GEOMETRIC RIGIDITY OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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Abstract. The classes of dynamically and geometrically tame functions meromorphic outside a small set are introduced. The Julia sets of geometrically tame functions are proven to be either geometrical circle (in $\overline{\mathbb{C}}$) or to have Hausdorff dimension strictly larger than 1. Vast classes of dynamically and geometrically tame functions are identified.

1. Introduction

We consider a meromorphic function $f : \overline{\mathbb{C}} \setminus T \to \overline{\mathbb{C}}$, where T is a closed subset of $\overline{\mathbb{C}}$ with linear (*i.e.* one-dimensional Hausdorff) measure zero. The Fatou set $F(f)$ is defined in the same manner as for transcendental meromorphic function of the complex plane i.e. $F(f)$ is the set of points $z \in \overline{\mathbb{C}}$ such that all the iterates of f are defined and form a normal family on a neighborhood of z. The Julia set $J(f)$ is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is are completely invariant while $f^{-1}(J(f)) \subset J(f) \setminus T$ and $f(J(f) \setminus T) = J(f)$.

Denote by $h = \mathrm{HD}(J(f))$ the Hausdorff dimension of the Julia set of f. Let H^h mean the h-dimensional Hausdorff measure on $\overline{\mathbb{C}}$ taken with respect to the spherical metric on $\overline{\mathbb{C}}$.

We call the function f *dynamically tame* if the following three conditions are satisfied:

(a) $\mathrm{H}^h(J(f)) < \infty$,
(b) there exists a Borel probability measure μ absolutely continuous with respect to the Hausdorff measure H^h on $J(f)$,
(c) $\mu \circ f^{-1} = \mu$ and μ is ergodic.

If in addition the following three conditions are satisfied:

(d) $J(f)$ is a Jordan curve,
(e) if A_0, A_1 are the two connected components of $\overline{\mathbb{C}} \setminus J(f)$, then $f^2(A_i) = A_i$, $i = 0, 1$,
(f) there are $a_i \in A_i$, $i = 0, 1$, satisfying $f^2(a_i) = a_i$,

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then f is called *geometrically tame.*

The main general result of this paper is the following.

Theorem A. *If f is a geometrically tame function, then either $J(f)$ is a geometric circle or* $\mathrm{HD}(J(f)) > 1$.

Remark 1. Although this theorem has a relatively short proof, it however appears to us clear and elegant, and it depends heavily on hard machineries developed in [8] and [11]. In Sections 3 and 4 we provide natural large classes of dynamically and geometrically tame functions including so concrete examples as the maps from the tangent family. Obviously Theorem A applies to them.

Remark 2. If in addition the map f is finely mixing meaning that for every subarc I of $J(f)$,

$$(*) \qquad \mathrm{HD}(J(f) \setminus \bigcup_{n=1}^{\infty} f^n(I)) \leq 1,$$

then the second alternative in Theorem A takes on the stronger form that $\mathrm{HD}(I) > 1$ for every subarc I of $J(f)$. In particular $J(f)$ contains then no differentiable arcs, the result strengthening in our setting Stallard's[1] Theorem D from [13]. Notice that f is finely mixing for instance if f has the local Picard property for some order $k = 1, 2, \ldots, \aleph_0$, at every point of T, which means that for every $z \in T$ and every $r > 0$, the set $\overline{\mathbb{C}} \setminus f(B(z, r))$ consists of at most k points. Finally observe that if T is countable then f has the local Picard property of order 2.

