\}_{n=1}^\infty$ converges to g uniformly on $g^{-1}([-s,+\infty))$) and since the sequence $\{\mu_n\}_{n=1}^\infty$ converges weakly to μ, we therefore get

$$\int_{g^{-1}([-s,+\infty))} g d\mu \geq \limsup_{n\to\infty} \int_{g^{-1}([-s,+\infty))} g_n d\mu_n \geq \limsup_{n\to\infty} s(g_n) \geq s(g), \tag{5.5}$$

where writing the last inequality sign we have used Lemma 5.6. Since $\{g^{-1}([-n,+\infty))\}_{n=0}^\infty$ is an ascending sequence of Borel sets and since $\bigcup_{n=0}^\infty g^{-1}([-n,+\infty)) = J$, applying (5.5), we get

$$\int_J g d\mu = \lim_{n\to\infty} \int_{g^{-1}([-n,+\infty))} g d\mu \geq s(g).$$

Hence $\mu \in \Sigma(g,\Omega,s(g))$ and we are done. ■

As a fact complementary to Lemma 5.7 we shall prove the following.

Lemma 5.8. *Let (J,ζ,Ω) be a maximizing triple. Suppose that $g, g_n \in [\zeta]$, $n \geq 1$, and that $\lim_{n\to\infty} g_n = g$ with respect to the metric $\overline{\rho}_\zeta$ on $[\zeta]$. If $\mu_n \in \Sigma(g_n,\Omega,s(g_n))$ for all $n \geq 1$, then the sequence $\{\mu_n\}_{n=1}^\infty$ is tight.*

Proof. Fix $\varepsilon > 0$. Let A be the proof of Lemma 5.7. Fix then $s > A+1$ so large that

$$\frac{1}{2}(s-A-1) > \frac{s}{4}, \ -\frac{4}{s}(s(g)-1) < \frac{\varepsilon}{2}, \ \frac{\sup(g)+1}{s} < \frac{\varepsilon}{8}. \tag{5.6}$$

Follow the proof of Lemma 5.7 verbatime from the beginning up to (5.4) included. Since $\mu_n \in \Sigma(g_n,\Omega,s(g_n))$ for all $n \geq 1$ and using the first inequality in (5.6) along with (5.4), we get for all $n \geq q$ that

$$\begin{aligned}
-\frac{s}{4}\mu_n\Big(g^{-1}((-\infty,-s))\Big) &\geq \int_{g^{-1}((-\infty,-s))} g_n d\mu_n \\
&= \int_J g_n d\mu_n - \int_{g^{-1}([-s,+\infty))} g_n d\mu_n = s(g_n) - \int_{g^{-1}([-s,+\infty))} g_n d\mu_n.
\end{aligned}$$

Hence

$$\mu_n\Big(g^{-1}((-\infty,-s))\Big) \leq -\frac{4s(g_n)}{s} + \frac{4}{s}\int_{g^{-1}([-s,+\infty))} g_n d\mu_n. \tag{5.7}$$

It follows from Lemma 5.6 that for all $n \geq 1$ large enough, say $n \geq q_1 \geq q$, we have $s(g_n) \geq s(g) - 1$. Since $g^{-1}([-s,+\infty))$ is a compact set, the sequence $\{g_n\}_{n=1}^\infty$ converges to g uniformly on $g^{-1}([-s,+\infty))$. In consequence $|g_n(z) - g(z)| \leq 1$ for all $z \in g^{-1}([-s,+\infty))$

and all $n \geq 1$ large enough, say $n \geq q_2 \geq q_1$. Therefore, we get from (5.7) that for all $n \geq q_2$

$$\begin{aligned}\mu_n\Big(g^{-1}((-\infty,-s))\Big) &\leq -\frac{4(s(g)-1)}{s} + \frac{4}{s}\int_{g^{-1}([-s,+\infty))}(\sup(g)+1)d\mu_n \\ &= -\frac{4(s(g)-1)}{s} + \frac{4}{s}(\sup(g)+1)\mu_n\Big(g^{-1}([-s,+\infty))\Big) \\ &\leq -\frac{4(s(g)-1)}{s} + 4\max\{0, s^{-1}(\sup(g)+1)\}\end{aligned}$$

Now, by the last two inequalities from (5.6), we get for all $n \geq q_2$ that $\mu_n\Big(g^{-1}((-\infty,-s))\Big) < \varepsilon$. Since $J \setminus g^{-1}((-\infty,-s)) = g^{-1}([-s,+\infty))$ is a compact set, the tightness of the sequnce $\{\mu_n\}_{n=1}^{\infty}$ is proved. ■

6. Uniqueness of Maximizing Measures

We say that a maximizing triple (J,ζ,Ω) is uniquely maximizing if Ω is a convex subset of $M(J)$. The main result of this section is the following theorem motivated by [4] and [2].

Theorem 6.1. *Let (J,ζ,Ω) be a uniquely maximizing triple. Suppose that $(\mathcal{H},|||\cdot|||)$ is Banach space contained densly in the normed space $(C_b(J),||\cdot||_\zeta)$ and that the inclusion map from $(\mathcal{H},|||\cdot|||)$ to $(C_b(J),||\cdot||_\zeta)$ is continuous. Then there exists a dense G_δ subset G of $\zeta+\mathcal{H}\subset[\zeta]$ (with the topology on $\zeta+\mathcal{H}$ induced by the metric $\rho_{\mathcal{H}}(\phi,\psi)=|||\psi-\phi|||$) such that each function in G has a unique maximizing measure in Ω.*

Proof. Put

$$\mathcal{H}_\zeta = \zeta + \mathcal{H}.$$

Since $\mathcal{H}$ is a dense subset of $C_b(J)$, there exists by Proposition 5.1 a sequence $\{\phi_n\}_{n=1}^{\infty}\subset\mathcal{H}$ forming a dense subset of $C_b(J)$. Let d_ζ be the metric on $\Sigma_>(\zeta,\Omega,-\infty)$ (see Lemma 5.2) given by (5.3) with the above sequence $\{\phi_n\}_{n=1}^{\infty}$. For every $\varepsilon>0$ put

$$\Re_\varepsilon = \{\phi\in[\zeta] : \mathrm{diam}_{d_\zeta}\Big(\Sigma(\phi,\Omega,s(\phi))\Big) < \varepsilon\}.$$

We shall prove that $\Re_\varepsilon$ is an open subset of $[\zeta]$ and $\Re_\varepsilon\cap\mathcal{H}_\zeta$ is dense in H_ζ (with respect to the metric $\rho_{\mathcal{H}}$). Suppose on the contrary that $I\!R_\varepsilon$ is not open. Then there exist $\psi 1\Re_\varepsilon$ and a sequence $\{\psi_n\}_{n=1}^{\infty}$ of functions from $[\zeta]$ such that $\lim_{n\to\infty}\psi_n=\psi$ and $\psi_n\notin\Re_\varepsilon$ for all $n\geq 1$. Hence

$$\mathrm{diam}_{d_\zeta}\Big(\Sigma(\psi_n,\Omega,s(\phi_n))\Big) \geq \varepsilon$$

for all $n\geq 1$. It therefore follows from Corollary 5.5(d) that for every $n\geq 1$ there are two measures $\mu_,\nu_n\in\Sigma(\psi_n,\Omega,s(\psi_n))$ such that

$$d_\zeta(\mu_n,\nu_n)\geq\varepsilon. \tag{6.1}$$

In view of Lemma 5.8 and Prokhorov's Theorem, we may assume without loss of generality that both sequences $\{\mu_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=1}^\infty$ converge weakly respectively to the measures μ and ν in $M(J)$. Since Ω is weakly closed, $\mu, \nu \in \Omega$. Now it follows from Lemma 5.7 that $\mu, \nu \in \Sigma(\psi, \Omega, s(\psi))$, whereas using (6.1), we conclude from Corollary 5.5(b) (the set $\{\mu, \nu\} \cup \{\mu_n, \nu_n : n \geq 1\}$ is weakly compact) that $d_\zeta(\mu, \nu) \geq \varepsilon$. But $\text{diam}_{d_\zeta}\Big(\Sigma(\psi, \Omega, s(\psi))\Big) < \emptyset$ as $\psi \in \Re_\varepsilon$. This contradiction finishes the proof that $\Re_\varepsilon$ is an open subset of $[\zeta]$.

