THE FINER GEOMETRY AND DYNAMICS OF SOME MEROMORPHIC FUNCTIONS

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Abstract. We deal with the meromorphic maps $f_\lambda : \mathbb{C} \to \overline{\mathbb{C}}$ of the form $\lambda(1-\exp(-2z))^{-1}$, $\lambda > 0$. We introduce as our main technical device the induced map F_λ defined on the infinite cylinder Q. Results about the map F_λ easily translate to the original map f_λ. The main object of our interest is the set $J_r(F_\lambda) \subset Q$ defined by (5.10), We prove that $h_\lambda \in (1,2)$, where h_λ is the Hausdorff dimension of $J_r(F_\lambda)$. Concerning finer geometry, we prove that the h_λ-dimensional Hausdorff measure of $J_r(f_\lambda)$ is positive and finite whereas the h_λ-dimensional packing measure of $J_r(f_\lambda)$ is locally infinite at each point of $J_r(f_\lambda)$. Concerning the dynamics of the map $F_\lambda : Q \to Q$, we prove the existence and uniqueness of a Borel probability F-invariant ergodic measure equivalent to the h_λ-dimensional Hausdorff measure on $J_r(f_\lambda)$. Finally, in the Appendix we show that the Lebesgue measure of the Julia sets $J(f_\lambda)$ and $J(F_\lambda)$ vanish.

1. Introduction

The Fatou set $F(f)$ of a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is the set of all points $z \in \mathbb{C}$ such that all the iterates are well-defined and form a normal family on a neighborhood of z. The Julia set $J(f)$ is the complement of $F(f)$ in $\mathbb{C}$. Thus, the Fatou set is open whereas $J(f)$ is closed, the Fatou set is completely invariant, while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f)) \subset J(f) \cup \{\infty\}$. This latter property enables us to consider the dynamical system $f : J(f) \to J(f) \cup \{\infty\}$. For a general description of the topological dynamics of meromorphic functions the reader may consult [5], [1]-[4]. It follows from Montel's criterion of normality that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ has at least one pole which is not an omitted value then

$$J(f) = \overline{\bigcup_{n\geq 1} f^{-n}(\infty)}.$$

In this paper we continue the extensive study (see [16], [17], [18], [19], [29], [30], [31], and [32]) of the finer geometric and dynamical structure of the Julia sets of transcendental entire

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and meromorphic functions. Namely, given $\lambda > 0$ we consider the function

$$f_\lambda(z) = \frac{\lambda}{1 - \exp(-2z)}. \tag{1.1}$$

The research presented in this article stems from [29] and [32]. In order to give an overview of what our paper is about, let us introduce first the equivalence relation $\sim$ in $\mathbb{C}$ by saying that $w \sim z$ if and only if $w - z \in \pi i \mathbb{Z}$. We then define the cylinder

$$Q := \mathbb{C}/\sim$$

and let

$$\Pi : \mathbb{C} \to Q$$

be the quotient map. Since each function f_λ is πi-periodic, it induces a unique meromorphic map $F_\lambda : Q \setminus \{0\} \to Q$ such that $\Pi \circ f_\lambda = F_\lambda \circ \Pi$. In order to simplify notation we declare $F_\lambda(0) = \infty$ so that we may consider the function

$$F_\lambda : Q \to Q \cup \{\infty\}.$$

Observe that 0 is a discontinuity point of F_λ. Note that $\Pi \circ f_\lambda^n = F_\lambda^n \circ \Pi$ for every integer $n \geq 1$, that

$$J(F_\lambda) = \Pi(J(f_\lambda))$$

and the map F_λ reflects essentially all interesting features of the dynamics of f_λ. The map F_λ will be the primary object of our interest in this paper and we will formulate actually all our results in terms of F_λ. We leave for the interested reader the task of translating those results in terms of the function f_λ itself. Equipped with the map F_λ, the concept of tightness and the $K(V)$ method from [10] (comp. Ch. 10 of [23]) we prove the existence and uniqueness of a probability conformal measure m (with an exponent greater than 1) for the map F_λ. This powerful tool enables us in turn to prove that h_λ, the Hausdorff dimension of the set $J_r(F_\lambda)$ (see (5.10) for its definition), is less than 2, that the h_λ-dimensional Hausdorff measure of $J_r(F_\lambda)$ is positive and finite, and that the h_λ-dimensional packing measure of $J_r(F_\lambda)$ is locally infinite at each point of $J_r(f_\lambda)$. All these geometric properties of the set $J_r(F_\lambda)$ clearly indicate that this is the right object to deal with. However, this is not the end. The dynamics of the map F_λ restricted to this set also has interesting features. For example, we prove that $m(J_r(F_\lambda)) = 1$ and then the existence and uniqueness of a Borel probability F-invariant ergodic measure equivalent to the conformal measure m. We do this by applying first the method of Marco Martens to show the existence of a σ-finite F-invariant conservative ergodic measure equivalent to the measure m and checking then that this measure is finite. Although all the facts mentioned above show how much the sets $J_r(F_\lambda)$ and $J(F_\lambda) \setminus J_r(F_\lambda)$ differ one from the other, there is one difference more to be added to this picture. Namely, it has been shown in [15] that $\mathrm{HD}(J(F_\lambda)) = 2$, which due to the fact that $h_\lambda < 2$, implies that $\mathrm{HD}(J(F_\lambda) \setminus J_r(F_\lambda)) = 2$. We also analyze the sets $J(F_\lambda)$ and $J(f_\lambda)$ from the "Lebesgue" point of view. We prove that the Lebesgue measure of the Julia set $J(f_\lambda)$ vanishes. Since this result lies rather aside of the mainstream considerations in this paper, we postponed its proof to the Appendix.

Although our paper shares the general strategy and some arguments with some papers quoted above, the function f_λ is meromorphic, its all poles belong to the Julia set, and we need to cope with the problems they cause. Our paper is evidence that the presence of poles may not affect properties of geometric (Hausdorff, packing) and invariant measures. The question of real analyticity (positively solved in [30] for the exponential family) of the function $\lambda \mapsto \mathrm{HD}(J_r(F_\lambda))$ remains however open.

Our paper is organized as follows. Section 2 provides a brief description of the topological structure of the Julia set $J(f_\lambda)$. It will not be used too heavily and more details can be found in [12] and [15]. Section 3 is devoted to collect useful little formulas involving the functions f_λ which will be frequently used from then on. In Section 4 we prove that for every $\lambda > 0$ the Hausdorff dimension of the set $J_{bd}(f_\lambda)$ (see formula (4.1)) is larger than one. Besides being interesting itself, this fact (actually its strengthening given in Lemma 4.1) is absolutely necessary for all further results of this paper to be provable. In Section 5 we establish the existence and uniqueness of conformal measures. We also prove there that the conformal measure is supported on $J_r(F_\lambda)$. In Section 6 we establish the existence of a unique Borel ergodic F_λ-invariant measure equivalent to the conformal measure. The Section 6 is occupied with geometric issues. We prove that the packing measure is locally infinite on $J_r(F_\lambda)$ whereas the Hausdorff measure is on $J_r(F_\lambda)$ finite and positive. We also show that the Hausdorff dimension of $J_r(F_\lambda)$ is in $(1, 2)$. Appendix contains the proof of the fact that the two-dimensional Lebesgue measure of the Julia set $J(f_\lambda)$ vanishes.

2. Basic topological properties and dynamics of the maps $f_\lambda(z)$

Consider the family f_λ of maps defined in (1.1). The function f_λ is periodic with period πi, and its Schwarzian derivative Sf_λ is equal to -2.

The singularities of f_λ^{-1} are $a_1 = 0$ and $a_2 = \lambda$. They are the transcendental singularities. Recall that a point α is said to be a *transcendental singularity* of f_λ^{-1} (or an asymptotic value of f) if there is a curve γ in $\mathbb{C}$ such that $f(z) \to \alpha$ on $f(\gamma)$ when $z \to \infty$ on γ. Let

$$W_1 = \{z : \mathrm{Re}(z) < 0\}, \quad W_2 = \{z : \mathrm{Re}(z) > 0\}.$$

In each sector W_i, $i = 1, 2$, f_λ has the following behavior: there is a disc B_i around a_i such that $f_\lambda^{-1}(B_i \setminus \{a_i\})$ contains a unique unbounded component $U_i \subset W_i$ and $f_\lambda : U_i \to B_i \setminus \{a_i\}$ is a universal covering. These U_i are called *exponential tracts*. It is seen from the graph of f_λ restricted to $\mathbb{R}$ that f_λ has two fixed points $q_i = q_i(\lambda), i = 1, 2$, with $q_1 < 0 < q_2$, q_1 is repelling while q_2 is attracting. Moreover, if $\mathrm{Re}(z) > 0$ then $f_\lambda^n(z) \to q_2$ as $n \to \infty$, hence $J(f_\lambda)$ is contained in the half-plane $\{z : \mathrm{Re}(z) \le 0\}$. As $\mathbb{R}^- \cup \{0\} \subset J(f_\lambda)$, all the preimages of $\mathbb{R}^-$ are in $J(f_\lambda)$. In particular the inverse-images $f_\lambda^{-2n}(\mathbb{R}^-)$ are contained in so called, *Cantor bouquet*. We recall its definition. Let Σ_N be the set of sequences of $s = (s_0, s_1, s_2, \dots)$, where the s_i are integers, $|s_i| \le N$. An invariant subset $\mathcal{C}_N$ of $J(f_\lambda)$ is called a *Cantor N-bouquet* for f_λ if there exists a homeomorphism $h : \Sigma_N \times [0, \infty) \to \mathcal{C}_N$ such that

$$\pi \circ h^{-1} \circ f_\lambda \circ h(s, t) = \sigma(s),$$

where $\pi : \Sigma_N \times [0,\infty) \to \Sigma_N$ is the projection map, σ is the shift automorphism defined by $\sigma(s_0, s_1, s_2, \dots) = (s_1, s_2, \dots)$ and $\lim_{t\to\infty} h(s,t) = \infty$, $\lim_{t\to\infty} f_\lambda^n \circ h(s,t) = \infty$ if $t \neq 0$ and $n \to \infty$. An N-bouquet $\mathcal{C}_N$ is included naturally in an $N+1$-bouquet $\mathcal{C}_{N+1}$ by considering only the sequences with entries less or equal to N in absolute value. The set

$$\mathcal{C} = \overline{\bigcup_{N\geq 0} \mathcal{C}_N}$$

is called a *Cantor bouquet* . Note that one of the asymptotic value 0 is also a pole of f_λ. Thus f_λ satisfies the assumptions of the following lemma proved in [12].

