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Thermodynamic Formalism and Multifractal Analysis of Conformal Infinite Iterated Function Systems

by

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Abstract. In this paper we develop the thermodynamic formalism for equilibrium states of strongly Hölder families of functions. These equilibrium states are supported on the limit set generated by iterating a system of infinitely many contractions. The theory of these systems was laid out in an earlier paper of the last two authors. The first five sections of this paper except Section 3 are devoted to developing the thermodynamic formalism for equilibrium states of Hölder families of functions. The first three sections provide us with the tools needed to carry out the multifractal analysis for the equilibrium states mentioned above assuming that the limit set is generated by conformal contractions. The theory of infinite systems of conformal contractions is laid out in [MU1]. The multifractal analysis is then given in Section 7. In Section 8 we apply this theory to some examples from continued fraction systems and Apollonian packing.

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§**1. Introduction, Preliminaries.** The multifractal formalism arose from various considerations in physics and mathematics (see e.g., [Man], [FP], [Gr] and [Ha]). In this last paper a formulation of the secnarios of multifractal theory was elaborated in which there were strong hints of parallels to the theory of statistical physics. Some of the first rigorous mathematical results concerning this formalism are in [CM] and [R]. Since then there have been many papers written verifying some aspects of this formalism. Recently, Pesin presented a general formulation of the setting for multifractal theory [Pe]. Also, many more references concerning this topic may be found in his book. Let us recall the general setting. Let m be a probability measure defined on the Borel subsets of a metric space X. For each number α, let K_α consist of those points x where the measure is scaling pointwise with parameter α or has pointwise dimension α. Let $f(\alpha)$ be the dimension of the set K_α. To say that m has pointwise dimension α at x can be defined in several different ways. One of the simplest is to say that $\lim_{\varepsilon\mapsto 0} \log m(B(x,r))/\log r = \alpha$. However, one may want to use the liminf or limsup instead of the limit(if it exists) or one may wish to take these limits only over some natural filtration of sets based at x. Also, one may use various notions of dimension. In this paper we will be concerned with $dim_H = HD$, Hausdorff dimension. As we have mentioned, the measures for which we carry out the multifractal analysis are equilibrium states for a natural potential function associated with a given infinite iterated function system and an associated family of Hölder continuous weights. This part of the theory, the beginning of the thermodynamic formalism, is given in Sections 2 and 4. In Section 3 we deal with parabolic systems and in Section 4 we compute the Hausdorff dimension of a projected measure by extending the well-known formula of Billingsley. Some parts of the material in the first two sections may also be found in [HU] and in [Ur]. In Section 5 and Section 6, we provide a further development of the thermodynamic formalism by investigating some aspects of the pressure function for potentials and equilibrium measures associated to families of weights which involve two complex parameters. In Section 7, we demonstrate how the multifractal formalism works. We show that for each α there is an auxiliary measure that witnesses the Hausdorff dimension of the set K_α and there is an auxiliary function T or "temperature" function such that the "$f(\alpha)$" curve is the Legendre transform of the temperature function. In Section 7 our considerations show some similarities to those in [PW] (comp. also [Pe] and [PU]). Section 8 contains a detailed analysis of examples coming from the continued fractions algorithm and Apollonian packing constructions.

In [MU1] we have provided the framework to study infinite conformal iterated function systems. We shall recall first this notion and some of its basic properties. Let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \le s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\ge 1} I^n$, the space of finite words, and for $\omega \in I^n$, $n \ge 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i \in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.
An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the Open Set Condition if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.

An iterated function system S satisfying the Open Set Condition is said to be conformal if $X \subset I\!R^d$ for some $d \geq 1$ and the following conditions are satisfied.

(1a) $U = \mathrm{Int}_{I\!R^d}(X)$.

(1b) There exists an open connected set$V$ with $X \subset V \subset I\!R^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V.

(1c) There exist $\gamma, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.

(1d) Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

We note that under these conditions we may exchange the order of the set operations:

$$J = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \phi_\omega(X).$$

In fact throughout the whole paper we will need one more condition which (comp. [MU1]) can be considered as a strengthening of (BDP).

(1e) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi'_i(y)| - |\phi'_i(x)|\big| \leq L||\phi'_i|||y - x|^\alpha.$$

for every $i \in I$ and every pair of points $x, y \in V$. The topological pressure function, $\mathrm{P}(t)$, for a conformal iterated function systems is defined as follows.

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} ||\phi'_\omega||^t$$

As it was shown in [MU1] there are two disjoint classes of conformal iterated function systems, regular and irregular. A system is regular if there exists $t \geq 0$ such that $\mathrm{P}(t) = 0$. Otherwise the system is irregular. Moreover the following property holds.

Theorem 1.1. If S is a conformal iterated function system, then

$$\mathrm{HD}(J) = \sup\{\mathrm{HD}(J_F) :\ F \subset I,\ F \text{ finite}\} = \inf\{t \geq 0 :\ P(t) \leq 0\},$$

where J_F is the limit set associated to the index set F. If a system is regular and $\mathrm{P}(t) = 0$ then $t = \mathrm{HD}(J)$.

§**2. Thermodynamic formalism for iterated function systems.** Let I be a countable alphabet and let $\Sigma = I^\infty$ be the symbolic coding space equipped with the product topology. Let $\sigma : \Sigma \to \Sigma$ be the left shift transformation (cutting out the first coordinate), $\sigma(\{x_n\}_{n=1}^\infty) = (\{x_n\}_{n=2}^\infty)$. Fix $\beta > 0$. In this section $S = \{\phi_i : X \to X : i \in I\}$ is a (hyperbolic) iterated function system and $F = \{f^{(i)} : X \to \mathbb{C} : i \in I\}$ is a family of continuous functions such that if we define for each $n \geq 1$,

$$V_n(F) = \sup_{\omega \in I^n} \sup_{x,y \in X} \{|f^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\} \mathrm{e}^{\beta(n-1)},$$

then the following is satisfied:

$$V_\beta(F) = \sup_{n\geq 1}\{V_n(F)\} < \infty \tag{2.1}$$

The collection F is called then a *Hölder family of functions* (of order β). Denote by $\|\cdot\|_0$ the supremum norm on the Banach space $C(X)$ and by $1\!\!1$ the function with constant value 1 on X. If in addition to (2.1) we have

$$\sum_{i\in I} ||\mathrm{e}^{f^{(i)}}||_0 < \infty \quad \text{or equivalently} \quad \mathcal{L}_F(1\!\!1) \in C(X), \tag{2.2}$$

where

$$\mathcal{L}_F(g)(x) = \sum_{i\in I} \mathrm{e}^{f^{(i)}(x)} g(\phi_i(x)), \quad g \in C(X),$$

is the associated Perron-Frobenius or transfer operator, then the family F is called a *strongly Hölder family of functions* (of order β). In this section we assume that F is a strongly Hölder family of real-valued functions of order β. We have made the conventions that the empty word $\emptyset$ is the only word of length 0 and $\phi_\emptyset = \mathrm{Id}_X$. Thus, $V_1(F) < \infty$ simply means the diameters of the sets $f^i(X)$ are uniformly bounded. Notice that $\mathcal{L}_F$ acts on $C(X)$ as a continuous operator and $||\mathcal{L}_F||_0 \le ||\mathcal{L}_F(1\!\!1)||_0$. Let $\mathcal{L}_F^* : C(X)^* \to C(X)^*$ be the dual operator and define the following map on the space of probability measures on X The map

$$\nu \mapsto \frac{\mathcal{L}_F^*(\nu)}{\mathcal{L}_F^*(\nu)(1\!\!1)}$$

is continuous and therefore in view of the Shauder-Tichonov theorem it has a fixed point, say m_F. Thus

$$\mathcal{L}_F^*(m_F) = \lambda m_F, \tag{2.3}$$

where $\lambda = \mathcal{L}_F^*(m_F)(1\!\!1)$. Following the classical thermodynamic formalism, we define the topological pressure of F by setting

$$\begin{aligned}\mathrm{P}(F) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \left\| \exp\left(\sum_{j=1}^n f^{\omega_j} \circ \phi_{\sigma^j\omega} \right) \right\|_0 \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\left(\sup_X \sum_{j=1}^n f^{\omega_j} \circ \phi_{\sigma^j\omega} \right).\end{aligned}$$

Notice also that the limit indeed exists since the logarithm of the partition function

$$Z_n(F) = \sum_{|\omega|=n} \left\| \exp\left(\sum_{j=1}^n f^{\omega_j} \circ \phi_{\sigma^j\omega} \right) \right\|_0$$

is subadditive. Moreover,

$$\mathrm{P}(F) = \inf_{n\ge 1} \left\{ \frac{1}{n} \log Z_n(F) \right\}.$$

Remark 2.1. With these definitions in mind we see that our development in [MU1] of infinite conformal iterated function systems is in fact a consideration of the Hölder systems of functions $tG = \{\log|\phi_i'|^t\}_{i\in I}$, $t \ge 0$. Notice $Z_n(tG) = \sum_{|\omega|=n} ||\phi_\omega'||^t$. In particular the pressure function introduced in [MU1] (see also Section 1) is the same as the pressure $\mathrm{P}(tG)$. We may refer to G as the natural volume potential.

Given $n \ge 1$ and $\omega \in I^n$ denote $\sum_{j=1}^n f^{(\omega_j)} \circ \phi_{\sigma^j\omega}$ by $S_\omega(F)$. Let us now show that $diam(S_\omega F(X))$ decreases to 0 at a uniform exponential rate.

Lemma 2.2. For each $\tau, \omega \in I^*$,

$$diam S_\omega F(\phi^\tau(X)) = \sup_{x,y\in\phi_\tau(X)} |S_\omega(F)(x) - S_\omega(F)(y)| \le \frac{V(F)\mathrm{e}^\beta}{1-\mathrm{e}^{-\beta}} \mathrm{e}^{-\beta|\tau|}.$$

Proof. Let $n = |\omega|$. Write $x = \phi_\tau(u)$, $y = \phi_\tau(w)$, where $u, w \in X$. By (2.1) we get

$$\begin{aligned}
\left|\sum_{j=1}^{n} f^{(\omega_j)}(\phi_{\sigma^j\omega}(x)) - \sum_{j=1}^{n} f^{(\omega_j)}(\phi_{\sigma^j\omega}(y))\right| &= \left|\sum_{j=1}^{n} f^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(u) - \sum_{j=1}^{n} f^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(w)\right| \\
&\le \sum_{j=1}^{n} \left| f^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(u) - f^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(w)\right| \\
&\le \sum_{j=1}^{n} V(F)\mathrm{e}^{-\beta(n+|\tau|-j-1)} \\
&\le \frac{V(F)\mathrm{e}^{\beta}}{1-\mathrm{e}^{-\beta}}\mathrm{e}^{-\beta|\tau|}
\end{aligned}$$

The proof is finished. ■

Set

$$Q = \exp\left(\frac{V(F)\mathrm{e}^{\beta}}{1-\mathrm{e}^{-\beta}}\right).$$

In Section 8 we will need a continuity property of topological pressure considered as a function on the space of strongly Hölder families of functions. This property is also interesting itself. So, we introduce a suitable topology in the space of Hölder families of functions by declaring that a sequence $\{F_n\}_{n=1}^{\infty}$ of Hölder families of functions on X converges to a Hölder family of functions F on X if $f_n^{(i)}$ converges uniformly to $f^{(i)}$ for every $i \in I$. We shall prove the following.

Proposition 2.3. The topological pressure defined on the space of all Hölder families of functions is lower semi-continuous.

Proof. Consider F, a Hölder family of functions, and $\{F_n\}_{n=1}^{\infty}$, a sequence of Hölder families of functions on X converging to F. Fix $k \ge 1$ and $\varepsilon > 0$. Suppose that $\mathrm{P}(F) < \infty$ (the proof is similar for the opposite case). There exists a finite set $E \subset I$ such that

$$\frac{1}{k}\log \sum_{\omega \in E^k} \exp(\sup S_\omega(F)) \ge \frac{1}{k}\log Z_k(F) - \varepsilon \ge \mathrm{P}(F) - \varepsilon.$$

Since $\lim_{n\to\infty} F_n = F$ and E is finite, for every n large enough, say $n \ge n_0$, we have

$$\begin{aligned}
\frac{1}{k}\log Z_k(F_n) &\ge \frac{1}{k}\log \sum_{\omega \in E^k} \exp(\sup S_\omega(F_n)) \\
&\ge \frac{1}{k}\log\left(\mathrm{e}^{-\varepsilon k} \sum_{\omega \in E^k} \exp(\sup S_\omega(F))\right) \ge \mathrm{P}(F) - 2\varepsilon.
\end{aligned}$$

Using Lemma 2.2 we therefore obtain for every $n \geq n_0$

$$\begin{aligned}\mathrm{P}(F_n) &= \lim_{p\to\infty} \frac{1}{pk} \log Z_{pk}(F_n) \geq \lim_{p\to\infty} \frac{1}{pk} \left(\log(Q^{-p} Z_k(F_n)^p\right) \\ &= \frac{-\log Q}{k} + \frac{1}{k}\log Z_k(F_n) \geq \frac{-\log Q}{k} + \mathrm{P}(F) - 2\varepsilon.\end{aligned}$$

Letting thus $k \nearrow \infty$ and $\varepsilon \searrow 0$, we finally get $\lim_{n\to\infty} \mathrm{P}(F_n) \geq \mathrm{P}(F)$. The proof is complete. ■

The orbit sums $S_\omega F$ yield a simple expression for $\mathcal{L}_F^n$ by a straightforward induction. For every $n \geq 1$,

$$\mathcal{L}_F^n(g) = \sum_{|\omega|=n} \exp S_\omega(F) \cdot (g \circ \phi_\omega).$$

We shall prove the following.

Lemma 2.4. The eigenvalue λ (see 2.3) of the dual Perron-Frobenius operator is equal to $\mathrm{e}^{\mathrm{P}(F)}$.

Proof. Iterating (2.3) we get

$$\begin{aligned}\lambda^n &= \lambda^n m_F(1\!\!1) = \mathcal{L}_F^{*n}(1\!\!1) = \int_X \mathcal{L}_F^n(1\!\!1) dm_F \\ &= \int_X \sum_{|\omega|=n} \exp(S_\omega(F)(x)) \leq \sum_{|\omega|=n} || \exp(S_\omega(F))||_0.\end{aligned}$$

So,

$$\log\lambda \leq \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} || \exp(S_\omega(F))||_0 = \mathrm{P}(F).$$

Fix now $\omega \in I^n$ and take a point x_ω where the function $S_\omega(F)$ takes on its maximum. In view of Lemma 2.2, for every $x \in X$ we have

$$\sum_{|\omega|=n} \exp(S_\omega(F)(x)) \geq Q^{-1} \sum_{|\omega|=n} \exp(S_\omega(F)(x_\omega)) = Q^{-1} \sum_{|\omega|=n} || \exp(S_\omega(F))||_0.$$

Hence, iterating (2.3) as before,

$$\lambda^n = \int_X \sum_{|\omega|=n} \exp(S_\omega(F)) dm_F \geq Q^{-1} \sum_{|\omega|=n} || \exp(S_\omega(F))||_0.$$

So, $\log\lambda \geq \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} || \exp(S_\omega(F))||_0 = \mathrm{P}(F)$. The proof is finished. ■

Let $\mathcal{L}_0$ denote the normalized Perron-Frobenius operator, i.e. $\mathcal{L}_0 = \mathrm{e}^{-\mathrm{P}(F)} \mathcal{L}_F$. We shall prove the following.

Proposition 2.5. $m_F(J) = 1$.

Proof. According to (2.3)

$$\mathcal{L}_0^*(m_F) = m_F \tag{2.4}$$

and consequently $\mathcal{L}_0^{*n}(m_F) = m_F$ for all $n \geq 0$. We have

$$\int_X \mathcal{L}_0^n g dm_F = \int_X \sum_{|\omega|=n} \exp\big(S_\omega(F) - \mathrm{P}(F)n\big) \cdot (g \circ \phi_\omega) dm_F = \int_X g dm_F \tag{2.5}$$

for all $n \geq 0$ and all continuous functions $g : X \to I\!\!R$. Since this equality extends to all bounded measurable functions g, we find for all $n \geq 0$, all $\omega \in I^n$, and all Borel sets $A \subset X$
(2.6)
$$m_F(\phi_\omega(A)) = \sum_{\tau \in I^n} \int \exp\big(S_\tau(F) - \mathrm{P}(F)n\big) \cdot 1\!\!1_{\phi_\omega(A)} \circ \phi_\tau dm_F \geq \int_A \exp\big(S_\omega(F) - \mathrm{P}(F)n\big) dm_F.$$

Now, for each $n \geq 1$, set $X_n = \bigcup_{|\omega|=n} \phi_\omega(X)$. Then $1\!\!1_{X_n} \circ \phi_\omega = 1\!\!1$ for all $\omega \in I^n$. Thus, applying (2.5) to the function $g = 1\!\!1_{X_n}$ and later to the function $g = 1\!\!1$, we obtain

$$\begin{aligned} m_F(X_n) &= \int_X \sum_{|\omega|=n} \exp\big(S_\omega(F) - \mathrm{P}(F)n\big) \cdot (1\!\!1_{X_n} \circ \phi_\omega) dm_F \\ &= \int_X \sum_{|\omega|=n} \exp\big(S_\omega(F) - \mathrm{P}(F)n\big) dm_F = \int 1\!\!1 dm_F = 1. \end{aligned}$$

Hence $m_F(J) = m_F\big(\bigcap_{n \geq 1} X_n\big) = 1$. The proof is complete. ■

Theorem 2.6. For all $n \geq 1$
$$Q^{-1} \leq \mathcal{L}_0^n(1\!\!1) \leq Q.$$

Proof. Given $n \geq 1$ by (2.5) there exists $x_n \in X$ such that $\mathcal{L}_0^n(1\!\!1)(x_n) \leq 1$. It then follows from Lemma 2.2 that for every $x \in X$, $\mathcal{L}_0^n(1\!\!1) \leq Q$. Similarly, by (2.5) there exists $y_n \in X$ such that $\mathcal{L}_0^n(1\!\!1) \geq 1$. It then follows from Lemma 2.2 that for every $x \in X$, $\mathcal{L}_0^n(1\!\!1) \geq Q^{-1}$. The proof is finished. ■

If $\omega \in I^*$ set $[\omega] = \{\tau \in I^\infty : \tau|_{|\omega|} = \omega\}$. We shall prove the following.

Lemma 2.7. There exists a unique Borel probability measure $\tilde{m}_F$ on I^∞ such that $\tilde{m}_F([\omega]) = \int \exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big) dm_F$ for all $\omega \in I^*$.

Proof. In view of (2.5), $\sum_{|\omega|=n} \int \exp\big(S_\omega(F) - \mathrm{P}(F)n\big) dm_F = 1$ for all $n \geq 1$ and therefore one can define a Borel probability measure m_n on C_n, the algebra generated by the cylinder sets of the form $[\omega]$, $\omega \in I^n$, by putting $m_n([\omega]) = \int \exp\big(S_\omega(F) - \mathrm{P}(F)n\big) dm_F$. Hence, applying (2.5) again we get for all $\omega \in I^n$.

$$\begin{aligned} m_{n+1}(\omega) &= \sum_{i\in I} m_{n+1}([\omega i]) = \sum_{i\in I}\int \exp\bigl(S_{\omega i}(F) - \mathrm{P}(F)(n+1)\bigr)dm_F \\ &= \int \sum_{i\in I} \exp\left(\sum_{j=1}^{n} f^{(\omega_j)}\circ\phi_{\sigma^j(\omega i)} - \mathrm{P}(F)n + f^{(i)} - \mathrm{P}(F)\right) dm_F \\ &= \int \sum_{i\in I} \exp\bigl(S_\omega(F)\circ\phi_i - \mathrm{P}(F)n\bigr)\exp\bigl(f^{(i)} - \mathrm{P}(F)\bigr)dm_F \\ &= \int \mathcal{L}_0\bigl(\exp(S_\omega(F) - \mathrm{P}(F)n)\bigr)dm_F = \int \exp\bigl(S_\omega(F) - \mathrm{P}(F)n\bigr)dm_F = m_n([\omega]) \end{aligned}$$

and therefore, in view of Kolmogorov's extension theorem, there exists a unique probability measure $\tilde m_F$ on I^∞ such that $\tilde m_F([\omega]) = \tilde m_{|\omega|}([\omega])$ for all $\omega \in I^*$. The proof is complete. ■

As an immediate consequence of this lemma we see that if R is a collection of incomparable words such that $\bigcup_{\omega\in R}[\omega] = I^\infty$, then we have
(2.7)

$$1 \le \sum_{\omega\in R} ||\exp\bigl(S_\omega(F) - \mathrm{P}(F)|\omega|\bigr)||_0 \le Q \text{ and } Q^{-1} \le \sum_{\omega\in R}\inf_X \exp\bigl(S_\omega(F) - \mathrm{P}(F)|\omega|\bigr) \le 1.$$

Lemma 2.8. The measures m_F and $\tilde m_F\circ\pi^{-1}$ are equal.

Proof. Let $A\subset J$ be an arbitrary closed subset of J and for every $n\ge 1$ let $A_n = \{\omega \in I^n : \phi_\omega(X)\cap A \ne \emptyset\}$. In view of (2.5) applied to the characteristic function $1\!\!1_A$ we have for all $n \ge 1$

$$\begin{aligned} m_F(A) &= \sum_{\omega\in I^n}\int \exp\bigl(S_\omega(F) - \mathrm{P}(F)|\omega|\bigr)(1\!\!1_A\circ\phi_\omega)\, dm_F \\ &= \sum_{\omega\in A_n}\int \exp\bigl(S_\omega(F) - \mathrm{P}(F)|\omega|\bigr)(1\!\!1_A\circ\phi_\omega)\, dm_F \\ &\le \sum_{\omega\in A_n}\int \exp\bigl(S_\omega(F) - \mathrm{P}(F)|\omega|\bigr)\, dm_F = \sum_{\omega\in A_n}\tilde m_F([\omega]) = \tilde m_F\bigl(\bigcup_{\omega\in A_n}[\omega]\bigr) \end{aligned}$$

Since the family of sets $\{\bigcup_{\omega\in A_n}[\omega] : n\ge 1\}$ is descending and $\bigcap_{n\ge1}\bigcup_{\omega\in A_n}[\omega] = \pi^{-1}(A)$, we therefore get $m_F(A) \le \lim_{n\to\infty}\tilde m_F\bigl(\bigcup_{\omega\in A_n}[\omega]\bigr) = m_F(\pi^{-1}(A))$. Since both measures m_F and $\tilde m_F\circ\pi^{-1}$ are regular (as J is a separable metric space), this inequality extends to the family of all Borel subsets of J. Since both measures are probabilistic we get $m_F = \tilde m_F\circ\pi^{-1}$. The proof is finished. ■

Recall that $\sigma : I^\infty \to I^\infty$ denotes the left shift map (cutting out the first coordinate) on the coding space I^∞. Also recall that a measure preserving endomorphism is said to be completely ergodic if and only if all its (positive) iterates are ergodic. Now we shall prove that the shift map $\sigma : I^\infty \to I^\infty$ has a unique invariant (completely ergodic) probability measure equivalent with $\tilde{m}_F$.

Theorem 2.9. There exists a unique σ-invariant probability measure $\tilde{\mu}_F$ absolutely continuous with respect to $\tilde{m}_F$. Moreover, $\tilde{\mu}_F$ is equivalent with $\tilde{m}_F$, $Q^{-1} \le d\tilde{\mu}_F/d\tilde{m}_F \le Q$ and the dynamical system $\sigma : I^\infty \to I^\infty$ is completely ergodic with respect to the measure $\tilde{\mu}_F$.