Now let us demonstrate that the set $\Re_\varepsilon \cap \mathcal{H}_\zeta$ is dense in H_ζ. In order to do it fix $\psi \in \mathcal{H}_\zeta$. For every $k \geq 1$ consider the continuous map $\pi_k : \Sigma_>(\zeta, \Omega, -\infty) \to I\!\!R^k$ defined by the formula

$$\pi_k(\mu) = \Big(\int \phi_1 d\mu, \int \phi_2 d\mu, \dots, \int \phi_k d\mu\Big).$$

It follows from (5.3) that

$$\text{diam}_{d_\zeta}\Big(\pi_k^{-1}(w)\Big) \leq 2^{-k} \tag{6.2}$$

for all $w \in I\!\!R^k$. Fix $n \geq 1$ so large that

$$2^{-n} < \varepsilon. \tag{6.3}$$

Since Ω is convex, so is the set $\Sigma(\psi, \Omega, s(\psi))$. By Corollary 5.5(d) this is also a compact subset of $\Sigma_>(\zeta, \Omega, -\infty)$. Hence $\pi_n\Big(\Sigma(\psi, \Omega, s(\psi))\Big)$ is a convex compact subset of $I\!\!R^n$. Thus by Straszewicz's Theorem this set has a strictly extreme point, i.e. a point $p^n = (p_1, p_2, \dots, p_n) \in \pi_n\Big(\Sigma(\psi, \Omega, s(\psi))\Big) \subset I\!\!R^n$ along with a vector $(a_1, a_2, \dots, a_n) \in I\!\!R^n$ such that

$$\sum_{i=1}^n a_i p_i > \sum_{i=1}^n a_i q_i \tag{6.4}$$

for all $(q_1, q_2, \dots, q_n) \in \pi_n\Big(\Sigma(\psi, \Omega, s(\psi))\Big)$. Since all the functions ϕ_j, $j \geq 1$, are in $C_b(J)$,

$$\phi = \sum_{i=1}^n a_i \phi_i \in C_b(J).$$

Put

$$\tilde{s}(\phi) = \sup\{\int \phi d\mu : \mu \in \Sigma(\psi, \Omega, s(\psi))\} \quad \text{and} \quad \tilde{\Sigma}(\phi) = \{\mu \in \Sigma(\psi, \Omega, s(\psi)) : \int \phi d\mu = \tilde{s}(\phi)\}.$$

Then using (6.4) we see that $\tilde{\Sigma}(\phi) \subset \pi_n^{-1}(p^n)$. It therefore follows from (6.2) and (6.3) that

$$\text{diam}_{d_\zeta}\Big(\tilde{\Sigma}(\phi)\Big) < \varepsilon. \tag{6.5}$$

We shall show that for all $t \in (0, 1)$ small enough

$$\psi_t := \psi + t\phi \in \Re_\varepsilon \cap \mathcal{H}_\zeta.$$

Indeed, for all $t \in I\!\!R$, $\psi_t - \zeta = (\psi - \zeta) + t\phi \in \mathcal{H}$ ($\psi - \zeta \in \mathcal{H}$ since $\psi \in \mathcal{H}_\zeta$ and $t\phi \in \mathcal{H}$ since $\mathcal{H}$ is linear and $\{\phi_j\}_{j=1}^\infty \subset \mathcal{H}$). We are therefore left to show that $\psi_t \in \Re_\varepsilon$ for all $t \in (0, 1)$

small enough. In view of (6.5) there exists an open set $U \subset \Sigma_>(\zeta, \Omega, -\infty)$ such that

$$\tilde{\Sigma}(\phi) \subset U \tag{6.6}$$

and

$$\mathrm{diam}_{d_\zeta}(U) < \varepsilon. \tag{6.7}$$

We shall show that for all $t \in (0,1)$ small enough

$$\Sigma\Big(\psi_t, \Omega, s(\psi_t)\Big) \subset U. \tag{6.8}$$

Indeed, suppose on the contrary that there exists a decreasing to 0 sequence $\{t_k\}_{k=1}^\infty \subset (0,1)$ such that $\Sigma\Big(\psi_{t_k}, \Omega, s(\psi_{t_k})\Big)$ is not contained in U for any $k \geq 1$. This means that for every $k \geq 1$ there exists a measure

$$\mu_k \in \Sigma\Big(\psi_{t_k}, \Omega, s(\psi_{t_k})\Big) \setminus U. \tag{6.9}$$

Since $\phi \in C_b(J)$, the sequence $\{\psi_{t_k}\}_{k=1}^\infty$ converges to ψ in the standard supremum metric on $[\zeta]$, and consequently, $\lim_{k\to\infty} \overline{\rho}_\zeta\Big(\psi_{t_k}, \psi\Big) = 0$. Hence, applying Lemma 5.8 and Prokhorov's Theorem, and passing to a subsequence if necessary, we may assume without loss of generality that the sequence $\{\mu_k\}_{k=1}^\infty$ converges weakly to a measure $\mu \in \Omega$. Making now use of Lemma 5.7, Corollary 5.5(a) and (6.9), we conclude that

$$\mu \in \Sigma\Big(\psi, \Omega, s(\psi)\Big) \setminus U. \tag{6.10}$$

Now take an arbitrary measure $\nu \in \Sigma\Big(\psi, \Omega, s(\psi)\Big)$. Then for every $k \geq 1$ we have that

$$\int \psi d\nu + t_k \int \phi d\nu = \int \psi_{t_k} d\nu \leq \int \psi_{t_k} d\mu_k = \int \psi d\mu_k + t_k \int \phi d\mu_k \leq \int \psi d\nu + t_k \int \phi d\mu_k.$$

Thus, $\int \phi d\nu \leq \int \phi d\mu_k$. Since the sequence $\{\mu_k\}_{k=1}^\infty$ converges weakly to μ and since $\phi \in C_b(J)$, we therefore get that $\int \psi d\nu \leq \int \phi d\mu$. This means that $\mu \in \tilde{\Sigma}(\phi)$. Along with (6.6) and (6.10) gives a contradiction and (6.8) is established. This formula and (6.7) show that $\mathrm{diam}_{d_\zeta}\Big(\Sigma\Big(\psi_{t_k}, \Omega, s(\psi_{t_k})\Big)\Big) < \varepsilon$ for all $t \in (0,1)$ sufficiently small. Consequently $\psi_t \in \Re_\varepsilon \cap \mathcal{H}_\zeta$ for all $t \in (0,1)$ sufficiently small. Since $\psi_t - \psi = t\phi \in \mathcal{H}$, we see that $\rho_{\mathcal{H}}(\psi_t, \psi) = |||t\phi||| = t|||\phi||| \to 0$, when $t \searrow 0$. The proof that $\Re_\varepsilon \cap \mathcal{H}_\zeta$ is dense in $\mathcal{H}_\zeta$ with respect to the topology induced by the metric $\rho_{\mathcal{H}}$ is finished. Putting $G = \bigcap_{n=1}^\infty \Re_{1/n} \cap \mathcal{H}_\zeta$ completes the proof of the whole theorem. ■

As an immediate consequence of this theorem we get the following.