Lemma 2.1. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic map. Suppose f has a polynomial Schwarzian derivative of degree $p-2$ and has an asymptotic value a_j which is a pole. Let W_j be the sector containing the exponential tract corresponding to a_j. Then for each $N > 0$, $J(f)$ contains a Cantor N-bouquet in W_j which is invariant under f^2.*

So, for $\lambda > 0$, $J(f_\lambda)$ is contained in the half-plane $\{z : \mathrm{Re}(z) \leq 0\}$ and contains a Cantor bouquet, while $F(f_\lambda)$ is an attractive basin of q_2. Let for $n \geq 1$

$$I_n(f_\lambda) = \{z \in \mathbb{C} : \lim_{k\to\infty} f^{kn}(z) = \infty\}.$$

Since f_λ has no wandering components (see [4]) and no Baker's domain (see [5], [24]) the set $I_n(f_\lambda) \subset J(f_\lambda)$ for all $n \in \mathbb{N}$. It was proved in [15] that $\mathrm{HD}(I_2(f_\lambda)) = 2$, and consequently $\mathrm{HD}(J(f_\lambda)) = 2$.

3. Useful Formulas

From now onward we put

$$f := f_\lambda \text{ and } F := F_\lambda.$$

It follows from (1.1) that

$$f'(z) = \frac{-2\lambda}{(e^z - e^{-z})^2} \tag{3.1}$$

For every $M \in \mathbb{R}$, let

$$\mathbb{C}_M = \{z \in \mathbb{C} : \mathrm{Re} z < M\} \quad \text{and} \quad Q_M = \{z \in Q : \mathrm{Re} z < M\}$$

By (3.1) there exists an integer $T_1 > 0$ so large that

$$\lambda e^{2\mathrm{Re}(z)} \leq |f'(z)| \leq 4\lambda e^{2\mathrm{Re}(z)} \tag{3.2}$$

for all $z \in \mathbb{C}_{-T_1}$. Since $\lim_{z\to 0} \frac{e^z - e^{-z}}{z} = 2$, it follows from (3.1) that there exists $R > 0$ so small that if $|z| \leq R$, then

$$\frac{\lambda}{4|z|} \leq |f(z)| \leq \frac{\lambda}{|z|} \tag{3.3}$$

and

$$\frac{\lambda}{4}|z|^{-2} \le |f'(z)| \le \lambda |z|^{-2}. \tag{3.4}$$

It immediately follows from the formula defining f that there exists an integer $T_2 \ge T_1$ such that

$$\frac{\lambda}{2} e^{2\mathrm{Re}(z)} \le |f(z)| \le 2\lambda e^{2\mathrm{Re}(z)}. \tag{3.5}$$

for all $z \in \mathbb{C}_{-T_2}$. It in turn immediately follows from this formula that there exists an integer $T_3 \ge T_2$ such that

$$f(\mathbb{C}_{-T_3}) \subset B(0,R). \tag{3.6}$$

Using the Chain Rule and combining (3.2) along with (3.4), (3.6) and (3.5), we see that for all $z \in \mathbb{C}_{-T_3}$

$$\begin{aligned}\frac{1}{16}e^{-2\mathrm{Re}(z)} = \frac{\lambda^2}{4}e^{2\mathrm{Re}(z)}(2\lambda)^{-2}e^{-4\mathrm{Re}(z)} &\le \frac{\lambda^2}{4}e^{2\mathrm{Re}(z)}|f(z)|^{-2} \le \\ &\le |(f^2)'(z)| \\ &\le 4\lambda^2 e^{2\mathrm{Re}(z)}|f(z)|^{-2} \le 4\lambda^2 4\lambda^{-2}e^{2\mathrm{Re}(z)}e^{-4\mathrm{Re}(z)} = 16e^{-2\mathrm{Re}(z)}.\end{aligned} \tag{3.7}$$

It follows from this and (3.4) that

$$\lim_{\mathrm{Re}(z)\to-\infty} |(f^2)'(z)| = \infty \tag{3.8}$$

and

$$\lim_{\mathrm{Re}(z)\to 0} |f'(z)| = \infty. \tag{3.9}$$

Notice that $f^{-1}(\mathbb{C}_0) \subset \mathbb{C}_0$ and $\mathbb{C}_0$ contains no asymptotic values (0 and λ). So, all the holomorphic inverse branches of f are well-defined on $\mathbb{C}_0$ and given by the formulas:

$$f_k^{-1}(z) = -\frac{1}{2}\log\left|1-\frac{\lambda}{z}\right| - \left(\frac{1}{2}\mathrm{Arg}\left(1-\frac{\lambda}{z}\right) + k\pi\right) i, \tag{3.10}$$

$k \in \mathbb{Z}$, where $\mathrm{Arg}w$ is the principal value of the argument i.e. $\mathrm{Arg}w \in [-\pi,\pi)$. Equivalently

$$f_k^{-1}(z) = -\frac{1}{2}\log_k\left(1-\frac{\lambda}{z}\right), \tag{3.11}$$

where the branch of logarithm $\log_k$ is determined by the requirement that $\log_k(1) = 2\pi i k$. Hence

$$(f_k^{-1})'(z) = \frac{-\lambda}{2z(z-\lambda)}. \tag{3.12}$$

Let

$$B_-(0,R) := \{z \in B(0,R) : \mathrm{Re}(z) < 0\}.$$

For $R > 0$ sufficiently small and all $z \in B_-(0, R)$, we have

$$\frac{2\lambda}{|z|} \geq 1 + \left|\frac{\lambda}{z}\right| \geq \left|1 - \frac{\lambda}{z}\right| \geq \left|\frac{\lambda}{z}\right| - 1 \geq \frac{\lambda}{2|z|}.$$

So

$$\log 2 + \log \lambda - \log |z| \geq \log \left|1 - \frac{\lambda}{z}\right| \geq \log \lambda - \log 2 - \log |z|. \tag{3.13}$$

Thus, if $z \in B_-(0, R)$ then for every $k \in \mathbb{Z}$

$$\mathrm{Re}(f_k^{-1}(z)) = -\frac{1}{2} \log \left|1 - \frac{\lambda}{z}\right| \leq \frac{1}{2} \log 2 - \frac{1}{2} \log \lambda + \frac{1}{2} \log |z| \leq c_1 + \frac{1}{2} \log R. \tag{3.14}$$

and

$$\mathrm{Re}(f_k^{-1}(z)) \geq c_2 + \frac{1}{2} \log |z|, \tag{3.15}$$

where $c_1 = \frac{1}{2} \log 2 - \frac{1}{2} \log \lambda$ and $c_2 = -\frac{1}{2} (\log 2 + \log \lambda)$. Therefore,

$$f_0^{-1}(B_-(0, R)) \subset P_{c_1 + \frac{1}{2} \log R} := \left\{ z \in \mathbb{C}\colon \ \mathrm{Re}(z) < c_1 + \frac{1}{2} \log R, \ -\frac{\pi}{2} < \mathrm{Im}(z) \leq \frac{\pi}{2} \right\}. \tag{3.16}$$

Given $0 < r < R$ set

$$A_-(0, r, R) = \{z \in \mathbb{C}\colon r < |z| < R, \ \mathrm{Re}(z) < 0\}.$$

It follows from (3.15) and (3.16) that

$$f_0^{-1}(A_-(0, r, R)) \subset P_{c_1 + \frac{1}{2} \log R} \setminus P_{c_2 + \frac{1}{2} \log r}. \tag{3.17}$$

It follows from (3.11) that

$$\frac{\lambda}{4|z|} \leq |f_0^{-1}(z)| \leq \frac{\lambda}{|z|} \tag{3.18}$$

for some $T_4 \geq T_3$ and all z with $|z| \geq T_4$. Recall that the dynamics of F is closely related to the dynamics of f, since $F^n(z) = \Pi(f^n(z))$ for all $n \geq 0$. Notice that

$$J(f) \subset \mathbb{C}_0 \cup \{\pi k i : k \in \mathbb{Z}\}, \quad J(F) = \Pi(J(f)), \quad F(J(F)) \subset J(F) \cup \{\infty\}) \tag{3.19}$$

$$\mathrm{dist}\Big(J(F), \Pi\Big(\{z \in \mathbb{C}\colon \mathrm{Im} z = \pi/2\}\Big)\Big) > 0 \tag{3.20}$$

and

$$J(F) \cap \Pi\Big(\mathbb{C}_{-T_5}\Big) \subset \left\{ z \in \Pi\Big(\mathbb{C}_{-T_5}\Big) : |\mathrm{Im} z| < \frac{\pi}{3} \right\}$$

with $T_5 \geq T_4$ sufficiently large. For every $M \in \mathbb{R}$ put

$$Q_M := Q \cap \Pi(\mathbb{C}_M).$$

4. Bounded Orbits

In this section we will prove the fact which besides being interesting itself is absolutely necessary for the proof of all further results of this paper. Let

$$J_{bd}(F) = \{z \in J(F) : \inf_{n\geq 1}\{\mathrm{Re}(F^n(z))\} > -\infty \text{ and } \inf_{n\geq 1}\{|F^n(z)|\} > 0\}. \tag{4.1}$$

The fact we just mentioned is this.

