ERGODIC OPTIMIZATION FOR NON-COMPACT DYNAMICAL SYSTEMS

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Abstract. We consider maximizing orbits and maximizing measures for continuous maps $T : X \to X$ and functions $f : X \to \mathbb{R}$, where X need not be compact. We give sufficient conditions for the function f to have a normal form, which allows a characterization of f-maximizing measures in terms of their support. For example if $T : X \to X$ is a countable state subshift of finite type and f has a Gibbs measure, then f has a normal form, and hence a maximizing measure.

1. Introduction

Let $T : X \to X$ be a continuous map on a topological space X. Given a continuous function $f : X \to \mathbb{R}$, an orbit $\{x, T(x), T^2(x), \ldots\}$ is called *f-maximizing* if the time average $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$ is larger than along any other orbit. If $\mathcal{M}$ denotes the set of T-invariant probability measures then the measure $\mu \in \mathcal{M}$ is called *f-maximizing* if $\int f\, d\mu = \sup_{m\in\mathcal{M}} \int f\, dm$. We can define f-minimizing orbits and measures in a similar way.

By *ergodic optimization* we mean the circle of problems relating to the search for maximizing (or minimizing) orbits and measures, and the determination of the maximum ergodic average. The subject evolved during the 1990s, and for much of this time researchers worked independently, unaware of others' contributions. In 1990 Coelho [C] studied zero temperature limits of equilibrium states, and their connection with certain maximizing measures. He showed that for Hölder functions f on subshifts of finite type, an f-maximizing measure cannot be fully supported unless f is cohomologous to a constant. In an unpublished manuscript from around 1993, Conze & Guivarc'h [CG] improved Coelho's result by proving that such an f has a *normal form*: there exists a

Date: September 2004.

The first author was partially supported by an EPSRC Advanced Research Fellowship. He acknowledges the warm hospitality of the University of North Texas, where this research was initiated. The second and third authors were supported in part by the NSF Grant no. DMS 0100078.

continuous function φ such that $\tilde{f} := f + \varphi - \varphi \circ T \leq \sup_{m \in \mathcal{M}} \int f\, dm$. The f-maximizing measures are thereby identified as precisely those invariant probability measures whose support lies in the set of global maxima of the normal form $\tilde{f}$.

Conze & Guivarc'h also gave a preliminary analysis of a specific non-trivial model for ergodic optimization: $T(x) = 2x \pmod 1$ and $f_\theta(x) = \cos 2\pi(x - \theta)$. A little later this parametrized family of functions was studied independently by Hunt & Ott [HO] and by Jenkinson [J1, J2], who proposed a complete characterisation of the maximizing measures for the family f_θ, in terms of Sturmian measures. This conjecture was proved by Bousch [B1], who had himself independently arrived at the same characterisation. Bousch also re-discovered the normal form theorem of Conze & Guivarc'h, exploiting it in a key way for his proof. An alternative proof of the normal form theorem was given by Contreras, Lopes & Thieullen [CLT], using techniques inspired by Mañé [M1, M2], who had established an analogous result in the context of Lagrangian systems. A subsequent strengthening of Mañé's result by Fathi [F1] is closer in spirit to the approach of Bousch, constructing the normal form from the fixed point of a certain nonlinear operator.

In the present article we consider ergodic optimization for dynamical systems $T : X \to X$ where X is not necessarily compact. In this general context a maximizing measure need not even exist, so it is convenient (cf. §2) to distinguish the successively weaker notions of maximizing measure, maximizing orbit, and limsup maximizing orbit. In §3 an appropriately strengthened notion of normal form is introduced, which in particular guarantees the existence of a maximizing measure. In §§4, 5 we proceed to derive sufficient conditions for a function to have such a strong normal form. In the very general context of §4 the sufficient condition is a rather abstract notion of *essential compactness*. Our main motivation for studying dynamical systems on non-compact spaces is the case of countable state subshifts of finite type, and the smooth systems, such as Gauss's continued fraction map, that they model. From §5 onwards we assume that $T : X \to X$ is a countable state subshift of finite type, and that f is bounded above. If X is compact, there exists a (strong) normal form whenever f has summable variations and T is topologically mixing [B2, J3]. In the non-compact setting this is not the case: some extra condition on f is required in order to guarantee a strong normal form. Our main result, Theorem 5.14, gives such a condition. The condition is an easily checkable one; in the case of the full shift on $I\!N$ it is that for some $I \in I\!N$,

$$\sum_{j=1}^{\infty} \mathrm{var}_j(f) < \inf f|_{[I]} - \sup f|_{[i]}$$

for all sufficiently large i (cf. Corollary 5.15), where $[k]$ denotes the (cylinder) set of sequences whose first entry is k, and $\mathrm{var}_j(f) = \sup\{f(x) - f(y) : \text{the first } j \text{ entries of } x \text{ and } y \text{ agree}\}$. In other words, the values of f on some set $[I]$ should be sufficiently larger than its values "at infinity", in the sense that the difference between these values dominates the total variation $\sum_{j=1}^{\infty} \mathrm{var}_j(f)$. In particular, if $\sup f|_{[i]} \to -\infty$ as $i \to \infty$ then f has a strong normal form (Corollary 5.16). It follows (Corollaries 5.17 and 5.18) that if f has an invariant Gibbs measure then it also has a strong normal form, and hence a maximizing measure; this is because the existence of a Gibbs measure implies that $\sup f|_{[i]} \to -\infty$ as $i \to \infty$.

2. Maximizing orbits and maximizing measures

Let X be a topological space, not necessarily compact. For a continuous transformation $T : X \to X$, let $\mathcal{M}$ denote the set of T-invariant Borel probability measures on X. In general $\mathcal{M}$ might be empty, though if X is a non-empty compact metrizable space then $\mathcal{M} \neq \emptyset$, by the Krylov-Bogolioubov Theorem ([Wa2], Cor. 6.9.1).

Definition 2.1. (Three types of maximum ergodic average)

By convention we define the supremum of the empty set to be $-\infty$.

If X is a topological space and $f : X \to I\!R$ is continuous then for $x \in X$ we define $\gamma(f, x) \in [-\infty, \infty]$ by

$$\gamma(f, x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) .$$

Then define the *maximum limsup time average* $\gamma(f)$ by

$$\gamma(f) = \sup_{x \in X} \gamma(f, x) .$$

A point $x \in X$ is called *regular* if $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$ exists (we allow divergence to either $-\infty$ or $+\infty$), and the set of regular points is denoted by $\mathrm{Reg}(X, T, f)$

Define the *maximum time average* $\beta(f)$ by

$$\beta(f) = \sup_{x \in \mathrm{Reg}(X,T,f)} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) .$$

The function f being continuous is Borel measurable, so for any Borel measure μ the integral $\int f\,d\mu \in [-\infty,\infty]$ is defined provided $\int f^+\,d\mu$ and $\int f^-\,d\mu$ are not both infinite, where $f^\pm := \max(\pm f, 0)$. Let $\mathcal{M}_f := \{\mu \in \mathcal{M} : \int f\,d\mu \text{ is defined}\}$, and define the *maximum space average* $\alpha(f)$ to be

$$\alpha(f) = \sup_{m\in\mathcal{M}_f} \int f\,dm\,.$$

Remark 2.2.
(a) If f is bounded either above or below then $\mathcal{M}_f = \mathcal{M}$.
(b) If X is compact then $\mathcal{M}_f = \mathcal{M}$ for all continuous functions f.
(c) If $\mathcal{M}_f$ is empty then $\alpha(f) = -\infty$. In particular, if $\mathcal{M}$ is empty then $\alpha(f) = -\infty$ for all continuous functions f.
(d) If $\mathcal{M}$ is non-empty, and $f \in L^1(\mu)$ for some $\mu \in \mathcal{M}$, then $\alpha(f) > -\infty$.
(e) If f is bounded above then $\alpha(f) < \infty$. For our main results (see §5) we always assume f to be bounded above.
(f) If f, g are continuous, and $f - g$ is bounded, then $\mathcal{M}_f = \mathcal{M}_g$.

Definition 2.3. (Maximizing orbits and measures)
Any $x \in X$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \gamma(f)$$

is called *limsup maximizing* for the function f. Its T-orbit $\mathcal{O}(x) = \{x, Tx, T^2x, \ldots\}$ is also called *limsup maximizing* for f.

If $x \in \mathrm{Reg}(X,T,f)$ satisfies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \gamma(f)\,,$$

we say that x, and its orbit $\mathcal{O}(x)$, are *f-maximizing.*

If there exists $\mu \in \mathcal{M}_f$ such that

$$\int f\,d\mu = \gamma(f)\,,$$

then μ is called an *f-maximizing measure*, or simply a *maximizing measure.* Let $\mathcal{M}_{\max}(f)$ denote the set of f-maximizing measures.

The above definition is formulated in terms of the maximum limsup time average $\gamma(f)$, though this convention is somewhat arbitrary; we might equally well have used $\alpha(f)$ or $\beta(f)$. In the case where X is a compact metrizable space this does not matter, since by the following Proposition 2.4 (ii) the three maximum ergodic averages coincide.

More generally if X is a Polish space, i.e. if it is separable and can be metrized by means of a complete metric[1], then $\gamma(f) \geq \beta(f) \geq \alpha(f)$, by Proposition 2.4 (i), which is some justification for our choice in Definition 2.3. In fact our main results in this paper, in sections 4 and 5, concern functions f for which $\alpha(f), \beta(f), \gamma(f)$ all coincide with a fourth quantity $c(f) = \limsup_n \sup_x \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ introduced in section 3, so for these results any arbitrariness in Definition 2.3 is immaterial.