Proof. The proof follows the argument given in [HU]. First notice that, using (2.6) and Lemma 2.2, for each $\omega \in I^*$ and each $n \ge 0$ we get

$$\begin{aligned}
\tilde{m}_F(\sigma^{-n}([\omega])) &= \sum_{\tau\in I^n} \tilde{m}_F([\tau\omega]) = \sum_{\tau\in I^n} \int |\exp\big(S_{\tau\omega}(F) - \mathrm{P}(F)|\tau\omega|\big)\, dm_F \\
&\ge \sum_{\tau\in I^n} Q^{-1}||\exp\big(S_\tau(F) - \mathrm{P}(F)|\tau|\big)||_0 \int \exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big)\, dm_F \\
&= Q^{-1}\int \exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big)\, dm_F \sum_{\tau\in I^n} ||\exp\big(S_\tau(F) - \mathrm{P}(F)|\tau|\big)||_0 \\
&\ge Q^{-1}\tilde{m}_F([\omega])\tilde{m}_F(I^\infty) = Q^{-1}\tilde{m}_F([\omega])
\end{aligned}$$

and

$$\begin{aligned}
\tilde{m}_F(\sigma^{-n}([\omega])) &= \sum_{\tau\in I^n} \tilde{m}_F([\tau\omega]) = \sum_{\tau\in I^n} \int \exp\big(S_{\tau\omega}(F) - \mathrm{P}(F)|\tau\omega|\big)\, dm_F \\
&\le \sum_{\tau\in I^n} ||\exp\big(S_\tau(F) - \mathrm{P}(F)|\tau|\big)||_0 \int \exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big)\, dm_F \\
&= \int \exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big)\, dm \sum_{\tau\in I^n} ||\exp\big(S_\tau(F) - \mathrm{P}(F)|\tau|\big)||_0 \\
&\le Q\tilde{m}_F([\omega]).
\end{aligned}$$

Let now L be a Banach limit defined on the Banach space of all bounded sequences of real numbers. For the definition and basic properties of Banach limits see for ex. the book [Co] by conway. We define $\mu([\omega]) = L\big((\tilde{m}_F(\sigma^{-n}([\omega])))_{n\ge 0}\big)$. Hence $Q^{-1}\tilde{m}_F([\omega]) \le \mu([\omega]) \le Q\tilde{m}_F([\omega])$ and therefore it is not difficult to check that the formula $\mu(A) = L\big((\tilde{m}_F(\sigma^{-n}(A)))_{n\ge 0}\big)$ defines a finite non-zero finitely additive measure on Borel sets of I^∞ satisfying $Q^{-1}\tilde{m}_F(A) \le \mu(A) \le Q\tilde{m}_F(A)$. Using now a theorem of Calderon (Theorem 4.13 of [Fr]) and its proof one constructs a Borel probability (σ-additive) measure $\tilde{\mu}_F$ on I^∞ satisfying the formula

$$Q^{-1}\tilde{m}_F(A) \le \tilde{\mu}_F(A) \le Q\tilde{m}_F(A)$$

with, perhaps, a larger constant Q. Thus, to complete the proof of our theorem we only need to show complete ergodicity of $\tilde{\mu}_F$ or equivalently of $\tilde{m}_F$. Toward this end take a

Borel set $A \in I^\infty$ with $\tilde m_F(A) > 0$. Using Lemma 2.7 and Lemma 2.2 it is straightforward to check that for every $\omega \in I^*$, $\tilde m_F(\omega A) \ge Q^{-1}||\exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big)||_0\tilde m_F(A) > 0$. Hence, since the nested family of sets $\{[\tau] : \tau \in I^*\}$ generates the Borel σ-algebra on I^∞, for every $n \ge 0$ and every $\omega \in I^n$ we can find a subfamily Z of I^* consisting of mutually incomparable words and such that $A \subset \bigcup\{[\tau] : \tau \in Z\}$ and $\sum_{\tau\in Z}\tilde m_F([\omega\tau]) \le 2\tilde m_F(\omega A)$, where $\omega A = \{\omega\rho : \rho \in A\}$. Then

$$
\begin{aligned}
\tilde m_F\big(\sigma^{-n}(A)\cap[\omega]\big) = \tilde m_F(\omega A) &\ge \frac12\sum_{\tau\in Z}\tilde m_F([\omega\tau]) = \frac12\sum_{\tau\in Z}\int|\exp\big(S_{\omega\tau}(F) - \mathrm{P}(F)|\omega\tau|\big)\,dm_F\\
&\ge \frac12 Q^{-1}||\exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big)||_0\sum_{\tau\in Z}\int|\exp\big(S_\tau(F) - \mathrm{P}(F)|\tau|\big)\,dm_F\\
&\ge \frac12 Q^{-1}\int\exp\big(S_\omega(F) - \mathrm{P}(F)|\omega|\big)\,dm_F\sum_{\tau\in Z}\tilde m_F([\tau])\\
&\ge \frac12 Q^{-1}\tilde m_F([\omega])\tilde m_F\big((\bigcup\{[\tau] : \tau \in Z\}\big) \ge \frac12 Q^{-1}\tilde m_F(A)\tilde m_F([\omega]).
\end{aligned}
$$

Therefore $\tilde m_F\big(\sigma^{-n}(I^\infty\backslash B)\cap[\omega]\big) = \tilde m_F\big([\omega]\backslash\sigma^{-n}(B)\cap[\omega]\big) = \tilde m_F([\omega]) - \tilde m_F\big(\sigma^{-n}(B)\cap[\omega]\big) \le \big(1-(2Q)^{-1}\tilde m_F(B)\big)\tilde m_F([\omega])$. Hence for every Borel set $B \subset I^\infty$ with $\tilde m_F(B) < 1$, for every $n \ge 0$, and for every $\omega \in I^n$ we get

$$
\tilde m_F(\sigma^{-n}(B)\cap[\omega]\big) \le \big(1-(2Q)^{-1}(1-\tilde m_F(B))\big)\tilde m_F([\omega]). \tag{2.8}
$$

In order to conclude the proof of the complete ergodicity of σ fix $r \ge 1$, and suppose that $\sigma^{-1}(B) = B$ with $0 < \tilde m_F(B) < 1$. Put $\gamma = 1-(2Q)^{-1}(1-\tilde m_F(B))$. Note that $0 < \gamma < 1$. In view of (2.8), for every $\omega \in (I^r)^*$ we get $\tilde m_F(B\cap[\omega]) = \tilde m_F(\sigma^{-|\omega|}(B)\cap[\omega]\big) \le \gamma\tilde m_F([\omega])$. Take now $\eta > 1$ so small that $\gamma\eta < 1$ and choose a subfamily R of $(I^r)^*$ consisting of mutually incomparable words and such that $B \subset \bigcup\{[\omega] : \omega \in R\}$ and $\tilde m_F\big(\bigcup\{[\omega] : \omega \in R\}\big) \le \eta\tilde m_F(B)$. Then $\tilde m_F(B) \le \sum_{\omega\in R}\tilde m_F(B\cap[\omega]) \le \sum_{\omega\in R}\gamma\tilde m_F([\omega]) = \gamma\tilde m_F\big(\bigcup\{[\omega] : \omega \in R\}\big) \le \gamma\eta\tilde m_F(B) < \tilde m_F(A)$. This contradiction finishes the proof. ■

Remark. Using the results about the Perron-Frobenius operator proven in Section 6, one can demonstrate similarly as in [Bo] that the dynamical system $(\sigma, \tilde\mu_F)$ is weakly-Bernoulli and the weak-Bernoulli generator is provided by the partition into initial cylinders of length 1. This property implies all kinds of mixing. For further stochstic features of the dynamical system $(\sigma, \tilde\mu_F)$ the reader may consult [Ur].

Theorem 2.10. m_F is the only probability measure m satisfying $\mathcal{L}_0^*(m) = m$.

Proof. Since m_F satisfies this equality we are only left to prove its uniqueness. So, let m_1 be another such a measure and let $\tilde m_1$ be the probability measure produced in Lemma 2.7 applied to the measure m_1. Then for every $\omega \in I^*$ we have $Q^{-1} \le \tilde m_1([\omega])/\tilde m_F([\omega]) \le Q$, whence $\tilde m_1$ and $\tilde m_F$ are equivalent and the Radon-Nikodym derivative ρ satisfies $Q^{-1} \le$

$\rho \le Q$. We also have $\tilde m_F([\sigma(\omega)]) = \int \exp\bigl(S_{\sigma(\omega)}(F) - \mathrm{P}(F)|\sigma(\omega)|\bigr)\, dm_F$ and

$$
\begin{aligned}
\tilde m_F([\omega]) &= \int \exp\bigl(S_\omega(F) - \mathrm{P}(F)|\omega|\bigr)\, dm_F \\
&= \int \exp\bigl(f^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - \mathrm{P}(F)\bigr) \exp\bigl(S_{\sigma(\omega)}(F)(x) - \mathrm{P}(F)|\sigma(\omega)|\bigr)\, dm_F(x)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\inf\{\exp\bigl(f^{(\omega_1)}(x) - \mathrm{P}(F)\bigr) : x \in \phi_{\sigma(\omega)}(X)\}\tilde m_F([\sigma(\omega)]) &\le \tilde m_F([\omega]) \\
&\le \sup\{\exp\bigl(f^{(\omega_1)}(x) - \mathrm{P}(F)\bigr) : x \in \phi_{\sigma(\omega)}(X)\}\tilde m_F([\sigma(\omega)]).
\end{aligned}
$$

Since $f^{(\omega_1)}$ is a continuous function on X we thus obtain that for every $\omega \in I^\infty$

$$
\lim_{n\to\infty} \frac{\tilde m_F\bigl([\omega|_n]\bigr)}{\tilde m_F\bigl([\sigma(\omega)|_{n-1}]\bigr)} = \exp\bigl(f^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(F)\bigr) \tag{2.9}
$$

and the same formula is true with $\tilde m_F$ replaced by $\tilde m_1$. In view of Theorem 2.9 there exists a set of points $\omega \in I^\infty$ with $\tilde m_F$ measure 1 for which the Radon-Nikodym derivatives $\rho(\omega)$ and $\rho(\sigma(\omega))$ both are defined. Let $\omega \in I^\infty$ be such a point. Then using (2.9) and its version for $\tilde m_1$ we obtain

$$
\begin{aligned}
\rho(\omega) = \lim_{n\to\infty}\left(\frac{\tilde m_1\bigl([\omega|_n]\bigr)}{\tilde m_F\bigl([\omega|_n]\bigr)}\right) &= \lim_{n\to\infty}\left(\frac{\tilde m_1\bigl([\omega|_n]\bigr)}{\tilde m_1\bigl([\sigma(\omega)|_{n-1}]\bigr)} \cdot \frac{\tilde m_1\bigl([\sigma(\omega)|_{n-1}]\bigr)}{\tilde m_F\bigl([\sigma(\omega)|_{n-1}]\bigr)} \cdot \frac{\tilde m_F\bigl([\sigma(\omega)|_{n-1}]\bigr)}{\tilde m_F\bigl([\omega|_n]\bigr)}\right) \\
&= \exp\bigl(f^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(F)\bigr)\rho(\sigma(\omega)) \exp\bigl(f^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(F) = \rho(\sigma(\omega))
\end{aligned}
$$

But since, in view of Theorem 2.9, σ is ergodic with respect to $\tilde m_F$, we conclude that ρ is $\tilde m_F$-almost everywhere constant. Since $\tilde m_1$ and $\tilde m_F$ are both probabilistic, $\tilde m_1 = \tilde m_F$. So, an application of Lemma 2.8 finishes the proof. ■

A Borel probability measure m is said to be F-conformal provided it is supported on J, for every Borel set $A \subset X$

$$
m(\phi_\omega(A)) = \int_A \exp\bigl(S_\omega(F) - \mathrm{P}(F)|\omega|\bigr)\, dm, \quad \forall \omega \in I^* \tag{2.10}
$$

and

$$
m(\phi_\omega(X) \cap \phi_\tau(X)) = 0 \tag{2.11}
$$

for all incomparable $\omega, \tau \in I^*$. A simple inductive argument shows that instead of (2.10) and (2.11) it is enough to require that for every Borel set $A \subset X$

$$
m(\phi_i(A)) = \int_A \exp\bigl(S_\omega(F) - \mathrm{P}(F)\bigr)\, dm, \quad \forall i \in I \tag{2.10'}
$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0 \tag{2.11'}$$

for all $i, j \in I$, $i \neq j$. A straightforward calculation shows that each F-conformal measure is a fixed point of the normalized dual operator $\mathcal{L}_0^*$. We shall now provide some sufficient conditions for the existence (and uniqueness) of F-conformal measures. In fact we shall show that every measure satisfying one of these conditions and fulfilling slightly weaker requirements than being a fixed point of the dual operator $\mathcal{L}_0^*$ is F-conformal. Our first condition comes from the following definition. Namely, we say that an iterated function system $\{\phi_i : i \in I\}$ satisfies the strong separation condition if $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i, j \in I$, $i \neq j$. Our second condition is just conformality.

Lemma 2.11. Suppose that the iterated function system $\{\phi_i : i \in I\}$ satisfies the strong separation condition or it is conformal. Then Borel probability measure ν on X is F-conformal if and only if

$$\nu(\phi_\omega(A)) \geq \int_A \exp\big(S_\omega(F) - \mathrm{P}(F)\big)\, d\nu \tag{2.12}$$

for all $\omega \in I^*$ and for all Borel subsets A of X.

Proof. That an F-conformal measure satisfies the requirements appearing in this lemma follows from its definition and Proposition 2.5. In order to prove the harder part, first we shall show that condition (2.11) is satisfied, then that $\nu(J) = 1$, and finally that (2.10) holds. If the system satisfies the strong separation condition, then (2.11) is immediate. So, assume that it is conformal and suppose on the contrary that $\nu(\phi_\rho(X) \cap \phi_\tau(X)) > 0$ for some two incomparable words $\rho, \tau \in I^*$. We may assume without loosing generality that ρ and τ are of the same length, say $q \geq 1$. Let $E = \phi_\rho(X) \cap \phi_\tau(X)$ and for every $n \geq 1$, let $E_n = \bigcup_{\omega \in I^n} \phi_\omega(E)$. Since each element of E_n admits at least two different codes of length $n + q$ which agree on the initial segment of length n, it follows from Lemma 2.6 of [MU1] that $\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n = \emptyset$. On the other hand, by (2.12) and Theorem 2.6, we get $\nu(E_n) \geq Q^{-2}\nu(E)$, thus $\nu\big(\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n\big) \geq Q^{-2}\nu(E) > 0$. This contradiction shows that

$$\nu(\phi_\rho(X) \cap \phi_\tau(X)) = 0 \tag{2.13}$$

for all incomparable words $\rho, \tau \in I^*$. From now on the proof runs simultaneously for conformal systems and those satisfying the strong separation condition. In order to show that $\nu(J) = 1$ suppose to the contrary that $\nu(X \setminus J) > 0$. In view of (2.13) for all $\omega \in I^*$ we have $\nu(\phi_\omega(X \setminus J) \cap J) = \nu\big(\bigcup_{\tau \in I^{|\omega|}} \phi_\omega(X \setminus J) \cap \phi_\tau(J)\big) \leq \sum_{\tau \in I^{|\omega|}} \nu(\phi_\omega(X \setminus J) \cap \phi_\tau(J)) = 0$. Hence setting $E_n = \bigcup_{\omega \in I^n} \phi_\omega(X \setminus J)$ we get $\nu\big(J \cap \bigcup_{n \geq 1} E_n\big) = 0$. On the other hand, by (2.12) and Theorem 2.6, $\nu(E_n) \geq Q^{-2}\nu(X \setminus J)$ and therefore $\nu\big(\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n\big) \geq Q^{-2}\nu(X \setminus J) > 0$. Moreover

$$\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n \subset \bigcap_{k=1}^\infty \Big(\bigcup_{n=k}^\infty \bigcup_{\omega \in I^n} \phi_\omega(X)\Big) = \bigcap_{k=1}^\infty \bigcup_{\omega \in I^k} \phi_\omega(X) = J.$$

Combining the formulae occuring at the ends of the last three sentences we fall into a contradiction which proves that $\nu(J)=1$.
Now we need and we are in position to prove, that the measure m_F is F-conformal. Indeed, m_F satisfies (2.12). Moreover, using (2.13), (2.4), and Lemma 2.7, given an integer $n\geq 1$, we can write $1=m_F(X)=m_F\big(\bigcup_{\omega\in I^n}\phi_\omega(X)\big)=\sum_{\omega\in I^n}m_F(\phi_\omega(X))\geq \sum_{\omega\in I^n}\int\exp\big(S_\omega(F)-\mathrm{P}(F)|\omega|\big)\,dm_F=1$. Therefore, $m_F(\phi_\omega(X))=\int\exp\big(S_\omega(F)-\mathrm{P}(F)|\omega|\big)\,dm_F$ for all $\omega\in I^n$. Fixing now $\omega\in E^*$, define two finite measures m_1 and m_2 on X in the following way: $m_1(A)=\int_A\exp\big(S_\omega(F)-\mathrm{P}(F)|\omega|\big)\,dm_F$ and $m_2(A)=m_F(\phi_\omega(A))$. Since we know that $m_1(X)=m_2(X)$ and $m_1(A)\leq m_2(A)$ for all Borel sets A, we conclude that $m_1=m_2$. Hence, conformality of m_F is proved.
Let us now return to the measure ν. We shall show that m_F is absolutely continuous with respect ν. Indeed, it follows from F-conformality of m_F and Lemma 2.2 that $Q^{-1}||\exp\big(S_\omega(F)-\mathrm{P}(F)|\omega|\big)||_0\leq m_F(\phi_\omega(X))\leq ||\exp\big(S_\omega(F)-\mathrm{P}(F)|\omega|\big)||_0$ for all $\omega\in I^*$. Since, by the assumptions, $\nu(\phi_\omega(X))\geq Q^{-1}\exp\big(S_\omega(F)-\mathrm{P}(F)|\omega|\big)$, we therefore obtain $m_F(\phi_\omega(X))\leq Q\nu(\phi_\omega(X))$. So, using (2.13), we conclude that m_F is absolutely continuous with respect to ν and $\rho=dm_F/d\nu\leq Q$ ν-a.e. Repeating essentially the argument from the proof of Theorem 2.10 to show that ρ is almost everywhere constant, we proceed as follows. In view of Lemma 2.8 and Theorem 2.9 there exists a set of points $\omega\in I^\infty$ with $\tilde m_F$ measure 1 for which the Radon-Nikodym derivatives $\rho\circ\pi(\omega)$ and $\rho\circ\pi(\sigma(\omega))$ both are defined. Let $\omega\in I^\infty$ be such a point. Then

$$
\begin{aligned}
\rho\circ\pi(\omega)&=\lim_{n\to\infty}\left(\frac{m_F(\phi_{\omega|_n}(X))}{\nu(\phi_{\omega|_n}(X))}\right)\\
&=\lim_{n\to\infty}\left(\frac{m_F(\phi_{\omega|_n}(X))}{m_F(\phi_{\sigma(\omega)|_{n-1}}(X))}\cdot\frac{m_F(\phi_{\sigma(\omega)|_{n-1}}(X))}{\nu(\phi_{\sigma(\omega)|_{n-1}}(X))}\cdot\frac{\nu(\phi_{\sigma(\omega)|_{n-1}}(X))}{\nu(\phi_{\omega|_n}(X))}\right)\\
&\leq\lim_{n\to\infty}\left(\frac{\int_{\phi_{\sigma(\omega)|_{n-1}}(X)}\exp(f^{(\omega_1)})\,dm_F}{m_F(\phi_{\sigma(\omega)|_{n-1}}(X))}\right)\cdot\rho(\pi(\sigma(\omega)))\cdot\\
&\quad\cdot\lim_{n\to\infty}\frac{\nu\big(\phi_{\sigma(\omega)|_{n-1}}(X)\big)}{\int_{\phi_{\sigma(\omega)|_{n-1}}(X)}\exp(f^{(\omega_1)}(x))\,d\nu(x)}\\
&=\exp(f^{(\omega_1)}(\pi(\sigma(\omega))\rho(\pi(\sigma(\omega)))\big(\exp(f^{(\omega_1)}(\pi(\sigma(\omega)))\big)^{-1}=\rho(\pi(\sigma(\omega)))
\end{aligned}
$$

So, by Birkhoff's ergodic theorem, $\rho\circ\pi(\omega)$ is $\tilde m_F$-a.e. constant and so is the Radon-Nikodym derivative $\rho:J\to[0,\infty)$. Keep the same symbol ρ for this value. Since both measures m and ν are probabilistic, $\rho\geq 1$. In the proof of the previous theorem we were done at this point concluding that $\rho=1$ since $\mu_F^{(1)}$ and $\mu_F^{(2)}$ were equivalent. Here an additional argument is needed. And indeed, if $\rho>1$, m_F-almost everywhere, define the set $Z=\{x\in J:\rho(x)=0\}$. Then $\nu(Z)=1-1/\rho>0$. We claim that

$$\nu((J\setminus Z)\cap\phi_\omega(Z))=0 \tag{2.14}$$

for all $\omega\in I^*$. Indeed, if $\nu((J\setminus Z)\cap\phi_\omega(Z))>0$ for some $\omega\in I^*$, then $m_F(\phi_\omega(Z))\geq m_F((J\setminus Z)\cap\phi_\omega(Z))=\nu((J\setminus Z)\cap\phi_\omega(Z))/\rho>0$ which by F-conformality of m_F implies that $m_F(Z) > 0$. This contradiction finishes the proof of (2.14). But now it follows from (2.14) that the probability measure $\nu|_Z/\nu(Z)$ satisfies (2.12). Hence, from what has been proved we conclude that m_F is absolutely continuous with respect to $\nu|_Z/\nu(Z)$. This however contradicts the definition of the set Z and finishes the proof. ■

Corollary 2.12. m_F is the only probability measure satisfying $\mathcal{L}_0^*(m_F) = m_F$ and m_F is F-conformal. Also, m_F-almost every point $x \in J$ has a unique representation in the form $x = \pi(\omega)$, $\omega \in I^\infty$, that is $\pi^{-1}(x)$ is a singleton. In particular in view of Theorem 2.9 and Lemma 2.8, the measure $\mu_F = \tilde{\mu}_F \circ \pi^{-1}$ is equivalent to m_F with bounded Radon-Nikodym derivatives.

Let us recall that in [MU1] we have in fact introduced the measures m_{hG}, μ_{hG}, $\tilde{m}_{hG}$ and $\tilde{\mu}_{hG}$, where tG as in Remark 2.1 is the family $\{t \log |\phi_i'|\}_{i \in I}$ and $h = \mathrm{HD}(J)$ is the Hausdorff dimension of the limit set J. We called there the measure m_{hG} simply h-conformal. Since $\mathrm{P}(hG) = 0$, the formula (2.10) takes on in this case the following form

$$m_{hG}(\phi_\omega(A)) = \int_A |\phi_\omega'|^h dm_{hG}. \tag{2.15}$$

We end this section by further investigating the measures $\tilde{m}_F$ and $\tilde{\mu}_F$. In order to do this we introduce a potential function or amalgamated function, f, induced by the family of functions F as follows: $f : I^\infty \to I\!R$ is defined by setting

$$f(\omega) = f^{(\omega_1)}(\pi(\sigma(\omega))).$$

Our convention will be to use lower case letters for the potential function corresponding to a given Hölder system of functions. Given $n \geq 1$ we set

$$S_n f = \sum_{j=0}^{n-1} f \circ \sigma^j.$$

It follows from Lemma 2.7, Theorem 2.9 and Lemma 2.2 that for every $\omega \in I^\infty$ and every $n \geq 1$

$$Q^{-1} \leq \frac{\tilde{m}_F([\omega|_n])}{\exp(S_n f(\omega) - \mathrm{P}(F)n)} \leq Q \tag{2.16}$$

and

$$Q^{-2} \leq \frac{\tilde{\mu}_F([\omega|_n])}{\exp(S_n f(\omega) - \mathrm{P}(F)n)} \leq Q^2 \tag{2.16'}$$

In fact (see Proposition 2.13 below) the measure $\tilde{\mu}_F$ is the only invariant measure satisfying a condition slightly weaker than (2.16').

Proposition 2.13. $\tilde{\mu}_F$ is the only shift-invariant measure on I^∞ satifying (2.16') with Q^2 replaced by an arbitrary constant $C \geq Q^2$ and $\mathrm{P}(F)$ replaced by an arbitrary constant P.

Proof. In view of (2.16') we only need to prove that if a shift-invariant measure μ satisfies (2.16') with a constant $C \geq 1$, then $\mu = \tilde{\mu}_F$. And indeed, we than have for every $\omega \in I^\infty$ and every $n \geq 1$

$$Q^{-2}C^{-1}\exp\big(\mathrm{P}-\mathrm{P}(F))n)\big) \leq \frac{\mu([\omega|_n])}{\tilde{\mu}_F([\omega|_n])} \leq Q^2 C\exp\big(\mathrm{P}-\mathrm{P}(F)n\big). \tag{2.17}$$

Suppose first that $\mathrm{P} < \mathrm{P}(F)$. We would then have

$$1 = \mu(I^\infty) = \sum_{|\tau|=n}\mu([\tau]) \leq Q^2 C\exp\big(\mathrm{P}-\mathrm{P}(F)n\big)$$

which gives contradiction for $n \geq 1$ large enough. So, $\mathrm{P} \geq \mathrm{P}(F)$. Similarly we demonstrate that $\mathrm{P}(F) \geq P$. Thus $\mathrm{P} = \mathrm{P}(F)$. But then (2.17) implies that the measures μ and $\tilde{\mu}_F$ are equivalent. Since, by Theorem 2.9, measure $\tilde{\mu}_F$ is ergodic, we finally conclude that $\mu = \tilde{\mu}_F$ and the proof is complete. ■

We now need the following technical result.

Lemma 2.14. The following three conditions are equivalent:

(a) $\int_{I^\infty} -f\,d\tilde{\mu}_F < \infty$.

(b) $\sum_{i\in I}\inf(-f|_{[i]})\exp(\inf f|_{[i]}) < \infty$.

(c) $\mathrm{H}_{\tilde{\mu}_F}(\alpha) < \infty$, where $\alpha = \{[i] : i \in I\}$ is the partition of I^∞ into initial cylinders of length 1.