Lemma 4.1. *We have* $\mathrm{HD}(J_{bd}(F)) > 1$. *Even more, there exists* $s > 1$ *such that* $\mathrm{HD}(K_n) \geq s$ *for all* $M > 0$ *large enough, where the sets* K_M *are defined by formula (5.1).*

Proof. Given $T_4 \leq r_1 < r_2$ it follows from (3.18) that

$$f_0^{-1}(A_-(0, r_1, r_2)) \subset A\left(0, \frac{\lambda}{4r_2}, \frac{\lambda}{r_1}\right). \tag{4.2}$$

The heart of our proof is to define and examine an appropriate iterated function system consisting of backward holomorphic branches of F^2. In order to do it we need some estimates. Fix $R > 0$ to be determined in the course of the proof. For every $r \in (0, R)$ and every $k \geq 0$ put

$$P_k(r, R) = \left\{z \in \mathbb{C}: -R < \mathrm{Re}(z) < -r,\ k\pi - \frac{\pi}{2} < \mathrm{Im}(z) < k\pi + \frac{\pi}{2}\right\}$$

and notice that

$$P_k(1, R) \subset A_-\left(0, \frac{1}{2}(1 + k\pi), R + \left(k + \frac{1}{2}\right)\pi)\right) \subset A_-\left(0, \frac{1}{2}(1 + k\pi), 2R + k\pi\right),$$

where the latter inclusion was written assuming that $R \geq \pi/2$. It then follows from (4.2) and (3.17) that for all $k \geq 1$ large enough, say $k \geq k_0$, we have

$$\begin{aligned} f_0^{-2}(P_k(1, R)) &\subset f_0^{-2}\left(A_-\left(0, \frac{1}{2}(1 + k\pi), 2R + k\pi\right)\right) \subset f_0^{-1}\left(A_-\left(0, \frac{\lambda}{4(2R + k\pi)}, \frac{2\lambda}{1 + k\pi}\right)\right) \\ &\subset \left\{z \in P : c_2 + \frac{1}{2}\log\left(\frac{\lambda}{4(2R + k\pi)}\right) < \mathrm{Re}z < c_1 + \frac{1}{2}\log\left(\frac{2\lambda}{1 + k\pi}\right)\right\} \end{aligned} \tag{4.3}$$

Fix an integer $p \geq 2$ so large that $\frac{2\lambda}{\pi}\exp\Big(2(c_1 + 1 + pj)\Big) \geq k_0$, say $p \geq p_0$. Fix also an integer $q \geq 2$. If

$$0 \leq k < \frac{\lambda}{4\pi}\exp\Big(2(c_2 + 1 + p(j + 1)\Big) - \frac{2(1 + pq)}{\pi},$$

then

$$c_2 + \frac{1}{2}\log\left(\frac{\lambda}{4(2(1 + pq) + k\pi)}\right) > -1 - p(j + 1)$$

Also, if

$$k > \frac{2\lambda}{\pi}\exp\Big(2(c_1 + 1 + pj)\Big),$$

then

$$c_1 + \frac{1}{2}\log\left(\frac{2\lambda}{1+k\pi}\right) < -1 - pj,$$

Utilizing therefore (4.3), we get for all $1 \le j \le q-1$ that

$$f_0^{-2}(P_k(1, 1+pq)) \subset P_0(1+pj, 1+p(j+1)). \tag{4.4}$$

For all integers $2 \le q \le \exp(3p)$, where $p \ge p_0$ is large enough, we have

$$\frac{\lambda}{8\pi}\exp\Big(2(c_2+1+p(j+1)\Big) < \frac{\lambda}{4\pi}\exp\Big(2(c_2+1+p(j+1)\Big) - \frac{2(1+pq)}{\pi} \tag{4.5}$$

For all $j = 1, \dots, q-1$ let $I_j := [a_j, b_j] \cap I\!N$, where $a_j = \frac{2\lambda}{\pi}\exp\Big(2(c_1+1+pj)\Big) = C_4 e^{2pj}$ and $b_j = \frac{\lambda}{8\pi}\exp(2(c_2+1+p(j+1)) = C_5 e^{2p} e^{2pj}$ with appropriate positive constants C_4 and C_5. Then

$$\sharp(I_j) > \frac{1}{2} C_5 e^{2p} e^{2pj}. \tag{4.6}$$

For every $z \in P_0(1+pj, 1+p(j+1))$ it follows from (3.7) that $|(f^2)'(z)| \le C_6 e^{2p(j+1)}$ (with an appropriate positive constant C_6), and consequently

$$|(f^2)'(z)|^{-t} \ge C_6^{-t} e^{-2p(j+1)t}. \tag{4.7}$$

Fixing $q = E(e^{3p})$ we now define the conformal iterated function system (see [21] for an account of the theory of conformal iterated function systems)

$$\mathcal{S}_p = \{\phi_{j,k} : P_0(1, 1+pq) \to P_0(1, 1+pq)\}_{1 \le j \le q-1, k \in I_j},$$

where $\phi_{j,k}(z) = f_0^{-2}(z + 2\pi i k)$. Fix $t \ge 0$ and let $\mathrm{P}(t)$ be the topological pressure corresponding to the parameter t (see [21] for the definition of $\mathrm{P}(t)$). Since, by (4.4), for every $j \in \{1, 2, \dots, q-1\}$ we have

$$\phi_{j,k}(P_0(1, 1+pq)) \subset f_0^{-2}(P_k(1, 1+pq)) \subset P_0(1+pj, 1+p(j+1)),$$

it follows from (4.6) and (4.7) that

$$\begin{aligned}
\mathrm{P}(t) &\ge \log \sum_{j=1}^{q-1} \sum_{k \in I_j} \inf\{|(F^2)'(z)|^{-t} : z \in P_0(1+pj, 1+p(j+1))\} \ge \log \sum_{j=1}^{q-1} \sharp(I_j) C_6^{-t} e^{-2p(j+1)t} \\
&\ge \log\left(\sum_{j=1}^{q-1} \frac{1}{2} C_5 C_6^{-t} e^{2p(j+1)} e^{-2p(j+1)t}\right) \ge \log \sum_{j=1}^{q-1} e^{2p(j+1)(1-t)} + C_7
\end{aligned}$$

with an appropriate constant $C_7 \in I\!R$ independent of p and $q = E(e^{3p})$. Therefore

$$\mathrm{P}(1) \ge \log(q-1) + C_7 > \log(e^{3p} - 2) + C_7 > 0 \tag{4.8}$$

for all $p \ge 2$ large enough. Let J_p be the limit set of the conformal iterated function system $\mathcal{S}_p$. It follows from Theorem 3.15 from [21] that $\mathrm{HD}(J_p) > 1$. Identifying the belt P with $\Pi(P) \subset Q$, it follows from the construction of S_p and the definition of the limit set J_p, that $F^2(J_p) \subset J_p$ and that J_p is the closure of the fixed (attracting) points of all compositions of all maps from $\mathcal{S}_p$. Now, for every $z \in J_p$, the orbit $\{F^{2n}(z)\}_{n\geq 0}$ is contained in $P_0(1,1+pq)$. Since in addition $\overline{J(F) \cap P_0(1,1+pq)}$ is a compact subset of Q_{-1}, we conclude that $J_p \subset K_M$ for all $M > 0$ large enough. We are done. ■

5. Conformal measures

Our first goal in this section is to prove the existence of a conformal measure and to examine in detail its properties. In order to do it we begin with the definition and analysis of the sets K_M, $m > 0$. Indeed, for every $M > 0$, let

$$W_M = \{z \in J(F) : \mathrm{Re}(z) \geq -M \text{ and } |z| \geq 1/M\}$$

and let

$$K_M = \bigcap_{k\geq 0} F^{-k}(W_M). \tag{5.1}$$

Since $F : J(F) \to J(F)$ is continuous, K_M is a forward F-invariant compact subset of $J(F)$. Notice also that if $z \in Q$, $j \geq 0$, $F^j(z) \in Q_0$ and $|F^j(z)| \geq 1/M$, then there exists a unique holomorphic inverse branch $F_z^{-j} : B(F^j(z), 1/M) \to Q$ of F^j, sending $F^j(z)$ to z. We shall prove the following.

Lemma 5.1. *For all $M > 0$ and every $\theta > 1$ there exists $q_M(\theta) \geq 1$ such that $|(F^k)'(x)| \geq \theta$ for all $x \in K_M$ and all $k \geq q_M(\theta)$.*

Proof. Fix $M > 0$ and suppose on the contrary that there exist a sequence $\{x_i\}_{i=1}^{\infty} \subset K_M$ and $\{n_i\}_{i=1}^{\infty}$, an unbounded increasing sequence of positive integers such that

$$|(F^{n_i})'(x_i)| < \theta. \tag{5.2}$$

Consider holomorphic inverse branches $F_i^{-n_i} : B(F^{n_i}(x_i), 1/n) \to Q$ of F^{n_i} sending $F^{n_i}(x_i)$ to x_i. Passing to a subsequence, we may assume that the limits $x := \lim_{i\to\infty} x_i \in K_M$ and $y := \lim_{i\to\infty} F^{n_i}(x_i) \in K_M$ exist. The formula (5.2) equivalently means that $|(F_i^{-n_i})'(F^{n_i}(x_i))| > 1/\theta$ for all $i \geq 1$. Applying now the $\frac{1}{4}$-Koebe distortion theorem we see that

$$F_*^{-n_i}(B(F^{n_i}(x_i), 1/n)) \supset B(x_i, (4\theta n)^{-1})$$

for all $i \geq 1$. Therefore, $F_*^{-n_i}(B(F^{n_i}(x_i), 1/n)) \supset B(x, (8\theta n)^{-1})$ for all $i \geq 1$ large enough. Hence $F^{n_i}(B(x, (8\theta n)^{-1})) \subset B(F^{n_i}(x_i), 1/n) \subset B(y, 2/n)$ for all $i \geq 1$ large enough. Thus the family $\{F^{n_i}|_{B(x,(8\theta n)^{-1})}\}_{i\geq 1}$ is normal. This however is a contradiction with the fact that $x_i \in K_M \subset J(F)$ and we are done. ■

Given $t \geq 0$ a Borel probability measure on Q is said to be t-conformal for $F : J(F) \to J(F)$ if and only if $m(J(F)) = 1$ and

$$m(F(A)) = \int_A |F'|^t dm \tag{5.3}$$

for every Borel set $A \subset J(F)$ such that $F_{|A}$ is one-to-one. First, following [10], for every $M > 0$ large enough, we shall build a probability Borel measure m_M, with the topological support contained in K_M, and which will be "almost conformal" for some $t_M \geq 0$, meaning that

$$m_M(F(A)) \geq \int_A |F'|^{t_M} dm_M \tag{5.4}$$

for every Borel set $A \subset Q$ such that $F|_A$ is 1-to-1, and (5.3) holds if we assume in addition that $A \cap \{z \in Q : \mathrm{Re}z \leq M \text{ or } |z| \leq 1/M\} = \emptyset$. In what follows throughout Corollary 5.4 we follow closely the appropriate reasoning from [32]. In the sequel, we will need to refer to some details of the construction, of the measure m_M, so we briefly describe it now. For every $M > 0$ large enough choose a finite set $E^M \subset K_M$ such that the $B(E^M, 1/2M) \supset K_M$ and that E^M contains the forward orbit of a periodic point ξ of F. Notice that, since K_M is F-forward invariant, the whole forward orbit of ξ is contained in K_M. The existence of such a periodic point follows from the density of periodic points in $J(F)$. Consider the function

$$c_M(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_M} \sum_{w \in \left(F|_{K_M}\right)^{-n}(x)} |(F^n)'|^{-t}(w).$$

The function $t \mapsto c_M(t)$, $t \in I\!R$, has three important properties. First, notice that it follows from Hölder's inequality that it is convex in $I\!R$, so it is continuous. Next, it follows easily from Lemma 5.1 that this function is strictly decreasing and $\lim_{t\to+\infty} c_M(t) = -\infty$. Finally, each set $\left(F|_{K_M}\right)^{-n}(E^M)$ is not empty as it contains a point from the forward orbit of ξ. In particular $c_M(t) \geq 0$ for all $t \in I\!R$. All these properties imply that there exists a unique value $t = t_M$ with $c_M(t_M) = 0$. Following the general construction described in [10] (see also [23], Chapter 10), with the sets $E_n = \left(F|_{K_M}\right)^{-n}(E^M)$ we obtain a measure m_M, for which $m_M(K_M) = 1$ and which is "almost conformal" with the exponent t_M. We continue on with the following two lemmas; the idea of their proofs comes from [32].

