Existence of Invariant Measures for Transcendental Subexpanding Functions

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Abstract

We consider the problem of the existence of absolutely continuous invariant measure for transcendental meromorphic functions. We prove sufficient conditions for a subexpanding meromorphic function f to have σ-finite absolutely continuous invariant measure μ and we find a class of functions satisfying these assumptions. We also show that for these functions any neigbourhood of the set of singular values $S(f)$ of f has infinite measure, i.e. $\mu(B(S(f),\epsilon)) = \infty$.

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1 Introduction

The orbits of points under iteration by a meromorphic function fall into three categories: they may be infinite, they may become periodic and hence consist of a finite number of distinct points or they may terminate at a pole of the function. Points in the last category are called *prepoles.* For transcendental meromorphic functions with more than one pole, it follows from Picard's theorem that there are infinitely many prepoles.

The *Fatou set* $F(f)$ of a meromorphic function $f:\mathbb{C}\to\overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z\in\mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f))\subset J(f)$ and $f(J(f)\setminus\{\infty\})=J(f)$. For description of the dynamics of meromorphic functions see e.g. [2]. We would however like to note that it easily follows from Montel's criterion of normality that if $f:\mathbb{C}\to\overline{\mathbb{C}}$ is either entire or has exactly one pole w and $w\notin f(\mathbb{C})$ (such functions f will be called subentire and for them $f:\mathbb{C}\setminus\{w\}\to\mathbb{C}\setminus\{w\}$ is well-defined), then there exists a set $E\subset\mathbb{C}$ consisting of at most one element and such that for every $z\in J(f)\setminus\{\infty\}$ if f is entire and for every $z\in J(f)\setminus\{w,\infty\}$ if f is subentire, every $r>0$ and every $q\geq 1$

$$\bigcup_{n\geq 1} f^{qn}(B(z,r))\supset\mathbb{C}\setminus E.$$

In the sequel E will be called the set of omitted values of f. It can be also defined for meromorphic functions which are not subentire. If f is meromorphic but not subentire nor entire, then (see [2])

$$J(f)=\overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

The *singular set* $S(f)$ of a meromorphic function f consists of those values at which f is not a regular covering. These are either critical values (algebraic singularities) or asymptotic values (transcendental singularities). The *postsingular set* $P(f)$ is the union of the forward orbits of all singular values, i.e.

$$P(f)=\bigcup_{n=0}^{\infty} f^n(S(f)).$$

It follows from Iversen's (see [3]) theorem that $E\subset S(f)$ and, and consequently, $E\subset\overline{P(f)}$. By l_2 we denote the Lebesgue measure on the plane and by m the measure induced by the spherical metric on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Note that both measures l_2 and m are equivalent in the sense that they have the same sets of measure zero. Let

$$I_\infty(f) = \{z : f^n(z) \to \infty\}.$$

M. Lyubich has proved in [6] that there is no σ-finite measure absolutely continuous with respect to the Lebesgue measure l_2 and invariant under the action of the map $z \mapsto e^z$. Aiming to give a positive contribution in the opposite direction we shall prove as our main result the following.

Theorem 1. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function satisfying the following conditions:*

a) $J(f) = \overline{\mathbb{C}}$

b) $l_2(I_\infty(f)) = 0$

c) $l_2(\{z : \omega(z) \subset \overline{P(f)}\}) = 0$

then there exists a σ-finite f-invariant measure μ equivalent with the Lebesgue measure l_2.

Note that due to Lyubich's result from [6] the condition (c) of Theorem 1 fails for the function $z \mapsto e^z$. The simplest examples of functions satisfying the assumptions of Theorem 1 are given by the formula $f(z) = 2\pi i e^z$ and $g(z) = \pi i \tan(z)$. We also present below a larger class of meromorphic functions satisfying the assumptions of this theorem.

Theorem 2. *If*

$$f(z) = \frac{Ae^{z^p} + Be^{-z^p}}{Ce^{z^p} + De^{-z^p}}$$

$p \in \mathbb{N}$, $AD - BC \neq 0$, *then* $l_2(I_\infty(f)) = 0$.

Set

$$\Theta^+(x) = \{f^n(x) : n \geq 1\}.$$

Theorem 3. *Let*

$$f(z) = \frac{Ae^{z^p} + Be^{-z^p}}{Ce^{z^p} + De^{-z^p}}$$

$p \in \mathbb{N}$, $AD - BC \neq 0$. *If f satisfies the following conditions*

i) $J(f) = \overline{\mathbb{C}}$

ii) $l_2(\{z : \omega(z) \subset \overline{P(f)}\}) = 0$

then there exists a σ-finite f-invariant measure μ equivalent with the Lebesgue measure m. If additionaly the asymptotic values $\frac{A}{C}, \frac{B}{D} \notin \overline{\Theta^+(\frac{A}{C}) \cup \Theta^+(\frac{B}{D}) \cup \Theta^+(0)}$, then for every $\epsilon > 0$

$$\mu\left(B\left(A/C, \epsilon\right)\right) = \infty \quad \textit{and} \quad \mu\left(B\left(B/D, \epsilon\right)\right) = \infty$$

2 Proof of Theorem 1

We start with the description of our setting. Let X be a compact metric space, m be a Borel measure such that $m(X) = 1$. Suppose $T : X \to X$ is a measurable map and m is a quasi-invariant measure, i.e. $m \circ T^{-1} << m$. In the proof of Theorem 1 we apply the following slightly modified result of M. Martens (see [7]) in the version presented in [8].