Proof. $(a) \Rightarrow (b)$. Suppose that $\int -f\,d\tilde{\mu}_F < \infty$. This means that $\sum_{i\in I}\int_{[i]} -f\,d\tilde{\mu}_F < \infty$ and consequently

$$\begin{aligned}
\infty &> \sum_{i\in I}\inf(-f|_{[i]})\int_{[i]} d\tilde{\mu}_F = \sum_{i\in I}\inf(-f|_{[i]})\int_{[i]}(d\tilde{\mu}_F/d\tilde{m}_F)d\tilde{m}_F \\
&\geq Q^{-1}\sum_{i\in I}\inf(-f|_{[i]})\tilde{m}_F([i]) = Q^{-1}\sum_{i\in I}\inf(-f|_{[i]})\int_X \exp(f^{(i)}(x)-\mathrm{P}(F))dm_F(x) \\
&= Q^{-1}\mathrm{e}^{-\mathrm{P}(F)}\sum_{i\in I}\inf(-f|_{[i]})\int_X \exp(f^{(i)}(x))dm_F(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\infty &> \sum_{i\in I}\inf(-f|_{[i]})\int_X \exp(f^{(i)}(x))dm_F(x) \geq \sum_{i\in I}\inf(-f|_{[i]})\exp(\inf_X(f^{(i)}) \\
&= \sum_{i\in I}\inf(-f|_{[i]})\exp(\inf f|_{[i]}).
\end{aligned}$$

$(b) \Rightarrow (c)$. Suppose that $\sum_{i\in I} \inf(-f|_{[i]}) \exp(\inf f|_{[i]}) < \infty$. We shall show that $\mathrm{H}_{\tilde{\mu}_F}(\alpha) < \infty$. By definition,

$$\mathrm{H}_{\tilde{\mu}_F}(\alpha) = \sum_{i\in I} -\tilde{\mu}_F([i]) \log \tilde{\mu}_F([i]) \le \sum_{i\in I} -Q^{-1}\tilde{m}_F([i])\big(\log \tilde{m}_F([i]) - \log Q\big).$$

But, $\sum_{i\in I} -Q^{-1}\tilde{m}_F([i])(-\log Q) = Q^{-1}\log Q$, so it suffices to show that

$$\sum_{i\in I} -\tilde{m}_F([i]) \log \tilde{m}_F([i]) < \infty.$$

However,

$$\begin{aligned}\sum_{i\in I} -\tilde{m}_F([i]) \log \tilde{m}_F([i]) &= \sum_{i\in I} -\tilde{m}_F([i]) \log \left(\int_X \exp\big(f^{(i)} - \mathrm{P}(F)\big) \right) dm_F \\ &\le \sum_{i\in I} -\tilde{m}_F([i])(\inf_X f^{(i)} - \mathrm{P}(F)).\end{aligned}$$

Since $\sum_{i\in I} \tilde{m}_F([i])\mathrm{P}(F) = \mathrm{P}(F)$, it suffices to show that $\sum_{i\in \mathbb{N}} -\tilde{m}_F([i]) \inf_X f^{(i)} < \infty$. And indeed,

$$\sum_{i\in I} -\tilde{m}_F([i]) \inf_X f^{(i)} = \sum_{i\in I} \tilde{m}_F([i]) \sup_X(-f^{(i)}) \le \sum_{i\in I} \tilde{m}_F([i])\big(\inf_X(-f^{(i)}) + \log Q\big).$$

Since $\sum_{i\in I} \tilde{m}_F([i]) \log Q = \log Q$, it is enough to show that $\sum_{i\in I} \tilde{m}_F([i]) \inf_X(-f^{(i)}) < \infty$. In fact,

$$\begin{aligned}\sum_{i\in I} \tilde{m}_F([i]) \inf_X(-f^{(i)}) &= \sum_{i\in I} \inf_X f^{(i)} - \mathrm{P}(F)) \inf_X(-f^{(i)}) \\ &\le \mathrm{e}^{-\mathrm{P}(F)} Q \sum_{i\in I} \exp(\inf_X(f^{(i)})) \inf_X(-f^{(i)}).\end{aligned}$$

But, since $\mathcal{L}_F(\mathbb{1}) \in C(X)$, $f^{(i)}$ are negative everywhere for all i large enough, say $i \ge k$. Then using Lemma 2.2 again we get

$$\sum_{i\ge k} \tilde{m}_F([i]) \inf_X(-f^{(i)}) \le \mathrm{e}^{-\mathrm{P}(F)} Q \sum_{i\ge k} \exp(\inf_X(f^{(i)})) \inf_X(-f^{(i)})$$

which is finite due to our assumption. Hence, $\sum_{i\in \mathbb{N}} \tilde{m}_F([i]) \inf_X(-f^{(i)}) < \infty$.

$(c) \Rightarrow (a)$. Suppose that $\mathrm{H}_{\tilde{\mu}_F}(\alpha) < \infty$. We need to show that $\int -f d\tilde{\mu}_F < \infty$. We have

$$\infty > \mathrm{H}_{\tilde{\mu}_F}(\alpha) = \sum_{i\in I} -\tilde{m}_F([i]) \log\big(\tilde{m}_F([i])\big) \le \sum_{i\in I} -\tilde{m}_F([i])\big(\inf(f|_{[i]} - \mathrm{P}(f) - \log Q)\big).$$

Hence, $\sum_{i\in I} -\tilde{m}_F([i])\inf(f|_{[i]}) < \infty$ and therefore

$$\int -f\,d\tilde{\mu}_F = \sum_{i\in I}\int_{[i]} -f\,d\tilde{\mu}_F \le \sum_{i\in I}\sup(-f|_{[i]})\tilde{m}_F([i]) = \sum_{i\in I} -\inf(f|_{[i]})\tilde{m}_F([i]) < \infty.$$

The proof is complete. ■

Let us note some further properties of the potential or amalgamation function. Since $F = \{f^{(i)} : i \in I\}$ is a Hölder system of functions of order β, the amalgamated function $f : I^\infty \to I\!R$ is Hölder continuous of order β meaning that

$$V_\beta(f) = \sup_{n\ge 1}\{e^{\beta n}V_n(f)\} < \infty,$$

where

$$V_n(f) = \sup\{|f(\omega) - f(\tau)| : \omega|_n = \tau|_n\}.$$

It is easy to see that

$$\mathrm{P}(F) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{|\omega|=n}\exp\left(\sup_{\tau\in[\omega]}\sum_{j=0}^{n-1} f\circ\sigma^j(\tau)\right)\right),$$

where the topological pressure $\mathrm{P}(F)$ has been defined at the beginning of the section. Frequently we will also write $\mathrm{P}(f)$ for $\mathrm{P}(F)$ as well as m_f, $\tilde{m}_f$, μ_f and $\tilde{\mu}_f$ for m_F, $\tilde{m}_F$, μ_F and $\tilde{\mu}_F$ respectively. It is also not difficult to check (see [HU]) that formula (2.15) gives the same value as the definition introduced by Sarig in [Sa]. Therefore, it follows from Theorem 3 of [Sa] that

$$\sup\{\mathrm{h}_\mu(\sigma) + \int f\,d\mu\} = \mathrm{P}(F) = \mathrm{P}(f), \tag{2.18}$$

where the supremum is taken over all σ-invariant probability measures such that $\int -f\,d\mu < \infty$. We call a σ-invariant probability measure μ an equilibrium state of the potential f, or equivalently of the family F, if $\int -f\,d\mu < +\infty$ and

$$\mathrm{h}_\mu(\sigma) + \int f\,d\mu = \mathrm{P}(F). \tag{2.19}$$

Given $\omega \in I^*$, say $\omega \in I^n$ by $\sigma_\omega^{-n} : I^\infty \to I^\infty$ we denote the map defined by the formula

$$\sigma_\omega^{-n}(\tau) = \omega\tau.$$

Notice that σ_ω^{-n} is a continuous branch of σ^n. Given a Borel probability shift-invariant measure μ on I^∞ we call the function $J_\mu : I^\infty \to [1,+\infty]$ the Jacobian of the shift map σ with respect to the measure μ if for every Borel set $A \subset I^\infty$

$$\mu(\sigma_i^{-1}(A)) = \int_A \frac{1}{J_\mu\circ\sigma_i^{-1}}\,d\mu.$$

By $L_\mu : L^\infty \to L^\infty$ we denote the Perron-Frobenius operator of the measure μ, i.e. the operator defined by the formula

$$L_\mu(g)(\omega) = \sum_{i\in I} J_\mu(i\omega)g(i\omega).$$

We shall prove the following.

Lemma 2.15. If μ is an equilibirum state for the shift map $\sigma : I^\infty \to I^\infty$ and potential f such that $\int -f d\mu < +\infty$, then

$$J_\mu = \frac{\rho\circ\sigma}{\rho} \cdot \exp(\mathrm{P}(f) - f)$$

μ almost everywhere, where $\rho = d\tilde{\mu}_F/d\tilde{m}_F$ is the density function introduced in Theorem 2.9.

Proof. Let H_0 be the space of all bounded real-valued continuous functions on I^∞ and let $L : H_0 \to H_0$ be the Perron-Frobenius operator defined by formula

$$L(g)(\omega) = \sum_{\tau\in\sigma^{-1}(\omega)} \exp\big(f(\tau) - \mathrm{P}(F)\big)f(\tau) = \sum_{i\in I} \exp\big(f(i\omega) - \mathrm{P}(F)\big)f(i\omega).$$

The density $\rho = d\tilde{mu}_F/d\tilde{m}_F$ existing due to Theorem 2.9 is its fixed point and according to Theorem 5.2 of [Ur] ρ has a version in H_0, even Hölder continuous. Therefore, using inequality $x \ge 1 + \log x$ we can write

$$\begin{aligned}
1 &= \int 1 d\mu = \int \frac{L(\rho)}{\rho} d\mu = \int L_\mu\left(\frac{\rho\cdot\exp(f - \mathrm{P}(F))}{J_\mu^{-1}\cdot\rho\circ\sigma}\right) d\mu \\
&= \int \frac{\rho\cdot\exp(f - \mathrm{P}(F))}{J_\mu^{-1}\cdot\rho\circ\sigma} d\mu \ge 1 + \int \log\left(\frac{\rho\cdot\exp(f - \mathrm{P}(F))}{J_\mu^{-1}\cdot\rho\circ\sigma}\right) d\mu \\
&= 1 + \int \log\rho d\mu - \int \log\rho\circ\sigma d\mu + \int (f - \mathrm{P}(F)) d\mu + \int \log J_\mu d\mu \\
&= 1 + \int f d\mu - \mathrm{P}(F) + \mathrm{h}_\mu(\sigma) = 1.
\end{aligned}$$

Notice that we were in position to write the inequality sign and the equality sign following it since by our assumptions $\int f d\mu$ is finite and since $\log J_\mu$ is a non-negative function. Since $x = 1 + \log x$ if and only if $x = 1$, we conclude from this display that $\frac{\rho\cdot\exp(f-\mathrm{P}(F))}{J_\mu^{-1}\cdot\rho\circ\sigma} = 1$ μ *a.e.*. The proof is complete. ■

Theorem 2.16. If $\sum_{i\in I}\inf(-f|_{[i]})\exp(\inf f|_{[i]}) < \infty$, then $\tilde{\mu}_F$ is a unique equilibrium state of the potential f.

Proof. It follows from Lemma 2.14 that $\int -f d\tilde{\mu}_F < \infty$. To show that $\tilde{\mu}_F$ is an equilibrium state of the potential f consider $\alpha = \{[i] : i \in I\}$, the partition of I^∞ into initial cylinders of length one. By Lemma 2.14, $\mathrm{H}_{\tilde{\mu}_F}(\alpha) < \infty$. Applying the Breiman-Shanon-McMillan theorem and the Birkhoff ergodic theorem, we find for $\tilde{\mu}_F$-a.e. $\omega \in \Sigma$

$$
\begin{aligned}
\mathrm{h}_{\tilde{\mu}_F}(\sigma) \geq \mathrm{h}_{\tilde{\mu}_F}(\sigma, \alpha) &= \lim_{n\to\infty} \frac{-1}{n} \log \tilde{\mu}_F([\omega|_n]) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log \left(\int \exp\big(S_\omega(F)(x) d\mu_F - \mathrm{P}(F)n\big) \right) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log \left(\int \exp\big(\sum_{j=0}^{n-1} f(\sigma^j(\omega|_n\tau)) d\mu_F(\tau) - \mathrm{P}(F)n\big) \right) \\
&\geq \limsup_{n\to\infty} \frac{-1}{n} \log \left(\int \exp\big(\sum_{j=0}^{n-1} f(\sigma^j(\omega)) + \log Q - \mathrm{P}(F)n\big) \right) \\
&= \lim_{n\to\infty} \frac{-1}{n} \sum_{j=0}^{n-1} f(\sigma^j(\omega)) + \mathrm{P}(F) = -\int f d\tilde{\mu}_F + \mathrm{P}(F).
\end{aligned}
$$

Hence $\mathrm{h}_{\mu_F}(\sigma) + \int f d\tilde{\mu}_F \geq \mathrm{P}(F)$, which in view of (2.18) implies that $\tilde{\mu}_F$ is an equilibrium state for the potential f. We shall now prove that μ is the only equilibrium state for f. So, suppose that μ is an equilibrium state. Fix $\omega \in I^*$, say $\omega \in I^n$. It the follows from Lemma 2.15, Lemma 2.2 and formula (2.16') that for every $\gamma \in [\omega]$ we get

$$
\begin{aligned}
\mu([\omega]) = \mu(\sigma_\omega^{-n}(I^\infty)) &= \int J_\mu^{-1}(\omega\tau) J_\mu^{-1}(\sigma(\omega\tau)) \ldots J_\mu^{-1}(\sigma^{n-1}(\omega\tau)) d\mu(\tau) \\
&= \int \frac{\rho(\omega\tau)}{\rho(\sigma(\omega\tau))} \exp(f(\omega\tau) - \mathrm{P}(F)) \frac{\rho(\sigma(\omega\tau))}{\rho(\sigma^2((\omega\tau))} \exp(f(\sigma(\omega\tau)) - \mathrm{P}(F)) \ldots \\
&\ldots \frac{\rho(\sigma^{n-1}(\omega\tau))}{\rho(\sigma^n((\omega\tau))} \exp(f(\sigma^{n-1}(\omega\tau)) - \mathrm{P}(F)n) d\mu(\tau) \\
&= \int \frac{\rho(\omega\tau)}{\rho(\tau)} \exp(S_n f(\omega\tau) - \mathrm{P}(F)n) \leq Q^2 \int \exp(S_n f(\omega\tau) - \mathrm{P}(F)n) \\
&\leq Q^3 \exp(S_n f(\gamma) - \mathrm{P}(F)n) \leq Q^5 \tilde{\mu}_F([\omega]).
\end{aligned}
$$

Hence, the invariant measure μ is absolutely continuous with respect to ergodic invariant measure $\tilde{\mu}_F$. The proof is finished. ■

We say that two amalgamated functions (of two Hölder systems of functions) $f, g : I^\infty \to I\!R$ are cohomologous in a class $\mathcal{H}$ if there exists a function : $I^\infty \to I\!R$ such that

$$g - f = u - u \circ \sigma.$$

We would like to end up this section with Theorem 2.17 whose proof analogous to the proof of Theorem 1.28 in [Bo] we provide for the sake of completeness.

Theorem 2.17. Suppose that $F = \{f^{(i)}\}_{i\in I}$ and $G = \{g^{(i)}\}_{i\in I}$ are two Hölder systems of functions. Then the following conditions are equivalent:

(1) $\tilde{\mu}_F = \tilde{\mu}_G$.

(2) There exists a constant R such that

$$S_nf(\omega) - S_ng(\omega) = nR$$

if only $\sigma^n(\omega) = \omega$.

(3) The difference $f - g$ is cohomologous to a constant in the class of bounded Hölder continuous functions.

(4) The difference $g - f$ is cohomologous to a constant in the class of bounded continuous functions.

(5) There exist constants S and T such that for every $\omega \in I^\infty$ and every $n \geq 1$

$$|S_nf(\omega) - S_ng(\omega) - Sn| \leq T.$$

If these conditions are satisfied then $R = S = \mathrm{P}(F) - \mathrm{P}(G)$.

Proof. (1) $\Rightarrow$ (2). It follows from (2.16') that

$$Q^{-4} \leq \frac{\exp\big(S_kf(\omega) - \mathrm{P}(F)k)\big)}{\exp\big(S_kg(\omega) - \mathrm{P}(g)k)\big)} \leq Q^4$$

for every $\omega \in I^\infty$ and every $k \geq 1$. Suppose now that $\sigma^n(\omega) = \omega$. Then for every $k = ln$, $l \geq 1$,

$$Q^{-4} \leq \exp\big(l(S_nf(\omega) - S_ng(\omega)) - (\mathrm{P}(F) - \mathrm{P}(G))n\big) \leq Q^4.$$

Hence, there exists a constant $T \geq 0$ such that

$$l|S_nf(\omega) - S_ng(\omega) - (\mathrm{P}(F) - \mathrm{P}(G))n| \leq T$$

and therefore, letting $l \nearrow \infty$, we conclude that $S_nf(\omega) - S_ng(\omega) = (\mathrm{P}(F) - \mathrm{P}(G))n$. Thus, putting $R = \mathrm{P}(F) - \mathrm{P}(G)$ completes the proof of the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Define

$$\eta = f - g - R$$

and fix a point $\tau \in I^\infty$ transitive for the shift map $\sigma : I^\infty \to I^\infty$. Put

$$\Gamma = \{\sigma^k(\tau) : k \geq 1\}$$

and define the function $u : \Gamma \to I\!R$ by setting

$$u(\sigma^k(\tau)) = \sum_{j=0}^{k-1} \eta(\sigma^j(\tau)).$$

Note that the function u is well-defined since all points $\sigma^k(\tau)$, $k \geq 1$, are mutually distinct. Taking the minimum of exponents we may assume that both families F and G form Hölder systems of functions of the same order β. Fix now $k \geq 1$ and consider periodic point $\omega = (\tau|_k)^\infty$. Then by our assumption

$$
\begin{aligned}
|u(\sigma^k(\tau)| &= \left|\sum_{j=0}^{k-1}\big(\eta(\sigma^j(\tau)) - (f(\sigma^j(\omega)) - g(\sigma^j(\omega))\big) + Rk\right| \\
&= \left|\sum_{j=0}^{k-1}\big((f(\sigma^j(\tau)) - f(\sigma^j(\omega))) - (g(\sigma^j(\tau)) - g(\sigma^j(\omega)))\big)\right| \\
&\leq \sum_{j=0}^{k-1} |f(\sigma^j(\tau)) - f(\sigma^j(\omega))| + \sum_{j=0}^{k-1} |g(\sigma^j(\tau)) - g(\sigma^j(\omega))| \\
&\leq \sum_{j=0}^{k-1} V_\beta(f)\mathrm{e}^{-\beta(k-j)} + \sum_{j=0}^{k-1} V_\beta(g)\mathrm{e}^{-\beta(k-j)} \\
&\leq (V_\beta(f) + V_\beta(g))\frac{\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}} < \infty. \qquad (2.20)
\end{aligned}
$$

Assume now $\sigma^k(\tau)|_r = \sigma^l(\tau)|_r$ for some $k < l$ and some $r \geq 1$. Let $\omega = \tau|_k(\sigma^k(\tau)|_{l-k})^\infty$. By our assumption $\sum_{j=k}^{l-1} \eta(\sigma^j(\omega)) = 0$. Hence,

$$
\begin{aligned}
|u(\sigma^l(\tau)) - u(\sigma^k(\tau))| &= \left|\sum_{j=k}^{l-1}\eta(\sigma^j(\tau))\right| = \left|\sum_{j=k}^{l-1}\eta(\sigma^j(\tau)) - \eta(\sigma^j(\omega))\right| \\
&\leq \sum_{j=k}^{l-1}\big(|f(\sigma^j(\tau)) - f(\sigma^j(\omega))| + |g(\sigma^j(\tau)) - g(\sigma^j(\omega))|\big) \\
&\leq \sum_{j=k}^{l-1}(V_\beta(f) + V_\beta(g))\mathrm{e}^{-\beta(r+l-j-1)} \\
&\leq \mathrm{e}^{-\beta r}(V_\beta(f) + V_\beta(g))\sum_{j=0}^{\infty}\mathrm{e}^{-\beta j} = \frac{V_\beta(f) + V_\beta(g)}{1-\mathrm{e}^{-\beta}}\mathrm{e}^{-\beta r} \qquad (2.21)
\end{aligned}
$$

In particular it follows from (2.21) that u is uniformly continuous on Γ. Since Γ is a dense subset of I^∞ we therefore conclude that u has a unique continuous extension on I^∞. Moreover, it follows from (2.20) and (2.21) that u is bounded and Hölder continuous. The proof of the implication (2) $\Rightarrow$ (3) is therefore complete.

Now, the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (1). It follows from (5) and (2.16') that for every $\omega \in I^*$, say $\omega \in I^n$

$$
(2.22)\quad Q^{-4}\mathrm{e}^{-T}\exp\big((S + \mathrm{P}(G) - \mathrm{P}(F))n\big) \leq \frac{\tilde{\mu}_F([\omega])}{\tilde{\mu}_G([\omega])} \leq Q^4\mathrm{e}^T\exp\big((S + \mathrm{P}(G) - \mathrm{P}(F))n\big).
$$

Suppose that $S \neq \mathrm{P}(F) - \mathrm{P}(G)$. Without loosing generality we may assume that $S < \mathrm{P}(F) - \mathrm{P}(G)$. But then it would follow from (2.22) that that for every $n \geq 1$

$$1 = \tilde{\mu}_F(I^\infty) = \sum_{|\omega|=n} \tilde{\mu}_F([\omega]) \leq Q^4 \mathrm{e}^T \exp\bigl((S + \mathrm{P}(G) - \mathrm{P}(F))n\bigr)$$

which gives contradiction for $n \geq 1$ large enough. Hence $S = \mathrm{P}(F) - \mathrm{P}(G)$. But then (2.22) implies that the measures $\tilde{\mu}_F$ and $\tilde{\mu}_G$ are equivalent. Since, in view of Theorem 2.9 these measures are ergodic, they must conincide. The proof of the implication (5) $\Rightarrow$ (1) and simultaneously of the whole Theorem 2.17 is complete. ■

§**3. Parabolic systems.** In this section we recall from [MU3] the concept of conformal parabolic iterated function systems and employing the construction of the associated hyperbolic system, we demonstarate how to reduce the theory of Hölder families of functions for parabolic systems to the corresponding theory of hyperbolic systems. The significance of this section lies on the level of geometric features of the measures m_F and μ_F and not on the level of their abstract properties. In Section 7 we will develop the multifractal analysis of conformal hyperbolic iterated function systems, and using the result this section, we will apply the results of Section 7 to study geometry of equilibrium states (μ_F) of parabolic systems in Section 8. Let us recall the setting from [MU3]. Let X be a compact connected subset of a Euclidean space $I\!R^d$. Suppose that we have countably many conformal maps $\phi_i : X \to X$, $i \in I$, where I has at least two elements satisfying the following conditions

(1) (Open Set Condition) $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$ for all $i \neq j$.

(2) $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic.

(3) $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then ϕ_ω extends conformally to an open connected set $V \subset I\!R^d$ and maps V into itself.

(4) If i is a parabolic index, then $\bigcap_{n \geq 0} \phi_{i^n}(X) = \{x_i\}$ and the diameters of the sets $\phi_{i^n}(X)$ converge to 0.

(5) (Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ $\forall x, y \in V$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \leq K.$$

(6) $\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.

(7) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.

(8) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi_i'(y)| - |\phi_i'(x)|\big| \leq L||\phi_i'|||y-x|^\alpha,$$

for every $i \in I$ and every pair of points $x, y \in V$.

We call such a system of maps $S = \{\phi_i : i \in I\}$ a subparabolic iterated function system. If $\Omega \neq \emptyset$, we call this system parabolic. I has been proved in [MU3] that $\lim_{n\to\infty} \sup_{|\omega|=n}\{\mathrm{diam}(\phi_\omega(X))\} = 0$. So, the projection $\pi : I^\infty \to X$ and consequently also the limit set $\pi(I^\infty)$, are well defined. Let us now recall the main construction from [MU3]. So, consider the system S^* generated by I_*, the set of maps of the form

$$\phi_{i^n j},$$

where $n \geq 1$, $i \in \Omega$, $i \neq j$, and the maps

$$\phi_k,$$

where $k \in I \setminus \Omega$. It immediately follows from our assumptions that the following is true.

Theorem 3.1. The system S^* is a hyperbolic conformal iterated function system.

We recall that J^* is the limit set generated by the system S^*. The proof of the following result (see [MU3]) is straightforward.

Lemma 3.2. The limit sets J and J^* of the systems S and S^* respectively differ only by a countable set. In fact, $J^* \subset J$ and $J \setminus J^* \subset \{\phi_\omega i^\infty : \omega \in I^*, i \in \Omega\}$.

Let now $F = \{f^{(i)}\}_{i\in I}$ be a Hölder family of functions of an order $\beta > 0$ for the iterated function system $S = \{\phi_i\}_{i\in I}$. We define the family F^* for the system S^* by setting

$$f_*^{(i)} = f^{(i)}$$

if i is a hyperbolic index of I and

$$f_*^{(i^n j)} = f^{(j)} + \sum_{k=0}^{n-1} f^{(i)} \circ \phi_{i^n j} - \mathrm{P}(F)(n+1)$$

if $n \geq 1$, $i \in \Omega$ and $j \in I \setminus \{i\}$. We start we with te following.

Theorem 3.3. If F is a Hölder family of functions of order $\beta > 0$ for the system S, then F^* is a Hölder family of functions of order $\beta > 0$ for the system S^* and $\mathrm{P}_{S^*}(F^*) = 0$.

