HAUSDORFF DIMENSION AND HAUSDORFF MEASURES OF JULIA SETS OF ELLIPTIC FUNCTIONS

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Abstract. It is proven that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function and q is the maximal multiplicity of all poles of f, then the Hausdorff dimension of the Julia set of f is greater than $2q/(q+1)$ and the Hausdorff dimension of the set of points which escape to infinity is less than or equal to $2q/(q+1)$. In particular the area of this latter set is equal to 0.

1. Introduction

Throughout the entire paper $f : \mathbb{C} \to \overline{\mathbb{C}}$ denotes a non-constant elliptic function. Every such function is doubly periodic and meromorphic. In particular there exist two vectors ω_1, ω_2, $\Im(\frac{\omega_1}{\omega_2}) \neq 0$, such that for every $z \in \mathbb{C}$ and $n, m \in \mathbb{Z}$,

$$f(z) = f(z + m\omega_1 + n\omega_2).$$

Let

$$\mathcal{R} = \{t_1\omega_1 + t_2\omega_2 : 0 \le t_1, t_2 \le 1\},$$

be the basic fundamental parallelogram of f. Then $f(\mathbb{C})$ is an open subset of $\overline{\mathbb{C}}$ and simultaneously $f(\mathbb{C}) = f(\mathcal{R})$ is a compact subset of $\overline{\mathbb{C}}$. Since the sphere $\overline{\mathbb{C}}$ is connected, this implies that

$$f(\mathbb{C}) = \overline{\mathbb{C}}. \tag{1.1}$$

It follows from periodicity of f that

$$f^{-1}(\infty) = \bigcup_{m,n\in\mathbb{Z}} \Big(\mathcal{R} \cap f^{-1}(\infty) + m\omega_1 + n\omega_2\Big).$$

For every pole b of f let q_b denote its multiplicity. We define

$$q := \max\{q_b : b \in f^{-1}(\infty)\} = \max\{q_b : b \in f^{-1}(\infty) \cap \mathcal{R}\}.$$

The *Fatou set* $F(f)$ of a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z \in \mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant

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while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f)\setminus\{\infty\}) = J(f)$. For a general description of the dynamics of meromorphic functions see e.g. [1]. We would however like to note that it easily follows from Montel's criterion of normality that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ has at least one pole which is not an omitted value then

$$J(f) = \overline{\bigcup_{n\ge 0} f^{-n}(\infty)}. \tag{1.2}$$

Since, by (1.1), the elliptic function f has no omitted values and since it has at least one pole (thus infinitely many), (1.2) is true for all elliptic functions. Let $\mathrm{Crit}(f)$ be the set of critical points of f i.e.

$$\mathrm{Crit}(f) = \{z : \ f'(z) = 0\}.$$

Its image, $f(\mathrm{Crit}(f))$, is called the set of critical values of f. Since $\mathcal{R} \cap \mathrm{Crit}(f)$ is finite and since $f(\mathrm{Crit}(f)) = f(\mathcal{R} \cap \mathrm{Crit}(f))$, the set of critical values $f(\mathrm{Crit}(f))$ is also finite. Let

$$I_\infty(f) = \{z \in \mathbb{C} : \ \lim_{n\to\infty} f^n(z) = \infty\}$$

be the set of points escaping to infinity under iterates of f. Let HD denote the Hausdorff dimension, H^h and l_2 denote respectively h-dimensional Hausdorff measure and 2-dimensional Lebesgue measure. The following two theorems constitute the main results of our paper.

Theorem 1.1. *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function, then*

$$\mathrm{HD}(J(f)) > \frac{2q}{1+q}.$$

This theorem generalizes the results of [2], stating that $\mathrm{HD}(I_\infty(f)) \ge \frac{2q}{1+q}$ for elliptic functions satisfying the condition that the closure of the postcritical set is disjoint from the set of poles.

Theorem 1.2. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic function. Then*

$$\mathrm{HD}(I_\infty(f)) \le \frac{2q}{1+q}.$$

As an immediate consequence of these two theorems we obtain the following two corollaries.

Corollary 1.3. *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function, $h := \mathrm{HD}(J(f))$, then $\mathrm{H}^h(I_\infty(f)) = 0$, and consequently $l_2(I_\infty(f)) = 0$.*

and

Corollary 1.4. *If L is a lattice in $\mathbb{C}$ and $\mathcal{K}_\mathcal{L}$ is the field of all elliptic functions with respect to L, then $\sup\{\mathrm{HD}(J(f)) : f \in \mathcal{K}_L\} = 2$.*

The following third corollary will be proven in the last section.

