On the uniqueness of the density for the invariant measure in an infinite hyperbolic iterated function system

by

R. Daniel Mauldin[1]
and
Mariusz Urbański[1]

Department of Mathematics
University of North Texas
Denton, Texas 76203-305118 U.S.A.
:
e-mail: mauldin@unt.edu; urbanski@unt.edu
:
homepage: www.math.unt.edu

September 8, 1998

ABSTRACT. We consider a regular infinite hyperbolic iterated function satisfying a property which guarantees that the associated Frobenius-Perron operator $\mathcal{L}$ is almost periodic. For such a system there is a unique invariant probablility measure μ supported on J, the limit set of the system and which is equivalent to the conformal measure m of the system. In this note we will demonstrate two properties of $d\mu/dm$. Firstly, we show that there is a unique positive continuous function on X, ρ such that $\mathcal{L}\rho = \rho$ and $\int \rho dm = 1$. This function is the density of μ with respect to m. Secondly, we show that $\{\mathcal{L}(1\!\!1_X)\}_{n=1}^{\infty}$ converges uniformly to ρ on X.

[1] Research supported by NSF Grant DMS-9801583.
AMS(MOS) subject classifications(1980). Primary 28A80; Secondary 58F08, 58F11, 28A78

Key words and phrases. Iterated function systems, conformal and invariant measures, density, almost periodic.

§**1. Introduction.** In [MU] we have provided a framework to study infinite (hyperbolic) conformal iterated function systems. We shall first recall this notion and some of its basic properties. Let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\geq 1} I^n$, the space of finite words, and for $\omega \in I^n$, $n \geq 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest is the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

and various natural measures and functions associated with it. Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property in general fails for infinite systems.

We consider a regular infinite hyperbolic iterated function system satisfying a property which guarantees that the associated Frobenius-Perron operator is almost periodic. For such a system there is a unique invariant probablility measure μ supported on J, the limit set of the system and which is equivalent to the conformal measure m of the system. In [MU] we showed that the density ρ of μ with respect to m is the unique positive continuous function on J such that $\mathcal{L}\rho = \rho$ and $\int \rho dm = 1$. In this note we will demonstrate two further properties of this density. Firstly, we show that ρ has a unique extension to a positive continuous function on X such that $\mathcal{L}\rho = \rho$ and $\int \rho dm = 1$. We also denote this extension by ρ. Let $\mathcal{L}$ be the Perron-Frobenius operator associated with this system. Secondly, we show that $\{\mathcal{L}(\mathbb{1}_X)\}_{n=1}^{\infty}$ converges uniformly to ρ on X.

§**2. Preliminaries.** By a hyperbolic iterated function system we will mean the following. Let X be a compact connected subset of a Euclidean space R^d. There is a countable family of conformal maps $\phi_n : X \to X$, $n \in I$, satisfying the following conditions

(C1) (Open Set Condition) $\phi_n(Int(X)) \cap \phi_m(Int(X)) = \emptyset$ for all $m \neq n$.

(C2) (Uniformly Contracting)$\exists s < 1$ $\forall i \in I$, $||\phi_i'|| \leq s$.

(C3) (Uniform Extension) There is an open connected set $V \supset X$ such that each ϕ_i, $i \in I$ extends to a $C^{1+\epsilon}$ diffeomorphism on V which is conformal on V and maps V into itself.

(C4) (Bounded Distortion Property) $\exists K \geq 1\ \forall n \geq 1\ \forall \omega = (\omega_1, ..., \omega_n) \in I^n\ \forall x, y \in V$, then

$$\frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} \leq K.$$

(C5) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset R^d$ there exists an open cone $\mathrm{Con}(x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.

Throughout the entire paper we will make two additional assumptions. The first assumption is that the system is regular.

(C6) (Regularity) There is a number $\delta \geq 0$ and a δ-conformal probability measure m for the system. This means $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$,

$$m(\phi_i(A)) = \int_A |\phi'_i|^\delta \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \neq j$.