Proof. Since $|\omega|_* \leq |\omega|$ for every $\omega \in I_*^*$, in order to check (2.1) we only need to estimate from above the numbers

$$|f^{(i^n j)}(\phi_\omega(x)) - f^{(i^n j)}(\phi_\omega(y))|$$

for all $n \geq 1$, all $i \in \Omega$, all $j \in I \setminus \{i\}$, and all $\omega \in I_*^*$. And indeed, applying Lemma 2.2 we get

$$|f^{(i^n j)}(\phi_\omega(x)) - f^{(i^n j)}(\phi_\omega(y))| = |S_{(i^n j)}(F)(\phi_\omega(x)) - S_{(i^n j)}(F)(\phi_\omega(y))|$$
$$\leq \frac{V_\beta(F)\mathrm{e}^\beta}{1-\mathrm{e}^{-\beta}}\mathrm{e}^{-\beta|\omega|} \leq \frac{V_\beta(F)\mathrm{e}^\beta}{1-\mathrm{e}^{-\beta}}\mathrm{e}^{-\beta|\omega|_*}.$$

So,

$$|f^{(i^n j)}(\phi_\omega(x)) - f^{(i^n j)}(\phi_\omega(y))|\mathrm{e}^{\beta|\omega|_*} \leq \frac{V_\beta(F)\mathrm{e}^\beta}{1-\mathrm{e}^{-\beta}}.$$

And therefore

$$V(F^*) \leq \frac{V_\beta(F)\mathrm{e}^\beta}{1-\mathrm{e}^{-\beta}}.$$

Let us now verify condition (2.2). In view of Lemma 2.7 and Lemma 2.2 we can write

$$\sum_{i \in I\setminus\Omega} ||\mathrm{e}^{f^{(i)}}||_0 + \sum_{i\in\Omega}\sum_{j\neq i}\sum_{n\geq 1} ||\mathrm{e}^{f_*^{(i^n j)}}||_0 =$$

$$= \sum_{i \in I\setminus\Omega} ||\mathrm{e}^{f^{(i)}}||_0 + \sum_{i\in\Omega}\sum_{j\neq i}\sum_{n\geq 1} \left\|\exp(f^{(j)} + \sum_{k=0}^{n-1} f^{(i)} \circ \phi_{i^n j} - \mathrm{P}(F)(n+1))\right\|_0$$
$$\leq \sum_{i \in I\setminus\Omega} ||\mathrm{e}^{f^{(i)}}||_0 + \sum_{i\in\Omega}\sum_{j\neq i}\sum_{n\geq 1} \left\|\exp\big(S_{i^n j}(F) - \mathrm{P}(F)(n+1)\big)\right\|_0$$
$$\leq \sum_{i \in I\setminus\Omega} ||\mathrm{e}^{f^{(i)}}||_0 + \sum_{i\in\Omega}\sum_{j\neq i}\sum_{n\geq 1} Q\tilde{m}_F([i^n j])$$
$$\leq \sum_{i \in I\setminus\Omega} ||\mathrm{e}^{f^{(i)}}||_0 + Q\tilde{m}_F([i^n j]) \leq \sum_{i \in I\setminus\Omega} ||\mathrm{e}^{f^{(i)}}||_0 + Q < \infty$$

So, F^* is a Hölder family of functions of order β and we are only left to show that $\mathrm{P}_{S^*}(F^*) = 0$. The proof uses the argument similar as above. First notice that if $\omega \in I_*^*$ and $\omega\overline{\omega}$ is the word ω written in the alphabet I, then

$$S_\omega(F^*) = S_{\overline{\omega}}(f) - \mathrm{P}(F)|\omega|. \tag{3.1}$$

Hence,

$$Z_n(F^*) = \log \sum_{|\omega|_*=n} ||\exp S_\omega(F^*)||_0 = \log \sum_{|\omega|_*=n} ||\exp\big(S_{\overline{\omega}}(f) - \mathrm{P}(F)|\omega|\big).$$

Therefore, using Lemma 2.2 and Lemma 2.7 we conclude that

$$Z_n(F^*) \leq \log Q + \sum_{|\omega|_*=n} \tilde{m}_F([\overline{\omega}]) \leq \log Q + m(I^\infty) \leq 1 + \log Q$$

and similarly $Z_n(F^*) \geq 1 - \log Q$. Thus

$$\mathrm{P}_{S^*}(F^*) = \lim_{n\to\infty} \frac{1}{n} Z_n(F^*) = 0.$$

The proof is complete. ■

Notice now that in Section 2 we have not really used the hyperbolicity of the iterated function system considered, but only the fact that the intersections $\bigcap_{n\geq 1}\phi_{\omega|_n}(X)$ are singletons. Therefore, given a parabolic system S and a Hölder family of functions F, we, in particular, can speak about the measures $\tilde{m}_F$, $\tilde{\mu}_F$ and m_F. Notice however that the measure m_F is defined as the fixed point of the normalized Perron-Frobenius operator and we do not claim that it is conformal. Since the map $I_*^\infty \to I^\infty$, $\omega \mapsto \overline{\omega}$, is injective, we can consider I_*^∞ as a subset of I^∞. Since $I^\infty \setminus I_*^\infty = \{\omega i^\infty : \omega \in I^*, i \in \Omega\}$ is a countable set, since the measure $\tilde{\mu}_F$ is ergodic (see Theorem 2.9), and since, by (2.16'), the topological support of μ_F is equal to I^∞, we conclude that $\tilde{\mu}_F(I^\infty \setminus I_*^\infty) = 0$. Thus, with regard to the measures $\tilde{\mu}_F, \tilde{m}_F, \tilde{\mu}_{F^*}, \tilde{m}_{F^*}$ we can identify the sets I^∞ and I_*^∞. We are now in position to prove the following.

Theorem 3.4. Suppose that $F = \{f^{(i)}\}_{i\in I}$ is a Hölder family of functions of the parabolic system S and that F^* is the corresponding family of functions for te system S^*. Then

(a) $\tilde{m}_F = \tilde{m}_{F^*}$.

(b) $m_F = m_{F^*}$.

(c) The measures $\tilde{\mu}_F$ and $\tilde{\mu}_{F^*}$ are equivalent with uniformly bounded Radon-Nikodym derivatives.

Proof. It immediately follows from Lemma 2.7, Lemma 2.2 and (3.1) that the measures $\tilde{m}_F$ and $\tilde{m}_{F^*}$ are equivalent, and moreover, the Radon-Nikodym derivative $\rho = \frac{d\tilde{m}_{F^*}}{d\tilde{m}_F}$ is uniformly bounded away from zero and infinity. The item (c) of our theorem is now an immediate consequence of Theorem 2.9. In view of ergodicity of the dynamical system $(\sigma^*, \tilde{\mu}_{F^*})$ in order to prove part (a), it is sufficient to show that the Radon-Nikodym derivative $\rho = \frac{dm_{F^*}}{d\tilde{m}_F}$ is constant on almost all forward trajectories under the shift map σ^*. The idea is to apply (2.9) for the maps σ^* and σ and to proceed further similarly as in the proof of Theorem 2.10. And indeed, applying first (2.9) for all $\omega \in I_*^\infty$ and the shift map σ^*, we get for every $n \geq 1$

$$\lim_{n\to\infty} \frac{\tilde{m}_{F^*}([\omega|_n^*])}{\tilde{m}_{F^*}([\sigma^*\omega|_{n-1}^*])} = \exp\big(f_*^{(\omega_1)}(\pi(\sigma^*\omega))\big) = \exp\big(S_{\omega_1}(f)(\sigma^{|\omega_1|}) - \mathrm{P}(F)|\omega_1|\big).$$

Applying in turn (2.9) $\omega_1|$ times to the shift map σ, we get for every $n \geq 1$

$$\lim_{n\to\infty} \frac{\tilde{m}_F([\omega|_{\overline{n}}])}{\tilde{m}_F([\sigma^|\omega_1|(\omega)|_{\overline{n-1}}])} = \exp\big(S_{\omega_1}(f)(\sigma^{|\omega_1|}) - \mathrm{P}(F)|\omega_1|\big)$$

where $\overline{n} = \sum_{i=1}^n |\omega_i|$, $i \in I_*$. Hence, for every $\omega \in I_*^\infty$

$$\frac{d\tilde{m}_{F^*}}{d\tilde{m}_{F^*}\circ\sigma^*}(\omega) = \frac{dm_F}{d\tilde{m}_F\circ\sigma^*}(\omega) \notin \{0,\infty\}.$$

Therefore,

$$\begin{aligned}\rho(\omega) &= \frac{d\tilde{m}_{F^*}}{d\tilde{m}_F}(\omega) = \frac{d\tilde{m}_{F^*}}{d\tilde{m}_{F^*}\circ\sigma^*}(\omega)\cdot\frac{d\tilde{m}_{F^*}\circ\sigma^*}{d\tilde{m}_F\circ\sigma^*}(\omega)\cdot\frac{d\tilde{m}_F\circ\sigma^*}{d\tilde{m}_F}(\omega)\\ &= \frac{d\tilde{m}_{F^*}}{d\tilde{m}_F}(\sigma^*\omega) = \rho(\sigma\omega)\end{aligned}$$

Thus the proof of item (a) is complete. Part (b) is an immediate application of (a) and Lemma 2.8. ■

§**4. Volume Lemma.** Throughout this section we assume that the system $F = \{\phi_i : i \in I\}$ is conformal. Recall that if ν is a finite Borel measure on X, then $\mathrm{HD}(\nu)$, the Hausdorff dimension of ν, is the minimum of Hausdorff dimensions of sets of full ν measure. By $\alpha = \{[i] : i \in I\}$, we denote the partition of I^∞ into initial cylinders of length 1. If μ is a Borel shift-invariant ergodic probability measure on I^∞, by $\mathrm{h}_\mu(\sigma)$ we denote its entropy with respect to the shift map σ and by $\chi_\mu(\sigma) = \int \zeta d\mu > 0$ its characteristic Lyapunov exponent, where

$$\zeta(\omega) = -\log|\phi'_{\omega_1}(\pi(\sigma(\omega)))|.$$

In this section we shall prove the following.

Theorem 4.1.(Volume Lemma) Suppose that μ is a Borel shift-invariant ergodic probability measure on I^∞ such that

$$\mu\circ\pi^{-1}\big(\phi_\omega(X)\cap\phi_\tau(X)\big) = 0 \tag{4.1}$$

for all incomparable words $\omega, \tau \in I^*$. If $\mathrm{H}_\mu(\alpha) < \infty$, then

$$\mathrm{HD}(\mu\circ\pi^{-1}) = \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)},$$

where $\mathrm{H}_\mu(\alpha)$ is the entropy of the partition α with respect to the measure μ and we put $\frac{\mathrm{h}_\mu(\sigma)}{\infty} = 0$.

Proof. Supposing the series $\sum_{i\in I} -\mu([i])\log(||\phi_i'||_0)$ converges, using (BDP), we conclude that the function ζ is integrable. Since $\mathrm{H}_\mu(\alpha) < \infty$ and α is a generating partition, the entropy $\mathrm{h}_\mu(\sigma) = \mathrm{h}_\mu(\sigma,\alpha) \le \mathrm{H}_\mu(\alpha)$ is finite. Thus, in view of the Birkhoff ergodic theorem and the Breimann-Shannon-McMillan theorem there exists a set $I_0 \subset I^\infty$ such that $\mu(I_0) = 1$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\zeta\circ\sigma^j(\omega) = \chi_\mu(\sigma) \quad\text{and}\quad \lim_{n\to\infty}\frac{-\log(\mu([\omega|_n])}{n} = \mathrm{h}_\mu(\sigma) \tag{4.2}$$

for all $\omega \in I_0$. Fix now $\omega \in I_0$ and $\eta > 0$. For $r > 0$ let $n = n(\omega, r) \ge 0$ be the least integer such that $\phi_{\omega|_n}(X) \subset B(\pi(\omega), r)$. Then $\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big) \ge \log\big(\mu\circ\pi^{-1}(\phi_{\omega|_n}(X))\big) \ge$

$\log(\mu([\omega|_n]) \geq -(\mathrm{h}_\mu(\sigma)+\eta)n$ for every r small enough (which implies that $n=n(\omega,r)$ is large enough) and $\operatorname{diam}\big(\phi_{\omega|_{n-1}}(X)\big) \geq r$. The last inequality implies that

$$
\begin{aligned}
\log r &\leq \log\big(\operatorname{diam}(\phi_{\omega|_{n-1}}(X))\big) \leq \log\big(D|\phi'_{\omega|_{n-1}}(\pi(\sigma^{n-1}(\omega)))|\big) \\
&\leq \log D + \sum_{j=1}^{n-1} \log|\phi'_{\omega_j}(\pi(\sigma^j(\omega)))| \leq \log D - (n-1)(\chi_\mu(\sigma)-\eta)
\end{aligned}
$$

for all r small enough. Therefore, for these r

$$
\begin{aligned}
\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega),r))\big)}{\log r} &\leq \frac{-(\mathrm{h}_\mu(\sigma)+\eta)n}{\log D-(n-1)(\chi_\mu(\sigma)-\eta)} \\
&= \frac{\mathrm{h}_\mu(\sigma)+\eta}{\frac{-\log D}{n}+\frac{n-1}{n}(\chi_\mu(\sigma)-\eta)}.
\end{aligned}
$$

Hence letting $r\to 0$, and consequently $n\to\infty$, we obtain

$$
\limsup_{r\to 0}\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega),r))\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\sigma)+\eta}{\chi_\mu(\sigma)-\eta}.
$$

Since η was an arbitrary positive number we finally obtain

$$
\limsup_{r\to 0}\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega),r))\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}
$$

for all $\omega\in I_0$. Hence (see [Ma], [PU]), as $\mu\circ\pi^{-1}(\pi(I_0))=1$, $\mathrm{HD}(\mu\circ\pi^{-1})\leq \mathrm{h}_\mu(\sigma)/\chi_\mu(\sigma)$. Let now $J_1\subset J$ be an arbitrary Borel set such that $\mu\circ\pi^{-1}(J_1)>0$. Fix $\eta>0$. In view of (4.2) and Jegorov's theorem there exist $n_0\geq 1$ and a Borel set $\tilde J_2\subset\pi^{-1}(J_1)$ such that $\mu(\tilde J_2)>\mu(\pi^{-1}(J_1))/2>0$,

$$
\mu([\omega|_n]) \leq \exp\big((-\mathrm{h}_\mu(\sigma)+\eta)n\big) \tag{4.3}
$$

and $|\phi'_{\omega|_n}(\pi(\sigma^n(\omega))|\geq \exp\big((-\chi_\mu(\sigma)-\eta)n\big)$ for all $n\geq n_0$ and all $\omega\in\tilde J_2$. Due to the (BDP), the last inequality implies that there exists $n_1\geq n_0$ such that

$$
\operatorname{diam}\big(\phi_{\omega|_n}(X)\big) \geq D^{-1}\mathrm{e}^{(-\chi_\mu(\sigma)-\eta)n} \geq \mathrm{e}^{-(\chi_\mu(\sigma)+2\eta)n} \tag{4.4}
$$

for all $n\geq n_1$ and all $\omega\in\tilde J_2$. Given now $0<r<\exp(-(\chi_\mu(\sigma)+2\eta)n_1\big)$ and $\omega\in\tilde J_2$ let $n(\omega,r)$ be the least number n such that $\operatorname{diam}\big(\phi_{\omega|_{n+1}}(X)\big)<r$. Using (4.4) we deduce that $n(\omega,r)+1>n_1$, hence $n(\omega,r)\geq n_1$ and $\operatorname{diam}\big(\phi_{\omega|_n}(X)\big)\geq r$. In view of Lemma 2.7 of [MU] there exists a universal constant $L\geq 1$ such that for every $\omega\in\tilde J_2$ and $0<r<\exp(-(\chi_\mu(\sigma)+2\eta)n_1\big)$ there exist $k\leq L$ points $\omega^{(1)},\dots,\omega^{(k)}\in\tilde J_2$ such that

$\pi(\tilde{J}_2)\cap B(\pi(\omega),r)\subset\bigcup_{j=1}^k\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}}(X)$. Let $\tilde{\mu}=\mu|_{\tilde{J}_2}$ be the restriction of the measure μ to the set $\tilde{J}_2$. Using (4.1), (4.3) and (4.4) we get

$$\begin{aligned}\tilde{\mu}\circ\pi^{-1}(B(\pi(\omega),r)) &\le \sum_{j=1}^k\mu\circ\pi^{-1}\big(\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}}(X)\big)=\sum_{j=1}^k\mu\big([\omega^{(j)}|_{n(\omega^{(j)},r)}]\big)\\
&\le\sum_{j=1}^k\exp\big((-\mathrm{h}_\mu(\sigma)+\eta)n(\omega^{(j)},r)\big)\\
&=\sum_{j=1}^k\Big(\exp\big(-(\chi_\mu(\sigma)+2\eta)(n(\omega^{(j)},r)+1)\big)\Big)^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1}\cdot\frac{-\mathrm{h}_\mu(\sigma)+\eta}{-(\chi_\mu(\sigma)+2\eta)}}\\
&\le\sum_{j=1}^k\mathrm{diam}\big(\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)+1}}(X)\big)^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1}\cdot\frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta}}\le\sum_{j=1}^k r^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1}\cdot\frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta}}\\
&\le Lr^{\frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}},\end{aligned}$$

where the last inequality sign was written assuming n_1 so large that $\frac{n_1}{n_1+1}\cdot\frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta}\ge\frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}$. Hence (see [Ma], [PU]) $\mathrm{HD}(J_1)\ge\mathrm{HD}(\pi(\tilde{J}_2))\ge\frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}$ and since η was an arbitrary number $\mathrm{HD}(J_1)\ge\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$. Thus $\mathrm{HD}(\mu\circ\pi^{-1})\ge\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$ and the proof is complete. ■

Remark 4.2. Note that proving $\mathrm{HD}(\mu\circ\pi^{-1})\le\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$ we did not use the property $\mu([\omega])=\mu\circ\pi^{-1}(\phi_\omega(X))$, $\omega\in I^*$, which is equivalent with (4.1).

Remark 4.3. Note that the proof of Theorem 4.1 actually shows that the measure $\mu\circ\pi^{-1}$ has pointwise dimension $\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$. This implies that every set with positive mesure has Hausdorff dimension at least this large.

Remark 4.4. Note that it is in fact sufficient to assume in Theorem 4.1 that $\mathrm{H}_\mu(\alpha^q)<\infty$ for some $q\ge1$.

As an immediate consequence of Theorem 4.1 and Remark 4.4 we get the following.

Corollary 4.5. If $F=\{f^{(i)}:i\in I\}$ is a strongly Hölder family of functions and the series

$$\sum_{\omega\in I^q}-\tilde{m}_F(\phi_\omega(X))\log(\tilde{m}_F(\phi_\omega(X)))$$

converges for some $q\ge1$, then

$$\mathrm{HD}(m_F)=\mathrm{HD}(\mu_F)=\frac{\mathrm{h}_{\tilde{\mu}_F}(\sigma)}{\chi_{\tilde{\mu}_F}(\sigma)}.$$

We would like to end this short section with the proof of the following.

Theorem 4.6. Suppose that $\{\phi_i\}_{i\in I}$ is a regular conformal system such that $\chi_{\tilde{\mu}_{-h\zeta}} < \infty$. Suppose also that μ is a Borel ergodic probability shift-invariant measure on I^∞ such that $\mathrm{H}_\mu(\alpha) < \infty$. If $\mathrm{HD}(\mu\circ\pi^{-1}) = h := \mathrm{HD}(J)$, then $\tilde{\mu} = \tilde{\mu}_{-h\zeta}$.

Proof. If $\chi_\mu = \infty$, then it follows from Remark 4.2 that $h = \mathrm{HD}(\mu\circ\pi^{-1}) = 0$ which is a contradiction. So, $\chi_\mu < \infty$ and it follows from Remark 4.2 that $\mathrm{h}_\mu - h\chi_\mu \geq 0$. Since, in view of Theorem 3.15 of [MU1], $\mathrm{P}(-h\zeta) = \mathrm{P}(h) = 0$, we therefore deduce from (2.18), Lemma 2.14 and Theorem 2.16 with $f = -h\zeta$, that $\mu = \mu_{-h\zeta}$. The proof is complete. ■

Corollary 4.7. Suppose that $\{\phi_i\}_{i\in I}$ is a regular conformal system such that $\chi_{\tilde{\mu}_{-h\zeta}} < \infty$. Suppose also that $F = \{f^{(i)} : i : i \in I\}$, is a strongly Hölder family of functions satisfying the asumptions of Corollary 4.5 (or equivalently $\mathrm{H}_{\tilde{\mu}_F}(\alpha) < \infty$). If $\mathrm{HD}(\mu_F) = h := \mathrm{HD}(J)$, then $\tilde{\mu}_F = \tilde{\mu}_{-h\zeta}$ and the difference between the amalgamated function $f : I^\infty \to I\!R$ and the function $-h\zeta : I^\infty \to I\!R$ is cohomologous to a constant in the class of bounded Hölder continuous functions on I^∞.

Proof. Since $\mu_F = \tilde{\mu}_F\circ\pi^{-1}$, all the asuumptions of Theorem 4.6 are satisfied. It therefore follows from this theorem that $\tilde{\mu}_F = \tilde{\mu}_{-h\zeta}$. As an immediate application of Theorem 2.17 we now conclude that $f + h\zeta$ is cohomologous to a constant in the class of bounded Hölder continuous functions. The proof is complete. ■

§**5. Ionescu-Tulcea and Marinescu inequality.** In this section we come back to the setting of general hyperbolic iterated function systems explored in Section 2. We first consider the natural extension of the operator $\mathcal{L}_0$ to the space of bounded measurable functions. Let

$$\mathcal{H}_0 = \{g : g \text{ is a bounded function on } J \text{ and } g\circ\pi \text{ is continuous on } I^\infty\}.$$

Note that $\mathcal{H}_0$ is a Banach space equipped with the uniform norm $||\cdot||_0$ and $\mathcal{H}_0 \supset C_b(J)$, the space of bounded continuous functions on J. It can happen that $\mathcal{H}_0 = C_b(J)$. This would be the case for example if the map π were an open map. However, the map π is generally not open. For instance, if π were open, then according to a classical theorem of Sierpinski, J would be a G_δ set and there are examples where J is not a G_δ set [MU1, Ex. 5.1]. Thus, $\mathcal{H}_0$ is, in general, a proper enlargement of $C_b(J)$. Let us note a simple property of $\mathcal{H}_0$. If $g \in \mathcal{H}_0$ and $\omega \in I^*$, then $g\circ\phi_\omega \in \mathcal{H}_0$, since for each $\tau \in I^\infty$, $(g\circ\phi_\omega)(\tau) = g\circ\pi(\omega\tau)$. Also, for every $\alpha > 0$, let

$$\mathcal{H}_\alpha = \{g \in \mathcal{H}_0 : V_\alpha(g) < \infty\},$$

where

$$V_\alpha(g) = \sup\left\{\frac{|g\circ\pi(\omega) - g\circ\pi(\tau)|}{d_\alpha(\omega,\tau)} : \omega,\tau \in I^\infty, \omega \neq \tau \text{ and } \omega_1 = \tau_1\right\}$$

and $d_\alpha(\omega,\tau) = \mathrm{e}^{-\alpha k}$, where k is the maximal integer such that $\omega|_k = \tau|_k$. Since $\pi : I^\infty \to J$ is a surjection each $\mathcal{H}_\alpha$ is a Banach space with norm $||\cdot||_\alpha$ defined by $||g||_\alpha = ||g||_0 + V_a(g)$.