2. Proof of Theorem A

Suppose that $\mathrm{HD}(J(f)) \leq 1$. We are to show that $J(f)$ is a geometric circle. Since $J(f)$ is connected, $h = \mathrm{HD}(J(f)) \geq 1$, and consequently $h = 1$. It follows from the assumption (a) that the Hausdorff measure $\mathrm{H}^1(J(f))$ is finite. Replacing f by f^2, we may assume without loss of generality that both a_0 and a_1 are fixed points of f and both A_0, A_1 are f-invariant. Let $\mathbb{D}$ be the unit disc, $S^1 = \partial \mathbb{D}$ be the unit circle. Consider two Riemann mappings $R_0 : \mathbb{D} \to A_0$, $R_1 : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to A_1$ such that $R_0(0) = a_0$, $R_1(\infty) = a_1$. Define two holomorphic maps

$$g_0 := R_0^{-1} \circ f \circ R_0 : \mathbb{D} \to \mathbb{D} \quad \text{and} \quad g_1 := R_1^{-1} \circ f \circ R_1 : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

By the choice of R_0 and R_1 we have $g_0(0) = 0$ and $g_1(\infty) = (\infty)$. Since the boundaries of A_0 and A_1 are Jordan curves equal to $J(f)$, in view of Caratheodory Theorem the Riemann maps R_0 and R_1 extend to homeomorphisms (denoted by the same symbols R_0 and R_1) of the closed disk $\mathbb{D}$ and $\overline{\mathbb{C}} \setminus \mathbb{D}$ respectively. This implies that g_0 (resp. g_1) has a continuous extension to $\overline{\mathbb{D}} \setminus R_0^{-1}(T)$ (resp. $(\overline{\mathbb{C}} \setminus \mathbb{D}) \setminus (R_1^{-1}(T))$. We keep the same notation g_0 and g_1

[1] We wish to thank Walter Bergweiler for bringing our attention to Stallard's result.

for these extensions. Observe that $g_0(S^1 \setminus R_0^{-1}(T)) \subset S^1$ and $g_1(S^1 \setminus R_1^{-1}(T)) \subset S^1$. Since $\mathrm{H}^1(J(f)) < \infty$, Riesz's theorem yields that the Riemann mappings R_0 and R_1 and their inverse maps are absolutely continuous with respect to the normalized Lebesgue measure l on S^1 and the Hausdorff measure H^1 on the Julia set $J(f)$. Since $H^1(T) = 0$ the measure l on S^1 of $R_0^{-1}(T)$ and $R_1^{-1}(T)$ thus vanishes. We therefore see that g_0 and g_1 (abusing a little bit the terminology in the latter case) are inner functions. It then follows from Theorem A in [8] that the measure l is g_0, g_1-invariant (remember that $g_0(0) = 0, g_1(\infty) = \infty$) and it is a unique (since it is ergodic) Borel probability g_0, g_1-invariant measure absolutely continuous with respect to the Lebesgue measure l on S^1. Applying Riesz's theorem again along with the assumption (b) we see that both measures $\mu \circ R_0$ and $\mu \circ R_1$ are absolutely continuous with respect to the measure l on S^1 and it follows from (c) that $\mu \circ R_0$ is g_0-invariant and $\mu \circ R_1$ is g_1-invariant. Hence, it follows from the above that three measures $\mu \circ R_0$, $\mu \circ R_1$ and l are equal. We consider the new map $R_1^{-1} \circ R_0 : S^1 \to S^1$. Then

$$l \circ (R_1^{-1} \circ R_0)^{-1} = l \circ R_0^{-1} \circ R_1 = \mu \circ R_0 \circ R_0^{-1} \circ R_1 = \mu \circ R_1 = l,$$

which means that $R_1^{-1} \circ R_0$ preserves the measure l on S^1. Since R_0 and R_1 are homeomorphisms of the circle S^1, so is $R_1^{-1} \circ R_0$. As it preserves the measure l, it is a rotation of S^1. Thus, modifying R_0 by composing it with an appropriate rotation, we can make the composition $R_1^{-1} \circ R_0$ to be the identity map. Consequently $R_1 = R_0$ on S^1. Thus the map defined by the formula

$$R(z) = \begin{cases} R_0(z) & \text{if} \quad z \in \overline{\mathbb{D}} \\ R_1(z) & \text{if} \quad z \in \mathbb{C} \setminus \mathbb{D} \end{cases}$$

is meromorphic on $\mathbb{D} \cup \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and continuous on $\overline{\mathbb{C}}$. A straightforward application of Morera's theorem shows that the map $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is meromorphic. As R is a homeomorphism, it therefore must be a Möbius map. Since $J(f) = R_0(S^1) = R(S^1)$, we get that $J(f)$ is a geometric circle. □