Theorem 2.1. *Let (X, m, T) be as above. Suppose we have a partition $\mathcal{A} = \{A_i : i \in \mathbb{N} \cup \{0\}\}$ of X such that A_i are Borel sets of positive measure, $m(X \setminus \bigcup_{n=0}^{\infty} A_i) = 0$ and they satisfy the following conditions:*

1. $\sum_{n=0}^{\infty} m(T^{-n}(A_0)) = \infty$
2. $\forall i,\, j \geq 0 \ \exists k \geq 0$ *such that up to measure zero* $T^k(A_i) \supset A_j$
3. $\forall i \geq 0 \ \exists K_i \geq 1$*, for all Borel sets $A, B \subset A_i$ and for all integers $n \geq 0$*
$$\frac{m(T^{-n}(A))}{m(T^{-n}(B))} \leq K_i \frac{m(A)}{m(B)}.$$

Then there exists σ-finite measure μ equivalent with m and such that

$$\mu \circ T^{-1} = \mu$$

and for every Borel set A

$$\mu(A) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} m(T^{-k}(A))}{\sum_{k=0}^{n} m(T^{-k}(A_0))}.$$

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that for

$$l_2(\{z : \omega(z) \subset \overline{P(f)}\}) = 0.$$

Obviously this assumption implies that

$$l_2(\overline{P(f)}) = 0. \tag{1}$$

First we construct the partition $\mathcal{A}$, next we check that it satisfies the assumptions of Theorem 2.1. We define a new metric on the plane $\mathbb{C}$ by putting

$$d_{min}(x, y) := \min\{1, |x - y|)\}.$$

Let $\xi \in \mathbb{C}$ be a periodic point of f with some period $p \geq 3$. We put

$$\overline{P_3(f)} = \overline{P(f)} \cup \{\xi, f(\xi), \ldots, f^{p-1}(\xi)\}.$$

We then define the family of balls

$$\left\{B\left(z, \frac{1}{2}d_{min}(z, \overline{P_3(f)})\right)\right\}_{z \in \mathbb{C} \setminus \overline{P_3(f)}}.$$

This family obviously covers $\mathbb{C} \setminus \overline{P_3(f)}$. Since $\mathbb{C} \setminus \overline{P_3(f)}$ is an open set, it is a Lindelöf space, and therefore we can choose a countable subcover of $\mathbb{C} \setminus \overline{P_3(f)}$, which we denote by

$$\left\{B\left(z_i, \frac{1}{2}d_{min}(z_i, \overline{P_3(f)})\right)\right\}_{i=1}^{\infty}.$$

We inductively define a partition $\mathcal{A} = \{A_i\}_{i=0}^{\infty}$ of $\mathbb{C} \setminus \overline{P_3(f)}$ as follows. Let

$$A_0 = \left\{B\left(z_0, \frac{1}{2}d_{min}(z_0, \overline{P_3(f)})\right)\right\}.$$

Assume that we have defined the set $A_1, \ldots, A_n$ such that

$$A_j \subset \left\{B\left(z_j, \frac{1}{2}d_{min}(z_j, \overline{P_3(f)})\right)\right\}$$

and

$$\mathrm{Int} A_j \neq \emptyset.$$

Then A_{n+1} we define as

$$A_{n+1} = \left\{B\left(z_{n+1}, \frac{1}{2}d_{min}(z_{n+1}, \overline{P_3(f)})\right)\right\} \setminus \bigcup_{j=1}^{n} A_j.$$

The set A_{n+1} is disjoint with the sets $A_1, \ldots, A_n$ and

$$A_{n+1} \subset B\left(z_{n+1}, \frac{1}{2}d_{min}(z_{n+1}, \overline{P_3(f)})\right) \setminus \bigcup_{j=1}^{n} B\left(z_j, \frac{1}{2}d_{min}(z_j, \overline{P_3(f)})\right).$$

Thus either $A_{n+1} = \emptyset$ or $\mathrm{Int} A_{n+1} \neq \emptyset$ and we remove all the empty sets.

Remark 1. *Since $\mathcal{A}$ is the partition of $\mathbb{C} \setminus \overline{P_3(f)}$, then $S(f) \cap A_j = \emptyset$ for each $j \in \mathbb{N} \cup \{0\}$.*

Lemma 2.2. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that $J(f) = \overline{\mathbb{C}}$ $l_2(\{z : \omega(z) \subset \overline{P(f)}\}) = 0$. If $\mathcal{A}$ is a partition defined above, then $l_2(\mathbb{C} \setminus \bigcup_{n=0}^{\infty} A_i) = 0$ and $\mathcal{A}$ satisfies the second assumption of Theorem 2.1 i.e.*

$$\forall i, j \geq 0 \ \ \exists k \geq 0 \ \text{such that up to measure zero} \ f^k(A_i) \supset A_j.$$

Proof: The equality $l_2(\mathbb{C} \setminus \bigcup_{n=0}^{\infty} A_i) = 0$ has been proved as formula (1). It follows from the construction of the sets $\{A_i\}_{i=0}^{\infty}$ that it suffices now to demonstrate that if $z \in \mathbb{C}$, $r > 0$ and $K \subset \mathbb{C} \setminus E$ (E is the set of omitted values) is a compact set, then there exists $n \geq 1$ such that

$$f^n\left(B(z,r) \setminus \bigcup_{k \geq 0} f^{-k}(\infty)\right) \supset K \setminus \bigcup_{k \geq 0} f^{-k}(\infty).$$

And indeed, suppose first that f is either entire or subentire. Since due to Baker's theorem (see [1], comp. [2]), the set of repelling periodic points is dense in the Julia set, there thus exists a periodic point $x \in B(z,r)$, say of period $q \geq 1$. Since x is repelling there exists $s > 0$ so small that $B(x,s) \subset B(z,r)$ and $f^q(B(x,s)) \supset B(x,s)$. Since $\bigcup_{j \geq 1} f^{qj}(B(x,s)) \supset \mathbb{C} \setminus E$, since K is a compact subset of $\mathbb{C} \setminus E$ and since $\{f^{qj}(B(x,s))\}_{j=1}^{\infty}$ is an increasing family of open sets, there thus exists $k \geq 1$ such that $f^{qk}(B(x,s)) \supset K$. So, we are done in this case. Assume in turn that f is not entire nor subentire. Then $\overline{\bigcup_{n \geq 1} f^{-n}(\infty)} = \mathbb{C}$ and fix a point

$$w \in B(z,r) \cap \bigcup_{n \geq 1} f^{-n}(\infty).$$

So, $w \in f^{-n}(\infty)$ for some $n \geq 1$ and there exists $t > 0$ so small that $B(w,t) \cap \bigcup_{j=1}^{n-1} f^{-j}(\infty) = \emptyset$. Hence $f^n(B(w,t))$ is well-defined and it forms an open neighbourhood of $\infty \in \overline{\mathbb{C}}$. Since ∞ is an essential singularity of f, by Picard's theorem the set $f(f^n(B(w,t)) \setminus \{\infty\})$ contains the whole $\overline{\mathbb{C}}$ without at most two points. This fact along with the observation that the set $\bigcup_{k \geq 0} f^{-k}(\infty)$ is countable, finish the proof. ■

Set

$$T(f) = \{z : \overline{\Theta^+(z)} = \mathbb{C}\}$$

To check that the first assumption of Theorem 2.1 is satisfied we need the following lemma.