Theorem 5.1. Let $F = \{f^{(i)} : i \in I\}$ be Hölder system of functions of order β. The normalized operator $\mathcal{L}_0 : \mathcal{H}_0 \to \mathcal{H}_0$ preserves the space $\mathcal{H}_\beta$. Moreover, there are constants $0 < \gamma < 1$ and $C > 0$ and some $q_0 \in N$ such that if $q \geq q_0$, then for every $g \in \mathcal{H}_\beta$

$$||\mathcal{L}_0^q(g)||_\beta \leq \gamma ||g||_\beta + C||g||_0.$$

Proof. Let $g \in \mathcal{H}_0$ and let $n \geq 1$, we have

$$\mathcal{L}_0^n(g)(x) = \sum_{\omega \in I^n} \exp(S_\omega(F) - \mathrm{P}(F))(x) g(\phi_\omega(x)).$$

So,

$$\mathcal{L}_0^n(g)(\pi(x)) = \sum_{\omega \in I^n} \exp(S_\omega(F) - \mathrm{P}(F)n)(\pi(x)) g(\phi_\omega(\pi(x))).$$

Since this is a uniformly convergent series of functions in $\mathcal{H}_0$, $\mathcal{L}_0^n(g) \in \mathcal{H}_0$. Now assume that $g \in \mathcal{H}_\beta$. Let $x = \pi(\tau), y = \pi(\rho)$, $\tau, \rho \in I^\infty$, $\tau|_k = \rho|_k$ and $\tau_{k+1} \neq \rho_{k+1}$ for some $k \geq 1$. Then for every $n \geq 1$

$$\begin{aligned}
&\mathcal{L}_0^n(g)(y) - \mathcal{L}_0^n(g)(x) \\
&\quad = \sum_{\omega \in I^n} \exp(S_\omega(F) - \mathrm{P}(F)n)(y) g(\phi_\omega(y)) - \sum_{\omega \in I^n} \exp(S_\omega(F) - \mathrm{P}(F)n)(x) g(\phi_\omega(x)) \\
&\quad = \sum_{\omega \in I^n} \exp(S_\omega(F) - \mathrm{P}(F)n)(y) \big(g(\phi_\omega(y)) - g(\phi_\omega(x))\big) \\
&\qquad + \sum_{\omega \in I^n} g(\phi_\omega(x)) \big(\exp(S_\omega(F) - \mathrm{P}(F)n)(y) - \exp(S_\omega(F) - \mathrm{P}(F)n)(x)\big)
\end{aligned}$$

But $|g(\phi_\omega(y)) - g(\phi_\omega(x))| \leq V_\beta(g) \mathrm{e}^{-\beta(n+k)}$ and therefore employing Theorem 2.6 and using Lemma 2.2 we obtain

$$\begin{aligned}
\sum_{\omega \in I^n} \exp(S_\omega(F) - \mathrm{P}(F)n)(y) \big(g(\phi_\omega(y)) - g(\phi_\omega(x))\big) &\leq V_\beta(g) \mathrm{e}^{-\beta(n+k)} Q \\
&\leq \mathrm{e}^{-\beta n} Q ||g||_\beta d_\beta(\rho, \tau) \qquad (5.1)
\end{aligned}$$

Now notice that there exists a constant $M \geq 1$ such that $|1 - \mathrm{e}^x| \leq M|x|$ for all x with $|x| \leq \mathrm{e}^{-\beta} \log Q$. Since by Lemma 2.2, $|S_\omega(F)(x) - S_\omega(F)(y)| \leq \mathrm{e}^{-\beta k} \log Q \leq \mathrm{e}^{-\beta} \log Q$, we can make the following estimates:

$$\begin{aligned}
&|\exp(S_\omega(F) - \mathrm{P}(F)n)(y) - \exp(S_\omega(F) - \mathrm{P}(F)n)(x)| \\
&\qquad = \exp(S_\omega(F) - \mathrm{P}(F)n)(y) |1 - \exp(S_\omega(F)(x) - S_\omega(F)(y))| \\
&\qquad \leq M \exp(S_\omega(F) - \mathrm{P}(F)n)(y) |S_\omega(F)(x) - S_\omega(F)(y)| \\
&\qquad \leq M \exp(S_\omega(F) - \mathrm{P}(F)n)(y) \log Q \mathrm{e}^{-\beta k} \\
&\qquad = M \log Q \exp(S_\omega(F) - \mathrm{P}(F)n)(y) d_\beta(\rho, \tau)
\end{aligned}$$

Thus, using Theorem 2.6,

$$\sum_{\omega\in I^n} g(\phi_\omega(x))\big(\exp(S_\omega(F)-\mathrm{P}(F)n)(y)-\exp(S_\omega(F)-\mathrm{P}(F)n)(x)\big)$$

$$\le M||g||_0\log Q d_\beta(\rho,\tau)\sum_{\omega\in I^n}\exp(S_\omega(F)-\mathrm{P}(F)n)(y)$$

$$\le MQ\log Q||g||_0 d_\beta(\rho,\tau)$$

Combining this inequality and (5.1) we finally get

$$|\mathcal{L}_0^n(g)(y)-\mathcal{L}_0^n(g)(x)|\le \mathrm{e}^{-\beta n}Q||f||_\beta d_\beta(\rho,\tau)+MQ\log Q||f||_0 d_\beta(\rho,\tau).$$

Taking now n so large that $\gamma=\mathrm{e}^{-\beta n}Q<1$ finishes the proof. ■

If the unit ball in $\mathcal{H}_\beta$ were compact as a subset of the Banach space $\mathcal{H}_0$ with the supremum norm $||\cdot||_0$, we could use now the famous Ionescu-Tulcea and Marinescu Theorem (see [ITM]) to establish some useful spectral properties of the Perron-Frobenius operator $\mathcal{L}_0$. But this ball is compact only in the topology of uniform convegence on compact subsets of E^∞ and we need to proceed in a different way. This has been done in Section 4 of [MU5] (see especially Theorem 4.4) and as a result we get the following.

Theorem 5.2. Let $F=\{f^{(i)}:i\in I\}$ be Hölder system of functions of order β and let $\mathcal{L}_0$ be the associated normalized operator, then

(a) $\lambda=1$ is the only eigenvalue of modulus 1 for $\mathcal{L}_0:\mathcal{H}_0\to\mathcal{H}_0$ and its eigenspace E has dimension 1. In fact, $\psi=\frac{d\tilde{\mu}_F}{d\tilde{m}_F\circ\pi^{-1}}$ (which is well defined m_F a.e.) has a version in $\mathcal{H}_\beta$ and $E=\mathbb{C}\psi$.

(b) $\mathcal{L}_0=P+S$, where $P:\mathcal{H}_0\to E$ is a projector from $\mathcal{H}_0$ to E, $P\circ S=S\circ P=0$ and $\sup_{n\ge1}||S^n||_0<\infty$.

(c) S acts on $\mathcal{H}_\beta$ and there exist constants $M>0$ and $0<\gamma_1<1$ such that

$$||S^n||_\beta\le M\gamma_1^n$$

for every $n\ge1$.

§6. Analytical properties of topological pressure and Perron-Frobenius operator.

In this section $S=\{\phi_i:X\to X:i\in I\}$ is a regular conformal iterated function system and $\Psi=\{\psi^{(i)}:X\to\mathbb{R}:i\in I\}$ is a Hölder family of functions of order $\beta>0$. We begin with the following.

Lemma 6.1. Suppose that $\Psi=\{\psi^{(i)}:X\to\mathbb{R}:i\in I\}$ is a Hölder family of functions of order β. Then for each $R>0$, there exists a constant $M=M(\Psi,R)>0$ such that

$$|\mathrm{e}^{z\psi^{(i)}(y)}-\mathrm{e}^{z\psi^{(i)}(x)}|\le M|\mathrm{e}^{z\psi^{(i)}(x)}||\psi^{(i)}(y)-\psi^{(i)}(x)|$$

for all $z\in B(0,R)\subset\mathbb{C}$, all $i\in I$ and all $x,y\in X$.

Proof. Considering the Taylor series expansion of the function e^w we see that there exists a constant $M_1(R)$ such that $|\mathrm{e}^w - 1| \le M_1(R)|w|$ for every $w \in B\big(0, RV_\beta(\Psi)\big)$. Since $|z\psi^{(i)}(y) - z\psi^{(i)}(x)| < RV_\beta(\Psi)$, we therefore get

$$|\mathrm{e}^{z\psi^{(i)}(y)} - \mathrm{e}^{z\psi^{(i)}(x)}| = |\mathrm{e}^{z\psi^{(i)}(x)}| \left|\mathrm{e}^{z\psi^{(i)}(y) - z\psi^{(i)}(x)} - 1\right| \le |\mathrm{e}^{z\psi^{(i)}(x)}| M_1(R)R|\psi^{(i)}(y) - \psi^{(i)}(x)|$$

Thus the proof is complete by setting $M = M_1(R)R$. ■

Lemma 6.2. If Ψ is a Hölder family of functions and $\sum_{i\in I} \mathrm{e}^{\sup(t\psi^{(i)})} < \infty$ for all $t > t_0 \ge 0$, then

$$\sum_{i\in I} ||(\psi^{(i)})||_0^2 ||\mathrm{e}^{t\psi^{(i)}}||_0 < \infty$$

for all $t > t_0$.

Proof. Fix $t_0 < t_1 < t$ and $x \in X$. By our assumption, $\lim_{i\to\infty} \sup(\psi^{(i)}) = -\infty$. Since $\sup_{i\in I}(\sup(\psi^{(i)}) - \inf(\psi^{(i)})) \le V_\beta(\Psi) < \infty$, except for finitely many $i \in I$, we have $||\psi^{(i)}||_0^2 = (\inf \psi^{(i)})^2 \le \mathrm{e}^{(t_1 - t) \inf \psi^{(i)}} \le \mathrm{e}^{(t-t_1)V_b(\Psi)} \mathrm{e}^{(t_1 - t) \sup \psi^{(i)}}$. Hence, for these i

$$||(\psi^{(i)})||_0^2 ||\mathrm{e}^{t\psi^{(i)}}||_0 = ||(\psi^{(i)})||_0^2 ||\mathrm{e}^{t \sup \psi^{(i)}} \le ||\mathrm{e}^{t_1 \sup \psi^{(i)}}$$

and the proof is complete. ■

Consider now two Hölder families of functions $F = \{f^{(i)} : X \to \mathbb{C}\}_{i\in I}$ and $\Psi = \{\psi^{(i)} : X \to \mathbb{R}\}_{i\in I}$ both of order β. Of course, for all $t \in \mathbb{C}$, the family $\{f^{(i)} + t\psi^{(i)}\}_{i\in I}$ is also Hölder with order β. Let

$$\theta(\Psi, F) = \max\Bigg\{ \inf\{t \in \mathbb{R} : \sum_{i\in I} \mathrm{e}^{t \sup \psi^{(i)}} < \infty\}, \\ \inf\{t \in \mathbb{R} : \sum_{i\in I} \exp\big(\sup(\mathrm{Re} f^{(i)}) + \sup(t\psi^{(i)})\big) < \infty \Bigg\}.$$

Then for all $t \in \mathbb{C}$ with $\mathrm{Re}(t) > \theta(\Psi)$, we may consider the operator $\mathcal{L}_t : \mathcal{H}_0 \to \mathcal{H}_0$ defined by the formula

$$\mathcal{L}_t(g)(x) = \sum_{i\in I} \exp\big(f^{(i)}(x) + t\psi^{(i)}(x)\big) g(\phi_i(x)).$$

By Theorem 5.1, $\mathcal{L}_t$ preserves the space $\mathcal{H}_\beta$. Denote by $L(\mathcal{H}_\beta)$ the Banach space of bounded linear opertors on $\mathcal{H}_\beta$. We shall prove the following.

Lemma 6.3. Suppose that $\{i : \sup(\psi_i) > 0\}$ is finite. Then for all t_0 with $\mathrm{Re}(t_0) > \theta(\Psi, F)$, the function

$$t \mapsto \mathcal{L}_t \in L(\mathcal{H}_\beta)$$

is well-defined and differentiable in a neighbourhood of $t_0 \in \mathbb{C}$. Moreover, the derivative $\frac{d\mathcal{L}_t}{dt}(t_0)$ is given by the formula

$$\frac{d\mathcal{L}_t}{dt}(t_0)(g)(x) = \sum_{i \in I} \exp\bigl(f^{(i)}(x) + t_0\psi^{(i)}(x)\bigr)\psi^{(i)}(x)g(\phi_i(x)).$$

Proof. Fix $t \in \mathbb{C}$ with $\mathrm{Re}(t) > \theta(\Psi, F) \geq 0$. Assume additionally that $|t - t_0| \leq \delta_1$, where $0 < \delta_1 \leq 1$ is so small that $\mathrm{Re}(t_0) - \delta_1 > \theta(\Psi)$. For each $x \in X$, we have

$$\begin{aligned}
\frac{\mathrm{e}^{t\psi^{(i)}(x)} - \mathrm{e}^{t_0\psi^{(i)}(x)}}{t - t_0} - \psi^{(i)}(x)\mathrm{e}^{t_0\psi^{(i)}(x)} &= \mathrm{e}^{t_0\psi^{(i)}(x)}\left(\frac{\mathrm{e}^{(t-t_0)\psi^{(i)}(x)} - 1}{t - t_0} - \psi^{(i)}(x)\right) \\
&= \mathrm{e}^{t_0\psi^{(i)}(x)}\left(\sum_{n=1}^{\infty} \frac{\psi^{(i)}(x)^{(n+1)}}{(n+1)!}(t - t_0)^n\right) \\
&= \mathrm{e}^{t_0\psi^{(i)}(x)}(t - t_0)\sum_{n=0}^{\infty} \frac{\psi^{(i)}(x)^{(n+2)}}{(n+2)!}(t - t_0)^n \\
&= \mathrm{e}^{t_0\psi^{(i)}(x)}(t - t_0)\sum_{n=0}^{\infty} \frac{(\psi^{(i)}(x))^2}{(n+1)(n+2)}\frac{\psi^{(i)}(x)^n}{n!}(t - t_0)^n. \qquad (6.1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\frac{\mathrm{e}^{t\psi^{(i)}(x)} - \mathrm{e}^{t_0\psi^{(i)}(x)}}{t - t_0} - \psi^{(i)}(x)\mathrm{e}^{t_0\psi^{(i)}(x)}\right| &\leq \mathrm{e}^{\sup(\mathrm{Re}(t_0\psi^{(i)}))}|t - t_0|||\psi^{(i)}||_0^2\sum_{n=0}^{\infty}\frac{||\psi^{(i)}||_0^n}{n!}|t - t_0|^n \\
&= |t - t_0|\mathrm{e}^{\sup(\mathrm{Re}(t_0\psi^{(i)}))}||\psi^{(i)}||_0^2\mathrm{e}^{||\psi^{(i)}||_0|t-t_0|}. \qquad (6.2)
\end{aligned}$$

By our assumptions there exists a finite set $F \subset I$ such that $\sup(\psi^{(i)}) \leq 0$ for all $i \in I \setminus F$. Then for every $i \in I \setminus F$ we get

$$\begin{aligned}
\mathrm{e}^{\mathrm{Re}(t_0)\sup(\psi^{(i)})}||\psi^{(i)}||_0^2\mathrm{e}^{||\psi^{(i)}||_0\delta_1} &\leq \mathrm{e}^{\mathrm{Re}(t_0)V_\beta(\Psi)}||\psi^{(i)}||_0^2\mathrm{e}^{\mathrm{Re}(t_0)\inf(\psi^{(i)})}\mathrm{e}^{-\inf(\psi^{(i)})\delta_1} \\
&= \mathrm{e}^{\mathrm{Re}(t_0)V_\beta(\Psi)}||\psi^{(i)}||_0^2\exp\bigl((\mathrm{Re}(t_0) - \delta_1)\inf(\psi^{(i)})\bigr).
\end{aligned}$$

Since $\mathrm{Re}(t_0) > \theta(\Psi, F)$, the set $\{i \in I : \sup(\mathrm{Re}f^{(i)}) + \sup(t\psi^{(i)}) > 0\}$ is finite. Since moreover, $\mathrm{Re}(t_0) - \delta_1 > \theta(\Psi)$, it therefore follows from Lemma 6.2 that the number

$$Z = \sum_{i \in I} \mathrm{e}^{\sup(\mathrm{Re}(f^{(i)}))}\mathrm{e}^{\mathrm{Re}(t_0)\sup(\psi^{(i)})}||\psi^{(i)}||_0^2\mathrm{e}^{||\psi^{(i)}||_0\delta_1} < \infty$$

is finite. Define now the operator ∂ by the formula

$$\partial(g)(x) = \sum_{i \in I} \exp\bigl(f^{(i)}(x) + t_0\psi^{(i)}(x)\bigr)\psi^{(i)}(x)g(\phi_i(x)).$$

By Lemma 6.2, ∂ acts on the space $\mathcal{H}_0$. As the first step towards proving Lemma 6.3 we shall show that ∂ is the partial derivative of the function $(q,t) \mapsto \mathcal{L}_t \in \mathcal{L}(\mathcal{H}_0)$ with respect to the variable t at the point t_0. To show this, first fix $\varepsilon > 0$ and then set

$$\delta_2 = \min\left\{\delta_1, \frac{\varepsilon}{2Z}\right\}.$$

Using (6.1) and the definition of Z, for any function $g \in \mathcal{H}_0$, every $x \in X$ and every $t \in \mathbb{C}$ with $|t - t_0| < \delta_2$, we get

$$\begin{aligned}
&\left|\frac{\mathcal{L}_t(g)(x) - \mathcal{L}_{t_0}(g)(x)}{t - t_0} - \partial(g)(x)\right| \\
&\quad = \left|\sum_{i\in I}\left(\frac{\exp\bigl(f^{(i)}(x) + t\psi^{(i)}(x)\bigr) - \exp\bigl(f^{(i)}(x) + t_0\psi^{(i)}(x)}{t - t_0}\right.\right. \\
&\qquad\quad \left.\left. - \exp\bigl(f^{(i)}(x) + t_0\psi^{(i)}(x)\bigr)\psi^{(i)}(x)\bigr)\right) g(\phi_i(x))\right| \\
&\quad = \left|\sum_{i\in I} e^{f^{(i)}(x)}\left(\frac{e^{t\psi^{(i)}(x)} - e^{t_0\psi^{(i)}(x)}}{t - t_0} - \psi^{(i)}(x)e^{t_0\psi^{(i)}(x)}\right) g(\phi_i(x))\right| \\
&\quad \le \sum_{i\in I} e^{(\sup \mathrm{Re}(f^{(i)}(x)))}\left|\frac{e^{t\psi^{(i)}(x)} - e^{t_0\psi^{(i)}(x)}}{t - t_0} - \psi^{(i)}(x)e^{t_0\psi^{(i)}(x)}\right| ||g||_0 \\
&\quad \le |t - t_0| Z ||g||_0 \le \delta_2 ||g||_0 \le \frac{\varepsilon}{2}||g||_0.
\end{aligned}$$

Hence,

$$\left|\left|\frac{\mathcal{L}_t(g) - \mathcal{L}_{t_0}(g)}{t - t_0} - \partial(g)\right|\right|_0 \le \frac{\varepsilon}{2}||g||_0. \tag{6.3}$$

We shall now deal with the technically more complicated task concerning the Hölder norm. As a byproduct we shall prove that ∂ acts on the space $\mathcal{H}_\beta$. So, fix $g \in \mathcal{H}_\beta$ and $x =$

$\pi(\omega), y = \pi(\tau) \in J$. We then have

$$\begin{aligned}
&\mathcal{L}_t(g)(y) - \mathcal{L}_{t_0}(g)(y) - \mathcal{L}_t(g)(x) + \mathcal{L}_{t_0}(g)(x) - (\partial(g)(y) - \partial(g)(x))(t - t_0) \\
&\quad = \sum_{i\in I} \Big(\exp\big(f^{(i)}(y) + t\psi^{(i)}(y)\big) - \exp\big(f^{(i)}(y) + t_0\psi^{(i)}(y)\big)\Big) g(\phi_i(y)) \\
&\qquad - \Big(\exp\big(f^{(i)}(x) + t\psi^{(i)}(x)\big) + \exp\big(f^{(i)}(x) + t_0\psi^{(i)}(x)\big)\Big) g(\phi_i(x)) \\
&\qquad + \Big(-\psi^{(i)}(y)\exp\big(f^{(i)}(y) + t_0\psi^{(i)}(y)\big)\Big) g(\phi_i(y)) \\
&\qquad + \psi^{(i)}(x)\exp\big(f^{(i)}(x) + t_0\psi^{(i)}(x)\big)\Big) g(\phi_i(x))\Big)(t - t_0)\Big) \\
&\quad = \sum_{i\in I} \mathrm{e}^{f^{(i)}(y)} g(\phi_i(y))\big(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0)\big) \\
&\qquad - \sum_{i\in I} \mathrm{e}^{f^{(i)}(x)} g(\phi_i(x))\big(\mathrm{e}^{t\psi^{(i)}(x)} - \mathrm{e}^{t_0\psi^{(i)}(x)} - \psi^{(i)}(x)\mathrm{e}^{t_0\psi^{(i)}(x)}(t - t_0)\big) \\
&\quad = \sum_{i\in I} \mathrm{e}^{f^{(i)}(y)} \big(g(\phi_i(y)) - g(\phi_i(x))\big)\big(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0)\big) \\
&\qquad + \sum_{i\in I} g(\phi_i(x))\big(\mathrm{e}^{f^{(i)}(y)}(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0)) \\
&\qquad\qquad - \mathrm{e}^{f^{(i)}(x)}(\mathrm{e}^{t\psi^{(i)}(x)} - \mathrm{e}^{t_0\psi^{(i)}(x)} - \psi^{(i)}(x)\mathrm{e}^{t_0\psi^{(i)}(x)}(t - t_0))\big)
\end{aligned} \tag{6.4}$$

Let $|t - t_0| < \delta_2$. Applying (6.2) along with the definition of Z, we can estimate from above the absolute value of the first sum of the last formula of (6.4) as follows.

$$\begin{aligned}
\Sigma_1 &= \Big|\sum_{i\in I} \mathrm{e}^{f^{(i)}(y)} \big(g(\phi_i(y)) - g(\phi_i(x))\big)\big(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0)\big)\Big| \\
&\le Z V_\beta(g) d_\beta(\omega, \tau)|t - t_0|^2.
\end{aligned} \tag{6.5}$$

Write now the second sum Σ_2 of the last formula of (6.4) as follows

$$\begin{aligned}
\Sigma_2 &= \sum_{i\in I} g(\phi_i(x))\big(\mathrm{e}^{f^{(i)}(y)} - \mathrm{e}^{f^{(i)}(x)}\big)\big(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0)\big) \\
&\quad + \sum_{i\in I} g(\phi_i(x))\mathrm{e}^{f^{(i)}(x)}\big(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0) \\
&\qquad - \big(\mathrm{e}^{t_0\psi^{(i)}(x)} - \psi^{(i)}(x)\mathrm{e}^{t_0\psi^{(i)}(x)}(t - t_0)\big)\big).
\end{aligned} \tag{6.6}$$

Now, using (6.2) and Lemma 6.1, the absolute value of the first sum in Σ_2 can be estimated from above as follows

$$\begin{aligned}
\Sigma_3 &:= \Big|\sum_{i\in I} g(\phi_i(x))\big(\mathrm{e}^{f^{(i)}(y)} - \mathrm{e}^{f^{(i)}(x)}\big)\big(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0)\big)\Big| \\
&\le ||g||_0 \sum_{i\in I} M\mathrm{e}^{\sup(\mathrm{Re}(f^{(i)}))}|f^{(i)}y - f^{(i)}x||\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t - t_0)| \\
&\le MZ||g||_0|V_\beta(F) d_\beta(\omega, \tau)|t - t_0|^2
\end{aligned} \tag{6.7}$$

where $M = M(F, 1)$ is the number produced in Lemma 6.1. Now, in view of (6.1), we can make the following estimate of the second sum in (6.6)

$$
\begin{aligned}
\Sigma_4 := &\sum_{i\in I} g(\phi_i(x))\mathrm{e}^{f^{(i)}(x)}\big(\mathrm{e}^{t\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(y)} - \psi^{(i)}(y)\mathrm{e}^{t_0\psi^{(i)}(y)}(t-t_0)\\
&\quad - \big(\mathrm{e}^{t\psi^{(i)}(x)} - \mathrm{e}^{t_0\psi^{(i)}(x)} - \psi^{(i)}(x)\mathrm{e}^{t_0\psi^{(i)}(x)}(t-t_0)\big)\big)\\
= &\sum_{i\in I} g(\phi_i(x))\mathrm{e}^{f^{(i)}(x)}\Bigg((t-t_0)^2\mathrm{e}^{t_0\psi^{(i)}(y)}\sum_{n=0}^{\infty}\frac{(\psi^{(i)}(y))^2}{(n+1)(n+2)}\cdot\frac{(\psi^{(i)}(y))^n}{n!}(t-t_0)^n\\
&\quad - (t-t_0)^2\mathrm{e}^{t_0\psi^{(i)}(x)}\sum_{n=0}^{\infty}\frac{(\psi^{(i)}(x))^2}{(n+1)(n+2)}\cdot\frac{(\psi^{(i)}(x))^n}{n!}(t-t_0)^n\Bigg)\\
= &\sum_{i\in I}(t-t_0)^2 g(\phi_i(x))\mathrm{e}^{f^{(i)}(x)}\Bigg(\Big(\mathrm{e}^{t_0\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(x)}\Big)\\
&\cdot(\psi^{(i)}(y))^2\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\frac{(\psi^{(i)}(y))^n}{n!}(t-t_0)^n\\
&+ \mathrm{e}^{t_0\psi^{(i)}(x)}\sum_{n=0}^{\infty}\frac{(\psi^{(i)}(y))^{n+2} - (\psi^{(i)}(x))^{n+2}}{(n+2)!}(t-t_0)^n\Bigg).
\end{aligned}
\tag{6.8}
$$

We next estimate the two sums in (6.8). Using Lemma 6.1, (6.1) and the definition of Z, we get with $P = M(\Psi, |t_0|)$

$$
\begin{aligned}
\Sigma_5 := &\Bigg|\sum_{i\in I}(t-t_0)^2 g(\phi_i(x))\mathrm{e}^{f^{(i)}(x)}\left(\mathrm{e}^{t_0\psi^{(i)}(y)} - \mathrm{e}^{t_0\psi^{(i)}(x)}\right)\cdot\\
&\cdot(\psi^{(i)}(y))^2\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\frac{(\psi^{(i)}(y))^n}{n!}(t-t_0)^n\Bigg| \le\\
\le &|t-t_0|^2||g||_0\sum_{i\in I}\mathrm{e}^{\sup(\mathrm{Re}(f^{(i)}(x))}P\mathrm{e}^{\sup(\mathrm{Re}(t_0\psi^{(i)}))}|\psi^{(i)}(y)-\psi^{(i)}(x)|||\psi^{(i)}||_0^2\sum_{n=0}^{\infty}\frac{|\psi^{(i)}(y))|^n}{n!}|t-t_0|^n\\
\le &|t-t_0|^2 V_\beta(\Psi)d_\beta(\omega,\tau)||g||_0 P\sum_{i\in I}\mathrm{e}^{\sup(\mathrm{Re}(f^{(i)}(x))}||\psi^{(i)}||_0^2\mathrm{e}^{\sup(\mathrm{Re}(t_0\psi^{(i)}))}\mathrm{e}^{|\psi^{(i)}(y)||t-t_0|}\\
\le &|t-t_0|^2 V_\beta(\Psi)d_\beta(\omega,\tau)||g||_0 P\sum_{i\in I}\mathrm{e}^{\sup(\mathrm{Re}(f^{(i)})}||\psi^{(i)}||_0^2\mathrm{e}^{\sup(\mathrm{Re}(t_0\psi^{(i)}))}\mathrm{e}^{||\psi^{(i)}||_0|t-t_0|}\\
\le &ZMP||g||_0 V_\beta(\Psi)d_\beta(\omega,\tau)|t-t_0|^2
\end{aligned}
$$