Lemma 5.2. $\mathrm{HD}(K_M) \geq t_M$.

Proof. Fix a point $x \in K_M$ and an integer $n \geq 1$. Let $F_x^{-n} : B(F^n(x), 1/M) \to Q$ be the holomorphic inverse branch of F^n sending $F^n(x)$ to x. Applying now the $\frac{1}{4}$-Koebe distortion theorem and the standard Koebe distortion theorem, it follows from (5.4) that

$$\begin{aligned} m_M\left(B\left(x, \frac{1}{4}|(F^n)'(x)|^{-1}\frac{1}{2M}\right)\right) &\leq m_M\left(F_x^{-n}\left(B\left(F^n(x), \frac{1}{2M}\right)\right)\right) \\ &\leq K^{t_M}|(F^n)'(x)|^{-t_M} m_M\Big(B(F^n(x), 1/2M)\Big) \\ &\leq (8KM)^{t_M}\left(\frac{1}{4}|(F^n)'(x)|^{-1}\frac{1}{2M}\right)^{t_M}. \end{aligned} \tag{5.5}$$

Since, by Lemma 5.1, $\lim_{n\to\infty} |(F^n)'(x)| = \infty$ uniformly on K_M, we conclude that for every $r > 0$ small enough there exists $n \geq 1$ such that

$$\frac{1}{4}|(F^{n+1})'(x)|^{-1}\frac{1}{2M} \leq r \leq \frac{1}{4}|(F^n)'(x)|^{-1}\frac{1}{2M}.$$

Using (5.5), we therefore get

$$\begin{aligned} m_M(B(x,r)) &\leq m_M\left(B\left(x, \frac{1}{4}|(F^n)'(x)|^{-1}\frac{1}{2M}\right)\right) \\ &\leq (8KM)^{t_M}\left(\frac{1}{4}|(F^{n+1})'(x)|^{-1}\frac{1}{2M}\right)^{t_M} |F'(F^n(x))|^{t_M} \\ &\leq \left[(8KTM)^{t_M}\right] r^{t_M}, \end{aligned}$$

where $T = \sup\{|F'(y)| : y \in K_M\}$ is finite since K_M is a compact subset of $Q \setminus \{0\}$. This inequality implies in a standard way that $\mathrm{HD}(K_M) \geq t_M$ (see e.g [23]). ■

Lemma 5.3. *For every M large enough there exists $p > 0$ such that* $\mathrm{HD}(K_M) \leq t_{M+p}$.

Proof. It easily follows from Lemma 5.1 and the absence of critical points of F in Q that

$$L = \inf\{|(F^n)'(w)| : w \in K_M, n \geq 1\} > 0 \quad \text{and} \quad \lim_{n\to\infty} |(F^n)'(z)| = \infty \tag{5.6}$$

for all $z \in K_M$. Fix $p > 0$ so large that $KL^{-1} < p(M(M+p))^{-1}$ and consider the set J_{M+p}. Following the construction of almost conformal measures described above, we choose a finite collection of points $E^{M+p} \subset K_{M+p}$ such that the balls $B(x, (2(M+p))^{-1})$, $x \in E^{M+p}$, cover the set K_{M+p}. Let $y \in K_M \subset K_{M+p}$. Given $n \geq 0$ there exists $x \in E^{M+p}$ such that $F^n(y) \in B(x, (2(M+p))^{-1})$. By our definition of the set K_{M+p}, all holomorphic branches of F^{-i}, $i \geq 0$, are well-defined on $B(x, 1/(M+p))$. Fix $0 \leq i \leq n$ and let F_y^{-i} be the holomorphic branch of F^{-i} sending $F^n(y)$ to $F^{n-i}(y)$. Then, by Koebe's distortion theorem, for all $z \in B(x, (2(M+p))^{-1})$, we get

$$\frac{|(F_y^{-i})'(z)|}{|(F_y^{-i})'(F^n(y))|} \leq K.$$

So, since $F^{n-i}(y) \in K_M$, using (5.6), we obtain $|(F_y^{-i})'(z)| \leq K|(F_y^{-i})'(F^n(y))| \leq KL^{-1}$. Thus, $|F_y^{-i}(x) - F^{n-i}(y)| \leq KL^{-1} < p(M(M+p))^{-1}$, and consequently, using the fact that $F^{n-i}(y) \in W_M$, we see that $F_y^{-i}(x) \in W_{M+p}$ for all $0 \leq i \leq n$. This implies that $F_y^{-n}(x) \in K_{M+p}$, i.e. $F_y^{-n}(x) \in \left(F|_{K_{M+p}}\right)^{-n}(x)$. Let $\mathcal{F}_n(x)$ be the collection of all holomorphic inverse branches F_ν^{-n} of F^n defined on $B(x, (M+p)^{-1})$, such that $F_\nu^{-n}(x) \in K_{M+p}$. It follows from the above considerations that

$$K_M \subset \bigcup_{x\in E^{M+p}} \bigcup_{\nu\in\mathcal{F}_n(x)} F_\nu^{-n}(B(x, (2(M+p))^{-1})). \tag{5.7}$$

In addition, in view of Lemma 5.1, $\mathrm{diam}(F_\nu^{-n}(B(x,(2(M+p))^{-1})) \to 0$ uniformly as $n \to \infty$, and for every $t \ge 0$

$$\sum_{x\in E^{M+p}} \sum_{\nu\in\mathcal{F}_n(x)} \Big(\mathrm{diam}(F_\nu^{-n}(B(x,\delta_{M+p}))\Big)^t \preceq \sum_{x\in E^{M+p}} \sum_{w\in\left(F|_{K_{M+p}}\right)^{-n}(x)} \frac{1}{|(F^n)'(w)|^t} \tag{5.8}$$

Fix now an arbitrary $t > t_{M+p}$. Then $c_{M+p}(t) < 0$ and

$$\sum_{x\in E^{M+p}} \sum_{w\in\left(F|_{K_{M+p}}\right)^{-n}(x)} \frac{1}{|(F^n)'(w)|^t} \le \exp\left(\frac{1}{2}c_{M+p}(t)n\right)$$

for all n large enough. Combining this, (5.7) and (5.8), we conclude that $\mathrm{H}_t(K_M) = 0$ for all $t > t_{M+p}$ and, consequently, $\mathrm{HD}(K_M) \le t_{M+p}$. ■

Corollary 5.4. *There exists $s > 1$ such that $t_M \ge s$ for all M large enough.*

Proof. In view of Lemma 4.1 $\mathrm{HD}(K_M) > 1$ for all M large enough. Fix one such M and put $s = t_M$. Choose p ascribed to this M according to Lemma 5.3. Then, by this lemma, $t_{M+q} \ge \mathrm{HD}(K_M) = s > 1$ for all $q \ge p$. We are done. ■

Suppose now that $r \in (0,R)$ with some $R > 0$ sufficiently small and that $z, w \in \{\xi \in \mathbb{C} : r \le |\xi| \le er\}$. It then follows from (3.3) that

$$|\mathrm{Im}(f(z)) - \mathrm{Im}(f(w))| \le \frac{\lambda}{r}.$$

We therefore get the following.

Lemma 5.5. *If $r \in (0,R)$, then the map F restricted to the annulus $\{z \in \mathbb{C} : r \le |z| \le er\}$ is at most $\frac{\lambda}{\pi r}$-to-one.*

Recall that a family $\mathcal{F}$ of Borel probability measures on a metric space Y is called tight if for every $\epsilon > 0$ there exist a compact set $Y_\epsilon \subset Y$ and a finite set $\mathcal{F}_\epsilon \subset \mathcal{F}$ such that $\nu(Y \setminus Y_\epsilon) \le \epsilon$ for all $\nu \in \mathcal{F} \setminus \mathcal{F}_\epsilon$. The first step in the actual construction of a conformal measure (with an exponent > 1) is provided by the following.

Lemma 5.6. *The sequence $\{m_n\}_{n=1}^\infty$ is tight in Q.*

Proof. For every $k \ge T_4$ put

$$X_k = \{z \in J(F) : -(k+1) \le \mathrm{Re}(z) < -k\} \subset Q_{-T_4}.$$

It follows from (3.5) that

$$f(X_k) \subset A\left(0, \frac{\lambda}{2}e^{-2(k+1)}, 2\lambda e^{-2k}\right).$$

It therefore follows from Lemma 5.5 that there exists a universal constant $C > 0$ such that $F|_{f(X_k)}$ is at most Ce^{2k}-to-one. Hence, for every $n \geq 1$, so large that $2 \geq t_n \geq s > 1$ (s given by Corollary 5.4), say $n \geq q$, we have using (3.7) that

$$\begin{aligned} 1 &\geq m_n(F^2(X_k)) \geq (Ce^{2k})^{-1}\int_{X_k}|(F^2)'|^{t_n}\,dm_n \geq C^{-1}e^{-2k}m_n(X_k)\left(\frac{e^{2k}}{16}\right)^{t_n} \\ &\geq (2^8C)^{-1}e^{2k(t_n-1)}m_n(X_k) \geq (2^8C)^{-1}e^{2k(s-1)}m_n(X_k). \end{aligned}$$

Hence $m_n(X_k) \leq 2^8Ce^{2k(1-s)}$, and therefore for every $n \geq q$ and $M \geq 1$

$$m_n(Q_M) = \sum_{k=M}^{\infty} m_n(X_k) \leq 2^8C\sum_{k=M}^{\infty} e^{2(1-s)k} = \frac{2^8Ce^{2(1-s)M}}{1-e^{2(1-s)}}.$$

Since the last expression in this formula converges to zero when $M \to \infty$, we therefore conclude that the sequence $\{m_n\}_{n=1}^{\infty}$ is tight.■

Since $F(Q_M)$ is a punctured neighbourhood of 0, the following lemma is a straightforward consequence of Lemma 5.6. However, we provide a simple direct proof, as one formula derived in the course of the proof will be used later.