Lemma 2.3. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that $J(f) = \overline{\mathbb{C}}$. If $l_2(I_\infty(f)) = 0$ and $l_2(\{z : \omega(z) \subset \overline{P(f)}\}) = 0$, then*

$$l_2(\mathbb{C} \setminus T(f)) = 0.$$

Proof: Let

$$\mathbb{C}_0 = \mathbb{C} \setminus \bigcup_{n \geq 1} f^{-n}(\infty).$$

Then $f(\mathbb{C}_0) \subset \mathbb{C}_0$ and we may consider the dynamical system $f : \mathbb{C}_0 \to \mathbb{C}_0$. Given $k \geq 1$, $\epsilon > 0$, and $w \in \mathbb{C}_0$ we define the set

$$F_{k,\epsilon,w} = \{z \in \mathbb{C}_0 : \ \forall n \geq k \ \ f^n(z) \notin B(w,\epsilon)\} = \mathbb{C}_0 \setminus \bigcup_{n \geq k} f^{-n}(B(w,\epsilon)).$$

Each set $F_{k,\epsilon,w}$ is obviously closed in $\mathbb{C}_0$ and forward invariant, i.e. $f(F_{k,\epsilon,w}) \subset F_{k,\epsilon,w}$. We shall show now that the set $F_{k,\epsilon,w}$ is nowhere-dense as a subset of the topological space $\mathbb{C}_0$. Assume on the contrary that $F_{k,\epsilon,w}$ is not nowhere-dense. Then there exists an open set $G \subset \mathbb{C}$ such that $G \cap \mathbb{C}_0 \subset F_{k,\epsilon,w}$. It follows from the proof of previous lemma that $f^n(G \cap \mathbb{C}_0) \supset \mathbb{C}$ but a countable set for some $n \geq 1$. Hence $F_{k,\epsilon,w} \supset f^n(F_{k,\epsilon,w}) \supset \mathbb{C}$ but a countable set. Therefore $F_{k,\epsilon,w}$ is a closed and dense subset of $\mathbb{C}_0$, and consequently $F_{k,\epsilon,w} = \mathbb{C}_0$. Now, on the one hand $f^k(F_{k,\epsilon,w}) = f^k(\mathbb{C}_0) = \mathbb{C}_0$ but at most two points. On the other hand $f^k(F_{k,\epsilon,w}) \cap B(w,\epsilon) = \emptyset$. This contradiction shows that $F_{k,\epsilon,w}$ is a nowhere-dense subset of $\mathbb{C}_0$. Denote by $\overline{F}_{k,\epsilon,w}$ the closure of $F_{k,\epsilon,w}$ in $\mathbb{C}$. We want to show that $\overline{F}_{k,\epsilon,w}$ is a nowhere-dense subset of $\mathbb{C}$, i.e. that $\mathrm{Int}(\overline{F}_{k,\epsilon,w}) = \emptyset$, where this interior is taken in the topological space $\mathbb{C}$. Suppose on the contrary that $\mathrm{Int}(\overline{F}_{k,\epsilon,w}) \neq \emptyset$. Since $\mathbb{C}_0$ is a dense subset of $\mathbb{C}$, we therefore get

$$\mathrm{Int}_{\mathbb{C}_0}(\overline{F}_{k,\epsilon,w}) = \mathrm{Int}(\overline{F}_{k,\epsilon,w}) \cap \mathbb{C}_0 \neq \emptyset,$$

where $\mathrm{Int}_{\mathbb{C}_0}$ denotes the interior operation in the topological space $\mathbb{C}_0$. Consequently

$$\mathrm{Int}_{\mathbb{C}_0}(F_{k,\epsilon,w}) = \mathrm{Int}_{\mathbb{C}_0}(\overline{F}^{\mathbb{C}_0}_{k,\epsilon,w}) = \mathrm{Int}_{\mathbb{C}_0}(F_{k,\epsilon,w} \cap \mathbb{C}_0) = \mathrm{Int}_{\mathbb{C}_0}(F_{k,\epsilon,w}) \cap \mathbb{C}_0 \neq \emptyset,$$

where $\overline{A}^{\mathbb{C}_0}$ denotes the closure of A in the topological space $\mathbb{C}_0$. This contradiction shows that $\overline{F}_{k,\epsilon,w}$ is a nowhere-dense subset of $\mathbb{C}$.

We shall now show that

$$l_2(F_{k,\epsilon,w}) = 0.$$

Suppose on the contrary that $l_2(F_{k,\epsilon,w}) > 0$. Since $l_2(I_\infty) = 0$ and $l_2(\{z : \omega(z) \subset \overline{P_3(f)}\}) = 0$, there exists a density point z_0 of the Lebesgue measure of $F_{k,\epsilon,w}$ such that $\Theta^+(z_0)$ does not tend to ∞ and $\Theta^+(z_0)$ has an accumulation point y which does not belong to $\overline{P_3(f)}$. Thus there is a sequence of points $z_{n_k} = f^{n_k}(z_0)$ satisfying $y = \lim_{k\to\infty} z_{n_k}$. Let $\delta = dist(y, \overline{P_3(f)})/2$. Then for every $k \geq 1$ there exists an analytic inverse branch $g_{n_k} : B(z_{n_k}, \delta) \to \overline{\mathbb{C}}$ which maps z_{n_k} to z_0. Since $F_{k,\epsilon,w}$ is closed nowhere dense set, the balls $B(y, \delta/2)$ have nonempty intersection with the complement of $F_{k,\epsilon,w}$. Thus there is a constant $\kappa > 0$ such that

$$\kappa < \frac{l_2\left(B(y, \frac{\delta}{2}) \cap (\mathbb{C} \setminus F_{k,\epsilon,w})\right)}{l_2\left(B(y, \frac{\delta}{2})\right)}$$