Finally, we estimate the second sum in (6.8):

$$\Big|\sum_{i\in I}(t-t_0)^2 g(\phi_i(x))\mathrm{e}^{f^{(i)}(x)}\mathrm{e}^{t_0\psi^{(i)}(x)}\sum_{n=0}^{\infty}\frac{(\psi^{(i)}(y))^{n+2}-(\psi^{(i)}(x))^{n+2}}{(n+2)!}(t-t_0)^n\Big|$$
$$\le |t-t_0|^2||g||_0\sum_{i\in I}\exp\big(\sup(\mathrm{Re}(f^{(i)}))+$$
$$+\sup(\mathrm{Re}(t_0\psi^{(i)}))\big)\sum_{n=0}^{\infty}|\psi^{(i)}(y)-\psi^{(i)}(x)|\frac{(n+2)||\psi^{(i)}||_0^{n+1}}{(n+2)!}|t-t_0|^n$$
$$\le |t-t_0|^2||g||_0\sum_{i\in I}\exp\big(\sup(\mathrm{Re}(q\phi^{(i)})+\sup(\mathrm{Re}(t_0\psi^{(i)})\big)V_\beta(\Psi)V_\beta(\Psi)||\psi^{(i)}||_0\sum_{n=0}^{\infty}\frac{||\psi^{(i)}||_0^n}{n!}|t-t_0|^n$$
$$\le |t-t_0|^2V_\beta(\Psi)d_\beta(\omega,\tau)||g||_0\sum_{i\in I}\exp\big(\sup(\mathrm{Re}(q\phi^{(i)})+\sup(\mathrm{Re}(t_0\psi^{(i)})\big)||\psi^{(i)}||_0\mathrm{e}^{||\psi^{(i)}||_0|t-t_0|}$$
$$\le Z'||g||_0V_\beta(\Psi)d_\beta(\omega,\tau)|t-t_0|^2,$$

where Z' has the same meaning as Z with $||\psi^{(i)}||_0^2$ replaced by $||\psi^{(i)}||_0$. Combining now the above estimates we get

$$\left|\left(\frac{\mathcal{L}_t(g)(y)-\mathcal{L}_{t_0}(g)(y)}{t-t_0}-\partial(g)(y)\right)-\left(\frac{\mathcal{L}_t(g)(x)-\mathcal{L}_{t_0}(g)(x)}{t-t_0}-\partial(g)(x)\right)\right|\le$$
$$\le \big(ZV_\beta(g)+MZ||g||_0V_\beta(F)+PZV_\beta(\Psi)||g||_0+V_\beta Z'||g||_0\big)d_\beta(\omega,\tau)|t-t_0|$$
$$\le (Z+MZV_\beta(F)+ZPV_\beta(\Psi)+V_\beta(\Psi)Z')||g||_\beta d_\beta(\omega,\tau)|t-t_0|$$

Hence,

$$V_\beta\left(\frac{\mathcal{L}_t(g)-\mathcal{L}_{t_0}(g)}{t-t_0}-\partial(g)\right)$$
$$\le \big(Z+MZV_\beta(F)+ZPV_\beta(\Psi)+V_\beta(\Psi)Z'\big)|t-t_0|||g||_\beta. \tag{6.9}$$

Taking now

$$\delta_3=\min\left\{\delta_2,\big(Z+MZV_\beta(F)+ZPV_\beta(\Psi)+V_\beta(\Psi)Z'\big)^{-1}\frac{\varepsilon}{2}\right\}$$

and combining (6.9) and (6.3) we conclude that for every $t\in\mathbb{C}$ with $|t-t_0|<\delta_3$ and every $g\in\mathcal{H}_\beta$

$$\left|\left|\frac{\mathcal{L}_t(g)-\mathcal{L}_{t_0}(g)}{t-t_0}-\partial(g)\right|\right|_\beta\le\varepsilon||g||_\beta.$$

Therefore, for these t,

$$\left|\left|\frac{\mathcal{L}_t-\mathcal{L}_{t_0}}{t-t_0}-\partial\right|\right|_\beta\le\varepsilon.$$

The proof is complete. ■

Combining Lemma 6.3, Theorem 5.2 (which tells us that 1 is an isolated simple eigenvalue) and the perturbation theory of analytic dependence of an isolated simple eigenvalue (see [Ka]), we get the following.

Theorem 6.4. If Ψ and F are two real-valued Hölder families of functions such that the sets $\{i \in I : \sup(\psi^{(i)}) > 0\}$ and $\{i \in I : \sup(f^{(i)}) > 0\}$ are finite, then the function $(q,t) \mapsto \mathrm{P}(q,t) = \mathrm{P}(qF + t\Psi)$, $q \in (\theta(F), \infty)$, $t \in (\theta(\Psi, qF), \infty)$ is real-analytic with respect to both variables q and t.

Our next aim is to calculate the partial derivative $\frac{\partial P(q,t)}{\partial q}$. This is done in the following.

Proposition 6.5. If $q_0 \in (\theta(F), \infty)$, F and Ψ are two Hölder families of functions such that $q_0 F + \Psi$ is strongly Hölder and $\int (|f| + |\psi|) d\tilde{\mu} < \infty$, where $\tilde{\mu} = \tilde{\mu}_{q_0 F + \Psi}$, then

$$\frac{d\mathrm{P}}{dq}(q_0) = \int f \, d\tilde{\mu}.$$

Proof. By Theorem 6.4 we know that the derivative $\frac{d\mathrm{P}}{dq}(q_0)$ exists. Since the function $q \mapsto \mathrm{P}(q)$ is convex, in order to complete the proof is therefore enough to demonstrate that

$$\mathrm{P}(q) \geq \mathrm{P}(q_0) + \int f d\tilde{\mu}(q - q_0)$$

on an open neighbourhood of q_0. An indeed, in view of our assumptions and Theorem 2.16, for every $q \in I\!R$ we have

$$\begin{aligned} \mathrm{P}(q) &\geq \mathrm{h}_{\tilde{\mu}} + \int (qf + \psi) d\tilde{\mu} = \mathrm{h}_{\tilde{\mu}} + \int (q_0 f + \psi) d\tilde{\mu} + \int f d\tilde{\mu}(q - q_0) \\ &= \mathrm{P}(q_0) + \int f d\tilde{\mu}(q - q_0). \end{aligned}$$

The proof is complete. ■

Fix now $F, \Psi \in \mathcal{H}_\beta$, $q \in (\theta(F), \infty)$, $t \in (\theta(\Psi, qF), \infty)$ and suppose that $\int (|f| + |\psi|) d\tilde{\mu}_{q,t} < \infty$, where $\tilde{m}u_{q,t} = \tilde{\mu}_{qF + t\Psi}$,

Using the notation, $\int g d\mu_{q,t} = \mu_{q,t}(g)$, set

$$\sigma^2_{q,t}(f, \psi) = \sum_{k=0}^{\infty} \left(\tilde{\mu}_{q,t}(f \cdot \psi \circ \sigma^k) - \tilde{\mu}_{q,t}(f) \tilde{\mu}_{q,t}(\psi) \right) = \sum_{k=0}^{\infty} \left(\tilde{\mu}_q(\psi \cdot f \circ \sigma^k) - \tilde{\mu}_{q,t}(f) \mu_q(\psi) \right).$$

If $f = \psi$ we simply write $\sigma^2_{q,t}(f)$ for $\sigma^2_{q,t}(f, f)$. The last result in this section is the following.

Proposition 6.6. If $q_0 \in (\theta(F), \infty)$, $t_0 \in (\theta(\Psi), \infty)$ and $\int(|f| + |\psi|)d\tilde{\mu}_{q_0,t_0} < \infty$, then

$$\frac{\partial^2 \mathrm{P}}{\partial q \partial t}|_{(q_0,t_0)} = \lim_{n\to\infty} \frac{1}{n} \int S_n(f - \int f d\tilde{\mu}_{q_0,t_0}) S_n(\psi - \int \psi d\tilde{\mu}_{q_0,t_0}) = \sigma^2_{q,t}(f,\psi).$$

Sketch of a proof. We introduce the Perron-Frobenius operator acting on the shift space I^∞ and using Theorem 5.2 we proceed similarly as in the case of a subshift of finite type over a finite alphabet (see [Ru], [PU]). ■

§**7. Multifractal Analysis.** In this section $S = \{\phi_i : X \to X : i \in I\}$ is a regular conformal iterated function system such that

$$\phi_i(X) \cap \phi_j(X) \text{ is at most countable} \tag{7.1}$$

for all $i \neq j \in I$ and $F = \{f^{(i)} : X \to I\!R : i \in I\}$ is a strongly Hölder family of functions. Subtracting from each of the functions $f^{(i)}$ the topological pressure of F we may assume that $\mathrm{P}(F) = 0$. We consider a two-parameter family of Hölder continuous families of functions

$$G_{q,t} = \{g^{(i)}_{q,t} := qf^{(i)} + t\log|\phi_i'|\}$$

Let

$$\mathrm{Fin}(F) = \{q \in I\!R : \mathcal{L}_{qF}(1\!\!1) < \infty\} = \{q \in I\!R : \mathrm{P}(qF) < \infty\} \text{ and } \theta(F) = \inf \mathrm{Fin}(F).$$

By the definition of strongly Hölder families of functions, $1 \in \mathrm{Fin}(F)$ and, in particular $\{i : \sup f^{(i)} > 0\}$ is finite. Before dealing with smoothness properties, we shall prove the following result which will be needed in the next section.

Lemma 7.1. The function $(q,t) \mapsto \mathrm{P}(q,t) := \mathrm{P}(G_{q,t})$ is decreasing with respect to both variables $q \geq 0$ and $t \geq 0$.

Proof. Consider now two pairs (q_1,t_1) and (q_2,t_2) such that $q_1 \leq q_2$ and $t_1 \leq t_2$. If $\mathrm{P}(q_1,t_1) = \infty$, there is nothing to be proved. So, suppose that $\mathrm{P}(q_1,t_1) < \infty$. Then G_{q_1,t_1} is a strongly Hölder family of functions. Since the set $\{i : \sup f^{(i)} > 0\}$ is finite and since all the functions $\log|\phi_i'|$ are negative, this implies that G_{q_2,t_2} also forms a strongly Hölder family of functions. It then follows from (2.18) that for every $\varepsilon > 0$ there exists a Borel probability measure μ on I^∞ such that $\int -(q_2 f - t_2\zeta)d\mu < \infty$ (which implies that $\int -(q_1 f - t_1\zeta)d\mu < \infty$) and

$$\begin{aligned}
P(q_2,t_2) &\leq \mathrm{h}_\mu + \int (q_2 f - t_2\zeta)d\mu + \varepsilon \\
&= \mathrm{h}_\mu + \int (q_1 f - t_1\zeta)d\mu + (q_2 - q_1)\int f d\mu + (t_1 - t_2)\int \zeta d\mu + \varepsilon \\
&\leq \mathrm{h}_\mu + \int (q_1 f - t_1\zeta)d\mu + \varepsilon \leq \mathrm{P}(q_1,t_1) + \varepsilon,
\end{aligned}$$

where the last inequality we wrote due to (2.18). Letting $\varepsilon \searrow 0$ we thus get $\mathrm{P}(q_2, t_2) \leq \mathrm{P}(q_1, t_1)$. The proof is complete. ■

Given $q \geq 0$ let

$$\mathrm{Fin}(q) = \{t : \mathcal{L}_{G_{q,t}}(\mathbb{1}) < \infty\} = \{t : \mathrm{P}(G_{q,t}) < \infty\} \text{ and let } \theta(q) = \inf \mathrm{Fin}(q).$$

So, $\theta(q) \leq \theta(S)$. Notice that if $q \in \mathrm{Fin}(F)$, then $0 \in \mathrm{Fin}(q)$. We assume that for every $q \in \mathrm{Fin}(F)$ there exists $u \in \mathrm{Fin}(q)$ such that

$$0 < \mathrm{P}(G_{q,u}) < \infty. \tag{7.2}$$

We shall prove the following.

Lemma 7.2. If $q \in \mathrm{Fin}(F)$, then there exists a unique $t = T(q)$ such that $\mathrm{P}(G_{q,T(q)}) = 0$. In addition $T(q) \in (\theta(q), \infty)$.

Proof. Fix $q > \theta(F)$. Since for every $n \geq 1$ the function $t \mapsto \sum_{|\omega|=n} ||\exp\big(\sum_{j=1}^n \phi_{q,t}^{\omega_j} \circ \phi_{\sigma^j\omega}\big)||_0$, $t \in \mathrm{Fin}(q)$, is logarithmic convex, the function $t \mapsto \mathrm{P}(G_{q,t})$ is convex and hence continuous in $(\theta(q), \infty)$. Since $0 < \mathrm{P}(G_{q,u}) < \infty$ for some $u \in \mathrm{Fin}(q)$, in order to conclude the proof it therefore suffices to show that the function $t \mapsto \mathrm{P}(G_{q,t})$ is strictly decreasing on $t \in (\theta(q), \infty)$ and $\lim_{t\to+\infty} \mathrm{P}(G_{q,t}) = -\infty$. But for every $t \geq u$

$$\begin{aligned}
\mathrm{P}(G_{q,t}) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\omega|=n} |||\phi_\omega'|^t \exp(S_\omega(qF))||_0 \right) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\omega|=n} ||\phi_\omega'||_0^{t-u} |||\phi_\omega'|^u \exp(S_\omega(qF))||_0 \right) \\
&\leq \liminf_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\omega|=n} s^{n(t-u)} |||\phi_\omega'|^u \exp(S_\omega(qF))||_0 \right) = (t-u)\log s + \mathrm{P}(G_{q,u})
\end{aligned}$$

Hence $t \in \mathrm{Fin}(q)$ and moreover, as $s < 1$, $\lim_{t\to+\infty} \mathrm{P}(G_{q,t}) = -\infty$. To prove that $\mathrm{P}(G_{q,t})$ is strictly decreasing consider $t > \theta(q)$ and $\delta > 0$. We then have

$$\begin{aligned}
\mathrm{P}(G_{q,t+\delta}) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\omega|=n} |||\phi_\omega'|^{t+\delta} \exp(S_\omega(q\phi))||_0 \right) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\omega|=n} ||\phi_\omega'||_0^{\delta} |||\phi_\omega'|^t \exp(S_\omega(qF))||_0 \right) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \left(s^{n\delta} |||\phi_\omega'|^t \exp(S_\omega(qF))||_0 \right) \\
&= \delta \log s + \mathrm{P}(G_{q,t}) < \mathrm{P}(G_{q,t}).
\end{aligned}$$

The proof is complete. ■

Given $q \in \mathrm{Fin}(F)$ and $t \in \mathrm{Fin}(q)$, let

$$\mu_{q,t} = \mu_{G_{q,t}},\ \tilde{\mu}_{q,t} = \tilde{\mu}_{G_{q,t}},\ m_{q,t} = m_{G_{q,t}},\ \tilde{m}_{q,t} = \tilde{m}_{G_{q,t}}$$

and

$$\mu_q = \mu_{q,T(q)},\ \tilde{\mu}_q = \tilde{\mu}_{q,T(q)},\ m_q = m_{q,T(q)},\ \tilde{m}_q = \tilde{m}_{q,T(q)}$$

and let

$$\alpha(q) = \frac{\int f d\tilde{\mu}_q}{-\int \zeta d\tilde{\mu}_q}$$

if $\int |f| d\tilde{\mu}_q < \infty$. By (7.1), $\pi : I^\infty \to J$ is 1-to-1, so given $x = \pi(\omega)$ we can speak about x_n and $x|_n$ respectively as ω_n and $\omega|_n$. Given μ, a Borel probability measure on J and $x \in J$ we define

$$\underline{D}_\mu(x) = \liminf_{n\to\infty} \frac{\log\big(\mu(\phi_{x|_n}(X))\big)}{\log\big(\mathrm{diam}(\phi_{x|_n}(X))\big)},$$

$$\overline{D}_\mu(x) = \limsup_{n\to\infty} \frac{\log\big(\mu(\phi_{x|_n}(X))\big)}{\log\big(\mathrm{diam}(\phi_{x|_n}(X))\big)},$$

and

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log\big(\mu(B(x,r)\big)}{\log r},$$

$$\overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log\big(\mu(B(x,r)\big)}{\log r},$$

If $\overline{D}_\mu(x) = \underline{D}_\mu(x)$ we denote the common value by $D_\mu(x)$ and if $\overline{d}_\mu(x) = \underline{d}_\mu(x)$, the common value is denoted by $d_\mu(x)$. Given $\alpha \geq 0$ we define

$$K_\mu(\alpha) = \{x \in J : D_\mu(x) = \alpha\}$$

and

$$f_\mu(\alpha) = \mathrm{HD}(K_\mu(\alpha)),$$

the Hausdorff dimension of the set $K_\mu(\alpha)$. Let k be a strictly convex map on an interval I, hence $k'' > 0$ wherever this second derivative exists. The Legendre transform of k is the function l of a new variable p defined by

$$l(p) = \max_I\{px - k(x)\}$$

everywhere where the maximum exists. it can be proved that the domain of l is either a point, an interval or a half-line. It is also easy to show that l is strictly convex and that the Legendre transform is involutive. We then say that the functions k and l form a Legendre transform pair. The following theorem (see [Ro] for example) gives a useful characterization of a Legendre transform pair.

Theorem 7.3. Two strictly convex differentiable functions k and l form a Legendre transform pair if and only if $l(-k'(q)) = k(q) - qk'(q)$.

Our main result in this section is the following.

Theorem 7.4. Suppose that condition (7.1) is satisfied for all $q \in \mathrm{Fin}(F)$. Suppose also that there exists an interval $\Delta_1 \subset \mathrm{Fin}(F)$ such that $1 \in \Delta_1$ and for every $q \in \Delta_1$ and all t in some neighbourhood of $T(q)$ (contained in $(\theta(q), \infty)$),

$$\int (|f|^{2+\gamma} + |\zeta|^{2+\gamma}) d\tilde{\mu}_q < \infty \quad \text{and} \quad \int (|f| + |\zeta|) d\tilde{\mu}_{q,t} < \infty.$$

for some $\gamma > 0$. Suppose finally that $\mathrm{h}_{\mu_q}(\sigma)/\chi_{\mu_q}(\sigma) > \theta(S)$ for all $q \in \Delta_2 \subset \Delta_1$ for some interval $\Delta_2 \subset \Delta_1$. Then

(a) The number $D_{\mu_F}(x)$ exists for μ_F-a.e. $x \in J$ and

$$D_{\mu_F}(x) = \frac{-\int f d\tilde{\mu}_F}{\int \zeta d\tilde{\mu}_F}.$$

(b) The function $T : \Delta_1 \to I\!\!R$ is real-analytic, $T(0) = \mathrm{HD}(J)$, and $T'(q) < 0$, $T''(q) \geq 0$ for all $q \in \Delta_1$.

(c) For every $q \in \Delta_2$, $f_{\mu_F}(-T'(q)) = T(q) - qT'(q)$.

(d) If $\tilde{\mu}_F \neq \tilde{\mu}_{-\mathrm{HD}(J)\zeta}$, then the function $\alpha \mapsto f_{\mu_F}(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$ is real-analytic, where the interval (α_1, α_2), $0 \leq \alpha_1 < \alpha_2 \leq \infty$ is the range of the function $-T'(q)$ defined on the interval Δ_2. Otherwise $T'(q) = \mathrm{HD}(J)$ for every $q \in (\theta(F), \infty)$.

(e) If $\tilde{\mu}_F \neq \tilde{\mu}_{-\mathrm{HD}(J)\zeta}$, then the functions $f_{\mu_F}(\alpha)$ and $T(q)$ form a Legendre transform pair.

(f) For every $q \in \Delta_1$ the number $T(q)$ is uniquely determined by the property that there exists a constant $C \geq 1$ such that for every $n \geq 1$

$$C^{-1} \leq \sum_{|\omega|=n} \tilde{\mu}_F^q([\omega]) \mathrm{diam}^{T(q)}(\phi_\omega(X)) \leq C.$$

Proof. Since $1 \in \Delta_1$ and $\int |f| d\tilde{\mu}_F < \infty$, part (a) is a combined consequence of Birkhoff's ergodic theorem (along with (BDP)), the Breimann-McMillan-Shanon theorem and the assumption that $\mathrm{P}(F) = 0$. We shall now prove part (b). And indeed, since by Proposition 6.5, $\frac{\partial P}{\partial t}|_{q,t} = -\int \zeta d\tilde{\mu}_{q,t} < 0$ for every $q \in \Delta_1$ and all t in a neighbourhood of q, and since $T(q)$ is uniquely determined by the condition $\mathrm{P}(q, T(q)) = 0$, it follows from Theorem 6.4 and the Implicit Function Theorem that the map $q \mapsto T(q)$ is real-analytic on Δ_1. Since the system F is regular, $\mathrm{P}(-\mathrm{HD}(J)\zeta)=0$ which means that $T(0)=\mathrm{HD}(J)$. It follows from Proposition 6.5 that for every $q\in\Delta_1$

$$0=\frac{d\mathrm{P}}{dq}(q,T(q))=\frac{\partial \mathrm{P}}{\partial q}|_{(q,T(q))}+\frac{\partial \mathrm{P}}{\partial t}|_{(q,T(q))}T'(q)=\int \phi d\tilde{\mu}_q-\int g d\tilde{\mu}_q T'(q)$$

and therefore

$$T'(q)=\frac{\int f d\tilde{\mu}_q}{\int \zeta d\tilde{\mu}_q}=-\alpha(q). \tag{7.3}$$

Since $\mathrm{P}(f)=0$ and $\int f d\tilde{\mu}_q<\infty$, we deduce from Theorem 2.16 that $\int f d\tilde{\mu}_q+\mathrm{h}_{\tilde{\mu}_q}(\sigma)\le 0$ and therefore it follows from (7.3) that $T'(q)\le -\mathrm{h}_{\tilde{\mu}_q}(\sigma)/\int\zeta d\tilde{\mu}_q\le 0$. Thus to prove that $T'(q)<0$ it suffices to demonstrate that $\mathrm{h}_{\tilde{\mu}_q}(\sigma)>0$. And indeed, in order to see it one can proceed similarly as in [Bo] to show that the dynamical system $(\sigma,\tilde{\mu}_q)$ is weakly-Bernoulli and consequently has a positive entropy. Using the results concerning the Perron-Frobenius operator proved in Section 6 one can also similarly as in Lemma 1.14 of [Bo] derive in our context its last display and then proceed inductively with fixed $n-s$ large enough to show that $\tilde{\mu}_q([\omega|_n])$ converges to zero uniformly exponentially fast which in view of Shanonn-Breimann-McMillan theorem implies that $\mathrm{h}_{\tilde{\mu}_q}(\sigma)>0$. Hence, to complete the proof of Theorem 7.4(b) it is left to show that $T''(q)\ge 0$ for all $q\in\Delta_1$. This is done in the following.

Lemma 7.5. The function $q\mapsto T(q)$, $q\in\Delta_1$ is convex. It is not strictly convex if and only if $\tilde{\mu}_f$ is equal to $\tilde{\mu}_{-\mathrm{HD}(J)\zeta}$.