Corollary 6.2. *If (J, ζ, Ω) is a uniquely maximizing triple, then there exists a dense G_δ subset G of $[\zeta]$ ($[\zeta]$ endowed with the complete supremum metric) such that each function $\phi \in G$ has a unique maximizing measure in Ω ($\Sigma\Big(\phi, \Omega, s(\phi)\Big)$ is a singleton).*

We now describe large classes of Banach spaces densly contained in $C_b(J)$. Indeed, given $a \in (0,1]$, $\delta > 0$ and $\phi \in C_b(J)$ let

$$v_\alpha(\phi) = \inf\{L \geq 0 : |\phi(y) - \phi(x)| \leq L\rho^\alpha(x,y) \, \forall_{x\in J} \, \forall_{y \in B(x,\delta)}\}.$$

Let

$$\mathrm{H}_\alpha = \{\phi \in C_b(J) : v_\alpha(\phi) < \infty\}. \tag{6.11}$$

Obviously H_α is a linear subspace of $C_b(J)$ and becomes a banach space when endowed with the norm $||\,||_\alpha$ determined by the formula

$$||\phi||_\alpha = ||\phi||_\infty + v_\alpha(\phi).$$

Observe that the set H_α does not depend on δ and all norms defined with various δs induce the same topology on H_α. Since H_a is a dense subset of $C_b(J)$ continuously (because of (6.11) embedded in $C_b(J)$, as an immediate consequence of Theorem 6.1, we get the following.

Corollary 6.3. *If (J, ζ, Ω) is a uniquely maximizing triple, then there exists a dense G_δ subset G_α of $\zeta + \mathrm{H}_\alpha$ such that each function $\phi \in G$ has a unique maximizing measure in Ω ($\Sigma\Big(\phi, \Omega, s(\phi)\Big)$ is a singleton).*

7. Dynamical Applications

Suppose that $\zeta : J \to I\!R$ is an escaping to $-\infty$ continuous function and that $T : J \to J$ is a continuous mapping. Then M_T, the set of all Borel probability T-ivariant measures on J is convex and weakly closed in $M(J)$. Suppose that M_T is ζ-acceptable, i.e. that $\int \zeta d\mu \in (-\infty, +\infty)$ for some $\mu \in M_T$. The triple (J, T, ζ) is then called dynamically maximizable and each maximizing measure of ζ with respect to M_T is called dynamically maximizing. Notice that this holds for instance if for instance M_T contains at least one measure with compact support; this in turn hold for instance if T has at least one periodic orbit. As immediate consequences of Corollary 4.2, Theorem 6.1, Corollary 6.2 and Corollary 6.3, we respectively get the following.

Corollary 7.1. *Suppose that $\zeta : J \to I\!R$ is an escaping to $-\infty$ continuous function and that $T : J \to J$ is a continuous mapping. If the triple (J, T, ζ) is dynamically maximizable, then ζ has at least one dynamically maximizing measure.*

Corollary 7.2. *Suppose that $\zeta : J \to I\!R$ is an escaping to $-\infty$ continuous function and that $T : J \to J$ is a continuous mapping. Suppose that $(\mathcal{H}, |||\cdot|||)$ is Banach space contained densly in the Banach space $(C_b(J), ||\cdot||_\zeta)$ and that the inclusion map from $(\mathcal{H}, |||\cdot|||)$ to $(C_b(J), ||\cdot||_\zeta)$ is continuous. If the triple (J, T, ζ) is dynamically maximizable, then there exists a dense G_δ*

subset G of $\zeta+\mathcal{H}\subset[\zeta]$ (with the topology on $\zeta+\mathcal{H}$ induced by the metric $\rho_{\mathcal{H}}(\phi,\psi)=|||\psi-\phi|||$) such that each function in G has a unique dynamically maximizing measure.

Corollary 7.3. *If (J,T,ζ) is a dynamically maximizing triple, then there exists a dense G_δ subset G of $[\zeta]$ ($[\zeta]$ endowed with the complete supremum metric) such that each function in G has a unique dynamically maximizing measure.*

Corollary 7.4. *Suppose that $\zeta : J\to I\!\!R$ is an escaping to $-\infty$ continuous function and that $T : J\to J$ is a continuous mapping. If the triple (J,T,ζ) is dynamically maximizable, then there exists a dense G_δ subset G_α of $\zeta+\mathrm{H}_\alpha$ such that each function in G has a unique dynamically maximizing measure.*

All the corollaries listed above apply to so big aboundance of cases that it is virtually impossible to list some special ones. We will however describe in the next section one large class of dynamical systems with non-compact phase space for which all the corollaries established in this section are true and which will be investigated in greater detail and from wider perspectives in the next section.

8. The Exponential Family

We first consider the family $\{f_\lambda : C\!\!\!\!I\to C\!\!\!\!I\}_{\lambda\in C\!\!\!\!I\setminus\{0\}}$ of entire maps of the form

$$f_\lambda(z)=\lambda\exp(z).$$

The Fatou set of f_λ consists of those points $z\in C\!\!\!\!I$ that admit an open neighbourhood U_z such that the family $\{f_\lambda^n|_U\}_{n=1}^\infty$ of iterates of f, restricted to U is normal. The Julia set $J_\lambda(f)$ is defined to be the complement of the Fatou set. Since the map f_λ is periodic with period $2\pi i$, we consider it rather on the cylinder than on $C\!\!\!\!I$. So, let Q be the quotient space (the cylinder),

$$Q=C\!\!\!\!I/\sim,$$

where $z_1\sim z_2$ if and only if $z_1-z_2=2k\pi i$ for some $k\in\boldsymbol{Z}$. Let $\pi : C\!\!\!\!I\to Q$ be the natural projection. Since the map $\pi\circ f_\lambda : C\!\!\!\!I\to Q$ is constant on equivalence classes of relation $\sim$, it canonically induces a conformal map

$$F_\lambda : Q\to Q.$$

The map $F_\lambda : Q\to Q$ will be the main object of our considerations. The Julia set of F_λ is defined to be

$$J(F_\lambda)=\pi(J(f_\lambda)). \tag{8.1}$$

and

$$F_\lambda(J(F_\lambda)) = J(F_\lambda) = F^{-1}(J(F_\lambda)).$$

The cylinder Q is canonically endowed with a Euclidean metric which without confusion will be denoted by the same symbol $|w-z|$ for all $z, w \in P$. For ever $x \in I\!R$ we set

$$Q_x = \{z \in Q : \mathrm{Re} z \le x\} \text{ and } J(F)_x = \{z \in J(F) : \mathrm{Re} z \le x\}.$$

We have thoroughly studied the fractal and dynamical properties of such maps in [8] and [9]. The papers [9] and [10] develop the appropriate versions of thermodynamic formalism and it is evident from them that the right class of potentials to deal with is formed by Hölder continuous functions (on the Julia set) lying within a bounded distance from the functions of the form $z \mapsto -\kappa \mathrm{Re} z$ ($\kappa > 0$). Here is the spot where we meet the content of the previous section. Indeed, notice that a continuous function $\zeta : J(F_\lambda) \to I\!R$ is escaping to infinity if and only if

$$\lim_{\mathrm{Re} z \to +\infty} \zeta(z) = -\infty.$$

Since the Julia set $J(F_\lambda)$ is equal to the closure of its periodic points, we see that the triple $(J(F_\lambda), F_\lambda, \zeta)$ (where ζ is escaping to infinity) is dynamically maximizable and we may therefore formulate the following.