It is shown in [MU] that there is only one conformal measure for the system, that $\delta = \dim_H(J)$ and m is the unique probability measure which is fixed by $\mathcal{L}^*$, the dual of the Perron-Frobenius operator $\mathcal{L} = \mathcal{L}_\delta$ where

$$\mathcal{L}_\delta(f)(x) = \sum_{i \in I} |\phi'_i(x)|^\delta f(\phi_i(x)).$$

We note that this positive operator preserves the space of continuous functions $C(X)$. It is also shown in Theorem 3.8 of [MU] that there exists a unique invariant probability measure μ supported on J equivalent with m. Invariant means for every measurable set A,

$$\mu(\bigcup_{i \in I} \phi_i(A)) = \mu(A).$$

The Radon-Nikodym derivative $\rho = d\mu/dm$ is bounded away from zero and infinity and ρ is a fixed point of the operator $\mathcal{L}$ when considered as an operator on the bounded measurable functions on J. Also, ρ is unique on J up to sets of m measure zero. However, we do not know whether this function ρ is continuous without some additional assumption.

Our second additional assumption can be considered as a strengthening of (BDP):

(C7) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi'_i(y)| - |\phi'_i(x)|\big| \leq L||\phi'_i|||y - x|^\alpha,$$

for every $i \in I$ and every pair of points $x, y \in V$.

In Lemma A.6 of [MU] it is shown that assumption (C7) guarantees us that the operator $\mathcal{L}$ is almost periodic: for every continuous function $f : X \to \mathbb{C}$, the family $\{\mathcal{L}^n(f) : X \to \mathbb{C} :$

$n \geq 1\}$ is equicontinuous. Actually for the results given here we could replace assumption (C7) by the assumption that the operator $\mathcal{L}$ is almost periodic.

§3. Results.

Lemma 1. There exists at most one continuous function $\rho : X \to [0, \infty)$ such that $\mathcal{L}\rho = \rho$ and $\int \rho dm = 1$.

Proof. Suppose that there are two such functions ρ_1 and ρ_2. By Theorem 3.8 from [MU1] $\rho_1|_{\overline{J}} = \rho_2|_{\overline{J}}$ and denote this common restriction by ρ. Fix now $\epsilon > 0$ and consider $\eta > 0$ so small that for each $i = 1, 2$, $|\rho_i(y) - \rho_i(x)| < \epsilon$ if $x, y \in X$ and $|y - x| \leq \eta$. Take an arbitrary $n \geq 1$ so large that $Ds^n \leq \eta$. Finally fix an arbitrary $z \in X$ and consider an $\omega \in I^n$. Then $\mathrm{diam}(\phi_\omega(X)) \leq Ds^n \leq \eta$. Choose $x \in J \cap \phi_\omega(X)$. Then

$$|\rho_2(\phi_\omega(z)) - \rho_1(\phi_\omega(z))| \leq |\rho_2(\phi_\omega(z)) - \rho(x)| + |\rho(x) - \rho_1(\phi_\omega(z))| \leq \epsilon + \epsilon = 2\epsilon.$$

Hence, using (2.15) from [MU], we get

$$\begin{aligned}
|\rho_2(z) - \rho_1(z)| &= |\mathcal{L}^n\rho_2(z) - \mathcal{L}^n\rho_1(z)| = |\mathcal{L}^n(\rho_2 - \rho_1)(z)| \\
&\leq \sum_{|\omega|=n} |\rho_2(\phi_\omega(z)) - \rho_1(\phi_\omega(z))| \cdot |\phi_\omega'(z)|^\delta \\
&\leq \sum_{|\omega|=n} 2\epsilon||\phi_\omega'||^\delta \leq 2K^\delta\epsilon.
\end{aligned}$$

Therefore, letting $\epsilon \to 0$ we conclude that $\rho_2(z) = \rho_1(z)$ and we are done. ■

We want to examine the behaviour of the sequence $\{\mathcal{L}^n(1\!\!1_X)\}_{n=1}^\infty$. We first note the following fact.