Proposition 2.4. *Let X be a Polish space. Suppose that $T : X \to X$ and $f : X \to \mathbb{R}$ are both continuous. Then*
(i) (General case)

$$\alpha(f) \leq \beta(f) \leq \gamma(f)\,.$$

(ii) (Compact case)
If furthermore X is compact then $\alpha(f) = \beta(f) = \gamma(f) \neq \pm\infty$, and $\mathcal{M}_{\max}(f)$ is non-empty.

Proof. (i) Suppose that $\alpha(f) > \beta(f)$. Then there exists a measure $\mu \in \mathcal{M}_f$ for which $\int f\, d\mu > \beta(f)$. In particular $\int f\, d\mu > -\infty$. Since X is a Polish space then the triple $(X, \mathcal{B}, \mu)$ is a Lebesgue space, where $\mathcal{B}$ is the completion of the Borel σ-algebra by μ ([Ro], p. 174). So $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure-preserving automorphism of a Lebesgue space, and consequently admits an ergodic decomposition ([Ro] pp. 178, 194, [Wa2] p. 34): there is a Borel probability measure P_μ on the set $\mathcal{E} \subset \mathcal{M}$ of T-ergodic measures, such that if $g \in L^1(\mu)$ then $g \in L^1(m)$ for P_μ almost every $m \in \mathcal{E}$, and

$$\int g\, d\mu = \int_{m\in\mathcal{E}} \int g\, dm\, dP_\mu(m)\,.$$

If $f \in L^1(\mu)$ this gives $\int f\, d\mu = \int_{m\in\mathcal{E}} \int f\, dm\, dP_\mu(m)$, where $f \in L^1(m)$ for P_μ-a.e. $m \in \mathcal{E}$. So there exists an ergodic measure μ' such that $\int f\, d\mu' \geq \int f\, d\mu > \beta(f)$ and $f \in L^1(\mu')$ (so in particular $\mu' \in \mathcal{M}_f$). By the ergodic theorem we know that μ'-almost every x satisfies $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \int f\, d\mu'$. In particular there is at least one $x \in \mathrm{Reg}(X, T, f)$ for which $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \int f\, d\mu' > \beta(f)$, contradicting the definition of $\beta(f)$. So in the case where $f \in L^1(\mu)$ we in fact have $\alpha(f) \leq \beta(f)$.

Next suppose $f \notin L^1(\mu)$. The assumption that $\alpha(f) > \beta(f)$ then implies that $\int f\, d\mu = +\infty$ and $\beta(f) < \infty$. The function $f_k(x) :=$

[1]The assumption that X be Polish is preferable to assuming it to be a complete separable metric space, since a Polish space need not have a particularly natural or simple complete metric; eg. $X = [0, 1)$ is Polish but the usual metric $d(x, y) = |x-y|$ is not complete.

$\min(k, f(x))$ is in $L^1(\mu)$, and as above there exists an ergodic measure $\mu_k \in \mathcal{M}_{f_k}$ with $\int f_k \, d\mu_k \geq \int f_k \, d\mu$. Note that each $\mu_k \in \mathcal{M}_f$, with $\int f \, d\mu_k > -\infty$, because $\mu_k \in \mathcal{M}_{f_k}$ and $f^- = f_k^-$.

By the ergodic theorem there exists $x_k \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f_j(T^i x_k) = \int f_j \, d\mu_k \quad \text{for all } j \in \mathbb{N} \,. \tag{1}$$

We claim that x_k can be chosen with the additional property that $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x_k) = \int f \, d\mu_k$, so that in particular $x_k \in \mathrm{Reg}(X, T, f)$. In the case where $f \in L^1(\mu_k)$ this is true by the ergodic theorem. In the case where $\int f \, d\mu_k = \infty$, note that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x_k) &\geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f_j(T^i x_k) \\ &= \int f_j \, d\mu_k \end{aligned} \tag{2}$$

holds for all j, hence the lefthand side of (2) equals $+\infty$. That is, $\frac{1}{n} \sum_{i=0}^{n-1} f(T^i x_k) \to \infty$ as $n \to \infty$, and therefore each $x_k \in \mathrm{Reg}(X, T, f)$.

So in both cases, since $f \geq f_k$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x_k) \geq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f_k(T^i x_k) = \int f_k \, d\mu_k \geq \int f_k \, d\mu \,.$$

Therefore

$$\begin{aligned}\beta(f) &= \sup_{x \in \mathrm{Reg}(X,T,f)} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \\ &\geq \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x_k) \geq \lim_{k\to\infty} \int f_k \, d\mu = \infty \,,\end{aligned}$$

contradicting the fact that $\beta(f) < \infty$. This contradiction means that in the case where $f \notin L^1(\mu)$ we again have $\alpha(f) \leq \beta(f)$.

It is immediate from their definitions that $\beta(f) \leq \gamma(f)$, so part (i) is proved.

(ii) To prove $\alpha(f) = \beta(f) = \gamma(f)$ it suffices, by (i), to show that $\alpha(f) \geq \gamma(f)$. This is the content of Lemmas 2.3 and 2.4 in [YH], but for completeness we give the argument here. The compactness of X means that $\mathcal{M}$ is compact for the weak* topology ([Wa2], Thm. 6.10). Consequently for every $x \in X$ there exists $\mu_x \in \mathcal{M}_f = \mathcal{M}$ such that $\gamma(f, x) = \int f \, d\mu_x$. This is because there is an increasing sequence (n_i) such that $\gamma(f, x) = \lim_i \frac{1}{n_i} \sum_{j=0}^{n_i - 1} f(T^j x) = \lim_i \int f \, d\mu_i$, where

$\mu_i = \frac{1}{n_i}\sum_{j=0}^{n_i-1}\delta_{T^j x}$, but (μ_i) converges weak* subsequentially to some $\mu_x \in \mathcal{M}$, so $\lim_i \int f\,d\mu_i = \int f\,d\mu_x$. Now $\gamma(f) = \sup_{x\in X}\gamma(f,x) = \sup_{x\in X}\int f\,d\mu_x = \lim_k \int f\,d\mu_{x_k}$ for some sequence $x_k \in X$, so if $\mu \in \mathcal{M}$ is any weak* accumulation point of (μ_{x_k}) then $\gamma(f) = \int f\,d\mu \le \alpha(f)$, as required.

The common maximum ergodic average is finite because f is bounded, and $\mathcal{M}_{\max}(f) \neq \emptyset$ because the map $\mu \mapsto \int f\,d\mu$ is continuous for the weak* topology. □

Remark 2.5.
If f has a maximizing measure then it also has a maximizing orbit, and hence a limsup maximizing orbit, by an argument similar to the proof of Proposition 2.4 (i). If f has a *compactly supported* maximizing measure μ then $\gamma(f) \neq \pm\infty$, since $\int f\,d\mu \neq \pm\infty$ by continuity of f. If a limsup maximizing orbit has compact closure then there exists a (compactly supported) maximizing measure, and hence a maximizing orbit.

If X is compact then Proposition 2.4 (ii) means the study of (limsup) maximizing orbits reduces to, and is more elegantly formulated as, the study of maximizing measures. If X is non-compact then this is not the case; the following examples illustrate that $\alpha(f)$, $\beta(f)$, $\gamma(f)$ need not coincide nor be finite, and (limsup) maximizing orbits or maximizing measures need not exist.

Example 2.6. (f has no limsup maximizing orbit)

If T is the identity map on $[0,1)$ and f is strictly increasing, say, then there are no limsup maximizing orbits; $\gamma(f) = \infty$ if and only if f is unbounded.

Example 2.7. ($\alpha(f) < \beta(f)$; f has a maximizing orbit, $\mathcal{M}_{\max}(f) = \emptyset$)

Let $T : \mathbb{R} \to \mathbb{R}$ be the translation $T(x) = x+1$, and $f : \mathbb{R} \to \mathbb{R}$ any function which only takes negative values and has $\lim_{x\to\infty} f(x) = 0$. Clearly $\beta(f) = \gamma(f) = 0$, while $\alpha(f) = -\infty$ because $\mathcal{M}$ is empty.

If T is modified so as to create some recurrence then $\mathcal{M}$ will be non-empty yet $\alpha(f) < \beta(f)$ may still occur. If $T(x) = x/2$ on $[-2,0]$, $T(x) = 2x$ on $[0,1]$, and $T(x) = x+1$ elsewhere then $\mathcal{M}$ contains only the Dirac measure at 0, but if f is as above then $\alpha(f) = \int f\,d\delta_0 = f(0) < 0 = \beta(f) = \gamma(f)$.

Example 2.8. ($\beta(f) < \gamma(f)$; f has a limsup maximizing orbit, but no maximizing orbit)

Equip $X = \mathbb{Z}_2 \times \mathbb{N}$ with the discrete topology, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Let $T((i,n)) = (i+1, n+1)$ if $n = 2^k$ and $T((i,n)) = (i, n+1)$ otherwise.

If $f((i,n)) = i$ for each $i \in \mathbb{Z}_2$ then $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \frac{1}{3}$ and $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \frac{2}{3}$ for all $x \in X$. Therefore $\gamma(f) = 2/3$, and every orbit is limsup maximizing. However $\mathrm{Reg}(X,T,f)$ is empty, so there are no maximizing orbits.