Remark 8.1. *All the conclusions of Corollaries 7.1-7.4 hold with the triple* (J, T, ζ) *replaced by* $(J(F_\lambda), F_\lambda, \zeta)$

From now on we assume that our exponential mapping $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic which means that f has an attracting periodic orbit. We then analyze in greater detail the dynamically maximizing measures of Hölder continuous functions. We single out from them the class of 0^+-tame functions and using the ideas from thermodynamic formalism, we demonstrate the existence of maximizing measures with compact support for such functions. As an outcome of our method of the proof, we provide a more constructive way of producing maximizing measures for 0^+-tame function. We start with the following two auxiliary results.

Lemma 8.2. *If* μ *is a Borel probability measure on the real line* $I\!R$*, then for every* $\delta > 0$*, every* $\alpha > 0$*, and every integer* $n \in \boldsymbol{Z}$ *there exists* $x_n \in [\delta n, \delta(n+1))$ *such that*

$$\sum_{n=-\infty}^{+\infty} \sum_{k=0}^{\infty} \mu\Big([x_n - e^{-\alpha k}, x_n + e^{-\alpha k}]\Big) < \infty.$$

Proof. For every $x \in I\!R$ and every $k \in \boldsymbol{Z}$ write

$$I_{x,k} = [x - e^{-\alpha k}, x + e^{-\alpha k})$$

and note that $\int_{-\infty}^{+\infty} \mu(I_{x,k}) dx = 2e^{-\alpha k}$. Let

$$g(x) = \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{\infty} \mu(I_{x+\delta n,k}).$$

Then

$$\int_0^\delta g(x)dx = \sum_{k=0}^{\infty}\sum_{n=-\infty}^{+\infty}\int_{\delta n}^{\delta(n+1)}\mu(I_{y,k})dy = \sum_{k=0}^{\infty}\int_{-\infty}^{+\infty}\mu(I_{y,k})dy = \sum_{k=0}^{\infty}2e^{-\alpha k} < +\infty.$$

Hence there is $x \in (0,\delta)$ such that $g(x) < +\infty$, and taking $x_n = x + \delta n$ completes the proof. ■

Lemma 8.3. *If μ is a Borel probability measure on the cylinder Q, then for every $\delta > 0$ there exists a partition α of Q by rectangles with all sides of length $\le \delta$ parallel to the coordinate axes such that for every $\beta > 0$*

$$\sum_{n=0}^{\infty}\sum_{A\in\alpha}\mu\Big(B(\partial A, e^{-\beta n})\Big) < \infty.$$

Proof. Let $p_1 : Q \to I\!R$ be the orthogonal projection onto the x-axis and let $p_2 : Q \to I\!R$ be the orthogonal projection onto the circle $I\!R/2\pi\boldsymbol{Z}$. In view of Lemma 8.2 there are points $[x_n \in \left[\frac{\delta}{2}n, \frac{\delta}{2}(n+1)\right] \subset I\!R$, $n \in \boldsymbol{Z}$, $y_1 < y_2 < \ldots < y_q \in I\!R/2\pi\boldsymbol{Z}$ such that $|y_{i+1} - y_i| < \delta$ and

$$\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{\infty}\mu\circ p_1^{-1}\Big([x_n - e^{-\beta k}, x_n + e^{-\beta k}]\Big) < \infty \text{ and } \sum_{j=1}^{q}\sum_{k=0}^{\infty}\mu\circ p_2^{-1}\Big([y_j - e^{-\beta k}, y_j + e^{-\beta k}]\Big) < \infty. \tag{8.2}$$

Let α be the partition formed by all the rectangles $[x_n, x_{n+1}] \times [y_j, y_{j+1(modq)}] \subset Q$, $n \in \boldsymbol{Z}$, $j = 1, 2, \ldots, q$. Then

$$\partial\alpha = \bigcup_{n=-\infty}^{+\infty}\bigcup_{j=1}^{q}\{x_n\}\times S^1)\cup(I\!R\times\{y_j\})$$

and

$$B\Big(\partial\alpha, e^{-\beta k}\Big) = \left(\bigcup_{n=-\infty}^{+\infty}B(x_n, e^{-\beta k})\times S^1\right)\cup\left(\bigcup_{j=1}^{q}I\!R\times B(y_j, e^{-\beta k})\right).$$

Hence, due to (8.2)

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{A\in\alpha}\mu\Big(B(\partial A, e^{-\beta k})\Big) &\le \sum_{n=-\infty}^{+\infty}\mu\Big(B(x_n, e^{-\beta k})\times S^1\Big) + \sum_{j=1}^{q}\mu\Big(I\!R\times B(y_j, e^{-\beta k})\Big)\\ &\le \sum_{n=-\infty}^{+\infty}\mu\circ p_1^{-1}\Big(B(x_n, e^{-\beta k})\Big) + \sum_{j=1}^{q}\mu\circ p_2^{-1}\Big(B(y_j, e^{-\beta k})\Big) < \infty\end{aligned}$$

We are done. ■

We now pass to deal directly with Hölder continuous functions. Let

$$\delta = \frac{1}{2}\min\left\{\frac{1}{2}, \mathrm{dist}\Big(J(F), \{F^n(\pi(0)) : n \ge 0\}\Big)\right\}$$

Since the map $f : J(f) \to J(f)$ is hyperbolic, we see that $\delta > 0$. For every $n \geq 1$ and every $v \in J(F)$ the map

$$F_v^{-n} : B(F^n(v), 2\delta) \to Q$$

is then defined to be the holomorphic inverse branch of F^n on $B(F^n(v), 2\delta)$ sending $F^n(v)$ to v. It was proven in [9] that there exist two constants $C \geq 1$ and $\kappa > 0$ such that

$$|(F_v^{-n})'(z)| \leq Ce^{-\kappa n} \tag{8.3}$$

for all $n \geq 0$, all $v \in J(F)$ and all $z \in B(F^n(v), \delta)$. Let ϕ be a real-valued Hölder continuous function defined on some Euclidean R-neighbourhood of the Julia set $J(F) \subset \mathbb{C}$, $R \in (0, \delta/2)$. Hölder continuous means here that

$$\exists_{(\alpha>0)} \forall_{r\in(0,R)} \exists_{H_r>0} \text{ if } |y-x| \leq r, \text{ then } |\phi(y) - \phi(x)| \leq H_r|y-x|^\alpha.$$

Let $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ be the real part function (projection onto the real axis).

Definition 8.4. *A Hölder continuous function $\phi : J(F) \to \mathbb{R}$ is called κ-tame, $\kappa \in \mathbb{R}$, if $\phi \in [-\kappa\mathrm{Re}]$, where $\mathrm{Re} : Q \to \mathbb{R}$ is the function ascribing to each point in the cylinder Q its real part. This function is called 0^+-tame if $\kappa > 0$ and 1^+-tame if $\kappa > 1$.*

For every function $g : J(F) \to \mathbb{R}$ and every $n \geq 1$ let

$$S_n g = \sum_{j=0}^{n-1} g \circ F^j.$$

The following three basic facts have been essentially proved proved in [9] for the special case $\phi(z) = -\kappa\mathrm{Re}z$, $\kappa > 1$. The complete proofs in the case of an arbitrary 1^+-tame function requires only minor straightforward modifications and can be found in [10]).