Proposition 2. Suppose the system satisfies conditions (C1)-(C7). Then the sequence $\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j(1\!\!1_X)$ converges uniformly on X to a continuous function ρ. Also, $\mathcal{L}\rho = \rho$ and $\int \rho dm = 1$.

Proof. Since the sequence $\{\mathcal{L}^n(1\!\!1_X)\}_{n=1}^\infty$ is uniformly bounded between $K^{-\delta}$ and K^δ and is equicontinuous, the sequence $\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j(1\!\!1_X)$ has the same properties. Let ρ be an accumulation point of this sequence of averages. Then obviously, ρ is continous and $\mathcal{L}\rho = \rho$ and $\int \rho dm = 1$. By Lemma 1, the sequence of averages can have only one accumulation point. So, it converges. ■

Problem We do not know whether Proposition 2 remains true if we only assume (C1)-(C6).

We now turn to the convergence of the sequence $\{\mathcal{L}^n(1\!\!1_X)\}_{n=1}^\infty$. An elementary approach would be to simply show the the limsup and the liminf of this sequence agree on X. This leads to the next proposition which is probably well known. Its proof is short and elementary, so we have decided to present it.

Proposition 3. If $\{g_n : Y \mapsto R\}_{n\geq 1}$ is an equicontinuous family of uniformly bounded functions defined on a compact metric space (Y,d), then the functions $\overline{g}, \underline{g} : Y \mapsto R$ defined respectively as $\limsup_{n\to\infty} g_n(y)$ and $\limsup_{n\to\infty} g_n(y)$ are continuous.

Proof. We will only prove continuity of the function $\overline{g}$. The proof for $\underline{g}$ is analogous. So, consider for every $n \geq 1$ the function

$$s_n = \sup\{g_n, g_{n+1}, \ldots\}.$$

For every $y \in Y$ the sequence $\{s_n(y)\}$ is non-increasing and since

$$\sup_{n\geq 1}\sup_{z\in Y}\{|g_n(z)|\} := T$$

is finite, $\{s_n(y)\}$ is bounded from below by $-T$. Thus the limit

$$\limsup_{n\to\infty} g_n(y) = \lim_{n\mapsto\infty} s_n(y) = \overline{g}(y)$$

exists and lies between $-T$ and T. In order to complete the proof it is therefore enough to prove equicontinuity of the family $\{s_n\}_{n\geq 1}$. To do this, fix $\epsilon > 0$ and take $\delta > 0$ so small that if $d(y,x) < \delta$, then $|g_n(y) - g_n(x)| \leq \epsilon/2$ for all $n \geq 1$. Fix such two x and y, $k \geq 1$, and choose $m \geq k$ such that $0 \leq s_k(y) - g_m(y) \leq \epsilon/2$. Then

$$s_k(x) - s_k(y) \geq g_m(x) - s_k(y) \geq g_m(x) - g_m(y) - \epsilon/2 \geq -\epsilon$$

or equivalently $s_k(y) - s_k(x) \leq \epsilon$. Similarly $s_k(x) - s_k(y) \leq \epsilon$ and therefore $|s_k(y) - s_k(x)| \leq \epsilon$. The proof is complete. ■

Let

$$\overline{\rho} = \limsup_{n\mapsto\infty} \mathcal{L}^n(1).$$

Combining Proposition 3, Lemma A.6 and Lemma 2.2 from [MU] we conclude that the function $\overline{\rho} : X \mapsto [0,\infty)$ is continuous. Also, $\mathcal{L}(\overline{\rho}) \geq \overline{\rho}$. This means that $\overline{\rho}|J$ is a fixed point of $\mathcal{L}|J$. However, it is not clear that this function is fixed on X. But, using now Lemma A.4, Lemma A.6 and Lemma 2.2 from [MU] from we conclude that the function

$$\overline{\rho}_\infty = \lim \mathcal{L}^n \overline{\rho}$$

is continuous and satisfies the equation $\mathcal{L}\overline{\rho}_\infty = \overline{\rho}_\infty$. Since by (2.15) from [MU], $1 \leq \overline{\rho}_\infty \leq K^\delta$, after dividing the function $\overline{\rho}_\infty$ by its integral with respect to the conformal measure m and invoking Lemma 1 we get another argument for the following.