Proof. Differentiating the formula

$$0=\frac{\partial \mathrm{P}(q,t)}{\partial t}|_{(q,T(q))}\cdot T'(q)+\frac{\partial \mathrm{P}(q,t)}{\partial q}|_{(q,T(q))}$$

and using Proposition 6.5 we obtain

$$\begin{aligned}T''(q)&=-\frac{T'(q)^2\frac{\partial^2\mathrm{P}(q,t)}{\partial t^2}+2T'(q)\frac{\partial^2\mathrm{P}(q,t)}{\partial q\partial t}+\frac{\partial^2\mathrm{P}(q,t)}{\partial q^2}}{\frac{\partial\mathrm{P}(q,t)}{\partial t}}\\&=\frac{T'(q)^2\frac{\partial^2\mathrm{P}(q,t)}{\partial t^2}+2T'(q)\frac{\partial^2\mathrm{P}(q,t)}{\partial q\partial t}+\frac{\partial^2\mathrm{P}(q,t)}{\partial q^2}}{\chi_{\tilde{\mu}_q}},\end{aligned}$$

where, let us recall, $\chi_{\tilde{\mu}_q}=\int\zeta d\tilde{\mu}_q$ is the Lyapunov charatceristic exponent of the measure $\tilde{\mu}_q$. Invoking Proposition 6.6 we see that

$$\frac{\partial^2\mathrm{P}}{\partial t^2}=\sigma^2_{\tilde{\mu}_q}(-\zeta),\ \frac{\partial^2\mathrm{P}}{\partial q\partial t}=\sigma^2_{\tilde{\mu}_q}(\zeta,f),\ \frac{\partial^2\mathrm{P}}{\partial q^2}=\sigma^2_{\mu_q}(f).$$

Hence, we can write

$$
\begin{aligned}
&T'(q)^2\frac{\partial^2 \mathrm{P}}{\partial t^2} + 2T'(q)\frac{\partial^2 \mathrm{P}}{\partial q\partial t} + \frac{\partial^2 \mathrm{P}}{\partial q^2} = \\
&= T'(q)^2\sum_{k=0}^{\infty}\big(\tilde\mu_q(\zeta\cdot\zeta\circ\sigma^k) - \chi_{\tilde\mu_q}^2\big) + T'(q)\sum_{k=0}^{\infty}\big(\tilde\mu_q(-\zeta\cdot f\circ\sigma^k) \\
&+ \chi_{\tilde\mu_q}\tilde\mu_q(f)\big) + T'(q)\sum_{k=0}^{\infty}\big(\tilde\mu_q(f(-\zeta\circ\sigma^k)) + \chi_{\tilde\mu_q}\tilde\mu_q(f)\big) + \sum_{k=0}^{\infty}\big(\tilde\mu_q(f\cdot f\circ\sigma^k) - \tilde\mu_q(f)^2\big) \\
&= \sum_{k=0}^{\infty}\tilde\mu_q\big(-T'(q)\zeta(-T'(q)\zeta\circ\sigma^k + f\circ\sigma^k)\big) \\
&+ \sum_{k=0}^{\infty}\tilde\mu_q\big(f(-T'(q)\zeta(-T'(q)\zeta\circ\sigma^k + f\circ\sigma^k)\big) - \sum_{k=0}^{\infty}\big(-T'(q)\chi_{\tilde\mu_q} + \tilde\mu_q(f)\big)^2 \\
&= \sum_{k=0}^{\infty}\tilde\mu_q\big((-T'(q)\zeta + f)(-T'(q)\zeta + f)\circ\sigma^k - (-T'(q)\chi_{\tilde\mu_q} + \tilde\mu_q(f))^2 \\
&= \sigma_{\tilde\mu_q}^2(-T'(q)\zeta + f).
\end{aligned}
$$

It follows then from (7.3) that $\int(-T'(q)\zeta + f)d\tilde\mu_q = 0$. In view of Proposition 6.6 $\sigma_{\tilde\mu_q}^2(-T'(q)\zeta + f) \geq 0$ and it follows from Lemma 6.7 in [Ur] that $\sigma_{\tilde\mu_q}^2(-T'(q)\zeta + f) = 0$ if and only if the function $-T'(q)\zeta + f$ is cohomologous to 0 in the class of bounded Hölder continuous functions. Therefore $T'(q)\zeta$ is cohomologous to f and, as $\mathrm{P}(f) = 0$, also $\mathrm{P}(T'(q)\zeta) = 0$. Thus, by Theorem 1.1, $T'(q) = -\mathrm{HD}(J)$ and consequently f is cohomologous to the function $-\mathrm{HD}(J)\zeta$. This implies that $\tilde\mu_f = \tilde\mu_{-\mathrm{HD}(J)\zeta}$, the latter being the equilibrium (invariant Gibbs) state of the potential $-\mathrm{HD}(J)\zeta$. The proof is complete. ■

So, item (b) of Theorem 7.4 is now an immediate consequence of Lemma 7.5. We shall now focus on a contribution toward the proof of part (c)-(e). Given $\alpha \geq 0$ we define

$$
\tilde K(\alpha) = \left\{x \in J : \lim_{n\to\infty}\frac{\sum_{j=0}^{n-1} f\circ\sigma^j(x)}{\sum_{j=0}^{n-1} -\zeta\circ\sigma^j(x)} = \alpha\right\}.
$$

We shall prove the following.

Lemma 7.6. For every $\alpha \geq 0$, $\tilde K(\alpha) = K_{\mu_f}(\alpha)$.

Proof. In order to prove this lemma it suffices to show that for all $x \in J$

$$
\lim_{n\to\infty}\frac{\log\big(\mu_f(\phi_{x|_n}(X))\big)}{\sum_{j=0}^{n-1} f\circ\sigma^j(x)} = 1 \text{ and } \lim_{n\to\infty}\frac{\log\big(\mathrm{diam}(\phi_{x|_n}(X))\big)}{\sum_{j=0}^{n-1} -\zeta\circ\sigma^j(x)} = 1.
$$

And in order to prove the first equality it suffices to demonstrate that

$$
\lim_{n\to\infty}\frac{\log\big(m_F(\phi_{x|_n}(X))\big)}{\sum_{j=0}^{n-1} f\circ\sigma^j(x)} = 1
$$

and this follows immediately from Corllary 2.12, (2.8), Lemma 2.2 and the fact that $\mathrm{P}(F) = 0$. The second inequality to be proved is an immediate consequence of (BDP). The proof is complete. ■

Lemma 7.7. If $x \in \tilde{K}(\alpha)$ and

$$\liminf_{n\to\infty} \frac{\log|\phi'_{x_n}(\sigma^n(x))|}{\log|\phi'_{x|_{n-1}}(\sigma^{n-1}(x))|} = 0,$$

then for every $q \in I\!R$

$$\liminf_{n\to\infty} \left(\frac{\log|\phi'_{x_n}(\sigma^n(x))|}{\log|\phi'_{x|_{n-1}}(\sigma^{n-1}(x))|} + \frac{qf(\sigma^{n-1}x)}{\log|\phi'_{x|_{n-1}}(\sigma^{n-1}(x))|} \right) \le 0,$$

Proof. If $q \le 0$, then our inequality follows immediately from (2.2), our assumption and the formula $\lim_{n\to\infty} \log||\phi'_{x|_{n-1}}|| = -\infty$. So, we may assume that $q > 0$. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing infinite sequence such that

$$\lim_{k\to\infty} \frac{\log|\phi'_{x_{n_k}}(\sigma^{n_k}(x))|}{\log|\phi'_{x|_{n_k-1}}(\sigma^{n_k-1}(x))|} = 0 \tag{7.4}$$

In order to conclude the proof it suffices to sow that

$$\lim_{k\to\infty} \frac{f(\sigma^{n_k-1}x)}{\log|\phi'_{x|_{n_k-1}}(\sigma^{n_k-1}(x))|} \le 0.$$

So, suppose on the contrary, that

$$\limsup_{k\to\infty} \frac{f(\sigma^{n_k-1}x)}{\log|\phi'_{x|_{n_k-1}}(\sigma^{n_k-1}(x))|} \ge 2b > 0$$

for some positive b. Passing to a subsequence of the sequence $\{n_k\}_{k=1}^{\infty}$ we may assume that the limit $\lim_{k\to\infty} \frac{f(\sigma^{n_k-1}x)}{\log|\phi'_{x|_{n_k-1}}(\sigma^{n_k-1}(x))|}$ exists and is greater than or equal to $2b$ (perhaps $+\infty$). This, (7.4) and the fact that $x \in \tilde{K}(\alpha)$ imply the existence of an integer $l_0 \ge 1$ such that for every $l \ge l_0$

$$\frac{\sum_{j=0}^{l} -f(\sigma^j x)}{\sum_{j=0}^{l} \zeta(\sigma^j x)} \ge \alpha - \frac{b}{3}, \quad \frac{f(\sigma^l x)}{\sum_{j=0}^{l} \zeta(\sigma^j x)} \ge b \text{ and } \frac{\sum_{j=0}^{l} \zeta(\sigma^j x)}{\sum_{j=0}^{l+1} \zeta(\sigma^j x)} \ge 1 - \delta,$$

where δ is so small that $(\alpha - \frac{b}{3})(1-\delta) \ge \alpha - \frac{b}{2}$. But then, taking k so large that $n_k - 2 \ge l_0$, we get

$$\begin{aligned}\frac{\sum_{j=0}^{n_k-1} -f(\sigma^j x)}{\sum_{j=0}^{n_k-1} \zeta(\sigma^j x)} &= \frac{\sum_{j=0}^{n_k-2} -f(\sigma^j x)}{\sum_{j=0}^{n_k-2} \zeta(\sigma^j x)} \cdot \frac{\sum_{j=0}^{n_k-2} \zeta(\sigma^j x)}{\sum_{j=0}^{n_k-1} \zeta(\sigma^j x)} + \frac{-f(\sigma^{n_k-1}x)}{\sum_{j=0}^{n_k-1} \zeta(\sigma^j x)} \\ &\ge (\alpha - \frac{b}{3})(1-\delta) + b \ge \alpha - \frac{b}{2} + b = \alpha + \frac{b}{2}.\end{aligned}$$

This however implies that

$$\liminf_{n\to\infty} \frac{\sum_{j=0}^n -f(\sigma^j x)}{\sum_{j=0}^n \zeta(\sigma^j x)} \geq \alpha + \frac{b}{2} > \alpha$$

which is a contradiction since $x \in \tilde{K}(\alpha)$. The proof is finished. ■

Lemma 7.8. With the same assumptions as in Theorem 7.4

(a) $\mu_q(K_{\mu_F}(\alpha(q)) = 1$ for all $q \in \Delta_1$.

(b) $\underline{d}_{\mu_q}(x) \leq T(q) + q\alpha(q)$ for all $q \in \Delta_1$ and for every $x \in K_{\mu_F}(\alpha(q))$ but a set of Hausdorff dimension $\leq \theta(S)$.

(c) $f_{\mu_F}(\alpha(q)) = T(q) + q\alpha(q)$ for every $q \in \Delta_2$.

Proof. Fix $q \in \Delta_1$. Since the functions $|f|$ and $|\zeta|$ are integrable with respect to the measure μ_q, the part (a) follows immediately from Lemma 7.6 and Birkhoff's ergodic theorem. In order to prove part (b) fix $x \in K_{\mu_F}(\alpha(q))$ and $r > 0$. Let $n = n(x,r)$ be the least integer such that $\phi_{x|_n}(X) \subset B(x,r)$. Then $\mu_q(B(x,r)) \geq \mu_q(\phi_{x|_n}(X)$ and $\phi_{x|_{n-1}}(X)$ is not contained in $B(x,r)$. The latter implies that $\mathrm{diam}\big(\phi_{x|_{n-1}}(X)\big) \geq r$. Hence, due to Lemma 2.2

$$\begin{aligned}\frac{\log\big(\mu_q(B(x,r))\big)}{\log r} &\leq \frac{\log\big(\mu_q(\phi_{x|_n}(X))\big)}{\log\big(\mathrm{diam}(\phi_{x|_{n-1}}(X))\big)} \\ &\leq \frac{T(q)\sum_{j=1}^n \log|\phi'_{x_j}(\sigma^j(x))| + q\sum_{j=0}^{n-1} f\circ\sigma^j(x) + M_1}{\sum_{j=1}^{n-1}\log|\phi'_{x_j}(\sigma^j(x))| + M_2}\end{aligned}$$

for some constants M_1 and M_2. Since the range of the function $r \mapsto n(x,r)$, $r \in (0,1]$, is of the form $I\!\!N \cap [A,\infty)$, it follows from the last inequality, Lemma 7.7 and Lemma 7.6 that if

$$\liminf_{n\to\infty} \frac{\log|\phi'_{x_n}(\sigma^n(x))|}{\log|\phi'_{x|_{n-1}}(\sigma^{n-1}(x))|} = 0,$$

then $\underline{d}_{\mu_q}(x) \leq T(q) + q\alpha(q)$. So, consider the set

$$\mathrm{Bad} = \left\{ x \in J : \liminf_{n\to\infty} \frac{\log|\phi'_{x_n}(\sigma^n(x))|}{\log|\phi'_{x|_{n-1}}(\sigma^{n-1}(x))|} > 0 \right\}.$$

We shall show that $\mathrm{HD}(\mathrm{Bad}) \leq \theta(S)$. So, given $\gamma > 0$ define

$$\mathrm{Bad}(\gamma) = \left\{ x \in J : \exists_{q\geq 1}\forall_{n\geq q} \frac{\log|\phi'_{x_n}(\sigma^n(x))|}{\log|\phi'_{x|_{n-1}}(\sigma^{n-1}(x))|} \geq \gamma \right\}$$

and given $n \geq 1$ put

$$\mathrm{Bad}_n(\gamma) = \left\{ x \in J : \frac{\log |\phi'_{x_k}(\sigma^k(x))|}{\log |\phi'_{x|_{k-1}}(\sigma^{k-1}(x))|} \geq \gamma \ \forall k \geq n \right\}.$$

Fix $\eta > \theta(S)$. By the definition of $\theta(S)$ there exists $k \geq 1$ so large that for all $l \geq k$

$$\sum_{\{i \in I : ||\phi'_i||_0 \leq K s^{\gamma l}\}} ||\phi'_i||_0^\eta \leq \frac{1}{2}. \tag{7.5}$$

Fix $n \geq 1$. For every $l \geq p = \max\{n-1, k\}$ let $\Omega_l = \{\omega \in I^l : \phi_\omega(X) \cap \mathrm{Bad}_n(\gamma) \neq \emptyset\}$. We shall prove by induction that for every $l \geq p$

$$\sum_{\omega \in \Omega_l} ||\phi'_i||_0^\eta \leq \left(\frac{1}{2}\right)^{l-p} \sum_{\omega \in \Omega_p} ||\phi'_i||_0^\eta, \tag{7.6}$$

where as $\eta > \theta(S)$,

$$\sum_{\omega \in \Omega_p} ||\phi'_i||_0^\eta \leq \sum_{\omega \in I^p} ||\phi'_i||_0^\eta < \infty. \tag{7.7}$$

Indeed, for $l = p$ we have even equality. So, suppose that (7.6) holds for some $l \geq p$. Fix $\omega \in \Omega_{l+1}$. Then $\omega|_l \in \Omega_l$ and there exists $x \in \phi_\omega(X) \cap \mathrm{Bad}_n(\gamma)$. Since $x = \phi_{x|_{l+1}}(\sigma^{l+1}(x))$, it follows from (7.1) that $\omega = x|_{l+1}$. Since $l \geq n-1$ and $x \in \mathrm{Bad}_n(\gamma)$, we therefore get

$$||\phi'_{\omega_{l+1}}||_0 \leq K|\phi'_{\omega_{l+1}}(\sigma^{l+1}(x))| = K|\phi'_{x_{l+1}}(\sigma^{l+1}(x))| \leq K|\phi_{x|_l}(\sigma^l(x))|^\gamma \leq K s^{\gamma l}.$$

Thus, using (7.5) and (7.6) for l we can write

$$\begin{aligned}
\sum_{\omega \in \Omega_{l+1}} ||\phi'_i||_0^\eta &\leq \sum_{\omega \in \Omega_l} \sum_{\{i \in I : ||\phi'_i||_0 \leq K s^{\gamma l}\}} ||\phi'_\omega||_0^\eta ||\phi'_i||_0^\eta \\
&= \sum_{\omega \in \Omega_l} ||\phi'_\omega||_0^\eta \sum_{\{i \in I : ||\phi'_i||_0 \leq K s^{\gamma l}\}} ||\phi'_i||_0^\eta \\
&\leq \frac{1}{2} \sum_{\omega \in \Omega_l} ||\phi'_\omega||_0^\eta = \left(\frac{1}{2}\right)^{l+1-p} \sum_{\omega \in \Omega_p} ||\phi'_i||_0^\eta.
\end{aligned}$$

The inductive proof of (7.6) is finished. By (BDP.2) of [MU1] we therefore get for all $l \geq k$

$$\sum_{\omega \in \Omega_l} \mathrm{diam}^\eta(\phi_\omega(X)) \leq D \left(\frac{1}{2}\right)^{l-p} \sum_{\omega \in \Omega_p} ||\phi'_i||_0^\eta$$

and using (7.7) we conclude that $\mathcal{H}^\eta(\mathrm{Bad}_n(\gamma)) = 0$. Thus $\mathrm{HD}(\mathrm{Bad}_n(\gamma)) \leq \eta$ which implies that $\mathrm{HD}(\mathrm{Bad}_n(\gamma)) \leq \theta(S)$. Since $\mathrm{Bad}(\gamma) = \bigcup_{n \geq 1} \mathrm{Bad}_n(\gamma)$, $\mathrm{HD}(\mathrm{Bad}(\gamma)) \leq \theta(S)$ and since $\mathrm{Bad} = \bigcup_{m\ge1}\mathrm{Bad}(1/m)$, $\mathrm{HD}(\mathrm{Bad}) \le \theta(S)$. The proof of (b) is complete. Since $\mu_q(K_{\mu_F}(\alpha(q))) = 1$, it follows from Corollary 4.5 that $f_{\mu_F}(\alpha(q)) = \mathrm{HD}(K_{\mu_F}(\alpha(q)) \ge \mathrm{HD}(\mu_q) = \mathrm{h}_{\mu_q}(\sigma)/\chi_{\tilde\mu_q}(\sigma)$. Since $\mathrm{P}(G_{q,T(q)}) = 0$, using Theorem 2.16, we continue writing

$$
\begin{aligned}
f_{\mu_F}(\alpha(q)) &\ge \frac{\mathrm{h}_{\tilde\mu_q}(\sigma)}{\chi_{\tilde\mu_q}(\sigma)} = \frac{-\int g_{q,T(q)}d\tilde\mu_q}{\chi_{\tilde\mu_q}(\sigma)} = \frac{\int(-T(q)\zeta + qf)d\tilde\mu_q}{-\chi_{\tilde\mu_q}(\sigma)} \\
&= \frac{-T(q)\chi_{\tilde\mu_q}(\sigma) + q\int f d\tilde\mu_q}{-\chi_{\tilde\mu_q}(\sigma)} = T(q) + q\alpha(q) \qquad (7.8)
\end{aligned}
$$

This proves one half of (c). If now $q \in \Delta_2$, then our assumptions give $\mathrm{h}_{\mu_q}(\sigma)/\chi_{\mu_q}(\sigma) > \theta(S)$. Applying this along with (a) and (b), it follows from a well-known theorem in the dimension theory (see [Ma], [PU]) that $f_{\mu_F}(\alpha(q)) = \mathrm{HD}(K\mu_F(\alpha(q))) \le T(q)+q\alpha(q)$. This proves the other part of (c). The proof of Lemma 7.8 is thus complete. ■

So, part (c) of Theorem 7.4 is an immediate consequence of Lemma 7.8(c) and formula (7.5). Part (d) is a combined consequence of Lemma 7.5 and item (c) of Theorem 7.4. Part (e) of Theorem 7.4 follows from Lemma 7.5, part (c) of Theorem 7.4 and Theorem 7.3. We end up the proof of Theorem 7.4 by demonstrating its item (f). And indeed, since the diameters of the images $\phi_\omega(X)$ tend to zero uniformly (exponentially) fast with respect to the length of ω, we conclude that there exists at most one value $t \in I\!R$ such that for some $C \ge 1$ and every $n \ge 1$

$$C^{-1} \le \sum_{|\omega|=n} \mu_F^q([\omega])\mathrm{diam}^t(\phi_\omega(X)) \le C.$$

So, we only need to show that the display appearing in item (f) of Theorem 7.4 is true. And indeed, if $\omega \in I^*$, say $|\omega| = n$ and $\rho \in [\omega]$, then it follows from the definition of measures $\tilde\mu_q$ and $\tilde\mu_F$ that

$$
\begin{aligned}
\tilde\mu_q([\omega]) &\asymp \exp\left(q\sum_{j=0}^{n-1} f\circ\sigma^j(\rho) - T(q)\sum_{j=0}^{n-1}\zeta\circ\sigma^j(\rho)\right) \\
&= \left(\exp\sum_{j=0}^{n-1} f\circ\sigma^j(\rho)\right)^q\left(\exp\left(-\sum_{j=0}^{n-1}\zeta\circ\sigma^j(\rho)\right)\right)^{T(q)} \\
&\asymp \tilde\mu_F^q([\rho|_n])\mathrm{diam}^{T(q)}(\phi|_{\rho|_n}(X)) = \tilde\mu_F^q([\omega])\mathrm{diam}^{T(q)}(\phi|_\omega(X)).
\end{aligned}
$$

Since $\sum_{|\omega|=n}\tilde\mu_q([\omega]) = 1$, summing up the above display over all $\omega \in I^n$ we obtain the desired inequalities. The proof of Theorem 7.4 is complete. ■

Let us recall that in [MU2] we have introduced the class of absolutely regular conformal iterated function systems S by the requirement that $\theta(S) = 0$. For these systems we can rewrite Theorem 7.4 relaxing the assumption $\mathrm{h}_{\tilde\mu_q}(\sigma)/\chi_{\tilde\mu_q} > \theta(S)$ since we already know (see the paragraph proceeding Lemma 7.5) that the entropy $\mathrm{h}_{\tilde{\mu}_q}(\sigma)$ is always positive. It then reads as follows.

Theorem 7.9. Suppose that condition (7.1) is satisfied for all $q \in \mathrm{Fin}(F)$. Suppose also that there exists an interval $\Delta_1 \subset \mathrm{Fin}(F)$ such that $1 \in \Delta_1$ and for every $q \in \Delta_1$ and all t in some neighbourhood of $T(q)$ (contained in $(\theta(q), \infty)$),

$$\int (|f|^{2+\gamma} + |\zeta|^{2+\gamma}) d\tilde{\mu}_q < \infty \quad \text{and} \quad \int (|f| + |\zeta|) d\tilde{\mu}_{q,t} < \infty.$$

Suppose finally that the system S is absolutely regular. Then

(a) The number $D_{\mu_F}(x)$ exists for μ_F-a.e. $x \in J$ and

$$D_{\mu_F}(x) = \frac{-\int f d\tilde{\mu}_F}{\int \zeta d\tilde{\mu}_F}.$$

(b) The function $T : \Delta_1 \to I\!\!R$ is real-analytic, $T(0) = \mathrm{HD}(J)$, and $T'(q) < 0$, $T''(q) \geq 0$ for all $q \in \Delta_1$.

(c) For every $q \in \Delta_2$, $f_{\mu_F}(-T'(q)) = T(q) - qT'(q)$.

(d) If $\tilde{\mu}_F \neq \tilde{\mu}_{-\mathrm{HD}(J)\zeta}$, then the function $\alpha \mapsto f_{\mu_F}(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$ is real-analytic, where the interval (α_1, α_2), $0 \leq \alpha_1 < \alpha_2 \leq \infty$ is the range of the function $-T'(q)$ defined on the interval Δ_2. Otherwise $T'(q) = \mathrm{HD}(J)$ for every $q \in (\theta(F), \infty)$.

(e) If $\tilde{\mu}_F \neq \tilde{\mu}_{-\mathrm{HD}(J)\zeta}$, then the functions $f_{\mu_F}(\alpha)$ and $T(q)$ form a Legendre transform pair.

(f) For every $q \in \Delta_1$ the number $T(q)$ is uniquely determined by the property that there exists a constant $C \geq 1$ such that for every $n \geq 1$

$$C^{-1} \leq \sum_{|\omega|=n} \mu_F^q([\omega]) \mathrm{diam}^{T(q)}(\phi_\omega(X)) \leq C.$$

§**8. Examples.** This section is devoted to explore some concrete examples. We apply the results obtained in Sections 7 and 3 to the infinite systems generated by continued fractions and Apollonian packing. But first we will need the following general result being a straightforward consequence of Theorem 2.4 and formula (7.6).

Lemma 8.1. If the function T determined by the condition $\mathrm{P}(q, T(q)) = 0$ is well-defined on a right-hand side deleted neighbourhood of zero, then

$$\lim_{q \searrow 0} T(q) = \mathrm{HD}(J).$$

Proof. Let $h = \mathrm{HD}(J)$. It easily follows from Lemma 7.1 and Lemma 7.2 that the function $T(q)$ is decreasing and $T(q) \le h$ for every $q \ge 0$. Hence, the limit $L = \lim_{q\searrow 0} T(q)$ exists and $L \le h$. Clearly $\lim_{q\searrow 0} G_{q,T(q)} = G_{0,L}$ (in the sense describing at the beginning of Section 2. Therefore, applying Proposition 2.3 we get $\mathrm{P}(G_{0,L}) \le \lim_{q\searrow 0} \mathrm{P}(q, T(q)) = 0$, whence $L \ge h$. Consequently $L = h$ and the proof is complete. ■

Although it will turn out that the examples generated by Apollonian packings are easier to deal with and require less delicate analysis than continued fractions, we start with these latter ones. So, we consider the maps $\{\phi_i : [0,1] \to [0,1]\}_{i\ge 1}$ given by the formulae

$$\phi_i(x) = \frac{1}{x+i}.$$

Their limit set coincides with the set of all irrational numbers contained in the interval $[0,1]$. We consider a probability vector $P = \{p_i\}_{i=1}^{\infty}$ such that $p_i > 0$ for all $i \ge 1$ and then the measure μ_P which is the projection of the Bernoulli measure $\tilde\mu_P$ from the coding space I^∞ to the interval $[0,1]$. Let F be the family of functions $f^{(i)} = \log p_i$. Of course F is a strongly Hölder family of functions of any order. Our first aim is notice the following.

Lemma 8.2. We have

$$\mu_P = \mu_F = m_F, \quad \tilde\mu_P = \tilde\mu_F = \tilde m_F \text{ and } \mathrm{P}(qF) = \log \sum_{i=1}^{\infty} p_i^q.$$

Proof. We shall first prove the last part of this lemma which is a straightforward calculation. Indeed,

$$\begin{aligned}\mathrm{P}(qF) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\big(\sup S_\omega(qF)\big) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} p_{\omega_1} p_{\omega_2} \cdots p_{\omega_n} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i=1}^{\infty} p_i^q\right)^n = \log \sum_{i=1}^{\infty} p_i^q.\end{aligned}$$

Since $\mathrm{P}(F) = 0$, it follows from Proposition 2.13 that $\tilde\mu_P = \tilde\mu_F$. Hence $\mu_P = \tilde\mu_P \circ \pi^{-1} = \tilde\mu_F \circ \pi^{-1} = \mu_F$. Since $\mu_F = \mu_P$ is obviously atomless, condition (2.11′) is satisfied by μ_P. Since $\pi : (I^\infty, \tilde\mu_P) \to ([0,1], \mu_P)$ is a metric isomorphism, it is straightforward to verify condition (2.10’) for the measure μ_P. The last equality left, $\tilde\mu_P = \tilde m_F$ follows now from Lemma 2.8 and the fact that π is a metric isomorphism. The proof is complete. ■

Since the measure μ_P is certainly not equal to the Gauss measure, as an immediate consequence of Theorem 4.6, we get the following.