Theorem 8.5. *Let ϕ be 1^+-tame. If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic, then For every $z \in J(F)$ the following limit exists and is independent of the point z.*

$$\mathrm{P}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in F^{-n}(z)} \exp\Big(S_n\phi(x)\Big).$$

The number $\mathrm{P}(\phi)$ is called the topological pressure of the potential ϕ

Theorem 8.6. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then there exists a unique Borel probability measure m_ϕ on $J(F)$ such that*

$$m_\phi\Big(F_v^{-n}\Big(B(z,\delta)\Big)\Big) = \int_{B(z,\delta)} \exp\Big(S_n\phi\Big(F_v^{-n}(w)\Big) - \mathrm{P}(\phi)n\Big)dm_\phi(w)$$

for all $n \geq 1$, all $z \in J(F)$ and all $v \in F^{-n}(z)$.

Theorem 8.7. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then there exists a unique Borel probability F-invariant measure μ_ϕ absolutely continuous with respect to m_ϕ. In addition μ_ϕ is ergodic, equivalent to m_ϕ and the Radon-Nikodym derivative $\psi = d\mu_\phi/dm_\phi$ has a continuous everywhere (on $J(F)$) positive and uniformly bounded version.*

For every Borel probability F-invariant measure μ on $J(F)$ let h_μ denote the measure-theoretic entropy of the measure μ with respect to the dynamical system $F : J(F) \to J(F)$. Our first result, crucial for investigation of dynamically maximizing measures of 0^+-tame functions and interesting on its own is this.

Theorem 8.8. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then the invariant measure μ_ϕ is an equilibrium state of the potential ϕ, that is*

$$\mathrm{P}(\phi) = \sup\{\mathrm{h}_\mu(F) + \int \phi d\mu\}$$

where the supremum is taken over all Borel probability F-invariant ergodic measures μ with $\int \phi d\mu > -\infty$, and

$$\mathrm{P}(\phi) = \mathrm{h}_{\mu_\phi} + \int \phi d\mu_\phi.$$

Proof. We shall show first that if μ is a Borel probability F-invariant ergodic measure on $J(F)$ with $\int \phi d\mu > -\infty$, then

$$\mathrm{P}(\phi) \geq \mathrm{h}_\mu + \int \phi d\mu.$$

Let α be the partition constructed in Lemma 8.3 with diameter $\leq \delta$. We shall demonstrate that there exists a Borel set $Z \subset J(F)$ such that $\mu(Z) = 1$ and

$$\mu_\phi(\alpha_n(x)) \leq \mu(\alpha_n(x)) \tag{8.4}$$

for all $x \in Z$ and all $n \geq 1$ large enough (depending on x). If μ and μ_ϕ are not mutually singular, then $\mu = \mu_\phi$ since both measures are ergodic, and (8.4) becomes obvious. So, we may assume that μ and μ_ϕ are mutually singular. This means that there exists a Borel set $Y \subset J(F)$ such that $\mu_\phi(Y) = 0$ and $\mu(Y) = 1$. Seeking contradiction suppose now that there exists a compact set $S \subset Y$ with the following two properties.

(a) $\mu(S) > 0$.

(b) For every $x \in S$ there exists an unbounded increasing sequence $\{n_j(x)\}_{j=1}^\infty$ such that

$$\mu_\phi(\alpha_{n_j(x)}(x)) > \mu(\alpha_{n_j(x)}(x))$$

for all $j \geq 1$.

Since the measure μ_ϕ is regular and $\mu_\phi(S) = 0$, there exists $\varepsilon > 0$ such that $\mu_\phi(B(S, \varepsilon)) < \mu(S)/2$. Since for every $x \in J(F)$ and every $n \geq 0$, $\alpha_n(x) \subset F_x^{-n}(B(F^n(x), \delta))$, looking at (8.3) we see that

$$\lim_{n\to\infty} \mathrm{diam}(\alpha_n(x)) = 0.$$

Fix $\varepsilon > 0$. For every $x \in S$ there thus exists $j(x) \geq 1$ such that $\operatorname{diam}\Big(\alpha_{n_{j(x)}}(x)\Big) < \varepsilon$. Since any two elements of the family $\Big\{\alpha_{n_{j(x)}}(x)\Big\}_{x\in S}$ are either disjoint or one is contained in the other, we can choose countably many points $\{x_k\}_{k=1}^\infty \subset S$ such that all the sets $A_k = \alpha_{n_{j(x_k)}}(x_k)$, $k \geq 1$, are mutually disjoint and $\bigcup_{k\geq 1} A_k \supset S$. Using the property (b) we then get

$$\mu(S) \leq \mu\left(\bigcup_{k=1}^\infty A_k\right) = \sum_{k=1}^\infty \mu(A_k) < \sum_{k=1}^\infty \mu_\phi(A_k) = \mu_\phi\left(\bigcup_{k=1}^\infty A_k\right) \leq \mu_\phi(B(S,\varepsilon)) < \frac{1}{2}\mu(S).$$

This contradiction finishes the proof of (8.4). In view of Lemma 8.3 we get for every $\beta > 0$ that

$$\sum_{n=1}^\infty \sum_{A\in\alpha} \mu\Big(F^{-n}\Big(B(\partial A, e^{-\beta n})\Big)\Big) = \sum_{n=1}^\infty \sum_{A\in\alpha} \mu\Big(B(\partial A, e^{-\beta n})\Big) < \infty.$$

Therefore $\mu(Z_1) = 1$, where Z_1 is the set of all those $z \in Z$ that $F^n(z) \in B(\partial A, e^{-\beta n})$ for finitely many n's only. Fix $z \in Z_1$ and let $q \geq 1$ be such that $F^n(z) \notin B(\partial A, e^{-\beta n})$ for all $n \geq q$. For every $k \in \{0, 1, 2, \dots, n\}$, put $F_*^{-k} := F_{F^{n-k}(z)}^{-k}$ (so, F_*^{-k} is the branch of F^{-k} sending the point $F^n(z)$ back th $F^{n-k}(z)$). Let $p \geq 0$ be an arbitrary real number such that $C^{-1}e^{-\kappa p} \leq e^{-\beta(n-p)}$. This equivalently means that $\log C + \kappa p \geq \beta n - \beta p$ or $(\kappa+\beta)p \geq \beta n - \log C$, and finally this means that $p \geq \frac{\beta}{\kappa+\beta}n - \frac{\log C}{\kappa+\beta}$. So, putting

$$p = p_n = E\left(\frac{\beta}{\kappa+\beta}n\right) + 1,$$

we see that p satisfies the required inequality; in fact

$$C^{-1}e^{-\kappa k} \leq e^{-\beta(n-k)}$$

for all $k \geq p$. This implies that $\alpha(F^{n-k}(z)) \supset F_*^{-k}\Big(B(F^n(z),\delta)\Big)$ for all $p \leq k \leq n-q$, since, by (8.3), the preimage $F^{-k}B(F^n(z),\delta)$ is contained in the ball $B(F^{n-k}(z), C\varepsilon^{-\kappa k})$ Hence

$$F^{-(n-k-q)}\Big(\alpha(F^{n-k}(z))\Big) \supset F_q^{-(n-k-q)} \circ F_*^{-k}\Big(B(F^n(z),\delta)\Big) = F_*^{-(n-q)}\Big(B(F^n(z),\delta)\Big),$$

where $F_q^{-(n-k-q)} : B(F^{n-k}(z), 2\delta) \to \mathbb{C}$ is the holomorphic inverse branch of F^{n-k-q} sending $F^{n-k}(z)$ to $F^q(z)$. Thus,

$$\alpha_{n-q-p}(z) = \bigcap_{k=p}^{n-q} F^{-(n-k-q)}\Big(\alpha(F^{n-k}(z))\Big) \supset F_*^{-(n-q)}\Big(B(F^n(z),\delta)\Big).$$