Example 2.9. $(-\infty < \alpha(f) < \beta(f) < \gamma(f) < \infty)$

Example 2.8 can easily be modified so as to obtain $-\infty < \alpha(f) < \beta(f) < \gamma(f) < \infty$. Let X be the set of non-negative integers, with $T(0) = 0$ and $T(n) = n+2$ otherwise. Then $f : X \to \mathbb{R}$ can be chosen to take only the values 0 or 1, with $f(0) = 0$, $\liminf_n \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(2)) = 0$, $\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(2)) = 1$, and the limit $\lim \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(1)) = 1/2$ existing. So $\alpha(f) = 0$, $\beta(f) = 1/2$, and $\gamma(f) = 1$.

Example 2.10. $(\alpha(f) = \beta(f) = \gamma(f) = -\infty)$

If $T(x) = x + 1$ on $\mathbb{R}$ and $\lim_{x\to\infty} f(x) = -\infty$ then every orbit is maximizing, with $\alpha(f) = \beta(f) = \gamma(f) = -\infty$.

3. Normal forms

For a non-empty set X, let $B(X)$ denote the set of all bounded real-valued functions on X, and $B^\wedge(X)$ the set of functions on X which are bounded above.

If X is a topological space, let $C(X)$ denote the space of continuous real-valued functions on X. The topology of uniform convergence on compact subsets makes this a complete topological vector space, though if X is non-compact then $C(X)$ is usually not metrizable.[2] For $\varphi \in C(X)$, and $K \subset X$ a compact subset, we shall write $||\varphi||_{\infty,K} := \sup_{x\in K} |\varphi(x)|$.

The space $CB(X)$ of bounded continuous functions will be equipped with the uniform metric $d(\varphi,\psi) = \sup_{x\in X} |(\varphi - \psi)(x)|$, which makes it a Banach space. Note that (for non-compact X) this is not the topology induced by $C(X)$; in fact $CB(X)$ is a dense subspace of $C(X)$. Let $CB^\wedge(X)$ denote the set of continuous functions on X which are bounded above.

If the topological space X is equipped with a compatible uniform structure, in particular if it is equipped with a metric which generates the topology, then we let $UC(X)$ denote the space of all uniformly continuous functions on X. Let $UCB(X)$ denote the space of all bounded uniformly continuous functions. This is a closed subspace of $CB(X)$, so is itself a Banach space when equipped with the uniform norm. Let $UCB^\wedge(X)$ denote the set of uniformly continuous functions on X

[2] If X is σ-compact then $C(X)$ is metrizable though non-normable; it is a Fréchet space.

which are bounded above. Of course if X is compact then the sets $C(X)$, $CB(X)$, $CB^\wedge(X)$, $UC(X)$, $UCB(X)$, $UCB^\wedge(X)$ all coincide.

Definition 3.1. (Dynamical cohomology and normal forms)
Let $T : X \to X$ be a continuous map on a topological space. If $\varphi \in CB(X)$ then a function of the form $\varphi - \varphi \circ T$ is called a *(continuous) coboundary*. Two functions f, g which differ by a coboundary are *cohomologous*, and we write $f \sim g$. This is an equivalence relation on $C(X)$, say; the corresponding equivalence classes are called *cohomology classes*.

A function $\tilde{f} \sim f$ is a *weak normal form* for f if $\tilde{f} \leq \gamma(f)$. It is a *strong normal form* for f if in addition $\tilde{f}^{-1}(\gamma(f))$ contains the support of some T-invariant probability measure. (Recall that the *support* of a measure μ, denoted $\mathrm{supp}(\mu)$, is by definition the smallest closed subset $Y \subset X$ with $\mu(Y) = 1$). Obviously if f has a strong normal form then $\gamma(f)$ is finite.

The set $\mathcal{M}_f$ only depends on the cohomology class of f, by Remark 2.2 (f), since if $\tilde{f}$ is cohomologous to f then $f - \tilde{f}$ is bounded. Ergodic averages are also well-defined on cohomology classes: $\int f \, d\mu$ (for $\mu \in \mathcal{M}_f$) and $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$ (if it exists) and $\overline{\lim}_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$ do not depend on the cohomology class representative f. Consequently $\alpha(f)$, $\beta(f)$ and $\gamma(f)$ are well-defined on cohomology classes, and (limsup) maximizing orbits and maximizing measures are the same for all functions in a given cohomology class. If the cohomology class has a strong normal form, the following result implies that a maximizing measure is completely characterised by its *support*.

Proposition 3.2. *Suppose $T : X \to X$ is a continuous map on the topological space X, and the continuous function $f : X \to \mathbb{R}$ has a strong normal form $\tilde{f}$. Then $\mathcal{M}_f = \mathcal{M}$, and*

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset \, .$$

Proof. The strong normal form $\tilde{f}$ is bounded above, hence $\mathcal{M}_{\tilde{f}} = \mathcal{M}$ (cf. Remark 2.2 (a)). But $f \sim \tilde{f}$, so $\mathcal{M}_f = \mathcal{M}_{\tilde{f}} = \mathcal{M}$.

Now $\tilde{f} \leq \gamma(f)$, so $\int f \, dm = \int \tilde{f} \, dm \leq \gamma(f)$ for all $m \in \mathcal{M}$. If $\mu \in \mathcal{M}$ satisfies $\mathrm{supp}(\mu) \subset \tilde{f}^{-1}(\gamma(f))$ then $\int f \, d\mu = \int \tilde{f} \, d\mu = \gamma(f)$, so μ is f-maximizing. But $\tilde{f}$ is a strong normal form, so there exists at least one such μ, hence there exists at least one f-maximizing measure.

If $m \in \mathcal{M}$ is such that $\mathrm{supp}(m) \not\subset \tilde{f}^{-1}(\gamma(f))$, then in fact $\int f \, dm = \int \tilde{f} \, dm < \gamma(f)$, because $\tilde{f} \leq \gamma(f)$ and $m(\{x : \tilde{f}(x) < \gamma(f)\}) > 0$, so m is not f-maximizing. $\square$

Proposition 3.2 clearly demonstrates the usefulness of finding strong normal forms, as they essentially resolve the ergodic optimization problem: maximizing measures are identified as those invariant probability measures whose support lies inside the set of maxima of the normal form. In particular the existence of a strong normal form guarantees the existence of a maximizing measure. Not every continuous function has a strong normal form, even in the case where X is compact; indeed the absence of strong normal forms is in a sense typical (cf. Remarque 7 in [B2], §3 in [BJ]). However in many interesting cases f does have a strong normal form: Bousch [B1], and independently Contreras, Lopes & Thieullen [CLT], proved this for Lipschitz functions f and expanding maps T on the circle. Jenkinson [J3] established the analogous fact for functions of summable variation defined on finite alphabet subshifts of finite type, and Bousch [B2] further extended the result to functions f satisfying Walters' condition (cf. [Wa1]) and T either weakly expanding or with weak local product structure. An unpublished manuscript of Conze & Guivarc'h [CG] seems to be the earliest treatment of normal forms for ergodic optimization, containing in particular a version of our Proposition 3.2. In all of the above articles the space X is assumed to be compact, whereas in this paper we seek sufficient conditions for f to have a strong normal form in the case where X is *not necessarily compact*.

For compact X, every weak normal form $\tilde{f}$ is actually a strong normal form: the existence of an f-maximizing measure μ (cf. Proposition 2.4 (ii)) forces $\mathrm{supp}(\mu) \subset \tilde{f}^{-1}(\gamma(f))$, since if not then $\int f\, d\mu = \int \tilde{f}\, d\mu$ is *strictly* less than $\gamma(f)$. In the non-compact case, a weak normal form need not be a strong normal form (see Example 5.3), so that weak normal forms are less useful for our purposes.

One approach to proving the existence of a strong normal form for f is to search for fixed points of a certain nonlinear operator M_f, which we now define. This operator was introduced by Bousch [B1] to study maximizing measures in the case where X is compact. In the context of Lagrangian flows an analogous construction is the Lax-Oleinik semigroup of operators, as studied by Fathi [F1]. Fixed points of M_f can be used to construct a different kind of normal form for f, the *essentially fixed point normal form* (see Definition 3.5).

Definition 3.3. Let $T : X \to X$ be a surjection on a non-empty set X, and $f : X \to I\!R$ any function. If $\varphi : X \to I\!R$ then for each $x \in X$, define $M_f\varphi(x) \in (-\infty, \infty]$ by

$$M_f\varphi(x) := \sup_{y \in T^{-1}x} (f + \varphi)(y). \tag{3}$$

If f is bounded above then (3) defines an operator $M_f : B^\wedge(X) \to B^\wedge(X)$, while if f is bounded then it defines an operator $M_f : B(X) \to B(X)$. Iterates of M_f can be expressed as

$$M_f^n \varphi(x) = \sup_{y \in T^{-n}(x)} \left(S_n f + \varphi\right)(y)\,,$$

where

$$S_n f := \sum_{i=0}^{n-1} f \circ T^i\,.$$

Define

$$c(f) = \limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} S_n f(x) \in [-\infty, \infty]\,.$$

If $c(f) \in [-\infty, \infty)$ then $\sup_{x\in X} S_n f(x)$ is finite for all sufficiently large n, and is a subadditive sequence of reals, so in fact the limit $\lim_n \frac{1}{n} \sup_{x\in X} S_n f(x) \in [-\infty, \infty)$ exists and equals $\inf_n \frac{1}{n} \sup_{x\in X} S_n f(x)$. Clearly $c(f) \geq \gamma(f) = \sup_{x\in X} \limsup_n \frac{1}{n} S_n f(x)$.

If X is a compact metrizable space then $c(f) = \gamma(f)$, since if $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x_n}$, where x_n is such that

$$\sup_{x\in X} \frac{1}{n} S_n f(x) = \frac{1}{n} S_n f(x_n) = \int f\, d\mu_n\,,$$

then the sequence (μ_n) has a weak* accumulation point μ, with $\int f\,d\mu = c(f)$. But μ is a T-invariant probability measure, so $\int f\,d\mu \leq \alpha(f) = \gamma(f)$. If X is not compact then it is possible that $c(f) > \gamma(f)$; this arises if each point has a preimage where f is large, but is itself homoclinic to a part of X where f is small (cf. Example 5.4).