Theorem 8.3. If the entropy $\mathrm{H}(P) = -\sum_{i=1}^{\infty} p_i \log p_i$ is finite, then $\mathrm{HD}(\mu_P) < 1$.

Our aim now is to provide a sufficient conditions for doing the multifractal analysis of the measure μ_P on some interval $\tilde{\Delta}$. We say that the probability vector P satisfies the condition (8.1) if for every $q \geq 0$ there exists $u \in I\!R$ such that

$$1 < \sum_{i=1}^{\infty} p_i^q (i+1)^{-2u} < \infty. \tag{8.1}$$

We say that the probability vector P satisfies the condition (8.2) if

$$M = \sup_{i,j \geq 1} \left\{ \frac{-\log(p_i p_j)}{\log(ij+1)} \right\} < \infty \tag{8.2}$$

and it satisfies condition (8.3) if

$$L = \inf_{i \geq 1} \left\{ \frac{-\log p_i}{\log(i+1)} \right\} > 0. \tag{8.3}$$

Theorem 8.4. If the probability vector $P = \{p_i\}_{i=1}^{\infty}$ satisfies the conditions (8.1)-(8.3) and $\theta(F) < (M-L)^{-1}$, then there exists a non-degenerate interval $\tilde{\Delta} \subset (0, \infty)$ such that the function $f_{\mu_F} : \tilde{\Delta} \to [0,1]$ is convex and real-analytic.

Proof. W shall verify the assumptions of Theorem 7.4 and construct an appropriate interval Δ_2. As $\tilde{\Delta}$, according to Theorem 7.4(d) we will then take the image $-T'(\Delta_2)$. So, we need to check first the assumptions of Lemma 7.2 for every $q \geq 0$. Towards this end take u as assured in (8.1). It follows that

$$\sum_{i \geq 1} \exp\big(q \log p_i + u \log ||\phi_i'||\big) = \sum_{i=1}^{\infty} p_i^q i^{-2u} \leq 2^{2u} \sum_{i=1}^{\infty} p_i^q (i+1)^{-2u} < \infty.$$

This implies that $u \in \mathrm{Fin}(q)$. We also have for all $n \geq 1$

$$\begin{aligned} Z_n(G_{q,u}) &\geq \sum_{|\omega|=n} p_{\omega_1}^q p_{\omega_2}^q \dots p_{\omega_n}^q (\omega_1+1)^{-2u} (\omega_2+1)^{-2u} \dots (\omega_n+1)^{-2u} \\ &= \left(\sum_{i=1}^{\infty} p_i^q (i+1)^{-2u} \right)^n \end{aligned}$$

Hence, due to (8.1),

$$\mathrm{P}(G_{q,u}) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(G_{q,u}) \geq \log \left(\sum_{i=1}^{\infty} p_i^q (i+1)^{-2u} \right) > 0$$

So, (7.1) satisfied for all $q \geq 0$. We shall now check that for all $q \in \mathrm{Fin}(F)$, all $t > \theta(q)$ and all $\eta > 0$

$$\int |f|^\eta d\tilde{\mu}_{q,t} < \infty \quad \text{and} \quad \int |\zeta|^\eta d\tilde{\mu}_{q,t} < \infty.$$

Since $\tilde{\mu}_{q,t}$ and $m_{q,t}$ are equivalent with Radon-Nikodym derivatives bounded away from zero and infinity, we may replace $\tilde{\mu}_{q,t}$ with $m_{q,t}$. So, fix $q \in \mathrm{Fin}(F)$, $t > \theta(q)$, and take $\theta(q) < s < t$. Then for all $i, j \geq 1$ for which ij is large enough, say $ij \geq A$, we have $(ij)^{-2(t-s)} \log^\eta(ij+1) \leq 1$. Obviously $\int |\zeta|^\eta d\tilde{\mu}_{q,t} < \infty$ if and only if

$$\sum_{i,j:ij\geq A} \log^\eta(ij+1)\tilde{m}_{q,t}([ij]) < \infty.$$

But

$$\begin{aligned}
\sum_{i,j:ij\geq A} \log^\eta(ij+1)\tilde{m}_{q,t}([ij]) &\asymp \sum_{i,j:ij\geq A} \log^\eta(ij+1) p_i^q p_j^q (ij)^{-2t} \mathrm{e}^{-2\mathrm{P}(qF)} \\
&= \mathrm{e}^{-2\mathrm{P}(qF)} \sum_{i,j:ij\geq A} \log^\eta(ij+1)(ij)^{-2(t-s)} p_i^q p_j^q (ij)^{-2s} \\
&\leq \mathrm{e}^{-2\mathrm{P}(qF)} \sum_{i,j:ij\geq A} \sum_{i,j:ij\geq A} p_i^q p_j^q (ij)^{-2s} < \infty.
\end{aligned}$$

The fact that $\int |f|^\eta d\tilde{\mu}_{q,t} < \infty$ follws now by applying (8.2). So, $\Delta 1 = (\theta(F), \infty)$ and the function $T : (\theta(F), \infty) \to I\!R$ is real-analytic. We are now only left to show that there exists a non-degenerate interval $\Delta_2 \subset \mathrm{Fin}(F)$ such that for $q \in \Delta_2$, $\mathrm{h}_{\tilde{\mu}_q}/\chi_{\tilde{\mu}_q} > \theta(S) = 1/2$. At the beginning of this proof we have shown that the function $T(q)$ (determined by the condition $\mathrm{P}(G_{q,T(q)}) = 0$) is well defined for every $q \geq 0$. In order to construct the interval Δ_2 we will need the following.

Lemma 8.5. If condition (8.2) is satisfied, then $T : (0, \infty) \to I\!R$ is Lipschitz continuous with a Lipschitz constant bounded from above by $M/2$.

Proof. Fix $q_0 \geq 0$. Consider first the case $q \geq q_0$. Then, by Lemma 7.1 $T(q) \leq T(q_0)$. Fix now $u = \frac{1}{2}M(q - q_0)$, where M is taken from condition (8.2). Then, using the bounded distortion property (1d) and (8.2), we can estimate for all $n \geq 1$ as follows.

$$\begin{aligned}
Z_{2n}(G_{q,T(q_0)-u}) &= \sum_{|\omega|=2n} \exp\bigl(S_\omega(qF)\bigr)||\phi'_\omega||^{T(q_0)-u} \\
&= \sum_{|\omega|=2n} \exp\bigl(S_\omega(q_0F)\bigr)||\phi'_\omega||^{T(q_0)} \exp\bigl(S_\omega((q-q_0)F)\bigr)||\phi'_\omega||^{-u} \\
&\geq \sum_{|\omega|=2n} \exp\bigl(S_\omega(q_0F)\bigr)||\phi'_\omega||^{T(q_0)} p_{\omega_1}^{q-q_0} p_{\omega_2}^{q-q_0} \dots p_{\omega_{2n}}^{q-q_0} K^{-u} \inf\{|\phi'_\omega|\}^{-u} \\
&\geq K^{-u} \sum_{|\omega|=2n} \exp\bigl(S_\omega(q_0F)\bigr)||\phi'_\omega||^{T(q_0)} \Pi_{j=1}^n (p_{\omega_{2j-1}} p_{\omega_{2j}})^{q-q_0} (\omega_{2j-1}\omega_{2j}+1)^{2u} \\
&\geq K^{-u} Z_{2n}(G_{q_0,T(q_0)}).
\end{aligned}$$

Hence, $\mathrm{P}(G_{q,T(q_0)-u}) \geq \mathrm{P}(G_{q_0,T(q_0)}) = 0$, and therefore, due to Lemma 7.1, $T(q) \geq T(q_0) - u = T(q_0) - \frac{1}{2}M(q - q_0)$. Thus $0 \leq T(q_0) - T(q) \leq \frac{1}{2}M(q - q_0)$, and we are done in this case.

Consider now the case $q \le q_0$. Then $T(q) \ge T(q_0)$. Similarly as in the previous case we get for every $n \ge 1$

$$\begin{aligned}
Z_{2n}(G_{q,T(q_0)+u}) &= \sum_{|\omega|=2n} \exp\big(S_\omega(q_0F)\big)||\phi'_\omega||^{T(q_0)} \exp\big(S_\omega((q-q_0)F)\big)||\phi'_\omega||^u \\
&\le \sum_{|\omega|=2n} \exp\big(S_\omega(q_0F)\big)||\phi'_\omega||^{T(q_0)} \Pi_{j=1}^n (p_{\omega_{2j-1}} p_{\omega_{2j}})^{q-q_0} (\omega_{2j-1}\omega_{2j}+1)^{-2u} \\
&\le Z_{2n}(G_{q_0,T(q_0)}).
\end{aligned}$$

Hence $\mathrm{P}(G_{q,T(q_0)+u}) \le \mathrm{P}(G_{q_0,T(q_0)}) = 0$, and therefore $T(q) \le T(q_0)+u = T(q_0)+\frac{1}{2}M(q-q_0)$. Thus $0 \le T(q)-T(q_0) \le \frac{1}{2}M(q-q_0)$, and we are done. ■

Lemma 8.6. If $q \ge q_0 \ge 0$, then $T(q)-T(q_0) \le -\frac{1}{2}L(q-q_0)$.
Proof. Put $u = \frac{1}{2}L(q-q_0)$. Then for every $n \ge 1$ and using (8.3),

$$\begin{aligned}
Z_n(G_{q,T(q_0)-u}) &= \sum_{|\omega|=n} \exp\big(S_\omega(q_0F)\big)||\phi'_\omega||^{T(q_0)} \exp\big(S_\omega((q-q_0)F)\big)||\phi'_\omega||^{-u} \\
&\le \sum_{|\omega|=n} \exp\big(S_\omega(q_0F)\big)||\phi'_\omega||^{T(q_0)} \Pi_{j=1}^n p_{\omega_j}^{q-q_0} (\omega_j+1)^{2u} \\
&\le Z_n(G_{q_0,T(q_0)}).
\end{aligned}$$

Hence, $\mathrm{P}(G_{q,T(q_0)-u}) \le \mathrm{P}(G_{q_0,T(q_0)}) = 0$, and therefore, due to Lemma 7.1, $T(q) \le T(q_0)-u = T(q_0)-\frac{1}{2}L(q-q_0)$. The proof is complete. ■

Concluding the proof of Theorem 8.4 take an arbitrary $q \in (\theta(F), (M-L)^{-1})$. Then it follows from Lemma 8.6 (see also Theorem 7.4(b)) that $T'(q) \le -L/2$. Combining this fact and (7.6) along with Lemma 8.5, we get for all $q \in (\theta(F), (M-L)^{-1})$

$$\begin{aligned}
\mathrm{h}_{\tilde{\mu}_q}/\chi_{\tilde{\mu}_q} &\ge T(0) - \frac{1}{2}Mq + \frac{1}{2}Lq = 1 - \frac{1}{2}(M-L)q \\
&> 1 - \frac{1}{2}(M-L)(M-L)^{-1} = 1 - \frac{1}{2} = \frac{1}{2} = \theta(S).
\end{aligned}$$

The proof of Theorem 8.4 is complete. ■

We shall now provide a class of probability vectors P for which all the assumptions of Theorem 8.4 are satisfied.

Example 8.7. For every $s > 1$ consider the probability vector $P_s = \{N_s^{-1}n^{-s}\}_{n\ge1}$, where $N_s = \sum_{n=1}^\infty n^{-s} < \infty$. We shall verify that all the assumptions of Theorem 8.4 are satisfied for all s large enough.
And indeed, in order to check condition (8.1), consider the series $\sum_{i=1}^\infty N_s^{-q} i^{-sq}(i+1)^{-(2u)}$. Since one can take $u \in I\!R$ such that $qs+2u > 1$ but the difference $qs+2u-1 > 0$ is as small as one wishes, condition (8.1) can be fulfilled. Since for all $i,j \ge 1$

$$\frac{-\log(N_s^{-1}i^{-s}N_s^{-1}j^{-s})}{\log(ij+1)} = \frac{2\log N_s + s\log(ij)}{\log(ij+1)} \le \frac{2\log N_s}{\log 2} + s < \infty,$$

condition (8.2) is also satisfied. In order to verify condition (8.3) and inequality $\theta(F) < (M-L)^{-1}$, we need to establish first some analytical properties of the functions

$$f_s(x) = \frac{A_s + s\log x}{\log(x+1)}, \quad s > 1, x \geq 1,$$

where $A_s > 0$ and $\lim_{s\to\infty} A_s = 0$. Obviously

$$\lim_{x\to\infty} f_s(x) = s. \tag{8.4}$$

Now,

$$\begin{aligned}
\frac{df_s}{dx} &= \frac{\frac{s}{x}\log(x+1) - \frac{1}{x+1}(A_s + s\log x)}{\log^2(x+1)} = \frac{s(x+1)\log(x+1) - A_s x - sx\log x}{x(x+1)\log^2(x+1)} \\
&= \frac{sx(\log(x+1) - \log x) + s\log(x+1) - A_s x}{x(x+1)\log^2(x+1)} = \frac{sx\overline{x}^{-1} + s\log(x+1) - A_s x}{x(x+1)\log^2(x+1)}
\end{aligned} \tag{8.5}$$

for some $\overline{x} \in [x, x+1]$. Hence for every x large enough (depending on s)

$$\frac{df_s}{dx}(x) < 0. \tag{8.6}$$

Also $\frac{df_s}{dx}(1) = \frac{2s\log 2 - A_s}{2\log^2 2}$. Hence, for every s large enough

$$\frac{df_s}{dx}(1) > 0. \tag{8.7}$$

Let us now look at the function

$$g_s(x) = s(x+1)\log(x+1) - A_s x - sx\log x$$

and its derivative $g_s'(x) = s\log\left(\frac{x+1}{x}\right) - A_s$. The derivative g_s' has at most two pieces of monotonicity, and consequently it may have at most two zeros. In view of (8.6) and (8.7) the function f_s has at least one local extremum and the first one must be a local maximum. Suppose that f_s has a second local extremum. It must be a local minimum and g_s can have no more local extrema. But this contradicts (8.6). Summarizing: For every $s > 1$ there exists $x_s \geq 1$ such that

(8a) The function f_s is increasing on the interval $[1, x_s]$.
(8b) At the point x_s the function f_s takes on its global maximum and $f_s(x_s) > s$.
(8c) The function f_s is decreasing on the interval $[x_s, \infty)$ and $f_s(x) > s$ for every $x \geq x_s$.

Since in addition $f_s(1) = \frac{A_s}{\log 2} < s$ for all s large enough (since $\lim_{s\to\infty} A_s = 0$), we conclude that for every s large enough

$$\inf_{x\geq 1}\{f_s(x)\} = f_s(1) = \frac{A_s}{\log 2}. \tag{8.8}$$

Coming back to our probability vector P_s, it follows from (8.8) that for all s large enough $L = \frac{\log N_s}{\log 2} > 0$ which in particular implies that condition (8.3) is satisfied. So, we are left to show that $\theta(F_s) < (M-L)^{-1}$, and since $\theta(F_s) = 1/s$, it suffices to show that for all s large enough

$$M < \frac{\log N_s}{\log 2} + s. \tag{8.9}$$

Fix towards this end $0 < \gamma < 1$. In view of (8.2) it is enough to demonstrate that for all s large enough and all $j \geq 1$

$$\frac{2\log N_s + s\log j}{\log(j+1)} \leq \gamma\frac{\log N_s}{\log 2} + s$$

or equivalently

$$2\log 2\log N_s + s\log 2\log j \leq \gamma\log N_s\log(j+1) + s\log 2\log(j+1)$$

which in turn equivalently means that

$$s\log 2(\log(j+1) - \log j) + \log N_s(\gamma\log(j+1) - 2\log 2) \geq 0. \tag{8.10}$$

Now, the first summand in this inequality is always positive. The second summand is ≥ 0 as long as $j \geq 4^{1/\gamma}-1$. And since $\lim_{s\to\infty}\log N_s = 0$ and $\lim_{s\to\infty}\min\{s(\log(j+1)-\log j) : 1 \leq j \leq 4^{1/\gamma}-1\} = \infty$, we conclude that (8.10) is satisfied for all s large enough. We are done.

Example 8.8. (Apollonian packing) Consider on the complex plane the three points $z_j = \mathrm{e}^{2\pi ij/3}$, $j = 0,1,2$ and the following additional three points $a_0 = \sqrt{3}-2$, $a_1 = (2-\sqrt{3})\mathrm{e}^{-\pi i/3}$ and $a_2 = (2-\sqrt{3})\mathrm{e}^{\pi i/3}$. Let ϕ_0, ϕ_1, and ϕ_2 be the Möbius transformations determined by the following requirements: $\phi_0(z_0) = z_0$, $\phi_0(z_1) = a_2$, $\phi_0(z_2) = a_1$, $\phi_1(z_0) = a_2$, $\phi_1(z_1) = z_1$, $\phi_1(z_2) = a_0$, $\phi_2(z_0) = a_1$, $\phi_2(z_1) = a_0$, and $\phi_2(z_2) = z_2$. Set $X = \overline{B}(0,1)$, the closed ball centered at the origin of radius 1. It is straightforward that the images $\phi_0(X)$, $\phi_1(X)$ and $\phi_2(X)$ are mutually tangent (at the points a_0, a_1 and a_2, respectively) disks whose boundaries pass through the triples (z_0,a_1,a_2), (z_1,a_0,a_2) and (z_2,a_0,a_1) respectively. Of course all the three disks $\phi_0(X)$, $\phi_1(X)$ and $\phi_2(X)$ are contained in X and are tangent to X at the points z_0, z_1 and z_2 respectively. Let $S = \{\phi_0,\phi_1,\phi_2\}$ be the iterated function system on X generated by ϕ_0, ϕ_1 and ϕ_2. Notice that all the maps ϕ_0, ϕ_1 and ϕ_2 are parabolic with parabolic fixed points z_0, z_1 and z_2 respectively. It is not difficult to check that all the requirements of a parabolic system are satisfied. Observe that the limit set J of the parabolic system S coincides with the residual set of the Apollonian packing generated by the curvilinear triangle with vertices z_0, z_1, z_2. In [MU4], using a slightly different iterated function system, we have dealt with geometrical properties of J proving that $1 < h = \mathrm{HD}(J) < 2$, $0 < \mathcal{H}^h(J) < \infty$ and $\mathcal{P}^h(J) = \infty$. In [MU4] we studied its dynamical properties, especially the invariant measure equivalent with conformal measure. Here we present its multifractal analysis. So, let us first notice that the system S^* is regular. Indeed, we proved in [MU4] that

$$\phi_0^n(z) = \frac{(\sqrt{3}-n)z+n}{-nz+n+\sqrt{3}}$$

and

$$(\phi_0^n)'(z) = \frac{3}{(-nz+n+\sqrt{3})^2}.$$

By the symmetry of the situation this implies the existence of a constant $C \geq 1$ such that for all $i \neq j$ and all $n \geq 1$

$$C^{-1}\frac{1}{n^2} \leq |(\phi_i^n \circ \phi_j)'(z)| \leq C\frac{1}{n^2} \tag{8.11}$$

In particular $\theta(S^*) = 1/2$. Consider now $P = (p_0, p_1, p_2)$ a probability vector such that $p_1p_2p_3 > 0$ and then consider the measure μ_P which is the projection of the Bernoulli measure $\tilde{\mu}_P$ generated by the vector P from the symbolic space $\{0,1,2\}^\infty$ to the limit set S. In [MU4] we have proved the following.

Theorem 8.9. $\mathrm{HD}(\mu_P) < \mathrm{HD}(J)$.

This theorem also follows from Theorem 4.6 and the results obtained in the proof of Theorem 8.11. Let F be the family of functions $f^{(i)} = \log p_i$, $i = 1,2,3$. Of course F is strongly Hölder family of functions of any positive order. Our first aim is to to prove the following

Lemma 8.10. $\tilde{\mu}_P = \tilde{\mu}_F = \tilde{m}_F$, $\mu_P = \mu_F = m_F$ and $\mathrm{P}(F) = 0$.

Proof. We repeat word by word (putting $q = 1$) the proof of Lemma 8.2. until the conclusion that condition (2.10') is satisfied for the measure μ_P. Then, in view of Lemma 2.11 and Lemma.11, $\mu_P = m_F$. The rest of the proof is the same as in the proof of Lemma 8.2. ∎

Theorem 8.11. If $P = (p_0, p_1, p_2)$ is a probability vector such that $p_1p_2p_3 > 0$, then there exists a non-degenerate interval $\tilde{\Delta} \subset (0,\infty)$ such that the function $f_{\mu_P} : \tilde{\Delta} \to (0,\infty)$ (defined with respect to the system S^*) is convex and real-analytic.

Proof. Let, according to Section 3, the family F^* consist of the functions

$$f_*^{(i^n j)} = \log p_j + n \log p_i, \ i \neq j, \ n \geq 1.$$

Since $\mu_F = \tilde{\mu}_F \circ \pi^{-1}$ and $\mu_{F^*} = \tilde{\mu}_{F^*} \circ \pi^{-1}$, it follows from Lemma 8.10 and Theorem 3.4 that $f_{\mu_P} = f_{\mu_{F^*}}$ and therefore it suffices to prove Theorem 8.11 for the measure μ_{F^*}. Notice that the corresponding Hölder families $G_{q,t}^{(i^n j)}$ are defined as follows.

$$G_{q,t}^{(i^n j)} = q \log p_j + qn \log p_i + t \log|(\phi_{i^n j})'|.$$

As in the proof of Theorem 8.4 we shall verify the assumptions of theorem 7.4 and construct an appropriate interval Δ_2. As $\tilde{\Delta}$, according to Theorem 7.4(d) we will then take the image $-T'(\Delta_2)$. So, we need to check first the assumptions of Lemma 7.2 for all $q \in \mathrm{Fin}(F^*)$. Since

$$\sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} ||\mathrm{e}^{f_*^{(i^n j)}}||_0 = \sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} p_j^q p_i^{qn} < \infty$$

for all $q > 0$, and is equal to $+\infty$ for $q = 0$, we conclude that $\mathrm{Fin}(F^*) = (0, \infty)$. Now, for every $q > 0$ and every $t \in I\!\!R$, using (8.11), we get

$$\begin{aligned} Z_1(G_{q,t}) &= \sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} p_j^q p_i^{qn} ||(\phi_{i^n j})'||_0^t \\ &\leq C^t \sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} p_j^q p_i^{qn} n^{-2t} < \infty, \end{aligned} \tag{8.12}$$

so that $\mathrm{P}(G_{q,t}) < \infty$. Also

$$Z_1(G_{q,t}) \geq C_2^{-1} \sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} p_j^q p_i^{qn} n^{-2t} \geq C_2^{-1} p_0^q p_2^{2q} 4^{-t}.$$

Hence $\lim_{u\to -\infty} Z_1(G_{q,t}) = +\infty$, and therefore also $\lim_{u\to -\infty} \mathrm{P}(G_{q,t}) = +\infty$. Thus the condition (7.1) is satisfied for all $q \in \mathrm{Fin}(F^*)$. We shall now check that

$$\int (|f|^\eta + |\zeta|^\eta) d\tilde{\mu}_{q,t} < \infty$$

for all $\eta > 0$, all $q \in \mathrm{Fin}(F^*)$ and all $t > \theta(q)$. In (8.12) we have shown that $\theta(q) = -\infty$ for all $q > 0$. So, fix $q \in \mathrm{Fin}(F^*) = (0, \infty)$ and $t \in I\!\!R$. Take $s < t$ so small that $2(t - s) > \eta$. Using this, the fact $s > \theta(q)$ and (8.11) again, we get

$$\begin{aligned} \int (|f|^\eta + |\zeta|^\eta) d\tilde{\mu}_q &\asymp \int (|f|^\eta + |\zeta|^\eta) d\tilde{m}_q \\ &\asymp \sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} \big((-q\log p_j - qn\log p_i)^\eta + (2t\log n)^\eta\big)\tilde{m}_{q,t}([i^n j]) \\ &\asymp \sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} \big((-q\log p_j - qn\log p_i)^\eta + (2t\log n)^\eta\big) p_j^q p_i^{qn} n^{-2t} \\ &\asymp \sum_{i=0}^{2}\sum_{i\neq j}\sum_{n\geq 1} \big((-q\log p_j - qn\log p_i)^\eta + (2t\log n)^\eta\big) n^{-2(t-s)} p_j^q p_i^{qn} n^{-2s} \\ &< \infty. \end{aligned}$$

Thus one can take $(0,\infty)$ as Δ_1. We are now only left to demonstrate that there exists a non-degenerate interval $\Delta_2 \subset \mathrm{Fin}(F^*) = (0,\infty)$ such that $\mathrm{h}_{\tilde\mu_q}/\chi_{\tilde\mu_q} > \theta(S^*) = 1/2$ for all $q \in \Delta_2$. And indeed, since $\mathrm{HD}(J) \ge 1$ (this is obvious; for the much stronger fact that $\mathrm{HD}(J) > 1$ see [MU4] and the references therein), it follows from Lemma 8.1 that there exists an open interval Δ_2 (having zero as its left-hand endpoint) such that $T(q) > 1/2$ for all $q \in \Delta_2$. Combining this fact, formula (7.6) and Theorem 7.4(b), we conclude that $\mathrm{h}_{\tilde\mu_q}/\chi_{\tilde\mu_q} \ge T(q) > 1/2 = \theta(S^*)$ for every $q \in \Delta_2$. The proof is complete. ■

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