Hence, utilizing Theorem 8.6 and Theorem 8.7, we obtain

$$\begin{aligned}
\mu_\phi\Big(\alpha_{n-q-p}(z)\Big) &\geq \mu_\phi\Big(F_*^{-(n-q)}\Big(B(F^n(z),\delta)\Big)\Big) \\
&\succeq \psi(F^q(z)) \exp\Big(S_{n-q}\phi(F^q(z)) - \mathrm{P}(\phi)n\Big) m_\phi\Big(B(F^n(z),\delta)\Big) \\
&\asymp \exp\Big(S_n\phi(z) - \mathrm{P}(\phi)n\Big) m_\phi\Big(B(F^n(z),\delta)\Big),
\end{aligned}$$

where $\psi = d\mu_\phi/dm_\phi$ and the comparability constant, call it C_1 appearing in the above formula depends on z but is independent of n. Consequently

$$\frac{1}{n}\log\Big(\mu_\phi\Big(\alpha_{n-q-p}(z)\Big)\Big) \ge \frac{\log(C_1)}{n} + \frac{1}{n}S_n\phi(z) - \mathrm{P}(\phi) + \frac{1}{n}\log\Big(m_\phi\Big(B(F^n(z),\delta)\Big)\Big). \tag{8.5}$$

In view of classical Birkhoff's ergodic theorem and the Breiman-McMillan-Shanon theorem (see [6], [11], comp. [7]) there exists an F-invariant Borel set $Z_2 \subset Z_1$ such that $\mu(Z_2) = 1$, and for all $z \in Z_2$

$$\liminf_{n\to\infty} \mathrm{Re}(F^n(z)) < \infty,\ \lim_{n\to\infty}\frac{1}{n}S_n\phi(z) = \int \phi d\mu,\ \lim_{n\to\infty}\frac{1}{n}\log\Big(\mu(\alpha_n(z))\Big) = -\mathrm{h}_\mu. \tag{8.6}$$

Fix $z \in Z_2$. There thus exists an unbounded increasing sequence $\{n_j\}_{j=1}^\infty$ of positive integers such that $\mathrm{Re}(F^{n_j}(z)) \le M$ for some $M > 0$ and all $j \ge 1$. Hence, there exists $T > 0$ such that $m_\phi\Big(B(F^{n_j}(z),\delta)\Big) \ge T$ for all $j \ge 1$. It therefore follows from (8.5) and (8.6) that for every $z \in Z_2$,

$$\liminf_{j\to\infty}\frac{1}{n_j}\log\Big(\mu_\phi\Big(\alpha_{n_j-q-p_{n_j}}(F^q(z))\Big)\Big) \ge \int\phi d\mu - \mathrm{P}(\phi).$$

Combining this along with (8.4) and the last part of (8.6), we get

$$\begin{aligned}-\mathrm{h}_\mu &\ge \liminf_{j\to\infty}\left(\frac{n_j}{n_j-q-p_{n_j}}\cdot\frac{1}{n_j}\log\Big(\mu_\phi\Big(\alpha_{n_j-q-p_{n_j}}(F^q(z))\Big)\Big)\right)\\ &\ge \liminf_{j\to\infty}\left(\frac{n_j}{n_j-q-p_{n_j}}\right)\Big(\int\phi d\mu - \mathrm{P}(\phi)\Big).\end{aligned}$$

But it follows from the definition of the sequence $\{p_n\}_{n=1}^\infty$ that

$$\liminf_{n\to\infty}\frac{n}{n-q-p_n} = \liminf_{n\to\infty}\left(\frac{1}{1-\frac{q}{n}-\frac{p_n}{n}}\right) = \lim_{n\to\infty}\left(\frac{1}{1-\frac{p_n}{n}}\right) = \frac{1}{1-\frac{\beta}{\kappa+\beta}} = \frac{\kappa+\beta}{\kappa}.$$

Hence

$$\int\phi d\mu - \mathrm{P}(\phi) \le -\mathrm{h}_\mu\frac{\kappa}{\kappa+\beta}.$$

Letting now $\beta \searrow 0$, we finally obtain that

$$\mathrm{P}(\phi) \ge \mathrm{h}_\mu + \int\phi d\mu.$$

We shall now prove the easier part of our theorem, that

$$\mathrm{P}(\phi) \le \mathrm{h}_{\mu_\phi} + \int\phi d\mu_\phi.$$

Indeed, let α be the same partition as in the first part of the proof. Since F restricted to each atom of the partition α is 1-to-1, it follows that $\alpha_n(x) \subset F_x^{-n}\Big(B(F^n(x),\delta)\Big)$ for all $x \in J(F)$ and all $n \ge 0$. Applying Birkhoff's ergodic theorem and the Breiman-McMillan-Shanon theorem for the F-invariant measure μ_ϕ and utilizing Theorem 8.6 along with Theorem 8.7, we therefore get for μ_ϕ-a.e. $x \in J(F)$ that

$$\begin{aligned}-\mathrm{h}_{\mu_\phi} &\le \liminf_{n\to\infty} \frac{1}{n}\log \mu_\phi\Big(F_x^{-n}\Big(B(F^n(x),\delta)\Big)\Big) \preceq \liminf_{n\to\infty}\frac{1}{n}\Big(\log(2\psi(x)) + S_n\phi(x) - \mathrm{P}(\phi)n\Big)\\ &= \lim_{n\to\infty}\frac{1}{n}S_n\phi(x) - \mathrm{P}(\phi) = \int \phi d\mu_\phi - \mathrm{P}(\phi).\end{aligned}$$

Thus $\mathrm{P}(\phi) \le \mathrm{h}_{\mu_\phi} + \int \phi d\mu_\phi$ and we are done. ■

Since ϕ is 1^+-tame there exists a unique $\kappa > 1$ such that ϕ is κ-tame. Our aim is to show that the family $\{\mu_{t\phi}\}_{t\ge1}$ is tight. This requires several lemmas. We start with the following.

Lemma 8.9. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then*

$$\chi := \sup_{t\ge1}\{|\int \phi d\mu_{t\phi}|\} < \infty.$$

Proof. It has been proved in [10] that $\int \phi d\mu_{t\phi} = \mathrm{P}'(t\phi)$ and that the function $t \mapsto \mathrm{P}(t\phi)$ is convex. The latter means that the function $t \mapsto \mathrm{P}'(t\phi)$ is non-decreasing. Hence $\mathrm{P}'(t\phi) \ge \mathrm{P}'(\phi)$ for every $t \ge 1$, and consequently $\int \phi d\mu_{t\phi} \ge \int \phi d\mu_\phi > -\infty$ for all $t \ge 1$. The observation that $\sup\{\phi(z) : z \in J(F)\} < +\infty$ finishes therefore the proof. ■

In the proofs of the following two lemmas we occasionally use some results from [10]. They always correspond to analogous results from [9] and the difference between [9] and [10] is that in [9] the potentials of the form const$+t\mathrm{Re}z$ are considered, while [10] deals with more general potentials $t\mathrm{Re}z$ plus a bounded Hölder continuous function. The estimates we need here for this more general class of potentials are straightforward modifications of the corresponding estimates from [9].