Lemma 3.4. *Let $T : X \to X$ be a surjection on a non-empty set X, and $f : X \to \mathbb{R}$ any function. If there exists $\varphi \in B(X)$ and $c \in \mathbb{R}$ such that*

$$M_f \varphi = \varphi + c\,,$$

then

$$c = c(f) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} S_n f(x)\,.$$

Proof. The equation $\varphi + c = M_f \varphi$ is equivalent to $M_{f-c}\varphi = \varphi$, which implies that

$$\varphi(x) = M_{f-c}^n \varphi(x) = -nc + \sup_{y\in T^{-n}(x)} \left(S_n f(y) + \varphi(y)\right)$$

for all $n \in \mathbb{N}$, $x \in X$. Now φ is bounded, and writing $a = \inf \varphi$, $b = \sup \varphi$ we have

$$\frac{a-b}{n} + c \ \leq \ \frac{1}{n} \sup_{y \in T^{-n}(x)} S_n f(y) \ \leq \ c + \frac{b-a}{n}$$

for all $n > 0$, $x \in X$. Therefore for all $n > 0$,

$$\frac{a-b}{n} + c \ \leq \ \frac{1}{n} \sup_{x \in X} \sup_{y \in T^{-n}(x)} S_n f(y) \ \leq \ c + \frac{b-a}{n} \, ,$$

which is equivalent to

$$\frac{a-b}{n} + c \ \leq \ \frac{1}{n} \sup_{y \in X} S_n f(y) \ \leq \ c + \frac{b-a}{n} \quad \forall n > 0 \, .$$

Letting $n \to \infty$ gives the result. □

Definition 3.5. (Essentially fixed point normal form)
Let X be a topological space. Suppose that $T : X \to X$ is a continuous surjection, and $f : X \to \mathbb{R}$ is continuous. A function $\varphi \in CB(X)$ which satisfies $M_f \varphi = \varphi + c(f)$ is called an *essentially fixed point* of M_f. The function $\tilde{f} := f + \varphi - \varphi \circ T$ is then called an *essentially fixed point normal form* for f.

If X is compact then every essentially fixed point normal form is a strong normal form (but not conversely); indeed the strategy of Bousch [B1, B2], and in a different context Fathi [F1], for establishing existence of strong normal forms was to show that M_f has an essentially fixed point $\varphi = M_f \varphi - c(f)$, where compactness of X means that necessarily $c(f) = \alpha(f) = \beta(f) = \gamma(f)$. For non-compact X, an essentially fixed point normal form need not in general even be a weak normal form (cf. Example 5.4). In Theorem 5.14 we prove that for *suitable* functions f over countable state subshifts of finite type, an essentially fixed point normal form does exist and is also a strong normal form. In the following §4 we introduce the intermediate concept of *essential compactness* as a sufficient condition for the existence of a strong normal form.

4. Essentially compact functions

The importance of a strong normal form for a function was demonstrated by Proposition 3.2: existence of a strong normal form implies existence of a maximizing measure, and gives a characterisation of maximizing measures in terms of their support. In this section we seek sufficient conditions for a function to have a strong normal form, in the general context of Polish spaces X. In §5 this question is pursued in the special case of countable state subshifts of finite type.

Definition 4.1. (Essentially compact functions)

Let $T : X \to X$ be a continuous surjection on the topological space X. A continuous function $f : X \to I\!R$ is *essentially compact* if there is an essentially fixed point $\varphi \in CB(X)$ for M_f, and a subset $Y \subset X$ such that

(a) $\widetilde{Y} := \cap_{n=0}^{\infty} T^{-n} Y$ is non-empty and compact,

(b) $T(Y) = X$,

(c) for each $x \in X$,

$$\varphi(x) + c(f) = \sup_{y \in T^{-1}(x) \cap Y} (f + \varphi)(y)\,. \tag{4}$$

Remark 4.2. It is often straightforward to find a subset Y satisfying conditions (a) and (b). For example if X is a suitable countable state subshift of finite type then Y can be chosen to be an appropriate finite union of length-one cylinders (see §5, in particular Remark 5.10 (d)). For suitable subshifts of finite type the class of essentially compact functions is a large one; for example it contains all functions to which the thermodynamic formalism applies (cf. Corollaries 5.17 and 5.18).

By definition every essentially compact function has an essentially fixed point normal form. The following result, a non-compact generalization of Lemme B in [B1], tells us more.

Theorem 4.3. *Let $T : X \to X$ be a continuous surjection on a Polish space X. If the continuous function $f : X \to I\!R$ is essentially compact, and $\varphi \in CB(X)$ is as in Definition 4.1, then $\tilde{f} = f + \varphi - \varphi \circ T$ is a strong normal form for f, and hence*

$$\mathcal{M}_{max}(f) \;=\; \Big\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \Big\} \;\neq\; \emptyset\,.$$

Proof. It suffices to show that $\tilde{f}$ is a strong normal form; the characterization of $\mathcal{M}_{\max}(f)$ then follows by Proposition 3.2. Now $c(f) \geq \gamma(f)$, so in particular $\gamma(f) < \infty$. To see that $\tilde{f}$ is a strong normal form for f we shall show that $\tilde{f} \leq \gamma(f)$, and that $\tilde{f}^{-1}(\gamma(f))$ contains a non-empty compact T-invariant set Z (i.e. $TZ \subset Z$), from which it follows that $\tilde{f}^{-1}(\gamma(f))$ contains the support of some T-invariant measure, by the Krylov-Bogolioubov Theorem ([Wa2], Cor. 6.9.1).

Let $x \in X$ be arbitrary. We can replace x by $T(x)$ in (4) to obtain

$$\begin{aligned} \varphi(Tx) + c(f) &= \sup_{y \in Y \cap T^{-1}(T(x))} (f + \varphi)(y) \\ &= \varphi(x) + f(x) + r_{\varphi}(x) \end{aligned} \tag{5}$$

where

$$r_\varphi(x) := \sup_{y\in Y\cap T^{-1}(T(x))} (f+\varphi)(y) - (f+\varphi)(x).$$

The function $r_\varphi = \varphi\circ T - \varphi + c(f) - f$ is continuous, since f, φ, and T are. Also, since φ is an essentially fixed point, $r_\varphi(x) = \sup_{y\in Y\cap T^{-1}(T(x))}(f+\varphi)(y) - (f+\varphi)(x) = \sup_{y\in T^{-1}(T(x))}(f+\varphi)(y) - (f+\varphi)(x) \geq (f+\varphi)(x) - (f+\varphi)(x) = 0$, therefore r_φ is non-negative. We shall see later that in fact $r_\varphi^{-1}(0) = \tilde f^{-1}(\gamma(f))$.

Let S denote the restriction of the map T to the nonempty compact invariant set $\widetilde Y = \cap_{n=0}^\infty T^{-n}(Y)$. We first claim that

$$Z := \bigcap_{n=0}^\infty S^{-n}(r_\varphi^{-1}(0)\cap \widetilde Y)$$

is a non-empty compact T-invariant set contained in the zero set $r_\varphi^{-1}(0)$. The fact that $Z\subset r_\varphi^{-1}(0)$ is clear from the definition, as is T-invariance, because $S = T|_{\widetilde Y}$, so we concentrate on showing that Z is non-empty and compact. We can write

$$Z = \bigcap_{N=0}^\infty Z_N, \quad \text{where } Z_N := \bigcap_{n=0}^{N-1} S^{-n}(r_\varphi^{-1}(0)\cap\widetilde Y).$$

Now $Z_1\supset Z_2\supset\ldots$, so that if each Z_N is non-empty and compact then the same will be true of Z. To prove the compactness of Z_N, note that $r_\varphi^{-1}(0)$ is closed, since r_φ is continuous, and $\widetilde Y$ is compact. So $r_\varphi^{-1}(0)\cap\widetilde Y$ is compact, and in particular closed. Hence each $S^{-n}(r_\varphi^{-1}(0)\cap\widetilde Y)$ is closed, since S^n is continuous. But $S = T|_{\widetilde Y}$, so each $S^{-n}(r_\varphi^{-1}(0)\cap\widetilde Y)$ is a subset of the compact set $\widetilde Y$, and therefore itself compact. Consequently the intersection $Z_N = \cap_{n=0}^{N-1} S^{-n}(r_\varphi^{-1}(0)\cap\widetilde Y)$ is also compact.