Hence for sufficiently large $n_k \to \infty$

$$\kappa' < \frac{l_2\left(B(z_{n_k}, \frac{\delta}{2}) \cap (\mathbb{C} \setminus F_{k,\epsilon,w})\right)}{l_2\left(B(z_{n_k}, \frac{\delta}{2})\right)}, \quad \kappa' > 0.$$

This implies that

$$\frac{l_2\left(g_{n_k}(B(z_{n_k}, \frac{\delta}{2})) \cap g_{n_k}(F_{\epsilon,k,w})\right)}{l_2\left(g_{n_k}(B(z_{n_k}, \frac{\delta}{2}))\right)} < 1 - \frac{\kappa'}{K(1/2)^2}. \tag{2}$$

Let

$$\delta_k = \frac{\delta}{8}|(g_{n_k})'(z_{n_k})|.$$

By Koebe's $\frac{1}{4}$-theorem applied to the numerator of (2) and the Koebe's distortion theorem applied to the denominator of (2), we get

$$\frac{l_2\left(B(z_0, \delta_k) \cap F_{k,\epsilon,w}\right)}{l_2\left(B(z_0, \delta_k)\right)} \leq 1 - \kappa'',$$

for some universal $\kappa'' > 0$. Thus z_0 cannot be a density point of the Lebesgue measure of $F_{k,\epsilon,w}$ and the proof of the formula $l_2(F_{k,\epsilon,w}) = 0$ is finished.

It remains to demonstrate that

$$T(f) \supset \mathbb{C} \setminus \bigcup_{k=1}^{\infty} \bigcup_{\epsilon\in\mathbb{Q}} \bigcup_{w\in\mathbb{Q}^2} F_{k,\epsilon,w}.$$

Take $z \in \mathbb{C} \setminus \bigcup_{k=1}^{\infty} \bigcup_{w\in\mathbb{Q}^2} F_{k,1/k,w}$. Fix $w \in \mathbb{Q}^2$. Then for each $k \geq 1$, $z \notin F_{k,1/k,w}$. It follows from the definition of the sets $F_{k,\epsilon,w}$ that

$$\forall k \geq 1\, \exists n_k > k \text{ such that } f^{n_k}(z) \in B(w, 1/k)$$

Thus $w = \lim_{k\to\infty} f^{n_k}(z)$. Hence $w \in \omega(z)$ and consequently $\mathbb{Q}^2 \subset \omega(z)$. Since $\mathbb{Q}^2$ is dense in $\mathbb{C}$ and $\omega(z)$ is closed this implies that $\omega(z) = \mathbb{C}$. We are done. ■

Lemma 2.4. *If f satisfies the assumptions of Lemma 2.3, then*

$$\sum_{n=0}^{\infty} m(f^{-n}(A_k)) = \infty$$

for each $A_k \in \mathcal{A}$.

Proof: Since $\mathrm{Int} A_k \neq \emptyset$ it follows from Lemma 2.3 that for *a.e.* $z \in \mathbb{C}$ the forward trajectory $\Theta^+(z)$ visits A_k infinitely many times. It means that for *a.e.* $z \in \mathbb{C}$ and for each $k \in \mathbb{N} \cup \{0\}$

$$\sum_{n \geq 0} \mathbf{1}_{f^{-n}(A_k)}(z) = \infty.$$

Thus applying Lebesgue's Convergence Theorem for positive functions, we get

$$\infty = \int \infty dm = \int \lim_{l \to \infty} \sum_{n=0}^{l} \mathbf{1}_{f^{-n}(A_k)} dm = \lim_{l \to \infty} \int \sum_{n=0}^{l} \mathbf{1}_{f^{-n}(A_k)} dm$$
$$= \lim_{l \to \infty} \sum_{n=0}^{l} \int \mathbf{1}_{f^{-n}(A_k)} dm = \lim_{l \to \infty} \sum_{n=0}^{l} m(f^{-n}(A_k)) = \sum_{n=0}^{\infty} m(f^{-n}(A_k)).$$

■

Lemma 2.5. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that for $l_2(\{z : \omega(z) \subset \overline{P(f)}\}) = 0$. If $\mathcal{A} = \{A_i\}_{i=0}^{\infty}$ is a partition defined above, then for every $i \geq 0$ there exists $K_i \geq 1$ such that for each $n \geq 0$ and all Borel sets $A, B \subset A_i \in \mathcal{A}$ with $m(B) > 0$, we have*

$$\frac{m(f_\nu^{-n}(A))}{m(f_\nu^{-n}(B))} \leq K_i \frac{m(A)}{m(B)}.$$

Proof: Fix $i \geq 0$. Since $A_i \subset B(z_i, 1/2 d_{min}(z_i, \overline{P_3(f)}))$, then for each $n \in \mathbb{N}$ all the holomorphic branches f_ν^{-n} of f^{-n} are well defined on A_i. Since they omit the set $\{\xi, f(\xi), \ldots, f^{p-1}(\xi)\}$ which consists of at least three points, it follows from Montel's theorem that these inverse branches defined on the ball $B(z_i, d_{min}(z_i, \overline{P_3(f)}))$ form a normal family $\mathcal{F}_i$. Since $J(f) = \overline{\mathbb{C}}$, all the limit functions of $\mathcal{F}_i$ are constant. Therefore there exists $r_i > 0$ such that if $f_\nu^{-n} \in \mathcal{F}_i$, then $f_\nu^{-n}(B(z_i, 3/4 d_{min}(z_i, \overline{P_3(f)}))$ is disjoint from a ball of radius r_i (with respect to the spherical metric). It therefore follows from Koebe's distortion theorem that there exists $\tilde{K}_i \geq 1$ such that