Lemma 8.10. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then the family $\{m_{t\phi}\}_{t\ge1}$ is tight and its every limit measure (as $t \to +\infty$) has a compact support.*

Proof. By Theorem 8.8 and Lemma 8.9,

$$\mathrm{P}(t\phi) \ge -\chi t \tag{8.7}$$

for all $t \ge 1$. Corollary 3.13 from [10] applied to the potential $t\phi$ states that

$$m_{t\phi}(Q_n^c) \le 4C_4|\lambda|^{t\kappa}(t\kappa-1)^{-1}\exp\Big(tA_\phi - \mathrm{P}(t\phi)\Big)e^{M(1-t\kappa)}e^{(1-t\kappa)n}, \tag{8.8}$$

where $C_4 > 0$ is an absolute constant and $A_\phi = ||\phi + \kappa\mathrm{Re}||_\infty$. (Recall that $Q_n^c = \{z \in Q : \mathrm{Re}z > n\}$).This corollary says in fact that the above estimate holds for for all $n \ge M(t)$, where $M(t)$ can be computed precisely:

$$M(t) = \Big(2C_2|\lambda|^{t\kappa}\exp\Big(tA_\phi - \mathrm{P}(t\phi)\Big)(t\kappa-1)^{-1}\Big)^{\frac{1}{t\kappa-1}},$$

where $C_2 > 0$ is an absolute constant (compare Proposition 3.8 and Corollary 3.9 in [9], where a slightly simpler case $\phi(z) = -\mathrm{Re} z$ is considered). We therefore have

$$\begin{aligned}\limsup_{t\nearrow+\infty} M(t) &= \limsup_{t\nearrow+\infty}\left(\frac{2C_2}{\kappa t-1}\right)^{\frac{1}{\kappa t-1}}\exp\left(\frac{tA_\phi - \mathrm{P}(t\phi + t\kappa\log|\lambda|)}{\kappa t-1}\right)\\ &\le \limsup_{t\nearrow+\infty}\left(\frac{2C_2}{\kappa t-1}\right)^{\frac{1}{\kappa t-1}}\exp\left(\frac{tA_\phi + \chi t + t\kappa\log|\lambda|)}{\kappa t-1}\right)\\ &= \exp\left(\frac{A_\phi + \chi + \kappa\log|\lambda|}{\kappa}\right) < \infty.\end{aligned}$$

Combining this and (8.8 we see that there exists $M > 0$ such that for all $t \ge 1$ and all $n \ge M$

$$m_{t\phi}(Q_n^c) \le \frac{4C_4}{\kappa t-1}e^{(1-\kappa t)n}\exp\Big(tA_\phi - \mathrm{P}(t\phi) + t\kappa\log|\lambda|\Big).$$

So, employing (8.7), we get that

$$m_{t\phi}(Q_n^c) \le \frac{4C_4}{\kappa t-1}\exp\Big((1-\kappa t)n + (A_\phi + \chi + \kappa\log|\lambda|)t\Big).$$

Fix $n_0 \ge M$ so large that $(\kappa t - 1)n_0 \ge (A_\phi + \chi + \kappa\log|\lambda|)t$ for every $t \ge 1$. Then for every $n \ge n_0$ and every $t \ge 1$, we get that

$$m_{t\phi}(Q_n^c) \le \frac{2C_4}{\kappa-1}\exp\Big((1-\kappa t)(n-n_0)\Big) \le \frac{4C_4}{\kappa-1}\exp\Big((1-\kappa)(n-n_0)\Big) \tag{8.9}$$

Hence

$$\lim_{n\to\infty}\Big(\sup_{t\ge1}\{m_{t\phi}(Q_n^c)\}\Big) = 0$$

and since each set Q_n is compact, the proof of the tightness of the family $\{m_{t\phi}\}_{t\ge1}$ is complete. It also follows from (8.9) that for every $t \ge 1$,

$$m_{t\phi}(Q_{n_0+1}^c) \le \frac{4C_4}{\kappa-1}\exp(1-\kappa t).$$

This implies that $\lim_{t\to+\infty} m_{t\phi}(Q_{n_0+1}^c) = 0$ and consequently, $m(Q_{n_0+1}) = 1$ for every limit measure m of the family $\{m_{t\phi}\}_{t\ge1}$. Since Q_{n_0+1} is a compact set, we are therefore done. ■

For every $t \ge 1$ let $\psi_t = d\mu_{t\phi}/dm_{t\phi}$. We will also need the following

Lemma 8.11. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{C}$ is a 1^+-tame potential, then there exists $x > 1$ such that*

$$\sup_{t\ge1}\sup_{w\in J(F)_x^c}\{\psi_t(w)\} \le 1.$$

Proof. Fix $w \in J(F)_1^c$, i.e $\mathrm{Re}w > 1$. Treating w as an element of the strip $\{z \in \mathbb{C} : -\pi < \mathrm{Im}z \leq \pi\}$, let for every $k \in \mathbb{Z}$, the point $w_k \in Q$ be the only element of $f^{-1}(w + 2\pi ik)$ (we treat here the function f as defined on the cylinder Q and taking values in $\mathbb{C}$.) Let $\hat{\mathcal{L}}_{t\phi} : C_b \to C_b$ be the normalized Perron-Frobenius operator induced by the potential $t\phi$. It is given by the formula

$$\hat{\mathcal{L}}_{t\phi}(g)(z) = e^{-\mathrm{P}(t\phi)} \sum_{k=-\infty}^{\infty} e^{t\phi(w_k)} g(w_k). \tag{8.10}$$

It has been proven in [10] (following the proof of Lemma 3.4 in [9]) that there exists a function $K_\phi : \mathbb{R} \to [0, \infty)$ such that for all $n \geq 0$, all y large enough and all $t \geq 1$ we have

$$\hat{\mathcal{L}}_{t\phi}^n(1\!\!1) \leq \frac{K_\phi^t(y)}{m_{t\phi}(Q_y)}.$$

It follows from (8.9) that $m_{t\phi}(Q_y) \geq 1/2$ for all y large enough and all $t \geq 1$. Hence, there exists $y > 1$ (in fact all y large enough are good) such that

$$\hat{\mathcal{L}}_{t\phi}^n(1\!\!1) \leq 2K_\phi^t(y) \tag{8.11}$$

for all $n \geq 0$ and all $t \geq 1$. Since ϕ is a 1^+-tame function, there is a unique $\kappa > 1$ such that ϕ is κ-tame. Recall that $A_\phi = ||\phi + \kappa\mathrm{Re}||_\infty$. It is straightforward to calculate that

$$\sum_{k=-\infty}^{\infty} e^{t\phi(w_k)} \leq 2e^{tA_\phi}(|\lambda|)^{\kappa t} \sum_{k=0}^{\infty}\Big(\mathrm{Re}w + 2\pi k\Big)^{-\kappa t} \leq 2e^{tA_\phi}(|\lambda|)^{\kappa t}\frac{\mathrm{Re}w^{1-\kappa t}}{\kappa t - 1}$$