Corollary 1.5. *If f is an elliptic function, $J(f) = \overline{\mathbb{C}}$ and $l_2\Big(\{z : \omega(z) \subset \overline{\bigcup_{n\ge 1} f^n(Crit(f))}\}\Big) = 0$, then there exists a σ-finite f-invariant measure μ equivalent with the Lebesgue measure l_2.*

More about geometry and dynamics of elliptic functions in the case when $J(f) \neq \overline{\mathbb{C}}$ will be contained in our subsequent paper.

The present paper is organized as follows. In the second section we prove Theorem 1.2. In section three we recall the concept of infinite iterated function systems (frequently abbreviated as i.f.s) and apply estimates of Hausdorff dimension of the limit set given in [4] to prove Theorem 1.1. Finally in the section four we show how Corollary 1.5 follows from Theorem 1.1, Theorem 1.2 and the results of [3].

In the sequel $f^\sharp$ and diam_s denote respectively the derivatives and diameters defined by means of the spherical metric.

2. Proof of Theorem 2

Let

$$B_R = \{z \in \overline{\mathbb{C}} \colon |z| > R\}.$$

For every pole b of f by $B_b(R)$ we denote the connected component of $f^{-1}(B_R)$ containing b. If $R > 0$ is large enough, say $R \geq R_0$, then B_R contains no critical values of f, all sets $B_b(R)$ are simply connected, mutually disjoint and for $z \in B_b(R)$

$$f(z) = \frac{G_b(z)}{(z-b)^{q_b}} \tag{2.1}$$

where $G_b : B_b(R) \to \mathbb{C}$ is a holomorphic function taking values out of some neighbourhood of 0. If $U \subset B_R \setminus \{\infty\}$ is an open simply connected set, then all the holomorphic inverse branches $f^{-1}_{b,U,1}, \dots, f^{-1}_{b,U,q_b}$ of f are well-defined on U and for every $1 \leq j \leq q_b$ and all $z \in U$ we have

$$|(f^{-1}_{b,U,j})'(z)| \asymp |z|^{-\frac{q_b+1}{q_b}}. \tag{2.2}$$

Therefore

$$|(f^{-1}_{b,U,j})^\sharp(z)| \asymp |z|^{-\frac{q_b+1}{q_b}} \frac{1+|z|^2}{1+|(f^{-1}_{b,U,j})(z)|^2} \asymp \frac{|z|^{\frac{q_b-1}{q_b}}}{|b|^2}, \tag{2.3}$$

where the second comparability sign we wrote assuming in addition that $|b|$ is large enough, say $|b| \geq R_1 > R_0$. Let M be an upper bound of the ratios of $|(f^{-1}_{b,U,j})^\sharp(z)|$ and $|z|^{\frac{q_b-1}{q_b}}|b|^{-2}$ with b, U, j as above. A straightforward calculation based on (2.1) shows that there exists a constant $L \geq 1$ such that for all poles b and all $R \geq R_1$ we have

$$\begin{aligned} \mathrm{diam}(B_b(R)) &\leq LR^{-\frac{1}{q_b}}, \\ \mathrm{diam}_s(B_b(R)) &\leq LR^{-\frac{1}{q_b}}|b|^{-2}. \end{aligned} \tag{2.4}$$

We take $R_2 \geq R_1$ so large that

$$LR^{-\frac{1}{q_b}} < R_0 \tag{2.5}$$

for all poles $b \in B_{R_2}$ and all $R \geq R_2$. Given two poles $b_1, b_2 \in B_{2R_2}$ we denote by $f^{-1}_{b_2,b_1,j}$: $B(b_1, R_0) \to \mathbb{C}$ all the holomorphic inverse branches $f^{-1}_{b_2,B(b_1,R_0),j}$. It follows from (2.4) and (2.5) that

$$f^{-1}_{b_2,b_1,j}\Big(B(b_1,R_0)\Big) \subset B_{b_2}(2R_2 - R_0) \subset B_{b_2}(R_2) \subset B(b_2,R_0) \tag{2.6}$$

Set

$$I_R(f) = \{z \in \mathbb{C} : \forall_{n\geq 0} |f^n(z)| > R\}.$$

Since the series $\sum_{b\in f^{-1}(\infty)\setminus\{0\}} |b|^{-s}$ converges for all $s > 2$, given $t > \frac{2q}{q+1}$ there exists $R_3 \geq R_2$ such that

$$qM^t \sum_{b\in B_{R_3}\cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t} \leq 1. \tag{2.7}$$