$$\frac{|(f_\nu^{-n})'(y)|_\rho}{|(f_\nu^{-n})'(x)|_\rho} \leq \tilde{K}_i$$

for all $f_\nu^{-n} \in \mathcal{F}_i$ and all $x, y \in B(z_i, 1/2d_{min}(z_i, \overline{P_3(f)}))$, where the subscript ρ indicates that the derivative is taken with respect to the spherical metric. Hence for all Borel sets $A, B \subset A_i$ we get

$$\frac{m(f_\nu^{-n}(A))}{m(f_\nu^{-n}(B))} = \frac{\int_A |(f_\nu^{-n})'|_\rho^2 dm}{\int_A |(f_\nu^{-n})'|_\rho^2 dm} \leq \frac{\sup_{A_i}\{|(f_\nu^{-n})'|_\rho\}^2 m(A)}{\inf_{A_i}\{|(f_\nu^{-n})'|_\rho\}^2 m(B)} \leq \tilde{K}_i^2 \frac{m(A)}{m(B)}$$

and we are done. ■

The proof of Theorem 1 follows from Lemma 2.2, Lemma 2.4. Lemma 2.5 and from Theorem 2.1.

3 Proof of Theorem 2

The idea of the proof is to obtain good estimates of the derivative of the function f around poles and to follow the scheme worked out in [5]. We can rewrite $f(z)$ in the form

$$f(z) = \frac{Ae^{2z^p} + B}{Ce^{2z^p} + D}.$$

It is easy to calculate that

$$f'(z) = \frac{2p(AD - BC)z^{p-1}e^{2z^p}}{(Ce^{2z^p} + D)^2} = \frac{2p(AD - BC)z^{p-1}e^{2z^p}(f(z))^2}{(Ae^{2z^p} + B)^2} \tag{3}$$

For $p = 1$ the function f has no critical points. Let $p > 1$. Then $f'(z) = 0$ iff $z = 0$ and $f(0) = \frac{A+B}{C+D}$ is a critical value. Note that the assumption $AD - BC \neq 0$ implies that either $C \neq 0$ or $D \neq 0$. Assume that $C = 0$ then f is a transcendental entire map with one finite asymptotic value $\frac{B}{D}$. Analogously, if $D = 0$ then f is also a transcendental entire map with one finite asymptotic value $\frac{A}{C}$. For transcendental entire function the theorem follows from Theorem 7 in [4]. So, suppose that $C \neq 0$ and $D \neq 0$. Then f is a transcendental meromorphic functions with infinitely many poles. The function f has two finite omitted values $\frac{A}{C}, \frac{B}{D}$. These are asymptotic values of f. Thus

$$S(f) \subset \left\{\frac{A+B}{C+D}, \frac{A}{C}, \frac{B}{D}\right\}$$

Because $S(f)$ is bounded, we can find $R > 0$ such that $S(f) \subset D(0, R)$. Let $A_R = \{z \in \overline{\mathbb{C}} : |z| \geq R\}$. Suppose now that $|f(z)| \geq R$ (R is very big). Then

$$|A||e^{2z^p}| + |B| \geq |Ae^{2z^p} + B| \geq |RCe^{2z^p} + RD| \geq R|C||e^{2z^p}| - R|D|.$$

Hence

$$|e^{2z^p}| \leq \frac{R|D| + |B|}{R|C| - |A|} =: K_1, \tag{4}$$

where $R > |A|/|C|$. Also

$$|A||e^{2z^p}| + |B| \geq |RCe^{2z^p} + RD| \geq R|D| - R|C||e^{2z^p}|.$$

Hence

$$|e^{2z^p}| \geq \frac{R|D| - |B|}{R|C| + |A|} =: K_2, \tag{5}$$

where $R > |B|/|D|$. Let now b be a solution to the equation $Ce^{2z} + D = 0$. Let

$$g(z) = \frac{Ae^{2z} + B}{Ce^{2z} + D}$$

and let W_b be the connected component of $g^{-1}(A_R)$ containing b. Then for every $\epsilon > 0$ there is $R > 0$ (independent of b) such that

$$W_b \subset B(b, \epsilon) \tag{6}$$

Let z_q be a pole of f and let V_q be the connected component of $f^{-1}(A_R)$ containing z_q. Then $b = z_q^p$ is a pole of g and $V_q = T(W_b)$, where T is the holomorphic branch of $z^{\frac{1}{p}}$ determined by the condition $T(b) = z_q$. Then for every $z \in V_q$ we have $|z^p - b| < \epsilon$ for R produced by (6) with a fixed $\epsilon > 0$. Then $|z|^p < |b| + \epsilon$ and therefore

$$|z|^{p-1} \leq (|b| + \epsilon)^{\frac{p-1}{p}} \tag{7}$$

and

$$|z|^{p-1} \geq (|b| - \epsilon)^{\frac{p-1}{p}} \tag{8}$$

We may assume that $0 < \epsilon < \frac{1}{2}|b|$. The infimum of moduli of poles of $g(z)$ might be equal to 0 if there is a pole b=0. But later we will consider only poles satisfying $|z_q| \geq R' >> R$, so $|b| = |z_q^p| \geq (R')^p$. Combining terms in (3), (5), (8) we get

$$|f'(z)| \geq \frac{2p|AD - BC|(\frac{1}{2}|b|)^{\frac{p-1}{p}} K_2}{|Ce^{2z^p} + D|^2} \tag{9}$$

We may also assume that R to be so large (and consequently ϵ so small) that

$$|Ce^{2z}+D| \le \frac{1}{2}\frac{|AD-BC|}{|A|}$$

for each pole b and every $z \in W_b$, if $A \neq 0$. Thus

$$|Ce^{2z^p}+D| \le \frac{1}{2}\frac{|AD-BC|}{|A|} =: K_3 \tag{10}$$

for each pole z_q and every $z \in V_q$, if $A \neq 0$. We may assume that $C \neq 0$ since otherwise f is entire. If $A = 0$ then $B \neq 0$ and in (10) we may put

$$|Ce^{2z^p}+D| \le \frac{1}{2}|BC| =: K_3 \tag{11}$$

It follows from (9), (10) and (11) that

$$|f'(z)| \ge \frac{2p|AD-BC|(\frac{1}{2}|b|)^{\frac{p-1}{p}}K_2}{K_3} \ge \frac{2p|AD-BC|(\frac{1}{2}R')^{p-1}K_2}{K_3} \tag{12}$$