Lemma 5.7. *If m is a weak accumulation point of the sequence $\{m_n\}_{n=1}^{\infty}$, then $m(\{0\}) = 0$.*

Proof. The idea of this proof is the same as the idea of the proof of Lemma 5.6. For every $k \geq 0$ let

$$R_k = \{z \in Q : Re^{-(k+1)} \leq |z| \leq Re^{-k}\}.$$

Similarly to the proof of the previous lemma, we see from Lemma 5.5 that $F_{|R_k}$ is at most Ce^k to one with some constant $C > 0$ independent of k. Hence for all $n \geq q$, q defined in the previous proof, using (3.4) and Lemma 5.5, we get that

$$\begin{aligned} 1 &\geq m_n(F(R_k)) = (Ce^k)^{-1}\int_{R_k}|F'|^{t_n}\,dm_n \geq (Ce^k)^{-1}(\lambda/4)^{t_n}(R^{-2}e^{2k})^{t_n}m_n(R_k) \\ &\succeq e^{-k}e^{2ks}m_n(R_k) = e^{(2s-1)k}m_n(R_k). \end{aligned}$$

Hence

$$m_n(R_k) \preceq e^{(1-2s)k}, \tag{5.9}$$

and therefore, for every $n \geq q$ and every $j \geq 0$ we get

$$m_n(B(0, Re^{-j})) = \sum_{k=j}^{\infty} m_n(R_k) \preceq \sum_{k=j}^{\infty} e^{(1-2s)k} = \frac{e^{(1-2s)j}}{1-e^{(1-2s)}}.$$

Letting now $j \to \infty$, we conclude that $m(0) = 0$. ■

Since, in view of Lemma 4.1, $t_n \in [s,2]$, with $s > 1$, for all $n \geq 1$ large enough, we can choose a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\{t_{n_k}\}_{k=1}^{\infty}$ converges. Denote its limit by $h \in [s,2]$. It follows from Lemma 5.6 and Prohorov theorem (see [6], Theorem 5.1, Section 5, p. 59) that passing yet to another subsequence, we may assume that the sequence $\{m_{n_k}\}_{k=1}^{\infty}$ converges weakly, say to a Borel probability measure m. Since all the measures m_{n_k}, $k \geq 1$, are supported on $J(F)$, so is m. Since there can be a problem with conformality of measures m_{n_k} only on sets $\{z \in J(F) : \mathrm{Re}(z) = -n_k \text{ or } |z| = 1/n\}$, since $\lim_{k\to\infty} n_k = \infty$, and since $F : J(F) \to J(F)$ is open and has no singular points except for 0 (which, because of Lemma 5.7, does not matter), proceeding with obvious modifications as in [10] (comp. [23]) and using Lemma 5.7 we obtain the following.

Theorem 5.8. *The weak limit measure m is h-conformal for $F : J(F) \to J(F)$ with some $h \in (1,2]$ and $m(0) = 0$.*

Since, for every non-empty open set $U \subset J(F)$, $\bigcup_{n\geq 0} F^n(U) \supset J(F)\setminus\{0\}$, we get the following immediate consequence of Theorem 5.8.

Proposition 5.9. *The h-conformal measure m is positive on non-empty open subsets of $J(F)$.*

We define now the main object of our interest.

$$J_r(F) := \{z \in Q : \omega(z) \setminus \{0, -\infty\} \neq \emptyset\}. \tag{5.10}$$

We need one related definition. Let for $T > 0$

$$J_{r,T}(F) = \{z \in Q : \omega(z) \text{ is not contained in } W_T^c\},$$

where, we recall,

$$W_T = \{z \in J(F) : \mathrm{Re}(z) \geq -T \text{ and } |z| \geq 1/T\}.$$

Obviously

$$J_r(F) = \bigcup_{n\geq 1} J_{r,n}(F).$$

Theorem 5.10. *There exists $M > 0$ such that if ν is t-conformal measure for $F : J(F) \to J(F)$ with $t > 1$, then $\nu(J_{r,M}(F)) = 1$. In particular $\nu(J_r(F)) = 1$.*

Proof. We shall prove first that there exists an integer $T_5 \geq 1$ so large that

$$\nu(F(A)) \geq 4\nu(A) \tag{5.11}$$

for every Borel set $A \subset B(0, Re^{-T_5}) \cap Q$. And indeed, suppose for a contrary that for every integer $k \geq 1$ there exists a Borel set $A_k \subset B(0, Re^{-k})$ such that $\nu(F(A_k)) < 4\nu(A_k)$. Proceeding in the same way as in Lemma 5.7, we get for every $j \geq k$ the following.

$$\begin{aligned}4\nu(A_k) > \nu(F(A_k)) \geq \nu(F(A_k \cap R_j)) &\geq (Ce^j)^{-1} \int_{A_k \cap R_j} |F'|^t d\nu \\ &\geq C^{-1} e^{-j} (\lambda/4)^t (R^{-2} e^{2j})^t \nu(A_k \cap R_j) \geq C_1 e^{j(2t-1)} \nu(A_k \cap R_j)\end{aligned}$$

with some universal constant $C_1 > 0$. Hence, $\nu(A_k \cap R_j) \leq 4C_1^{-1} e^{j(1-2t)} \nu(A_k)$. Therefore, summing over all $j \geq k$, we get

$$\nu(A_k) = \sum_{j=k}^{\infty} \nu(A_k \cap R_j) \leq 4C_1^{-1} \nu(A_k) \sum_{j=k}^{\infty} e^{j(1-2t)} = \frac{4e^{k(1-2t)}}{C_1(1 - e^{(1-2t)})} \nu(A_k).$$

Since $\nu(A_k) > 0$, we therefore get

$$1 \leq \frac{4e^{(1-2t)k}}{C_1(1 - e^{(1-2t)})}.$$

Since the right-hand side of this inequality converges to 0 when $k \to \infty$, we get a contradiction, and formula (5.11) is proven. Proceeding now similarly as above and similarly as in the proof of Lemma 5.6 we shall demonstrate that there exists an integer T_6 so large that

$$\nu(F^2(A)) \geq 4\nu(A) \tag{5.12}$$

for every Borel set $A \subset Q_{-T_6}$. And indeed, suppose for a contrary that for every $k \geq T_6$ there exists a Borel set $E_k \subset Q_{-k}$ such that $\nu(F^2(E_k)) < 4\nu(E_k)$. Proceeding in the same way as in Lemma 5.6, we get for every $j \geq k$ that

$$\begin{aligned}4\nu(E_k) > \nu(F^2(E_k)) \geq \nu(F^2(E_k \cap X_j)) &\geq (Ce^{2j})^{-1} \int_{E_k \cap X_j} |(F^2)'|^t d\nu \\ &\geq C^{-1} e^{-2j} \left(\frac{1}{16} e^{2j}\right)^t \nu(E_k \cap X_j) \geq C^{-1} e^{2j(t-1)} \nu(E_k \cap X_j)\end{aligned}.$$

The rest of the argument is exactly the same as the corresponding part of the proof of (5.11). Now set

$$T_7 = \max\{T_5, T_6\}$$

and for every $k \geq T_7$ put

$$I_k = I_k(F) = \bigcap_{n \geq 0} F^{-n}(W_k^c) = \{z \in J(F) : \forall n \geq 0, \ \mathrm{Re}(F^n(z)) < -k \quad \text{or} \quad |F^n(z)| < 1/k\}.$$

Note that $F(I_k) \subset I_k$. We shall prove that there is $M > 0$ so large that for all $k \geq M$, $\nu(I_k) = 0$. Indeed, set

$$I_k^\infty := I_k \cap Q_{-k}, \quad I_k^0 := I_k \cap B(0, 1/k).$$

Then

$$I_k = I_k^\infty \cup I_k^0.$$

For every $A \subset I_k$ define

$$G(A) := F^2(A \cap I_k^\infty) \cup F(A \cap I_k^0). \tag{5.13}$$

Applying (5.12) and (5.11), we get for all $k \geq 1$ large enough that

$$\begin{aligned} \nu(G(A)) &\geq \frac{1}{2}\Big(\nu(F^2(A \cap I_k^\infty)) + \nu(F(A \cap I_k^0))\Big) \\ &\geq \frac{1}{2}(4\nu(A \cap I_k^\infty) + 4\nu(A \cap I_k^0) = 2\nu(A). \end{aligned} \tag{5.14}$$

Since $A \subset I_k$ and $F(I_k) \subset I_k$ (so also $F^2(I_k) \subset I_k$), it follows from (5.13) that $G(A) \subset I_k$. Hence $G^n(A) \subset I_k$ for all $n \geq 1$. We apply now (5.14) to the set $A = I_k$ and obtain

$$\nu(G^n(I_k)) \geq 2^n \nu(I_k).$$

for all $n \geq 1$. Since ν is a probability measure, $\nu(I_k) = 0$ for all $k \geq 1$ large enough. In particular $\nu(I_M) = 0$, and consequently

$$\nu(\bigcup_{j=0}^{\infty} F^{-j}(I_M)) = 0. \tag{5.15}$$

Since

$$J_r(F) \setminus J_{r,M}(F) \subset \{z \in \mathbb{C} \colon \exists k \geq 0 \ F^k(z) \in I_M \ \} = \bigcup_{k \geq 0} F^{-k}(I_M),$$

applying (5.15), we get $\nu(J_r(F) \setminus J_{r,M}(F)) = 0$. Hence $\nu(J_{r,M}(F)) = 1$ and we are done. ■

Now fix $T > 0$. It follows from the definition of the set $J_{r,T}(F)$ that there exists an unbounded increasing sequence $\{n_k(z)\}_{k=1}^\infty$ such that $y(z) := \lim_{k\to\infty} F^{n_k(z)}(z)$ exists and

$$F^{n_k(z)}(z) \in Q_{-2T}^c \setminus B\left(0, \frac{1}{2T}\right).$$

for all $k \geq 1$. We shall prove the following.