Now we show that each Z_N is non-empty. Let z be any point in $\widetilde Y = \cap_{n=0}^\infty T^{-n}Y$. Then $T^N(z)\in\widetilde Y$ as well. Now in general if $A\subset\widetilde Y$ then $T^{-1}A\cap Y\subset\widetilde Y$, so that $T^{-1}A\cap Y = T^{-1}A\cap\widetilde Y$. In particular $T^{-1}(T^N(z))\cap Y = T^{-1}(T^N(z))\cap\widetilde Y$, so is compact, and also non-empty by Definition 4.1 (b), so there exists $x_{N-1}\in T^{-1}(T^N(z))\cap Y$ such that

$$(f+\varphi)(x_{N-1}) = \max_{y\in T^{-1}(T^N(z))\cap Y}(f+\varphi)(y).$$

But $T^N(z) = T(x_{N-1})$, so x_{N-1} satisfies

$$(f+\varphi)(x_{N-1}) = \max_{y\in T^{-1}(T(x_{N-1}))\cap Y}(f+\varphi)(y).$$

In other words,

$$r_\varphi(x_{N-1}) = 0\,.$$

But $x_{N-1} \in T^{-1}(T^N(z)) \cap Y \subset \widetilde{Y}$ so in fact

$$x_{N-1} \in r_\varphi^{-1}(0) \cap \widetilde{Y}\,.$$

Now $T^{-1}(x_{N-1}) \cap Y = T^{-1}(x_{N-1}) \cap \widetilde{Y}$ is compact and non-empty, so there exists $x_{N-2} \in T^{-1}(x_{N-1}) \cap Y \subset \widetilde{Y}$ such that

$$(f+\varphi)(x_{N-2}) = \max_{y \in T^{-1}(x_{N-1}) \cap Y} (f+\varphi)(y).$$

But $x_{N-1} = T(x_{N-2})$, so x_{N-2} satisfies

$$(f+\varphi)(x_{N-2}) = \max_{y \in T^{-1}(T(x_{N-2})) \cap Y} (f+\varphi)(y).$$

That is,

$$r_\varphi(x_{N-2}) = 0\,.$$

So we have that

$$x_{N-2} \in r_\varphi^{-1}(0) \cap \widetilde{Y} \quad , \quad \text{and} \ \ T(x_{N-2}) = x_{N-1}\,.$$

Continuing in this way we find a finite sequence $x_0, x_1, \ldots, x_{N-1}$ of points with the property that

$$x_n \in r_\varphi^{-1}(0) \cap \widetilde{Y} \quad , \quad \text{and} \ \ x_n = T^n(x_0) = S^n(x_0)$$

for all $0 \le n \le N-1$. Therefore $T^n(x_0) \in r_\varphi^{-1}(0) \cap \widetilde{Y}$ for all $0 \le n \le N-1$. So $Z_N = \cap_{n=0}^{N-1} S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$ contains the point x_0, and in particular is non-empty, as required. It follows that $Z := \cap_{n=0}^{\infty} S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$ is a non-empty compact T-invariant subset of $r_\varphi^{-1}(0)$, by the argument given above.

But any non-empty compact T-invariant set has a T-invariant probability measure. That is, there exists a measure $m \in \mathcal{M}$ with

$$\operatorname{supp}(m) \subset Z \subset r_\varphi^{-1}(0) \cap \widetilde{Y}. \tag{6}$$

If we integrate equation (5) with respect to any invariant measure $\mu \in \mathcal{M}_f$ we obtain

$$\int f\, d\mu = c(f) - \int r_\varphi\, d\mu \,\le\, c(f)\,,$$

whereas setting $\mu = m$ gives

$$\int f\, dm = c(f)\,.$$

It follows that $c(f) = \alpha(f)$. But $c(f) \ge \gamma(f)$, and $\gamma(f) \ge \alpha(f)$ by Proposition 2.4 (i), so in fact $c(f) = \gamma(f)$. Therefore m is an f-maximizing measure. Moreover, equation (5) now reads

$$\gamma(f) - r_\varphi = \tilde{f}\,, \tag{7}$$

so that $\tilde{f} \leq \gamma(f)$, and $\tilde{f}$ is a weak normal form for f. Moreover (7) gives

$$r_\varphi^{-1}(0) = \tilde{f}^{-1}(\gamma(f)),$$

and combining this with equation (6) shows that $\mathrm{supp}(m) \subset \tilde{f}^{-1}(\gamma(f))$, so that $\tilde{f}$ is indeed a strong normal form for f. $\square$

5. Countable state subshifts of finite type

Definition 5.1. The space $\Sigma = I\!N^{I\!N}$, equipped with the product topology, is the *full shift* on the alphabet[3] $I\!N$. Of course, the map we are considering is the *shift map* $T : \Sigma \to \Sigma$ defined by $(T\omega)_n = \omega_{n+1}$. Given an *adjacency matrix* $A : I\!N \times I\!N \to \{0, 1\}$, the associated *subshift of finite type* Σ_A is the subspace of Σ defined by

$$\Sigma_A = \{\omega \in \Sigma : A(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \geq 1\}$$

and the map we consider is the shift restricted to the invariant set Σ_A. If Σ_A is a subshift of finite type then $T(\Sigma_A) \subset \Sigma_A$, and we again write $T : \Sigma_A \to \Sigma_A$ to denote the corresponding restriction of the shift map. All subshifts of finite type are Polish spaces, and we shall always use the complete metric $\delta(x, y) = 2^{-\min\{n: x_n \neq y_n\}}$.

A finite word $w \in I\!N^n$ is A-*admissible* if $A(w_i, w_{i+1}) = 1$ for all $1 \leq i \leq n - 1$. Given $w \in I\!N^n$ and $\omega \in \Sigma_A$, the *concatenation* $w\omega$ is the sequence defined by

$$(w\omega)_i = \begin{cases} w_i & \text{if } 1 \leq i \leq n \\ \omega_{i-n} & \text{if } i \geq n+1 \end{cases}.$$

In general $T : \Sigma_A \to \Sigma_A$ need not be surjective. Clearly it is surjective if and only if A has the property that for every $\omega \in \Sigma_A$ there is some $i \in I\!N$ such that $A(i, \omega_1) = 1$. The class of matrices A studied in this paper all have this property.

For any $n \in I\!N$ we define $\Pi_{A,n} : \Sigma_A \to I\!N^n$ by $\Pi_{A,n}(\omega) = (\omega_1, \ldots, \omega_n)$ (projection onto the first n coordinates). If $w \in I\!N^n$ then the corresponding *cylinder set* in Σ_A is defined by

$$[w] = [w]_A = \Pi_{A,n}^{-1}(w) = \{x \in \Sigma_A : \Pi_{A,n}(x) = w\}.$$

Define

$$I\!N_A = \{i \in I\!N : [i]_A \neq \emptyset\},$$

the set of those symbols which actually appear as an entry in some element of Σ_A. If $I\!N_A$ is finite then Σ_A is compact, and called a *finite*

[3]Here, as throughout the article, $I\!N$ denotes the set of *strictly* positive integers.

state subshift of finite type. If $I\!N_A$ is infinite then Σ_A is non-compact, and called a *(strictly) countable state subshift of finite type*[4].

Henceforth our Polish space X will be a subshift of finite type Σ_A for which the shift map $T : \Sigma_A \to \Sigma_A$ is surjective.

Lemma 5.2. *Let Σ_A be a subshift of finite type for which the shift map $T : \Sigma_A \to \Sigma_A$ is surjective. Suppose f is uniformly continuous and bounded above (i.e. $f \in UCB^\wedge(\Sigma_A)$). Then the formula*

$$M_f\varphi(x) = \sup_{y \in T^{-1}x} (f + \varphi)(y)$$

defines an operator $M_f : UCB^\wedge(\Sigma_A) \to UCB^\wedge(\Sigma_A)$.

Proof. Let $\varphi \in UCB^\wedge(\Sigma_A)$. Clearly $M_f\varphi$ is bounded above. Since $M_f\varphi = M_0(f + \varphi)$, it suffices to show that if $g \in UCB^\wedge(\Sigma_A)$ then $M_0g \in UC(\Sigma_A)$. Let $\varepsilon > 0$. Choose $n \in I\!N$ such that if $u, v \in \Sigma_A$ satisfy $\Pi_{A,n}(u) = \Pi_{A,n}(v)$, then $|g(u) - g(v)| < \varepsilon/2$. Fix $x, y \in \Sigma_A$ such that $\Pi_{A,n}(x) = \Pi_{A,n}(y)$. Choose $a \in I\!N$ such that ax is A-admissible and $g(ax) \le M_0g(x) < g(ax) + \varepsilon/2$. Since ay is A-admissible, $g(ay) \le M_0(g)(y)$, so $-\varepsilon < g(ay) - g(ax) - \varepsilon/2 < M_0g(y) - M_0g(x)$. Interchanging x and y gives $-\varepsilon < M_0g(x) - M_0g(y)$, so $|M_0g(x) - M_0g(y)| < \varepsilon$ and therefore M_0g is uniformly continuous. □

The main results of this section concern conditions on f which guarantee the existence of an essentially fixed point normal form, which is then shown to be also a strong normal form. To motivate the need for such conditions, consider the following example.

Example 5.3. (essentially fixed point normal form but no strong normal form)

Let $T : \Sigma \to \Sigma$ be the full shift on the alphabet $I\!N$, and let $f : \Sigma \to I\!R$ be constant on length-2 cylinder sets, with $f[m, n] = \frac{-1}{n(n+1)}$ if $m = n + 1$ and $f[m, n] = -1$ otherwise. First we claim that $\alpha(f) = \beta(f) = \gamma(f) = 0$, and that there exist f-maximizing orbits but no f-maximizing measures. Certainly $\gamma(f) \le 0$, since $f < 0$. If ν_n denotes the unique invariant measure supported on the periodic orbit generated by $\omega^{(n)} := \overline{(n, n-1, \ldots, 1)}$, then $\sup_{n \ge 0} \int f \, d\nu_n = 0$, so that $\alpha(f) = 0$. Therefore $\alpha(f) = \beta(f) = \gamma(f) = 0$, by Proposition 2.4 (i). If $\omega = (1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots)$, formed by concatenating, in order of increasing n, all words of the form $(n, n-1, \ldots, 1)$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega) = 0$, so that ω has an f-maximizing orbit. Clearly f has no f-maximizing measures, since $f < 0$ implies that

[4]Alternative terminology for Σ_A is a *countable Markov shift* or *countable state topological Markov chain.*

$\int f\, dm < 0$ for any (invariant) probability measure m. Therefore, by Proposition 3.2, f does not have a strong normal form.