Thus there exists a universal (independent of b and z_q) constant $\mathcal{K} = \frac{2p|AD-BC|(\frac{1}{2}R')^{p-1}K_2}{K_3} > 1$ such that for all $z \in V_q$

$$|f'(z)| \ge \mathcal{K}. \tag{13}$$

Let $A_{R,R_1} = \{z \in \mathbb{C} : R < |z| < R_1\}$. Suppose now that $R < |f(z)| < R_1$, where $R_1 >> R'$. Let U_b be a connected component of $g^{-1}(A_{R,R_1})$ such that $U_b \subset W_b$. By $\tilde{V}_{q,R_1}$ we denote the connected component of $f^{-1}(A_{R,R_1})$ containing z_q. Thus $\tilde{V}_{q,R_1} = T(U_b)$ is an annulus, where T is the holomorphic branch of $z^{\frac{1}{p}}$ determined by the condition $T(b) = z_q$.

We want to find an upper bound on $|f'(z)|$ for $z \in \tilde{V}_{q,R_1}$. It depends on the pole $b = z_q^p$ but the distortion of f in $\tilde{V}_{q,R_1}$ doesn't depend on pole. If $R > 0$ is sufficiently large then for every $z \in \tilde{V}_{q,R_1}$ we have $|Ce^{2z^p}+D| \le \frac{1}{2}\frac{|AD-BC|}{|A|}$ and consequently

$$\begin{aligned} |Ae^{2z^p}+B| &= \left|\frac{A}{C}\right|\left|Ce^{2z^p}+\frac{BC}{A}\right| = \left|\frac{A}{C}\right|\left|(Ce^{2z^p}+D)+(\frac{BC}{A}-D)\right| \qquad (14) \\ &\ge \left|\frac{A}{C}\right|\left(\left|\frac{BC-AD}{A}\right| - \left|Ce^{2z^p}+D\right|\right) \\ &\ge \frac{|A|}{|C|}\frac{1}{2}\frac{|AD-BC|}{|A|} \end{aligned}$$

$$= \frac{1}{2}\frac{|AD - BC|}{|C|} =: K_4$$

Combining formulae (3), (4), (7), (14), we obtain the following upper bound on $|f'(z)|$.

$$|f'(z)| \leq 2p|AD - BC|(2|b|)^{\frac{p-1}{p}} K_1 \frac{R_1^2}{K_4^2} =: \mathcal{K}_1(R_1) \tag{15}$$

Now we estimate the distortion of f in $\tilde{V}_{q,R_1}$. By (13) and (15) we obtain that

$$L = \frac{\sup_{z\in\tilde{V}_{q,R_1}} |f'(z)|}{\inf_{z\in\tilde{V}_{q,R_1}} |f'(z)|} = \frac{\mathcal{K}_1(R_1)}{\mathcal{K}} \tag{16}$$

Since the map g is the composition of a Möbius transformation and the map $z \mapsto e^{2z}$ for every q large enough and since each holomorphic branch of $z^{1/p}$ sending the point z_q^p to z_q is univalent on the balls containing z_q^p, so big that applying Koebe's distortion theorem produces some radius γ_q such that

$$B(z_q, \gamma_q/4) \subset V_q \subset B(z_q, \gamma_q/2). \tag{17}$$

Let $\mathcal{V} = f^{-1}(A_R)$. A straightforward calculations show that

$$\lim_{R_1\to\infty} \frac{l_2(A_{R,R_1} \setminus \mathcal{V})}{l_2(A_{R,R_1})} = 1 \tag{18}$$

and

$$\lim_{R_1\to\infty} \frac{l_2\big(\tilde{V}_{q,R_1}\big)}{l_2(V_q)} = 1 \tag{19}$$

uniformly with respect to q. Therefore for every $R_1 > 0$ large enough

$$\frac{l_2(A_{R,R_1} \setminus \mathcal{V})}{l_2(A_{R,R_1})} \geq \frac{1}{2} \quad \text{and} \quad \frac{l_2\big(\tilde{V}_{q,R_1}\big)}{l_2(V_q)} \geq \frac{1}{2}. \tag{20}$$

We want to show that for every q we have

$$\frac{l_2(V_q \setminus f^{-1}(\mathcal{V}))}{l_2(V_q)} \geq (4L^2)^{-1}, \tag{21}$$

where L is the upper bound on distortion given by (16). And indeed, using (16) and (20), we get

$$
\begin{aligned}
\frac{l_2(V_q \setminus f^{-1}(\mathcal{V}))}{l_2(V_q)} &= \frac{l_2([(V_q \setminus \tilde{V}_{q,R_1}) \setminus f^{-1}(\mathcal{V})] \cup [\tilde{V}_{q,R_1} \setminus f^{-1}(\mathcal{V})])}{l_2(V_q)} \\
&\geq \frac{l_2(\tilde{V}_{q,R_1} \setminus f^{-1}(\mathcal{V}))}{l_2(V_q)} = \frac{l_2(f_q^{-1}(A_{R,R_1} \setminus \mathcal{V}))}{l_2(f_q^{-1}(A_{R,R_1}))} \cdot \frac{l_2(\tilde{V}_{q,R_1})}{l_2(V_q)} \\
&\geq \frac{1}{2} L^{-2} \frac{l_2(A_{R,R_1} \setminus \mathcal{V})}{l_2(A_{R,R_1})} \geq (4L^2)^{-1}.
\end{aligned}
$$