Consider $R \geq 2R_3$. Put

$$I = f^{-1}(\infty) \cap B_{(R/2)}$$

Since $R/2 + R_0 \leq R/2 + R_3 < R/2 + R/2 = R$, it follows from (2.6), (2.4) and (2.5) that for every $l \geq 1$ the family W_l defined as

$$\Big\{f^{-1}_{b_l,b_{l-1},j_l} \circ f^{-1}_{b_{l-1},b_{l-2},j_{l-1}} \circ f^{-1}_{b_2,b_1,j_2} \circ f^{-1}_{b_1,b_0,j_1}\Big(B_{b_0}(R/2)\Big) : b_i \in I : 1 \leq j_i \leq q_{b_i}, i = 0,1,\ldots,l\Big\}$$

is well-defined and covers $I_R(f)$. Applying (2.3) and (2.4) we may now estimate as follows.

$\Sigma_l =$

$$= \sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I} \operatorname{diam}_s^t\left(f^{-1}_{b_l,b_{l-1},j_l} \circ f^{-1}_{b_{l-1},b_{l-2},j_{l-1}} \circ f^{-1}_{b_2,b_1,j_2} \circ f^{-1}_{b_1,b_0,j_1}\Big(B_{b_0}(R/2)\Big)\right)$$

$$\leq \sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I} ||\Big(f^{-1}_{b_l,b_{l-1},j_l} \circ f^{-1}_{b_{l-1},b_{l-2},j_{l-1}} \circ f^{-1}_{b_2,b_1,j_2} \circ f^{-1}_{b_1,b_0,j_1}\Big)^\sharp|_{B_{R_0}}||_\infty^t \operatorname{diam}_s^t\Big(B_{b_0}(R/2)\Big)$$

$$\leq \sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I} M^{lt}\left(\frac{|b_{l-1}|^{\frac{q_{b_l}-1}{q_{b_l}}}}{|b_l|^2}\right)^t\cdot\left(\frac{|b_{l-2}|^{\frac{q_{b_{l-1}}-1}{q_{b_{l-1}}}}}{|b_{l-1}|^2}\right)^t\cdots\left(\frac{|b_0|^{\frac{q_{b_1}-1}{q_{b_1}}}}{|b_1|^2}\right)^t L^t\left(\frac{R}{2}\right)^{-\frac{t}{q_{b_0}}}\frac{1}{|b_0|^{2t}}$$

$$= L^t\left(\frac{2}{R}\right)^{\frac{t}{q}} M^{lt}\sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I}|b_l|^{-2t}\Big(|b_{l-1}|^{-\frac{q+1}{q}t}\ldots|b_0|^{-\frac{q+1}{q}t}\Big)$$

$$\leq L^t\left(\frac{2}{R}\right)^{\frac{t}{q}} M^{lt}\sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I}\Big(|b_l|^{-\frac{q+1}{q}t}|b_{l-1}|^{-\frac{q+1}{q}t}\ldots|b_0|^{-\frac{q+1}{q}t}\Big)$$

$$\leq L^t\left(\frac{2}{R}\right)^{\frac{t}{q}} M^{lt}\left(\sum_{b\in I}|b|^{-\frac{q+1}{q}t}\right)^l q^l$$

$$\leq L^t\left(\frac{2}{R}\right)^{\frac{t}{q}}\left(qM^t\sum_{b\in B_{R_3}\cap f^{-1}(\infty)}|b|^{-\frac{q+1}{q+}t}\right)^l$$

Applying (2.7) we therefore get $\Sigma_l \le L^t(2/R)^{t/q}$. Since the diameters (in the spherical metric) of the sets of the covers W_l converge uniformly to 0 when $l \searrow \infty$, we therefore infere that $\mathrm{H}_s^t(I_R(f)) \le L^t(2/R)^{t/q}$, where the subscript s indicates that the Hausdorff measure is considered with respect to the spherical metric. Consequently $\mathrm{HD}(I_R(f)) \le t$ and if we put

$$I_{R,e}(f) := \left\{ z \in \mathbb{C} : \liminf_{n\to\infty} |f^n(z)| > R \right\} = \bigcup_{k\ge1} f^{-k}(I_R(f)),$$

then also $\mathrm{HD}(I_\infty(f)) \le \mathrm{HD}(I_{R,e}(f)) = \mathrm{HD}(I_R(f)) \le t$. Letting now $t \searrow \frac{2q}{q+1}$ finishes the proof. ■