The function f is already in weak normal form: $f \leq \gamma(f)$. To see that f has an essentially fixed point normal form, let $\varphi \in CB(\Sigma)$ be constant on length-1 cylinder sets, defined by $\varphi([n]) = -1/n$ for all $n \in \mathbb{N}$. A short calculation reveals that $f + \varphi - \varphi \circ T = 0 = \gamma(f)$ on cylinder sets of the form $[n+1, n]$, whereas $(f + \varphi - \varphi \circ T)([m, n]) = -1 - \frac{1}{m} + \frac{1}{n} < 0 = \gamma(f)$ if $m \neq n+1$, so φ is an essentially fixed point for M_f, and $c(f) = \gamma(f) = 0$.

The dynamics in Example 5.3 is topologically mixing, and the continuous function f is bounded and locally constant (in particular it has summable variations, cf. Definition 5.7). For *compact* subshifts of finite type these conditions are enough to ensure that f does have a strong normal form (see [B2, J3]). In the non-compact setting, however, an extra condition is required: the values of f "at infinity" should not be too large in comparison to its value on some "finite" part of the space. This condition will be formalised as (13), and in the case of the full shift has a simpler form (22). It fails to be satisfied in Example 5.3 because f is too large on cylinder sets of the form $[n+1, n]$ for arbitrarily high values of n.

The following example is in a sense even more striking than the last one: the function f has an essentially fixed point normal form, but not even a *weak* normal form. However this is simply because the value $c(f) = \limsup_n \frac{1}{n} \sup_x S_n f(x)$ is strictly larger than $\gamma(f) = \sup_x \limsup_n \frac{1}{n} S_n f(x)$, a phenomenon which can only arise if the space is non-compact (cf. the discussion prior to Lemma 3.4). It should be noted that the subshift of finite type below is not topologically mixing (i.e. its adjacency matrix is not primitive, cf. Definition 5.9), whereas for our main results (Lemma 5.11 onwards) we shall always assume topological mixing.

Example 5.4. ($c(f) > \gamma(f)$; essentially fixed point normal form but no weak normal form)

It is notationally convenient to take $\mathbb{Z}$ rather than $\mathbb{N}$ as the alphabet for the following subshift of finite $\Sigma_A \subset \mathbb{Z}^{\mathbb{N}}$. Define the adjacency matrix A by $A(i, i+1) = 1\ \forall i \in \mathbb{Z}$, $A(0, i) = 1\ \forall i \geq 1$, and $A(i, j) = 0$ otherwise. Define f to be constant on cylinder sets of length one, with $f[i]_A = 0$ for $i \leq 0$ and $f[i]_A = -1$ otherwise. Clearly $\lim_n \frac{1}{n} S_n f(x) = -1$ for all $x \in \Sigma_A$, so $\gamma(f) = -1$. But $c(f) = 0$; indeed f is already in essentially fixed point normal form, since $M_f 0 = 0$. However f does not have a weak normal form: if $\tilde{f} \sim f$ with $\tilde{f} \leq \gamma(f) = 0$ then

$c(\tilde{f}) \leq 0$, whereas in fact $c(\tilde{f}) = c(f) = -1$ since it is easily seen that $c(\cdot)$ is well-defined on cohomology classes.

Obviously a prerequisite for proving that $f : \Sigma_A \to \mathbb{R}$ has an essentially fixed point normal form is the existence of an essentially fixed point for the operator M_f. The first step in proving this existence is to show that M_f preserves the Banach space $UCB(\Sigma_A)$ (see Lemma 5.6), thereby improving on Lemma 5.2. For this we need the extra assumption that f is bounded below when restricted to some set Z with the property that $T(Z) = \Sigma_A$. Clearly f satisfies this assumption if it is bounded on the whole of Σ_A. It also satisfies the assumption if there exists a *finite* sub-alphabet $\mathbb{L} \subset \mathbb{N}$ such that every $x \in \Sigma_A$ has a preimage ix for some $i \in \mathbb{L}$; in this case we can define $Z = \Pi_{A,1}^{-1}(\mathbb{L})$. It follows easily (Lemma 5.6) that $M_f : UCB(\Sigma_A) \to UCB(\Sigma_A)$ has *approximate* fixed points: for each $0 \leq \lambda < 1$ the equation $\varphi = M_f(\lambda\varphi)$ has a unique solution in $UCB(\Sigma_A)$, since M_f is 1-Lipschitz for the uniform distance. Later, under extra hypotheses on f and A, it will be shown that any accumulation point (in a suitable topology) of the family of approximate fixed points is in fact an essentially fixed point for M_f. This general strategy is patterned on the proof of Théorème 1 in [B2]. As we shall soon see, however, the non-compactness of Σ_A complicates matters considerably.

The approximate fixed points for M_f are actual fixed points for the following operators:

Definition 5.5. Let $f \in B^\wedge(X)$. For $0 \leq \lambda \leq 1$, define the (nonlinear) operator $M_{f,\lambda} : B^\wedge(\Sigma_A) \to B^\wedge(\Sigma_A)$ by

$$M_{f,\lambda}\varphi(x) = \sup_{y \in T^{-1}(x)} (f + \lambda\varphi)(y)\,,$$

so that $M_{f,1} = M_f$. The iterates of $M_{f,\lambda}$ can be expressed as

$$M_{f,\lambda}^n\varphi(x) = \sup\{S_{\lambda,n}f(w_x x) + \lambda^n\varphi(w_x x) : w_x \in \mathbb{N}^n, w_x x \in \Sigma_A\}\,,$$

where $S_{\lambda,n}f(z) := \sum_{j=0}^{n-1} \lambda^{n-1-j} f(T^j z)$.

Lemma 5.6. *Let Σ_A be a subshift of finite type. Suppose f is uniformly continuous and bounded above, and* $\inf f|_Z > -\infty$ *for a subset Z with $T(Z) = \Sigma_A$.*

For each $0 \leq \lambda \leq 1$, the operator $M_{f,\lambda}$ preserves the Banach space $UCB(\Sigma_A)$.

If $0 \leq \lambda < 1$ then $M_{f,\lambda}$ is a contraction on $UCB(\Sigma_A)$, hence has a unique fixed point $\varphi_\lambda \in UCB(\Sigma_A)$.

Proof. Since $M_{f,\lambda}$ is the composition of M_f and the homothety $\varphi \mapsto \lambda\varphi$, and since $f \in UCB^\wedge(\Sigma_A)$, Lemma 5.2 implies that $M_{f,\lambda}$ preserves the space $UCB^\wedge(\Sigma_A)$. To show that $M_{f,\lambda}$ preserves $UCB(\Sigma_A)$, it remains to check that if $\varphi \in UCB(\Sigma_A)$ then $M_{f,\lambda}\varphi$ is bounded below, but this is the case because $\inf M_{f,\lambda}\varphi \geq \inf f|_Z + \lambda \inf \varphi > -\infty$.

The operator $M_{f,\lambda} : UCB(\Sigma_A) \to UCB(\Sigma_A)$ is λ-Lipschitz with respect to the complete metric $d(\varphi, \psi) = \sup_{x\in\Sigma_A} |(\varphi - \psi)(x)|$, so for $0 \leq \lambda < 1$ it has a unique fixed point $\varphi_\lambda \in UCB(\Sigma_A)$. □

Our aim is to extract an accumulation point from the family of approximate fixed points $(\varphi_\lambda)_{0\leq\lambda<1}$. To this end it will be useful to prove that the family is equicontinuous (Lemma 5.8), and that its global oscillation is bounded independently of λ (Lemma 5.11). These results will be obtained by imposing further control on the modulus of continuity of the function f, something stronger than uniform continuity.

Definition 5.7. For $n \geq 1$ the *n-th variation* of $f : \Sigma_A \to I\!\!R$ is defined by

$$\mathrm{var}_n(f) = \sup_{\Pi_{A,n}(x)=\Pi_{A,n}(y)} \{f(x) - f(y)\}.$$

Note that $f \in UC(\Sigma_A)$ if and only if $\mathrm{var}_n(f) \to 0$ as $n \to \infty$.
We say f has *summable variations* if

$$\sum_{n=1}^{\infty} \mathrm{var}_n(f) < \infty.$$

The 0*-th variation* is defined as $\mathrm{var}_0(f) = \sup_{x,y\in\Sigma_A}\{f(x)-f(y)\}$. Note that it is not included in the above sum, so summable variations does not imply boundedness.