Lemma 5.11. *For every $T \geq 1$ and every $z \in J_{r,T}(F)$*

$$\lim_{k\to\infty} |(F^{n_k(z)})'(z)| = +\infty.$$

Proof. Put $n_k = n_k(z)$, $z \in J_{r,T}(F)$, $k \geq 1$. Suppose on the contrary that

$$\liminf_{k\to\infty} |(F^{n_k})'(z)| < +\infty.$$

Without loss of generality we may assume that $\tau := \lim_{k\to\infty} |(F^{n_k})'(z)| < +\infty$ and that $F^{n_k}(z) \in B(y(z), (6T)^{-1})$ for all $k \geq 1$. Consider the family $\{F_z^{-n_k} : B(y(z), (2T)^{-1}) \to Q\}_{k\geq 1}$ of holomorphic inverse branches of F^{n_k} sending $F^{n_k}(z)$ to z. Applying the $\frac{1}{4}$-Koebe's distortion theorem, we see that $F_z^{-n_k}\Big(B(y(z), (2T)^{-1})\Big) \supset B(z, (8\tau)^{-1})$ for all $k \geq 1$ large enough. Thus $F^{n_k}(B(z, (8\tau)^{-1})) \subset B(y(z), (2T)^{-1})$, and consequently the family of maps

$\{F^{n_k} : B(z, (8\tau)^{-1}) \to Q\}_{k=1}^{\infty}$ is normal, which contradicts the fact that $z \in J(F)$ and finishes the proof. ■

Lemma 5.12. *Let m be the weak-limit measure whose h-conformality is established in Theorem 5.8. Then for every non-empty open set $U \subset Q$, we have*

$$\limsup_{n\to\infty} m(F^n(U)) = 1.$$

Proof. Let $\tilde{U}$ be a connected component of $\Pi^{-1}(U)$. Since periodic points of f are dense in $J(f)$, $\tilde{U}$ contains a repelling periodic point ω. Denote the period of ω by p. There then exists an open ball $W \subset \tilde{U}$ centered at ω such that $f^p(W) \supset W$. Since $\omega \in J(f^p)$, $\bigcup_{n\geq 0} f^{pn}(W) = \mathbb{C} \setminus \{0, \lambda\}$. Hence for every $n \geq 1$

$$F^{pn}(\Pi(W)) = \Pi(f^{pn}(W)) \supset \Pi(W)$$

and

$$\bigcup_{n\geq 0} F^{pn}(\Pi(W)) = \bigcup_{n\geq 0} \Pi(f^{pn}(W)) = \Pi\left(\bigcup_{n\geq 0} f^{pn}(W)\right) = \Pi(\mathbb{C} \setminus \{0, \lambda\}) = Q. \tag{5.16}$$

Thus

$$\lim_{n\to\infty} m\Big(F^{pn}(\Pi(W))\Big) = m\left(\bigcup_{n\geq 0} F^{pn}(\Pi(W))\right) = m(Q) = 1.$$

Since $\Pi(W) \subset \Pi(\tilde{U}) = U$, we are done. ■

The next theorem is the (important) starting point for the developing the ergodic theory of the dynamical system $F : J(F) \to J(F)$ with respect to the h-conformal measure m. The first elements of its proof go back to [26], the method matured in [27] and [28]. Its versions also can be for example found in [8], [19], [29], [31], and [32]. The full proof of Theorem 5.13 can be carried out with obvious minor modifications similarly to the proofs of Theorem 4.23 in [19] and Theorem 3.15 in [31].

Theorem 5.13. *The h-conformal measure m is a unique t-conformal probability measure, with $t > 1$, for $F : J(F) \to J(F)$. In addition, m is conservative and ergodic.*

6. An invariant measure equivalent to the conformal measure m

In this section we show the existence and uniqueness of a Borel probability F-invariant measure equivalent to m. We first prove the following.

Lemma 6.1. *Up to a multiplicative constant there exists a unique F-invariant, σ-finite measure μ, which is conservative, ergodic and equivalent to the h-conformal measure m.*

The idea of the proof of Lemma 6.1 is to apply a general sufficient condition for the existence of σ-finite absolutely continuous invariant measure proven in [20]. In order to formulate this condition suppose that X is a σ-compact metric space, m is a Borel probability measure on X, positive on open sets, and that a measurable map $T : X \to X$ is given with respect to which measure m is quasi-invariant, i.e. $m \circ T^{-1} << m$. Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure m. We also assume that $m(X \setminus \bigcup_{n\geq 0} A_n) = 0$, and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$m(T^{-k}(A_m) \cap A_n) > 0, \tag{6.1}$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [20] says the following.

Theorem 6.2. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure m. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1} \frac{m(A)}{m(A_n)} \leq \frac{m(T^{-k}(A))}{m(T^{-k}(A_n))} \leq K_n \frac{m(A)}{m(A_n)}, \tag{6.2}$$

then T has a σ-finite T-invariant measure μ absolutely continuous with respect to m. Additionally μ is equivalent with m, conservative and ergodic, and unique up to a multiplicative constant.

Notice that the tempting trivial choice of α consisting of $A_0 = X$ and $A_n = \emptyset$ for all $n \geq 1$ may make (6.2) fail to hold. If however (6.2) is true for all Borel subsets A of X (and $A_0 = X$), then the resulting measure μ is finite.

Proof of Lemma 6.1 (sketch). Since in the sequel we will not only need Lemma 6.1 but a bit more, namely the way in which the invariant measure claimed in Theorem 6.2 is produced, we shall also describe this procedure briefly. Following Martens, one considers the following sequences of measures

$$S_k m = \sum_{i=0}^{k-1} m \circ T^{-i} \quad \text{and} \quad Q_k m = \frac{S_k m}{S_k m(A_0)}. \tag{6.3}$$

It is proven in [20] that each weak limit μ of the sequence $Q_k(m)$ has the properties required in Theorem 6.2, where a sequence $\{\nu_k : k \geq 1\}$ of measures on X is said to converge weakly if for all $n \geq 1$ the measures ν_k converge weakly on all compact subsets of A_n. In fact it turns out that the sequence $Q_k m$ converges and

$$\mu(F) = \lim_{n\to\infty} Q_k m(F)$$

for every Borel set $F \subset X$. Making use of (6.1) and (6.2) one proves (see Lemma 2.4 in [20]) the following.

Lemma 6.3. *For every $n \geq 0$ we have $0 < \mu(A_n) < \infty$. Furthermore, the Radon-Nikodym derivative $\frac{d\mu}{dm}$ is bounded above and below on A_n.*

Let us pass now to our map $F : Q \backslash \{0\} \to Q$. The ergodicity and conservativity of the measure m is proven in Theorem 5.13. Thus, we only need to construct an irreducible partition α with property (6.2). Indeed, set $Y = J(F) \setminus \{0\}$ and for every $y \in Y$ consider a ball $B(y, r(y)) \subset Q$ such that $r(y) > 0$, $m(\partial B(y, r(y))) = 0$, and $r(y) < (1/2)\min\{\pi/2, \mathrm{dist}(y, 0)\}$. The balls $B(y, r(y))$, $y \in Y$, cover Y and since Y is a metric separable space, one can choose a countable cover, say $\{\tilde{A}_n : n \geq 0\}$, from them. We may additionally require that the family $\{\tilde{A}_n : n \geq 0\}$ is locally finite that is that each point $x \in Y$ has an open neighborhood intersecting only finitely many balls $\tilde{A}_n$, $n \geq 0$. We now define the family $\alpha = \{A_n : n \geq 0\}$ inductively by setting

$$A_0 = \tilde{A}_0 \quad \text{and} \quad A_{n+1} = \tilde{A}_{n+1} \setminus \bigcup_{k=1}^{n} \overline{\tilde{A}_n}$$

(and throwing away empty sets). Obviously α is a disjoint family and

$$\bigcup_{n \geq 1} A_n \supset J(F) \setminus \{0\} \setminus \bigcup_{n \geq 0} \partial \tilde{A}_n.$$

Hence $m\Big(\bigcup_{n \geq 0} A_n\Big) = 1$. The distortion condition (6.2) follows now from Koebe's distortion theorem with all constants $K_n = K$, and irreducibility of partition α follows from openness of the sets A_n and Lemma 5.12. ■

We are now ready to prove the main theorem of this section.

Theorem 6.4. *The σ-finite F-invariant measure μ equivalent to the h-conformal measure m proven to exists in Theorem 6.1, is finite.*

Proof. It suffices to show that there is $M > 0$ such that $\mu(Q_{-M}) < \infty$ and $\mu(B(0, Re^{-M})) < \infty$. Let us start with inequality $\mu(Q_{-M}) < \infty$. For every $k \geq T_4$ the sets

$$X_k = \{z \in J(F) : -(k+1) \leq \mathrm{Re}(z) < -k\} \subset \mathbb{C}_{-T_4}$$

have the same meaning as in the proof of lemma 5.6. Exactly the same argument as that used in the proof of Lemma 5.6 shows that

$$m(X_k) \leq 2^8 C e^{2k(1-h)}. \tag{6.4}$$

We may assume without loss of generality that $A_0 = X_{T_4}$. Fix $k \geq T_4$. Let

$$S_k = [-(k+2), -2] \times [-k/2, k/2] \subset \mathbb{C}_0.$$

The set $\{z \in \mathbb{C} \colon \mathrm{Im} z = (\pi/2)\}$ is canonically imbedded into $\mathbb{C}$, and therefore each holomorphic inverse branch $F_*^{-j} : Q \setminus \Pi(\{z \in \mathbb{C} : \mathrm{Im} z = (\pi/2)\}) \to Q$ of F^j, $j \geq 1$, can be treated as defined on a subset of the complex plane $\mathbb{C}$. This map restricted to X_k, extends in a holomorphically univalent fashion to the set S_k. It therefore follows from Koebe's distortion theorem that there exists a constant $C_1 > 0$ such that for every $j \geq 1$, every $x \in X_{T_3} = A_0$ and every $y \in X_k$, we have

$$\frac{|(F_*^{-j})'(y)|}{|(F_*^{-j})'(x)|} \leq C_1 k^3.$$

Therefore, using in addition Lemma 6.4, we obtain

$$\frac{m(F_*^{-j}(X_k))}{m(F_*^{-j}(A_0))} \leq C_1^h k^{3h} \frac{m(X_k)}{m(A_0)} \leq 2^8 C_1^h k^{3h} C m(A_0)^{-1} e^{2(1-h)k}.$$

Hence

$$\frac{m(F^{-j}(X_k))}{m(F^{-j}(A_0))} \leq 2^8 C C_1^h m(A_0)^{-1} k^{3h} e^{2(1-h)k},$$

and consequently, for every $n \geq 0$,

$$\frac{\sum_{j=0}^n m(F^{-j}(X_k))}{\sum_{j=0}^k m(F^{-j}(A_0))} \leq 2^8 C C_1^h m(A_0)^{-1} k^{3h} e^{2(1-h)k}.$$

Thus, applying Theorem 6.2 we get

$$\mu(X_k) = \lim_{k\to\infty} \frac{\sum_{j=0}^k m(F^{-j}(X_k))}{\sum_{j=0}^k m(F^{-j}(A_0))} \leq 2^8 C C_1^h m(A_0)^{-1} k^{3h} e^{2(1-h)k}.$$

Hence

$$\mu(Q_{-T_3}) \leq \sum_{k=T_3}^{\infty} \mu(X_k) \leq \sum_{k=T_3}^{\infty} 2^8 C C_1^h m(A_0)^{-1} k^{3h} e^{2(1-h)k} < \infty.$$