Theorem 4. There exists a unique continuous function $\rho : X \mapsto [0,\infty)$ such that $\mathcal{L}\rho = \rho$ and $\int \rho dm = 1$.

We now want to show that $\mathcal{L}^n(1)$ converges uniformly to ρ on X. The argument follows one given in [DU]. We recall that since the operator $\mathcal{L}$ is almost periodic on $E = C(X)$, considered as the space of complex valued continuous functions on X provided with the uniform norm, we have the following decomposition

$$E = E_0 \bigoplus E_u,$$

where $E_0 = \{f : ||\mathcal{L}^n(f)|| \mapsto 0\}$ and E_u is the closed span of $\{f : \mathcal{L}(f) = \lambda f$ for some$|\lambda| = 1\}$, [L]. We also need that following fact which is not proved here , but the proof follows the argument given in [MU] for Theorem 3.8 with minor modifications. A detailed proof may be found in [HMU].

Lemma 5. For each positive integer r, σ^r is ergodic with resect to μ^* the lift of the measure μ to the symbol space.

Lemma 6. $E_u = \{c\rho : c \in \mathbb{C}\}$.

Proof. Suppose $\mathcal{L}(\psi) = \lambda\psi$ with $|\lambda| = 1\}$. Since $\mathcal{L}$ is a positive operator on the Banach lattice, $C(X)$, it follows from Lemma 18, Theorem 4.9 and Exercise 2 in [S](p. 326/327) that the spectrum of $\mathcal{L}$ meets the unit circle in a cyclic compact group. Therefore, the group is finite and there is some positive integer r such that $\lambda^r = 1$. Thus, $\mathcal{L}^r(\psi) = \psi$ and $\mathcal{L}^r(Re\psi) = Re\psi\ \mathcal{L}^r(Im\psi) = Im\psi$. Let us suppose $Re\psi \neq 0$. Fix $M \in R$ so large that $Re\psi + M\rho > 0$. But, by lemma 5, σ^r is ergodic with respect to μ. This means ρdm is the only invariant measure for σ^r equivalent to m. Therefore, there is a constant c such that $Re\psi + M\rho = c\rho$. So, $Re\psi = (c - M)\rho$ and $\int Re\psi dm = c - M$. Repeating this argument for $Im\psi$, we have $\psi = (\int \psi dm)\rho$. This implies E_u consists of the scalar multiples of ρ. ■

Theorem 7. The sequence $\{\mathcal{L}^n(1_X)\}_{n=1}^{\infty}$ converges uniformly to ρ on X.

Proof. Let $\mathbb{1} = \mathbb{1}_X$. By the decomposition theorem and lemma 4, $\mathbb{1} = \mathbb{1}_u + \mathbb{1}_0 = c\rho + (\mathbb{1} - c\rho)$. But, note that if $f \in E_0$, then $\int f dm = 0$ since the operator $\mathcal{L}$ preserves integration with respect to m. But, this means $c = 1$. Therefore, $1 = \rho + \big(1 - \rho\big)$. So, $||\mathcal{L}^n(\mathbb{1}) - \rho|| = ||\mathcal{L}^n(\mathbb{1} - \rho)|| \to 0$. ■

References

[DU] M. Denker, M. Urbański, Ergodic theory of equilibrium states for rational maps, Nonlinearity 4 (1991), 103-134.

[HMU] P. Hanus, R. D. Mauldin, M. Urbański, Multifractal analysis of equilibrium states in conformal iterated functions systems, in preparation.

[L] V. Ljubich, Entropy properties of rational endomorphisms of the Riemann sphere, Ergod. Th. Dynam. Sys. 5 (1983), 27-43.

[MU] R. D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc. (3) 73 (1996) 105-154.

[S] H. Schaefer, Banach lattices and positive operators. (1974), Springer, New York.