Lemma 5.8. *Let Σ_A be a subshift of finite type. Suppose f is bounded above, with summable variations, and* $\inf f|_Z > -\infty$ *for a subset Z with $T(Z) = \Sigma_A$. Then for all $0 \leq \lambda < 1$, the fixed point $\varphi_\lambda \in UCB(\Sigma_A)$ of $M_{f,\lambda}$ satisfies*

$$var_i(\varphi_\lambda) \leq \sum_{j=i+1}^{\infty} var_j(f) \quad \textit{for all } i \geq 1\,. \tag{8}$$

Proof. Let $\varphi \in UCB(\Sigma_A)$. For $j \geq 1$ we shall consider the j^{th} variation of $M_{f,\lambda}\varphi$. Suppose $x, y \in \Sigma_A$ satisfy $\Pi_{A,j}(x) = \Pi_{A,j}(y)$. For any $\varepsilon > 0$ we can find $i \in I\!\!N$ with $ix \in \Sigma_A$ such that

$$M_{f,\lambda}\varphi(x) < \varepsilon + (f + \lambda\varphi)(ix)\,.$$

On the other hand we have $M_{f,\lambda}\varphi(y) \geq (f + \lambda\varphi)(z)$ for all $z \in T^{-1}(y)$. In particular we may choose $z = iy$: we know that $iy \in \Sigma_A$ because

$ix \in \Sigma_A$ and $\Pi_{A,1}(x) = \Pi_{A,1}(y)$, and Σ_A is a subshift of finite type. Therefore

$$M_{f,\lambda}\varphi(x) - M_{f,\lambda}\varphi(y) < \varepsilon + (f + \lambda\varphi)(ix) - (f + \lambda\varphi)(iy)\,,$$

from which we deduce that

$$\begin{aligned} M_{f,\lambda}\varphi(x) - M_{f,\lambda}\varphi(y) &< \varepsilon + \mathrm{var}_{j+1}(f + \lambda\varphi) \\ &\le \varepsilon + \mathrm{var}_{j+1}(f) + \mathrm{var}_{j+1}(\varphi)\,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we in fact have

$$\mathrm{var}_j(M_{f,\lambda}\varphi) \le \mathrm{var}_{j+1}(f) + \mathrm{var}_{j+1}(\varphi)\,,$$

so in particular

$$\mathrm{var}_j(\varphi_\lambda) \le \mathrm{var}_{j+1}(f) + \mathrm{var}_{j+1}(\varphi_\lambda)\,. \tag{9}$$

Now $\mathrm{var}_{j+1}(\varphi_\lambda) \to 0$ as $j \to \infty$ since φ_λ is uniformly continuous, so iteration of (9) yields

$$\mathrm{var}_i(\varphi_\lambda) \le \sum_{j=i+1}^{\infty} \mathrm{var}_j(f) \quad \text{for all } i \ge 1\,,$$

which is the required inequality (8). □

We require the following assumption on the matrix A in order to control, uniformly in λ, the 0-th variation of the fixed points φ_λ.

Definition 5.9. An adjacency matrix A, and the corresponding subshift of finite type Σ_A, are called *primitive* if there exists an integer $N \ge 0$, and a non-empty subset $\mathbb{M} \subset I\!N$, such that for all $x \in \Sigma_A$ and all $i \in I\!N_A$ there exists $w \in \mathbb{M}^N$ with $iwx \in \Sigma_A$. Any such pair $(N, \mathbb{M})$ is called a *primitive pair* for A; N is a *primitive constant* and $\mathbb{M}$ a *primitive alphabet*. If there exists a finite primitive alphabet $\mathbb{M}$ then we say that A and Σ_A are *finitely primitive*.

Remark 5.10.
(a) If A is primitive then $T : \Sigma_A \to \Sigma_A$ is surjective: for any primitive alphabet $\mathbb{M}$, the set $Z = \Pi_{A,1}^{-1}(\mathbb{M})$ is such that $T(Z) = \Sigma_A$.
(b) Primitivity is equivalent to *topological mixing* of $T : \Sigma_A \to \Sigma_A$ (i.e. for all non-empty open sets $Y, Z \subset \Sigma_A$ there exists $M \in I\!N$ such that $T^{-m}Y \cap Z \ne \emptyset$ for all $m \ge M$). If $I\!N_A = I\!N$ then A is primitive if and only if there exists $n \in I\!N$ such that every entry of the matrix A^n is strictly positive.
(c) Sarig [Sa] defines the big images and preimages property as the existence of a finite $\mathbb{M} \subset I\!N$ such that for all $j \in I\!N$ there exists $i, k \in \mathbb{M}$ with $A(i, j) = 1 = A(j, k)$. If A is primitive and $I\!N_A = I\!N$ then this condition is equivalent to finite primitivity.

(d) Suppose A has the big images and preimages property, with $\mathbb{M} \subset I\!N$ the finite set as in (c) above. If we let $Y = \cup_{i\in\mathbb{M}}[i]_A$, then $T(Y) = \Sigma_A$, and $\widetilde{Y} = \cap_{n=0}^{\infty} T^{-n}(Y)$ is a closed nonempty subset of the compact set $\mathbb{M}^{I\!N}$. Therefore conditions (a) and (b) of Definition 4.1 (defining an essentially compact function) are satisfied.

Lemma 5.11. *Suppose, in addition to the hypotheses of Lemma 5.8, that Σ_A is primitive. Then for all $0 \le \lambda < 1$, the fixed point $\varphi_\lambda \in UCB(\Sigma_A)$ of $M_{f,\lambda}$ satisfies*

$$\mathrm{var}_0(\varphi_\lambda) \le N\left(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})}\right) + \sum_{j=1}^{\infty} \mathrm{var}_j(f) \tag{10}$$

for any primitive pair $(N, \mathbb{M})$ for Σ_A.

Proof. Let $m \ge 0$ and write $n = N + m$. For any $\varphi \in UCB(\Sigma_A)$, $x \in \Sigma_A$, and $\varepsilon > 0$, we can find words $v \in I\!N^N$ and $u \in I\!N^m$ such that $uvx \in \Sigma_A$, and

$$M_{f,\lambda}^n \varphi(x) < S_{\lambda,n} f(uvx) + \lambda^n \varphi(uvx) + \varepsilon\,.$$

For any $y \in \Sigma_A$, primitivity means we can find $w \in \mathbb{M}^N$ such that $uwy \in \Sigma_A$. Clearly

$$M_{f,\lambda}^n \varphi(y) \ge S_{\lambda,n} f(uwy) + \lambda^n \varphi(uwy)\,,$$

and therefore

$$\begin{aligned} &M_{f,\lambda}^n \varphi(x) - M_{f,\lambda}^n \varphi(y) \\ &< S_{\lambda,n} f(uvx) - S_{\lambda,n} f(uwy) + \lambda^n \left[\varphi(uvx) - \varphi(uwy)\right] + \varepsilon\,. \end{aligned} \tag{11}$$

Now

$$\lambda^n \left[\varphi(uvx) - \varphi(uwy)\right] < \mathrm{var}_m(\varphi)\,,$$

and

$$\begin{aligned} S_{\lambda,n} f(uvx) - S_{\lambda,n} f(uwy) &= \sum_{i=0}^{n-1} \lambda^{n-1-i}\left[f(T^i uvx) - f(T^i uwy)\right] \\ &\le \sum_{i=0}^{m-1} \lambda^{n-1-i}\mathrm{var}_{m-i}(f) + \sum_{i=m}^{n-1} \lambda^{n-1-i}(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})}) \\ &\le \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})})\,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary in (11), we deduce

$$M_{f,\lambda}^n \varphi(x) - M_{f,\lambda}^n \varphi(y) \le \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})}) + \mathrm{var}_m(\varphi)\,.$$

In particular, choosing φ to be the fixed point φ_λ of $M_{f,\lambda}$ gives

$$\varphi_\lambda(x) - \varphi_\lambda(y) \le \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})}) + \mathrm{var}_m(\varphi_\lambda)\,.$$

But $\mathrm{var}_m(\varphi_\lambda) \to 0$ as $m \to \infty$, since φ_λ is uniformly continuous, so

$$\mathrm{var}_0(\varphi_\lambda) \le \sum_{j=1}^{\infty} \mathrm{var}_j(f) + N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})})\,,$$

as required. □

The family $(\varphi_\lambda)_{0\le\lambda<1}$ of approximate fixed points need not itself be uniformly bounded. However, if we define $\varphi_\lambda^* := \varphi_\lambda - \inf \varphi_\lambda$ then $\inf \varphi_\lambda^* = 0$ for each λ, and $\mathrm{var}_0(\varphi_\lambda^*) = \mathrm{var}_0(\varphi_\lambda)$, so inequality (10) implies that $(\varphi_\lambda^*)_{0\le\lambda<1}$ *is* bounded independently of λ. It is this family which will provide an accumulation point. The constants $\inf \varphi_\lambda$ are in general non-zero, which is why the accumulation point will be an *essentially* (rather than bona fide) fixed point for M_f.

Although $UCB(\Sigma_A)$ was a convenient space in which to find the approximate fixed points φ_λ, it is not an appropriate space in which to find accumulation points of the associated family $(\varphi_\lambda^*)_{0\le\lambda<1}$, which a priori is not pre-compact in $UCB(\Sigma_A)$ (equipped with the uniform distance). On the other hand $(\varphi_\lambda^*)_{0\le\lambda<1}$ is equicontinuous by (8), and uniformly bounded by the above discussion, so the Ascoli-Arzela Theorem implies it is pre-compact in the space $C(\Sigma_A)$ (equipped with the topology of uniform convergence on compact subsets). So there exists an accumulation point $\varphi_1^* \in C(\Sigma_A)$ as $\lambda \nearrow 1$. Indeed $\varphi_1^* \in UCB(\Sigma_A)$, since from (8), (10) it follows that $\mathrm{var}_i(\varphi_1^*) \le \sum_{j=i+1}^{\infty} \mathrm{var}_j(f) \to 0$ as $i \to \infty$, and

$$\mathrm{var}_0(\varphi_1^*) \le N\left(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})}\right) + \sum_{j=1}^{\infty} \mathrm{var}_j(f) < \infty\,. \tag{12}$$

We would now like to show that φ_1^* is an essentially fixed point of M_f. For $0 \le \lambda < 1$ the fixed point equation $M_{f,\lambda}\varphi_\lambda = \varphi_\lambda$ is equivalent to

$$M_{f,\lambda}\varphi_\lambda^* = \varphi_\lambda^* + (1-\lambda)\inf \varphi_\lambda\,,$$

and we wish to let $\lambda \nearrow 1$ along an appropriate subsequence in order to deduce that $M_f\varphi_1^* = \varphi_1^* + c(f)$. The (subsequential) convergence of $M_{f,\lambda}\varphi_\lambda^*$ to $M_f\varphi_1^*$ is not immediately obvious, however. We would like both $M_f\varphi_1^* - M_f\varphi_\lambda^*$ and $M_f\varphi_\lambda^* - M_{f,\lambda}\varphi_\lambda^*$ to become small in $C(\Sigma_A)$. The smallness of the first term would follow if $M_f : C(\Sigma_A) \to C(\Sigma_A)$ were continuous, and the smallness of the second would follow were it true that for all $\psi \in C(\Sigma_A)$, and all compact $K \subset \Sigma_A$, $\|(M_f - M_{f,\lambda})\psi\|_{\infty,K} \to 0$ as $\lambda \nearrow 1$. Unfortunately, a priori *neither* of these properties holds.[5] If M_f were continuous then for every compact subset $K \subset \Sigma_A$ there would exist another compact subset L such that, for all $\psi \in C(\Sigma_A)$, the restriction $M_f\psi|_K$ could be expressed as a (continuous) function of $\psi|_L$. In general this is not the case, since $M_f\psi(x)$ is defined by taking a supremum over the (non-compact) set $T^{-1}(x)$. For the same reason $\|(M_f - M_{f,\lambda})\psi\|_{\infty,K}$ does not in general converge to zero as $\lambda \nearrow 1$.