3. Classes of examples of dynamically tame functions

The first class of tame functions we want to bring reader's attention to is provided by Walters expanding conformal maps introduced and thoroughly investigated in [11]. In particular these maps were proven in [11] to satisfy all the requirements for dynamical tameness.

Recall that non-Möbius meromorphic function $f : \overline{\mathbb{C}} \setminus T \to \overline{\mathbb{C}}$ is called a Bolsch function (said to belong to the class $\mathcal{S}$ in the original Bolsch's paper [6]) if $T \subset \overline{\mathbb{C}}$ is a closed countable set and f cannot be meromorphically extended to any open set containing $\mathbb{C} \setminus T$. We denote by $Sing(f^{-1}) \subset \overline{\mathbb{C}}$ the set of all such points $w \in \overline{\mathbb{C}}$ that for every $r > 0$ there is a connected component G of $f^{-1}(B(w,r))$ such that $f : G \to B(w,r)$ is not a covering map onto $B(w,r)$. The Bolsch function $f : \overline{\mathbb{C}} \setminus T \to \overline{\mathbb{C}}$ is called topologically hyperbolic if $Sing(f^{-1})$ is contained in the Fatou set $F(f)$ and

$$J(f) \cap \overline{\bigcup_{n=0}^{\infty} f^n(Sing(f^{-1}))} = \emptyset.$$

Since $J(f) \neq \overline{\mathbb{C}}$, there exists a Möbius transformation of $\overline{\mathbb{C}}$ which conjugates f to a function f^*, whose Julia set is a compact subset of the complex plane $\mathbb{C}$. It was remarked in [11] on p. 636 that if T is a singleton, then the proof of theorem 4.7 in [11] can be easily modified to demonstrate that f^* is a Walters expanding conformal map. In our present case when T is allowed to be compact and countable, the same is true. Consequently f is a dynamically tame map. We would like to make it explicit that Bolsch maps form a special subclass of function meromorphic outside a small set; see [1] and [2] for its definition.

Now we want to describe two natural large subclasses of topologically hyperbolic Bolsch functions, called Barański maps and post-Barański maps. These maps were introduced and thoroughly investigated in [11] and were given their names afterword in [12].

Because of evident importance of these two classes of functions and for the convenience of the reader we now recall briefly their definitions.

Consider functions of the form

$$f(z) = H(\exp(Q(z)))$$

where Q and H are non-constant rational functions. Let $Q^{-1}(\infty) = \{d_j : j = 1, \dots , m\}$ be the set of poles of Q. Then

$$f(z) : \bar{\mathbb{C}} \setminus \{d_j; j = 1, \dots , m\} \to \bar{\mathbb{C}} \setminus \{H(0), H(\infty)\}.$$

Assume in addition that there is at least one pole d_i of Q different from $H(0)$ and $H(\infty)$. Assume without loss of generality that $d_i = d_1$. Let $Crit(f)$ denote the set of critical points of f. Then the set of asymptotic values $Asymp(f)$ of f is equal to $\{H(0), H(\infty)\}$.

The function f is then called a Barański map, if the following conditions are satisfied:

(1) $J(f) \cap \overline{\bigcup_{n=0}^{\infty} f^n\left(Crit(f) \cup Asymp(f)\right)} = \emptyset$,
(2) if $a \in Crit(Q)$, then $\exp(Q(a))$ is not a pole of H,
(3) if H has a multiple pole, then $Q(\infty) \neq \infty$.