3. Proof of Theorem 1

Keep the constants R_0 R_1 and R_2 with the same meaning as in the proof of Theorem 2. Fix a pole $a \in B_{R_2}$ with $q_a = q$. For every pole $b \in B_{R_2} \cap f^{-1}(\infty)$ with $q_b = q$ fix inverse branches

$$f_{b,a,1}^{-1} : \overline{B}(a, R_0) \to \mathbb{C} \text{ and } f_{a,b,1}^{-1} : \overline{B}(b, R_0) \to \mathbb{C}$$

of f, where by $\overline{B}(x, r)$ we mean the closed ball centered at x and with the (Euclidean) radius r. In view of (2.6)

$$f_{b,a,1}^{-1}\Big(\overline{B}(a, R_0)\Big) \subset \overline{B}(b, R_0) \text{ and } f_{a,b,1}^{-1}\Big(\overline{B}(b, R_0)\Big) \subset \overline{B}(a, R_0).$$

Since in addition, in exactly the same way one can prove these last two inclusions with R_0 replaced by $R_1 > R_0$, the family

$$S = \{f_{a,b,1}^{-1} \circ f_{b,a,1}^{-1} : \overline{B}(a, R_0) \to \overline{B}(a, R_0)\}_{b \in B_{R_2} \cap f^{-1}(\infty)}$$

forms a conformal infinite iterated function system in the sense of [4]. We set $\phi_b = f_{a,b,1}^{-1} \circ f_{b,a,1}^{-1}$ and given $\omega \in (B_{R_2} \cap f^{-1}(\infty))^n$, $n \ge 1$, we say that $|\omega| = n$ and we put

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \ldots \circ \phi_{\omega_n}.$$

The set

$$J_S = \bigcap_{n\ge0} \sum_{|\omega|=n} \phi_\omega(\overline{B}(a, R_0))$$

is called the limit set of the iterated function system S. Since J_S is contained in the closure of all fixed points of ϕ_ω, $\omega \in \bigcup_{n\ge1}(B_{R_2} \cap f^{-1}(\infty))^n$, which are repulsive periodic points of f, we conclude that $J_S \subset J(f)$. Given $t \ge 0$ we consider the function

$$\psi(t) = \sum_{b \in B_{R_2} \cap f^{-1}(\infty)} ||\phi_b'||^t$$

and the number

$$\theta_S = \inf\{t \ge 0 : \psi(t) < \infty\}.$$

Our proof is based on demonstrating that $\theta_S = \frac{2q}{q+1}$ and $\psi(\theta_S) = \infty$. In view of (2.2) we can write.

$$\psi(t) \asymp \sum_{b \in B_{R_2} \cap f^{-1}(\infty)} |a|^{-\frac{q+1}{q}t} |b|^{-\frac{q+1}{q}t} \asymp \sum_{b \in B_{R_2} \cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t}$$

But the series $\sum_{b\in B_{R_2}\cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t} < \infty$ if and only if $t > \frac{2q}{q+1}$ and therefore the equalities $\theta_S = \frac{2q}{q+1}$ and $\psi(\theta_S) = \infty$ are proven. The latter equality means in the terminology of [4] that the system S is hereditarily regular and it therefore follows from Theorem 3.20 in [4] that $\mathrm{HD}(J_S) > \theta_S = \frac{2q}{q+1}$. Since $J_S \subset J(f)$, we are therefore done. ■

4. Remarks

In [3] we have provided sufficient conditions for a subexpanding meromorphic function f to have σ-finite absolutely continuous invariant measure μ. We have proved the following theorem.

Theorem 4.1. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function satisfying the following conditions:*

a) $J(f) = \overline{\mathbb{C}}$
b) $l_2(I_\infty(f)) = 0$
c) $l_2(\{z : \omega(z) \subset \overline{P(f)}\}) = 0$

then there exists a σ-finite f-invariant measure μ equivalent with the Lebesgue measure l_2.

Thus Theorem 4.1 and Corollary 1.3 imply

Corollary 4.2. *If $J(f) = \overline{\mathbb{C}}$ is an elliptic function and $l_2(\{z : \omega(z) \cap \overline{P(f)} \neq \emptyset\}) = 0$, then there exists a σ-finite f-invariant measure μ equivalent with the Lebesgue measure l_2.*

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