In particular if $x \geq 1$ and $w \in J(F)_x^c$, then

$$\sum_{k=-\infty}^{\infty} e^{t\phi(w_k)} \leq 2e^{tA_\phi}(|\lambda|)^{\kappa t}\frac{x^{1-\kappa t}}{\kappa t - 1}$$

It therefore follows from (8.7) and (8.10) that

$$\hat{\mathcal{L}}_{t\phi}(1\!\!1)(w) \leq 2e^{\chi t}e^{tA_\phi}(|\lambda|)^{\kappa t}\frac{x^{1-\kappa t}}{\kappa t - 1} \leq \frac{2}{\kappa t - 1}e^{(\chi + A_\phi)t}(|\lambda|)^{\kappa t}x^{1-\kappa t}. \tag{8.12}$$

Since

$$\limsup_{t\to+\infty} \log\left(\frac{2}{\kappa t - 1}e^{(\chi+A_\phi)t}(|\lambda|)^{\kappa t}x^{1-\kappa t}\right) =$$
$$= \limsup_{t\to+\infty}\left(\log\left(\frac{2}{\kappa t - 1}\right) + (\chi + A_\phi)t + \kappa\log(|\lambda|)t + \log x - \kappa t\log x\right) = -\infty$$

if only $\kappa\log x > \chi + A_\phi + \kappa\log(2|\lambda|)$, we therefore see from (8.12) that if $x \geq y > 1$ is large enough, then

$$\hat{\mathcal{L}}_{t\phi}(1\!\!1)(w) \leq \Big(2K_\phi^t(y)\Big)^{-1} \tag{8.13}$$

for all $t \geq 1$. Applying the operator $\hat{\mathcal{L}}_{t\phi}$ to inequality (8.11), utilizing its monotonicity (following from its positivity, which in turn follows immediately from (8.10)), and using (8.13), we get

$$\hat{\mathcal{L}}_{t\phi}^{n+1}(1\!\!1)(w) \leq \hat{\mathcal{L}}_{t\phi}\Big(2K_\phi^t(y)1\!\!1\Big)(w) = 2K_\phi^t(y)\hat{\mathcal{L}}_{t\phi}(1\!\!1)(w) \leq 1$$

It therefore follows from the results proven in [10] (comp. Theorem 4.4 in [9]) that $\psi_t(w) \leq 1$ for all $t \geq 1$ and all $w \in J(F)_x^c$. ■

Combining this lemma and Lemma 8.10, we get the following.

Proposition 8.12. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then the family $\{\mu_{t\phi}\}_{t\geq1}$ is tight and its every limit measure (as $t \to +\infty$) has a compact support.*

It therefore follows from Prokhorov's theorem that the set M_ϕ of all weak limit points of the family $\{\mu_{t\phi}\}_{t\geq1}$ (as $t \to +\infty$) is non-empty. The significance of the set M_ϕ is explained by the following main result of this section.

Theorem 8.13. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to \mathbb{R}$ is a 1^+-tame potential, then the non-empty set M_ϕ is contained in the set of all dynamicall maximizing measures for ϕ. In particular this latter set contains measures with compact supports.*

Proof. In view of Proposition 8.12 and the paragraph following it, we are left to prove that each measure in M_ϕ is maximizing for the function ϕ. So, fix a measure $\mu \in M_\phi$. There then exists an increasing sequence $\{t_n\}_{n=1}^\infty$ diverging to $+\infty$ such that the sequence $\{\mu_{t_n\phi}\}_{n=1}^\infty$ converges weakly to μ. We shall show first that

$$\limsup_{n\to\infty} \int \phi d\mu_{t_n\phi} \leq \int \phi d\mu. \tag{8.14}$$

Indeed, let $\phi_k \searrow \phi$ be a sequence of bounded continuous functions converging pointwise to to ϕ (for example $\phi_k = \max\{\phi, -k\}$). Fix any number $T > \int \phi d\mu$ (note that we have not ruled out the possibility that $\int \phi d\mu = -\infty$. It follows from Lebesgue's monotone convergence theorem that $T > \int \phi_k d\mu$ for all $k \geq 1$ large enough. Fix one such k. Fix also $\varepsilon > 0$. Since ϕ_k is a bounded continuous function and since the sequence $\{\mu_{t_n\phi}\}_{n=1}^\infty$ converges weakly to μ, we have $\int \phi_k d\mu > \int \phi_k d\mu_{t_n\phi} - \varepsilon$ for all $n \geq 1$ large enough. But $\int \phi_k d\mu_{t_n\phi} \geq \int \phi d\mu_{t_n\phi}$ since $\phi_k \geq \phi$. Combining all these inequalities together, we get that

$$T > \int \phi_k d\mu > \int \phi_k d\mu_{t_n\phi} - \varepsilon > \int \phi d\mu_{t_n\phi} - \varepsilon$$

for all $n \geq 1$ large enough. Hence $\limsup_{n\to\infty} \int \phi d\mu_{t_n\phi} \leq T + \varepsilon$, and letting $\varepsilon \searrow 0$ and $T \searrow \int \phi d\mu$, formula (8.14) follows. We have already established in the proof of Lemma 8.9

that the function $t \mapsto \mathrm{P}'(t\phi)$ is non-decreasing. In particular, the limit $\lim_{t\nearrow+\infty} \mathrm{P}'(t\phi)$ exists, and looking at (8.14), we see that

$$\lim_{t\nearrow+\infty} \mathrm{P}'(t\phi) \le \int \phi d\mu. \tag{8.15}$$

Seeking contradiction suppose now that μ is not a maximizing measure for the function ϕ. Then there exists a Borel probability F-invariant measure ν on $J(F)$ such that $\int \phi d\nu > \int \phi d\mu$. In particular $\int \phi d\nu$ is a finite number and fix any $\in I\!R$ such that $\int \phi d\mu < R < \int \phi d\nu$. Since $\mathrm{P}(\phi) < \infty$, it therefore follows from Theorem 8.8 that $\mathrm{h}_\nu(F) < \infty$. So, we may consider the linear function $l_\nu(t) = \mathrm{h}_\nu(F) + t\int \phi d\nu$. It then follows from (8.15) that $l_\nu'(t) = \int \phi d\nu > R > \mathrm{P}'(t\phi)$ for all $t \ge 1$ large enough. Consequently $l_\nu(t) > \mathrm{P}(t\phi)$ for all $t \ge 1$ sufficiently large. But this contradicts Theorem 8.8 and finishes the proof. ■

Corollary 8.14. *If $f : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(F) \to I\!R$ is a 0^+-tame potential, then the set of all maximizing measures contains at least one measure with compact support.*

Proof. The function ϕ is κ-tame with some $\kappa > 0$. So, the function $\frac{2}{\kappa}\phi$ is 2-tame, and since both functions ϕ and $\frac{2}{\kappa}\phi$ have the same set of maximizing measures, an application of Theorem 8.13 completes the proof. ■

Since $-\log|F'(z)| = -\log|\lambda| + \log|e^z| = -\log|\lambda| + \mathrm{Re}z$, the function $-\log|F'|$ is 1-tame and consequently, 0^+-tame. Therefore, the following result follows immediately from Theorem 8.13.

Corollary 8.15. *There exists a Borel probability F-invariant measure with μ with compact support that minimizes the Lyapunov exponent $\chi_\mu = \int \log|F'|d\mu$.*

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