As we already stated Barański maps form a subclass of topologically hyperbolic Bolsch maps, and are therefore dynamically tame. We would like however to remark (see the tangent family below) that not all of these maps are Walters expanding conformal maps.

Given a Barański map $f(z) = H(\exp(Q(z)))$, the map

$$\tilde{f}(z) = \exp(Q(H(z)))$$

is called the post-Barański map derived from f. It was shown in [11] that all post-Barański maps are simultaneously Walters expanding conformal maps and topologically hyperbolic Bolsch maps. So, they are dynamically tame.

4. Classes of examples of geometrically tame functions

It was proved in [4] that each meromorphic function in the class $\mathcal{S}$ (here $f \in \mathcal{S}$ if and only if $\mathrm{Sing}(f^{-1})$ is finite) has at most two completely invariant domains. Bergweiler and Eremenko proved in [5] that each meromorphic function with two completely invariant domains and with no rationally indifferent periodic points is (with our terminology) topologically hyperbolic Bolsch function, and in addition, it satisfies conditions (d)-(f) of the definition of geometrically tame function. Since topologically hyperbolic Bolsch functions are dynamically tame, we infer that each such function is geometrically tame. Therefore, in view of theorem A, its Julia set is either geometric circle or it has Hausdorff dimension strictly greater than 1.

In order to get really concrete examples of geometrically tame transcendental meromorphic functions consider the family

$$\mathcal{F} = \{f_\lambda(z) = \lambda \tan z, \quad \lambda \in \mathbb{C} \setminus \{0\}, z \in \mathbb{C}\}.$$

Speaking about topology in $\mathcal{F}$ we will identify its members f_λ with parameters λ. The subfamily

$$\mathcal{H} = \{\lambda \in \mathbb{C} \setminus \{0\} : f_\lambda \text{ has an attracting periodic cycle}\}.$$

called, the hyperbolic subfamily of $\mathcal{F}$, consists precisely of topologically hyperbolic Bolsch functions from $\mathcal{F}$, which were proven to be dynamically tame. Let $\Omega_1 \subset \mathcal{H}$ be the subset of $\mathcal{H}$ composed of maps f_λ that have two distinct attracting fixed points and let $\Omega_2 \subset \mathcal{H}$ be formed by maps with exactly one periodic cycle of period 2. L. Keen and the first author proved in [10] that Ω_1 and Ω_2 are topological open disks. They also proved in [9] that all members Ω_1 and Ω_2 satisfy conditions (d)-(f) of the definition of geometrically tameness (for Ω_1 it also follows from the later paper [5]). So $\Omega_1 \cup \Omega_2$ consist entirely of geometrically tame mappings. Therefore, our theorem A yields the following.

Theorem B. *If* $\lambda \in (\Omega_1 \cup \Omega_2)$, *then* $\mathrm{HD}(J(f_\lambda)) > 1$ *unless* $\lambda \in \mathbb{R}$ *and then* $J(f_\lambda) = \mathbb{R} \cup \{\infty\}$ *(a geometric circle passing trough* ∞*).*

Proof The fact that $J(f_\lambda) = \mathbb{R} \cup \{\infty\}$ if $\lambda \in (\Omega_1 \cup \Omega_2) \cap \mathbb{R}$ was established in [3]. If $\lambda \in (\Omega_1 \cup \Omega_2) \setminus \mathbb{R}$, then $\{(k + \frac{1}{2})\pi : k \in \mathbb{Z}\} \cup f_\lambda^{-1}(\frac{\pi}{2}) \subset J(f_\lambda)$ and $f_\lambda^{-1}(\frac{\pi}{2}) \cap \mathbb{R} = \emptyset$. Therefore $J(f_\lambda)$ can not coincide with any geometric circle and consequently $\mathrm{HD}(J(f_\lambda)) > 1$. We are done. $\square$

Some less standard interesting concrete examples of geometrically tame transcendental meromorphic functions can be found in section 3 of [5].

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