Suppose now on the contrary that $l_2(I_\infty(f)) > 0$. Since

$$
I_\infty(f) \subset \bigcap_{n \geq 1} \bigcup_{k \geq n} \bigcap_{l \geq k} f^{-l}(A_R)
$$

there in particular exists $k \geq 1$ such that $l_2\left(I_\infty(f) \cap \bigcap_{j \geq k} f^{-j}(A_R)\right) > 0$. Let ξ_0 be a density point of the Lebesgue measure of the set $I_\infty(f) \cap \bigcap_{j \geq k} f^{-j}(A_R)$. For every $n \geq 0$ put

$$
\xi_n = f^n(\xi_0).
$$

Since $\lim_{n \to \infty} \xi_n = \infty$, for every n large enough there exists $q(n)$ such that $\xi_n \in V_{q(n)}$, $\lim_{n \to \infty} |\xi_n^p - z_{q(n)}^p| = 0$ and $\lim_{n \to \infty} q(n) = \infty$. Hence, using Koebe's distortion theorem we deduce that for all n large enough $|\xi_n - z_{q(n)}| \leq \gamma_{q(n)}/8$ (γ_q are the numbers defined in (17)) and combining this with (17), we conclude that

$$
B(\xi_n, \gamma_{q(n)}/8) \subset V_{q(n)} \subset B(\xi_n, \gamma_{q(n)}) \tag{22}
$$

Since each ξ_n is also a density point of the set $\bigcap_{j \geq k} f^{-j}(A_R)$, we may if needed replace ξ_0 by an appropriate iterate ξ_n and assume that all the numbers $q(n)$, $n \geq 1$ are so large as one wishes. Since for every q the map $f : V_q \to A_R$ is univalent, there exists its inverse map $g^{(q)} : A_R \to V_q$. Since in addition for all $n \geq 1$, $V_{q(n)} \subset A_R$, therefore the composition

$$
g_n = g^{q(0)} \circ g^{q(1)} \circ g^{q(2)} \ldots \circ g^{q(n)} : A_R \to A_R
$$

is well-defined (and obviously univalent). Moreover $g_n(V_{q(n)}) \subset V_{q(0)}$. By Koebe's $\frac{1}{4}$-distortion theorem and the standard version of Koebe's distortion theorem, we get

$$
B\left(\xi_0, \frac{1}{4}|g_n'(\xi_n)|\gamma_n\right) \subset g_n(B(\xi_n, \gamma_n)) \subset B(\xi_0, K|g_n'(\xi_n)|\gamma_n),
$$

where we abbreviated $\gamma_{q(n)}$ to γ_n. Using Koebe's distortion theorem again along with (22), we can estimate as follows.

$$
\begin{aligned}
&\frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \cap f^{-(n+2)}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} \\
&= 1 - \frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \setminus f^{-(n+2)}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} \\
&\leq 1 - \frac{l_2(g_n(B(\xi_n, \gamma_n)) \setminus f^{-(n+2)}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} = 1 - \frac{l_2(g_n(B(\xi_n, \gamma_n) \setminus f^{-1}(\mathcal{V})))}{(4K)^2 l_2\big(B(\xi_0, \frac{1}{4}|g_n'(\xi_n)|\gamma_n)\big)} \\
&\leq 1 - \frac{1}{16K^2}\frac{l_2(g_n(B(\xi_n, \gamma_n) \setminus f^{-1}(\mathcal{V})))}{l_2\big(g_n(B(\xi_n, \gamma_n))\big)} \leq 1 - \frac{1}{16K^4}\frac{l_2\big(B(\xi_n, \gamma_n) \setminus f^{-1}(\mathcal{V})\big)}{l_2\big(B(\xi_n, \gamma_n)\big)} \\
&\leq 1 - \frac{1}{2^{10}K^4}\frac{l_2\big(V_{q(n)} \setminus f^{-1}(\mathcal{V})\big)}{l_2\big(B(\xi_n, \gamma_n/8)\big)} \leq 1 - \frac{1}{2^{10}K^4}\frac{l_2\big(V_{q(n)} \setminus f^{-1}(\mathcal{V})\big)}{l_2\big(V_{q(n)}\big)} \\
&\leq 1 - \frac{1}{2^{12}K^4L^2},
\end{aligned}
$$

where writing the last inequality we have used (21). Hence, for every $n \geq k$

$$
\begin{aligned}
\frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \cap \bigcap_{j\geq k} f^{-j}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} &\leq \frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \cap f^{-(n+2)}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} \\
&\leq 1 - \frac{1}{2^{12}K^4L^2}.
\end{aligned}
$$

Thus ξ_0 is not a density point of the set $\bigcap_{j\geq k} f^{-j}(A_R)$ and consequently not a density point of the set $I_\infty(f) \cap \bigcap_{j\geq k} f^{-j}(A_R)$. This contradiction finishes the proof of our theorem.

4 Proof of Theorem 3

Proof: In view of Theorem 1 and Theorem 2 we only need to demonstrate that for every $\epsilon > 0$, $\mu(B(A/C, \epsilon)) = \infty$ and $\mu(B(B/D, \epsilon)) = \infty$. Let us deal with the asymptotic value A/C. The reasoning for B/D is the same. We know that $f^{-1}(B(A/C, \epsilon))$ is the union of domains $H_0, H_1, \dots, H_{p-1}$, whose boundaries are asymptotically tangent respectively to the rays

$$L_k = re^{i(\frac{1+4k}{2p})\pi}, \quad L_k' = re^{i(\frac{3+4k}{2p})\pi},$$

$k = 0, \dots, p-1$ and $r \in \mathbb{R}^+$. In particular the domain H_0 is contained in the region

$$V_0 = \left\{ z : |\arg z| < \frac{\pi}{2p} \right\}.$$

By our assumptions the asymptotic values $\frac{A}{C}, \frac{B}{D} \notin \overline{\Theta^+(\frac{A}{C}) \cup \Theta^+(\frac{B}{D}) \cup \Theta^+(0)}$. It therefore follows that for all $\epsilon > 0$ sufficiently small

$$\begin{aligned} H_0 \cap \overline{\Theta^+(A/C) \cup \Theta^+(B/D) \cup \Theta^+(0)} &= \\ = \left[f^{-1}(B(A/C,\epsilon)) \cap V_0\right] \cap \overline{\Theta^+(A/C) \cup \Theta^+(B/D) \cup \Theta^+(0)} &= \emptyset \end{aligned} \tag{23}$$