Let us now prove that $\mu(B(0, Re^{-M}) < \infty$ for some M large enough. We may assume without loss of generality that $A_0 = R_M$, where M will be determined later in the course of the proof and R_M was defined at the beginning of the proof of Lemma 5.7. Fix $j \geq 0$, assume that M is large enough, and for all $l \in \mathbb{Z} \setminus \{0\}$ consider all the holomorphic inverse branches $F_*^{-j} : B_-(0, Re^{-M}) \to Q$ of F^j such that $f^j(F_*^{-j}(B(0, Re^{-M})) = B(\pi l i, Re^{-M})$ or equivalently $f^j \circ F_*^{-j}(z) = z + \pi i l$, which slightly informally means that constructing F_*^{-j} we first translate the ball $B(\pi l i, Re^{-M})$ (treated as a subset of $\mathbb{C}$) by the vector $\pi i l$, which brings us far away ($l \neq 0$) from the singularity zero, and then we take any inverse branch of f^j composed at the last step with the canonical projection Π. The Koebe's distortion theorem for all those inverse branches F_*^{-j} applies and we get for every $k \geq M$ that

$$\frac{m(F_*^{-j}(R_k))}{m(F_*^{-j}(A_0))} \leq K^h \frac{m(R_k)}{m(A_0)} \tag{6.5}$$

Recall that $K \geq 1$ comes from Koebe's distortion theorem. In exactly the same way as (5.9) we get the following

$$m(R_k) \leq Le^{(1-2h)k} \tag{6.6}$$

for some constant $L > 0$ and all $k \geq 0$. Combining this with (6.5), we obtain

$$\frac{m(F_*^{-j}(R_k))}{m(F_*^{-j}(A_0))} \leq \frac{LK^h}{m(A_0)}e^{(1-2h)k} \tag{6.7}$$

Let now $F_0^{-j} : B_-(0,R) \to Q$ be a holomorphic branch of F^j such that, abusing slightly notation, $f^j \circ F_0^{-j}(R_k)) = R_k$. It follows from (3.10) (applied for f_0^{-1}) that

$$F^{j-1}(F_0^{-j}(\bigcup_{s=M}^{k} R_s)) \subset \{z \in Q : \zeta - \frac{1}{2}(k+1) \leq \mathrm{Re}(z) \leq -2\}$$

with some $\zeta > 0$ and every M large enough. Now, the same argument as used in the first part of this proof gives that

$$\frac{m(F_0^{-j}(R_k))}{m(F_0^{-j}(A_0))} \leq C_1 k^{3h} \frac{m(F^{j-1}(F_0^{-j}(R_k)))}{m(F^{j-1}(F_0^{-j}(A_0)))}$$

with some constant $C_1 > 0$. Using now (3.12) and (6.6) we conclude that with M large enough

$$\frac{m(F_0^{-j}(R_k))}{m(F_0^{-j}(A_0))} \leq C_2 k^{3h} \frac{e^{kh} m(R_k)}{e^{Mh} m(A_0)} = \frac{C_2}{e^{Mh} m(A_0)} k^{3h} e^{kh} m(R_k) \leq \frac{LC_2}{e^{Mh} m(A_0)} k^{3h} e^{(1-h)k}.$$

Combining this with (6.7), we see that for every $j \geq 1$ and every $k \geq M$

$$\frac{m(F^{-j}(R_k))}{m(F^{-j}(A_0))} \leq C_3 k^{3h} e^{(1-h)k}$$

with some universal constant $C_3 > 0$. Summing up over all $k \geq M$, we get

$$\frac{m(F^{-j}(B(0, Re^{-M})))}{m(F^{-j}(A_0))} \leq C_3 \sum_{k=M}^{\infty} k^{3h} e^{(1-h)k} < \infty.$$

Thus, for every $n \geq 0$

$$\frac{\sum_{j=0}^{n} m(F^{-j}(B(0, Re^{-M})))}{\sum_{j=0}^{n} m(F^{-j}(A_0))} \leq C_3 \sum_{k=M}^{\infty} k^{3h} e^{(1-h)k}.$$

Hence, applying Theorem 6.2 and the construction of the measure μ given by (6.3), we conclude that

$$\mu(B(0, Re^{-M})) \leq C_3 \sum_{k=M}^{\infty} k^{3h} e^{(1-h)k} < \infty.$$

We are done. ■

7. Hausdorff and packing measures and dimensions

Let H^h and P^h be respectively the h-dimensional Hausdorff and packing measures (see [25], comp. [23] for example, for its definition and some basic properties). The results of this section provide in a sense a complete description of the geometrical structure of the sets $J_r(F)$ and $J_r(f)$ and also they exhibit the geometrical meaning of the h-conformal measure m. The short proof of the first result improves on the argument from the proof of Proposition 4.9 from [29].

Proposition 7.1. *We have* $\mathrm{P}^h(J_r(f)) = \mathrm{P}^h(J_r(F)) = \infty$. *In fact* $\mathrm{P}^h(G) = \infty$ *for every open nonempty subset* G *of* $J_r(f)$.

Proof. Since $m(J_r(F) \cap Q_M) > 0$ for every $M \in \mathbb{R}$, it follows from ergodicity and conservativity of the measure m (see Theorem 5.13) that there exists a set $E \subset J_r(F)$ such that $m(E) = 1$ and $\liminf_{n\to\infty} \mathrm{Re}F^n(z) = -\infty$ for every $z \in E$. Fix $z \in E$. This formula means that there exists an unbounded increasing sequence $\{n_k\}_{k=1}^{\infty}$, depending on z, such that $\{F^{n_k}(z)\}_{k=1}^{\infty} \subset Q_{-2}$ and

$$\lim_{k\to\infty} \mathrm{Re}(F^{n_k}(z)) = -\infty. \tag{7.1}$$

Fix $k \geq 1$ and consider the ball $B(z, K^{-1}|(F^{n_k})'(z)|^{-1})$. Then

$$B(z, K^{-1}|(F^{n_k})'|(z)|^{-1}) \subset F_z^{-n_k}(B(F^{n_k}(z), 1)),$$

where $F_z^{-n} : B(F^n(z), 1) \to \mathbb{C}$ is the analytic inverse branch of F^n mapping $F^n(z)$ to z. Applying Koebe's distortion theorem and conformality of the measure m, we obtain

$$\begin{aligned} m(B(z, K^{-1}|(F^{n_k})'(z)|^{-1}) &\leq K^h|(F^{n_k})'(z)|^{-h} m(B(F^{n_k}(z), 1)) \\ &\leq K^{2h}(K^{-1}|(F^{n_k})'(z)|^{-1})^h m(Q_{\mathrm{Re}F^{n_k}(z)+1}) \end{aligned}$$

Since by (7.1), $\lim_{k\to\infty} m(Q_{\mathrm{Re}F^{n_k}(z)+1}) = 0$, we see that

$$\liminf_{r\to 0} \frac{m(B(z,r))}{r^h} = 0.$$

Since $m(G \cap J_r(F)) > 0$ for every non-empty open subset of $J_r(F)$, this implies (see an appropriate Converse Frostman's Type Theorem in [11], Theorem E(1) p.58, or [23]) that $\mathrm{P}^h(G) = \infty$. Since $J_r(f) = \bigcup_{k\in\mathbb{Z}}(J_r(F) + 2\pi ik)$, we are therefore done. ■

Theorem 7.2. *We have* $0 < \mathrm{H}^h(J_r(F)) < \infty$.

Proof. Let $M > 0$ be given by Theorem 5.10. Fix an integer $T \geq 1$ and a point $z \in J_{r,T}$. Consider the holomorphic inverse branches $F_z^{-n_k(z)} : B(y(z), (2T)^{-1}) \to Q$ sending $F^{n_k(z)}(z)$

to z. Since, by $\frac{1}{4}$-Koebe's distortion theorem and the standard version of Koebe's distortion theorem,

$$F_z^{-n_k(z)}\left(B\left(y(z),\frac{1}{2T}\right)\right) \supset F_z^{-n_k(z)}\left(B\left(F^{n_k(z)}(z),\frac{1}{3T}\right)\right) \supset B\left(z,\frac{1}{12T}|(F^{n_k(z)})'(z)|^{-1}\right)$$

and

$$\begin{aligned} F_z^{-n_k(z)}\left(B\left(y(z),\frac{1}{24KT}\right)\right) &\subset F_z^{-n_k(z)}\left(B\left(F^{n_k(z)}(z),\frac{1}{12KT}\right)\right) \\ &\subset B\left(z,\frac{1}{12T}|(F^{n_k(z)})'(z)|^{-1}\right). \end{aligned}$$

Using the conformality of the measure m along with the standard version of Koebe's distortion theorem, and the fact that $\inf\{m(B(w,(12KT)^{-1}) : w \in Q_{-2T}^c\} > 0$, we deduce that

$$B_T^{-1} r_k(z)^h \le \nu(B(z,r_k(z))) \le B_T r_k(z)^h, \tag{7.2}$$

where $r_k(z) = (12T)^{-1}|(F^{n_k(z)})'(z)|^{-1}$ and B_T is independent of z and k. It follows from (7.2) that $\mathrm{H}^h|_{J_{r,T}(F)} \prec m$ for every $T \ge 1$ and that $\mathrm{H}^h(J_{r,M}(F)) < \infty$. Since $m(J_{r,T}(F)\setminus J_{r,M}(F)) \le m(J_r(F)\setminus J_{r,M}(F)) = 0$ and since $J_r(F) = \bigcup_{n=0}^\infty J_{r,M+n}(F)$, we therefore conclude that $\mathrm{H}^h(J_r(F)) = \mathrm{H}^h(J_{r,M}(F)) < \infty$.

We shall prove now that $\mathrm{H}^h(J_r(F)) > 0$. Put

$$\kappa = \min\left\{1, R, \frac{\lambda}{2}e^{-2T_2}\right\}.$$

Fix $z \in J_r(F)$ and $r \in (0, \kappa(2^8K)^{-1})$. Since, by Lemma 5.11, $\limsup_{n\to\infty}|(f^n)'(z)| = +\infty$, there exists a least $n = n(z,r) \ge -1$ such that

$$r|(f^{n+1})'(z)| > \kappa(2^8K)^{-1}. \tag{7.3}$$

Thus $n \ge 0$ and

$$r|(f^n)'(z)| \le \kappa(2^8K)^{-1}.$$

Suppose that the holomorphic inverse branch of f^n defined on $B(f^n(z), 32r|(f^n)'(z)|)$ and sending $f^n(z)$ to z, does not exist. Then $n \ge 1$ and let $1 \le k \le n$ be the largest integer such that the holomorphic inverse branch of $f^{n-(k-1)}$ defined on $B(f^n(z), 32r|(f^n)'(z)|)$ and sending $f^n(z)$ to $f^{k-1}(z)$ does not exist. This implies that $0 \in f_k^{-(n-k)}(B(f^n(z), 32r|(f^n)'(z)|))$, where $f_k^{-(n-k)} : B(f^n(z), 32r|(f^n)'(z)|) \to \mathbb{C}$ is the holomorphic inverse branch of f^{n-k} sending $f^n(z)$ to $f^k(z)$. In addition, $n = k$ since $0 \notin f^{-1}(\overline{\mathbb{C}})$. Hence $|f^n(z)| < 32Kr|(f^n)'(z)| \le \kappa \le R$. It now follows from (3.14) that

$$\mathrm{Re}(f^{n-1}(z)) \le \frac{1}{2}\log\left(\frac{2}{\lambda}\right) + \frac{1}{2}\log\kappa \le -T_2.$$

We therefore get by (3.2) and (3.5) that $|f'(f^{n-1}(z))| \le 4\lambda \exp\Big(2\mathrm{Re}(f^{n-1}(z))\Big) \le 8|f^n(z)|$. Consequently, since $r|(f^{n-1})'(z)| < \kappa(2^8K)^{-1}$, we conclude that

$$\begin{aligned} 32Kr|(f^n)'(z)| &= 32Kr|(f^{n-1})'(z)|\cdot|f'(f^{n-1}(z))| \le 32Kr|(f^{n-1})'(z)|\cdot 8|f^n(z)| \\ &= 2^8Kr|(f^{n-1})'(z)|\cdot|f^n(z)| < |f^n(z)|. \end{aligned}$$

This contradiction shows that the holomorphic inverse branch $f_z^{-n} : B(f^n(z), 32r|(f^n)'(z)|) \to \mathbb{C}$ of f^n sending $f^n(z)$ to z, is well-defined. Now, the map f restricted to $B(f^n(z), 32r|(f^n)'(z)|)$ is 1-to-1, and by Koebe's $\frac{1}{4}$-Theorem

$$f(B(f^n(z), 32r|(f^n)'(z)|)) \supset B\Big(f^{n+1}(z), 8r|(f^{n+1})'(z)|\Big).$$

Hence there exists a unique holomorphic inverse branch $f_z^{-(n+1)} : B\Big(f^{n+1}(z), 8r|(f^{n+1})'(z)|\Big) \to \mathbb{C}$ of f^{n+1} mapping $f^{n+1}(z)$ to z. Applying Koebe's $\frac{1}{4}$-Theorem again, we see that

$$f_z^{-(n+1)}\Big(B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)\Big) \supset B(z,r). \tag{7.4}$$

Since the ball $B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)$ intersects at most $\frac{1}{\pi}4r|(f^{n+1})'(z)|+2 \preceq r|(f^{n+1})'(z)|$ horizontal strips of the form $2\pi ij + (\mathbb{R}\times[0,2\pi))$, $j \in \mathbb{Z}$, using (7.4), Koebe's Distortion Theorem, h-conformality of the measure m and, at the end, (7.3), we get

$$\begin{aligned} r^{-h}(m(B(z,r)) &\preceq r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)m\Big(\Pi\Big(B\Big(f^{n+1}(z), 4r|(f^{n+1})'(z)|\Big)\Big)\Big) \\ &\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|) \\ &= K^h(r|(f^{n+1})'(z)|)^{1-h} < K^h(2^8K)^{h-1}, \end{aligned}$$

We are done by applying an appropriate Converse Frostman's Type Theorem (see e.g. Theorem D(2) or [11] or [23]). ■

We end this section with the following.

Theorem 7.3. *We have* $\mathrm{HD}(J_r(f)) = \mathrm{HD}(J_r(F)) = h \in (1,2)$.

Proof. The first equality sign is obvious. The second one is an immediate consequence of Theorem 7.2. The inequality $h > 1$ has been established in Theorem 5.8. We are left to show that $h < 2$. And indeed, suppose on the contrary that $h = 2$. Then $\mathrm{P}^h = \mathrm{P}^2$ is a constant multiple of the two-dimensional Lebesgue measure on $\mathbb{C}$ and we have a contradiction with Theorem 7.1. ■

8. Appendix; the Lebesgue Measure of the Julia Set

Let l_2 denote the 2-dimensional Lebesgue's measure on $\mathbb{C}$. The post-singular set of a meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ i.e. the closure in $\overline{\mathbb{C}}$ of the forward orbit of the set of singularities of f^{-1}, is denoted by $P(f)$. If z is a point which belongs to the domain of the definition of each iterate of f, we denote by $\omega(z)$ the set of cluster points of the sequence

$(f^n(z))_{n\in\mathbb{N}}$ in $\overline{\mathbb{C}}$. Recall that $\mathrm{dist}(z, A)$ denotes the distance from the point z to the set $A \subset \overline{\mathbb{C}}$ with respect to the spherical metric on $\overline{\mathbb{C}}$. In ([7], Theorem 6.2) the following characterization of the (Lebesgue) typical behavior of points in the Julia set is provided.

Theorem 8.1. *If f is a rational function of degree $d \geq 2$ or a transcendental meromorphic function then at least one of the following holds.*

(a) $\lim_{n\to\infty}$ *dist*$(f^n(z), P(f)) = 0$ *for almost all* $z \in J(f)$.

(b) $J(f) = \overline{\mathbb{C}}$ *and the Lebesgue measure is conservative meaning that for every Borel set* $A \subset \overline{C}$ *with* $l_2(A) > 0$ *the set* $\{n \geq 0 : f^n(z) \in A\}$ *is infinite for Lebesgue-almost all points in* $z \in \overline{\mathbb{C}}$.

We have proved in [16] that $l_2(I_1(f_\lambda)) = 0$ for all $\lambda \in \mathbb{C}$. Now we prove the following stronger result in the case of positive parameters λ.

Theorem 8.2. *If $\lambda > 0$, then $l_2(J(f_\lambda)) = 0$.*

Proof. Fix $\lambda > 0$ and put

$$f := f_\lambda.$$

Suppose that on the contrary that $l_2(J(f)) > 0$. It then follows from Theorem 8.1 that for Lebesgue's almost every point $z \in J(f)$, $\lim_{n\to\infty} \mathrm{dist}(f^n(z), P(f)) = 0$, where $P(f)$ is the closure of the forward orbit of the singular set of f^{-1} and distance is taken with respect to the spherical metric on $\overline{\mathbb{C}}$. Let z be such a point, which is simultaneously a Lebesgue density point of $J(f)$. Since $P(f) \cap J(f) = \{0, \infty\}$, there exists $\{n_k\}_{k=1}^\infty$, an unbounded increasing sequence of positive integers such that $\lim_{k\to\infty} f^{n_k}(z) = \infty$. We shall show that also

$$\lim_{k\to\infty} (f_\lambda^{n_k})'(z) = \infty. \tag{8.1}$$

Let $r_k = |f^{n_k}(z)|/2$ and let $B_k = B(f^{n_k}(z), r_k)$. Since $P(f)\setminus\{\infty\}) = \{0\}$ and since $B_k \cap \{0\} = \emptyset$, for every $k \geq 1$, there exists a unique holomorphic inverse branch $f_z^{-n_k} : B_k \to \mathbb{C}$ of f^{n_k} mapping $f^{n_k}(z)$ onto z. By Koebe's $\frac{1}{4}$-theorem and since f is periodic with period πi, we get

$$B\left(z, \frac{|f^{n_k}(z)|}{8|(f^{n_k})'(z)|}\right) \subset B(z, \pi/2). \tag{8.2}$$

Since $|f^{n_k}(z)| \to \infty$, this implies that $|(f^{n_k})'(z)| \to \infty$ and the formula (8.1) is proven. We shall now prove that

$$\lim_{k\to\infty} \mathrm{Re}(f^{n_k}(z)) = -\infty \tag{8.3}$$

Suppose on the contrary that this is not the case. Then passing to a subsequence, we may assume without loss of generality that

$$\mathrm{Re}(f^{n_k}(z)) > -p$$

for some $p > 0$ and all $k \ge 1$. Let $H^+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ be the right half-plane. For every $k \ge 1$ let S_k be a square centered at $f^{n_k}(z)$ and with the sides parallel to the coordinate axes of length equal to $4p$. Since H^+ is a subset of the Fatou set of f, we get

$$\frac{l_2(S_k \cap J(f))}{l_2(S_k)} \le 1 - \frac{l_2(S_k \cap H^+)}{l_2(S_k)} \le 1 - \frac{1}{4} = \frac{3}{4}.$$

Since $\lim_{k\to\infty} f^{n_k}(z) = \infty$, we see that $S_k \subset B_k$ for all $k \ge 1$ sufficiently large, and that the maps $f_z^{-n_k}$ restricted to S_k, have distortion converging to 1 when $k \to \infty$. Thus, there is a universal constant $\alpha \in (0,1)$ such that

$$\frac{l_2\Big(f_z^{-n_k}\Big(S_k \cap J(f)\Big)\Big)}{l_2\Big(f_z^{-n_k}\Big(S_k\Big)\Big)} < \alpha < 1 \tag{8.4}$$

for all $k \ge 1$ large enough. Since $\lim_{k\to\infty}(f^{n_k})'(z) = \infty$, we get that $\lim_{k\to\infty} \mathrm{diam}\Big(f_z^{-n_k}\Big(S_k\Big)\Big) = 0$, and therefore (8.4) contradicts the choice of the point z. So, the formula (8.3) holds. Let S_k' be the square centered at $f_\lambda^{n_k}(z)$ and with the sides parallel to the coordinate axes of length equal to $\pi/2$. Looking at $f^{-1}(H^+)$ we see that there exists $\epsilon > 0$ and such that

$$\bigcup_{k=-\infty}^{+\infty} \{z \in \mathbb{C}_0 : (2k+1)\frac{\pi}{2} - \epsilon < \mathrm{Im}(z) < (2k+1)\frac{\pi}{2} + \epsilon\} \cap J(f) = \emptyset.$$

This implies that for all $k \in \mathbb{Z}$

$$\frac{l_2(S_k' \cap J(f))}{l_2(S_k')} \le 1 - \frac{l_2(S_k' \cap S)}{l_2(S_k')} \le 1 - \frac{2\epsilon}{\pi}. \tag{8.5}$$

Since $f^{n_k}(z) \to \infty$, for every large k we have $S_k' \subset B_k$, and the maps $f_z^{-n_k}$ restricted to S_k, have distortion converging to 1 when $k \to \infty$. Now we reach the contradiction in the same way as in the previous case. We are done. ■

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