Since H_0 is asymptotically tangent to the rays forming V_0, there exists $\zeta > 0$ such that

$$T = \left\{ z \in \mathbb{C} : |\arg(z-\zeta)| < \frac{\pi}{2p} \right\} \subset H_0. \tag{24}$$

Since in the proof of Theorem 1 we were in fact very flexible in constructing the partition $\mathcal{A} = \{A_j\}_{j=0}^{\infty}$ (see the considerations following the formulation of Theorem 2.1) we may assume that

$$\zeta + 3 \in A_0, \ \operatorname{diam}(A_0) \le 1, \ \text{and } A_0 \subset T. \tag{25}$$

For every $R > 2\zeta + 5$ set

$$T_R = \left\{ z \in \mathbb{C} : R \le \operatorname{Re}(z) \le 2R \text{ and } |\arg(z-R)| < \frac{\pi}{2p} \right\} \subset T.$$

Since

$$\begin{aligned} A_0 \cup T_R &\subset \left\{ z \in \mathbb{C} : \zeta + 1 \le \operatorname{Re}(z) \le 2R \text{ and } |\arg(z-(\zeta+1)| < \frac{\pi}{2p} \right\} \\ &\subset \left\{ z \in \mathbb{C} : \zeta \le \operatorname{Re}(z) \le 2R+1 \text{ and } |\arg(z-\zeta)| < \frac{\pi}{2p} \right\}, \end{aligned}$$

we conclude from Koebe's distortion theorem that there exists a constant $C \ge 1$ such that

$$\frac{|G'(u)|}{|G'(v)|} \le CR \tag{26}$$

for all univalent analytic functions

$$G : \left\{ z \in \mathbb{C} : \zeta \le \operatorname{Re}(z) \le 2R+1 \text{ and } |\arg(z-\zeta)| < \frac{\pi}{2p} \right\} \to \mathbb{C}$$

and all $u, v \in A_0 \cup T_R$. By (23) and (24) for every $n \ge 1$ all the holomorphic inverse branches of f^n are well-defined on T. Let $f_\nu^{-n} : T \to \mathbb{C}$ be such a branch. To prove Theorem 3 we compare first $m(f_\nu^{-n}(A_0)))$ and $m(f_\nu^{-n}(T_R))$.

Choose $y_n \in A_0$ and $x_n \in T_R$ such that $|(f_\nu^{-n})'(y_n)|_\rho$ is maximal, while $|(f_\nu^{-n})'(x_n)|_\rho$ is minimal. Then

$$\begin{aligned} \frac{m(f_\nu^{-n}(A_0))}{m(f_\nu^{-n}(T_R))} &\le \frac{|(f_\nu^{-n})'(y_n)|_\rho^2}{|(f_\nu^{-n})'(x_n)|_\rho^2} \frac{m(A_0)}{m(T_R)} \\ &= \frac{m(A_0)}{m(T_R)} \frac{|(f_\nu^{-n})'(y_n)|^2}{|(f_\nu^{-n})'(x_n)|^2} \frac{[1+|y_n|^2]^2}{[1+|x_n|^2]^2} \frac{[1+|(f_\nu^{-n})(x_n)|^2]^2}{[1+|(f_\nu^{-n})(y_n)|^2]^2} \end{aligned} \tag{27}$$

In order to find an appropriate upper bound we estimate each term on the right-hand side of (27). Since $|x_n| \ge R$,

$$\frac{1}{[1+|x_n|^2]^2} \le R^{-4} \tag{28}$$

Obviously $l_2(T_R) = \tan\left(\frac{\pi}{2p}\right) R^2$ and

$$m(T_R) \asymp \frac{\tan\left(\frac{\pi}{2p}\right) R^2}{R^4} \asymp \tan\left(\frac{\pi}{2p}\right) R^{-2} \asymp R^{-2} \tag{29}$$

for all R large enough. Since each holomorphic branch of f^{-1} is of the form

$$w \mapsto T\left(\frac{1}{2} L\left(\frac{Dw - B}{A - Cw}\right)\right),$$

where L is a branch of the logarithm and T is an inverse branch of the map $z \mapsto z^p$, and since images of all branches of logarithm are contained in horizontal strips of widh 2π, using Koebe's distortion theorem it is not difficult to demonstarate that there exists a constant $d > 0$ such that

$$\operatorname{diam}(f_\nu^{-n}(T_R)) \le d$$

for all $R > 0$ and for all inverse branches of positive iterates of f. Using (26), we conclude that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \frac{[1+|(f_\nu^{-n})(x_n)|^2]^2}{[1+|(f_\nu^{-n})(y_n)|^2]^2} &\le \frac{[1+(|f_\nu^{-n}(y_n)| + |f_\nu^{-n}(x_n) - f_\nu^{-n}(y_n)|)^2]^2}{[1+|(f_\nu^{-n})(y_n)|^2]^2} \\ &\le (1 + |f_\nu^{-n}(x_n) - f_\nu^{-n}(y_n)| + |f_\nu^{-n}(x_n) - f_\nu^{-n}(y_n)|^2)^2 \\ &\le C_1 d^4, \end{aligned} \tag{30}$$

Combining (27), (28), (29), (30), (25) and (26), we obtain

$$\frac{m(f_\nu^{-n}(A_0))}{m(f_\nu^{-n}(T_R))} \preceq m(A_0) R^2 C^2 R^2 (1 + (\zeta + 4)^2)^2 R^{-4} C_1 d^4 \asymp 1.$$

By the procedure applied to the construction of the invariant measure μ this implies that $\mu(T_R) \geq \kappa$ for some $\kappa > 0$. Since $T_R \subset T$, since by (24), $T \subset H_0$, and since the sets T_{4^n}, $n \geq 0$, are mutually disjoint, we conclude that $\mu(H_0) = \infty$. Thus $\mu(f^{-1}(B(A/C, \epsilon))) = \infty$, and, since μ is f-invariant, $\mu(B(A/C, \epsilon)) = \infty$. The proof is complete. ■

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