We therefore require an additional hypothesis on f, a certain quantitative control on its variations. This will ensure that, for any $0 \le \lambda \le 1$, only *finitely many* preimages $y \in T^{-1}(x)$ can contribute to the supremum defining $M_f\varphi_\lambda^*(x)$. The condition on f is that for some primitive pair $(N, \mathbb{M})$, and some set Z with $T(Z) = \Sigma_A$,

$$N(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})}) + \sum_{j=1}^{\infty} \mathrm{var}_j(f) < \inf f|_Z - \sup f|_{[i]_A} \qquad (13)$$

for all sufficiently large $i \in I\!N$. That is, the values of f on Z are sufficiently larger than its values "at infinity". Of course a necessary condition for (13) to hold is that $\inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})} > -\infty$ and $\inf f|_Z > -\infty$. Note that if Σ_A is the full shift then (13) simplifies: we may take $N = 0$, and choose Z to be any length-one cylinder set $[I]$, so that (13) holds when

$$\sum_{j=1}^{\infty} \mathrm{var}_j(f) < \inf f|_{[I]} - \sup f|_{[i]}$$

for all sufficiently large $i \in I\!N$.

Lemma 5.12. *Suppose, in addition to the hypotheses of Lemma 5.11, that the inequality (13) holds for some primitive pair* $(N, \mathbb{M})$ *and some set* Z *with* $T(Z) = \Sigma_A$. *Then there exists* $J \in I\!N$ *such that for all* $0 \le \lambda, \mu \le 1$,

$$M_{f,\mu}\varphi_\lambda^*(x) = \max_{y \in T^{-1}(x) \cap (\cup_{i=1}^{J}[i]_A)} (f + \mu\varphi_\lambda^*)(y) \quad \textit{for all } x \in \Sigma_A\,. \qquad (14)$$

[5]By contrast in $UCB(\Sigma_A)$ the analogues of both these facts *are* true: M_f is (1-Lipschitz) continuous, and $\|(M_f - M_{f,\lambda})\psi\|_\infty \le (1-\lambda)\|\psi\|_\infty$. However we cannot work in this space, because the (subsequential) convergence $\varphi_\lambda^* \to \varphi_1^*$ is not guaranteed.

Proof. Inequalities (10), (12) and (13) imply that

$$\begin{aligned}\mu\mathrm{var}_0(\varphi_\lambda^*) &= \mu\mathrm{var}_0(\varphi_\lambda) \\ &\le \mathrm{var}_0(\varphi_\lambda) \\ &\le N\left(\sup f - \inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})}\right) + \sum_{j=1}^{\infty}\mathrm{var}_j(f) \\ &< \inf f|_Z - \sup f|_{[i]_A}\end{aligned} \tag{15}$$

for all i sufficiently large, $i > J$ say. In particular

$$\inf f|_Z - \sup f|_{[i]_A} > 0 \quad \text{for all } i > J\,,$$

so if we define $Y := \cup_{i=1}^{J}[i]_A$ then $Z \subset Y$.

Let $x \in \Sigma_A$ be arbitrary, and suppose that $ix \in \Sigma_A$ for some $i > J$. Now x has at least one preimage $jx \in Z \subset Y$, and by (15) we know that

$$\begin{aligned}f(jx) - f(ix) &> \mu\,\mathrm{var}_0(\varphi_\lambda^*) \\ &\ge \mu\left(\varphi_\lambda^*(ix) - \varphi_\lambda^*(jx)\right)\,.\end{aligned}$$

That is,

$$(f + \mu\varphi_\lambda^*)(jx) - (f + \mu\varphi_\lambda^*)(ix) > 0\,,$$

so the supremum $\sup_{y\in T^{-1}(x)}(f + \mu\varphi_\lambda^*)(y) = M_{f,\mu}\varphi_\lambda^*(x)$ must be attained by one of the finitely many preimages of x lying in Y. That is,

$$M_{f,\mu}\varphi_\lambda^*(x) = \sup_{y\in T^{-1}(x)}(f + \mu\varphi_\lambda^*)(y) = \max_{y\in T^{-1}(x)\cap Y}(f + \mu\varphi_\lambda^*)(y)$$

for all $x \in \Sigma_A$, as required. □

Notice that for $\lambda = \mu = 1$ the equation (14), asserting that we need only check finitely many preimages $y \in T^{-1}(x)$ in order to compute $M_f\varphi_1^*(x)$, is reminiscent of the definition of essential compactness. Indeed once we have proved, in Theorem 5.14, that φ_1^* is an essentially fixed point of M_f, (14) will provide the important condition (c) of Definition 4.1 from which we deduce that f is essentially compact.

Before that, (14) is an important ingredient in the proof of the next lemma. Recall that the need for bounds such as the following (18), (19) were our motivation for introducing condition (13).

Lemma 5.13. *Under the same hypotheses as Lemma 5.12, for every compact subset $K \subset \Sigma_A$, and for all $0 \le \lambda, \lambda', \mu, \mu' \le 1$,*

$$||(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*||_{\infty,K} \le |\mu - \mu'|\,||\varphi_\lambda^*||_{\infty,L}\,, \tag{16}$$

and

$$\|M_{f,\mu}\varphi_\lambda^* - M_{f,\mu}\varphi_{\lambda'}^*\|_{\infty,K} \le \mu\|\varphi_\lambda^* - \varphi_{\lambda'}^*\|_{\infty,L}\,, \tag{17}$$

where $L = L(K) := T^{-1}K \cap (\cup_{i=1}^J [i]_A)$ *is compact, and* $J \in I\!N$ *is as in Lemma 5.12.*

In particular,

$$||(M_{f,\lambda} - M_f)\varphi_\lambda^*||_{\infty,K} \le (1-\lambda)||\varphi_\lambda^*||_{\infty,L}\,, \tag{18}$$

and

$$||M_f\varphi_1^* - M_f\varphi_\lambda^*||_{\infty,K} \le ||\varphi_1^* - \varphi_\lambda^*||_{\infty,L}\,. \tag{19}$$

Proof. Let $Y = \cup_{i=1}^J [i]_A$, where $J \in I\!N$ is as in Lemma 5.12. If $K \subset \Sigma_A$ is compact and $x \in K$ then Lemma 5.12 implies we can find $y \in T^{-1}(x) \cap Y \subset T^{-1}K \cap Y =: L$ such that $M_{f,\mu}\varphi_\lambda^*(x) = (f + \mu\varphi_\lambda^*)(y)$. Also $M_{f,\mu'}\varphi_\lambda^*(x) \ge (f + \mu'\varphi_\lambda^*)(y)$, so

$$\begin{aligned}(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*(x) &\le (f + \mu\varphi_\lambda^*)(y) - (f + \mu'\varphi_\lambda^*)(y)\\ &= (\mu - \mu')\varphi_\lambda^*(y)\\ &\le |\mu - \mu'|\,||\varphi_\lambda^*||_{\infty,L}\,.\end{aligned}$$

Reversing the roles of μ, μ', an analogous argument gives

$$(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*(x) \ge -|\mu - \mu'|\,||\varphi_\lambda^*||_{\infty,L}\,,$$

and since $x \in K$ was arbitrary,

$$||(M_{f,\mu} - M_{f,\mu'})\varphi_\lambda^*||_{\infty,K} \le |\mu - \mu'|\,||\varphi_\lambda^*||_{\infty,L}\,,$$

which is the required inequality (16).

Lemma 5.12 also implies that

$$\begin{aligned}M_{f,\mu}\varphi_\lambda^*(x) - M_{f,\mu}\varphi_{\lambda'}^*(x) &= \max_{y\in T^{-1}(x)\cap Y}(f + \mu\varphi_\lambda^*)(y) - \max_{z\in T^{-1}(x)\cap Y}(f + \mu\varphi_{\lambda'}^*)(z)\\ &\le \max_{w\in T^{-1}(x)\cap Y}((f + \mu\varphi_\lambda^*)(w) - (f + \mu\varphi_{\lambda'}^*)(w))\\ &= \mu \max_{w\in T^{-1}(x)\cap Y}(\varphi_\lambda^* - \varphi_{\lambda'}^*)(w)\\ &\le \mu||\varphi_\lambda^* - \varphi_{\lambda'}^*||_{\infty,L}\,,\end{aligned}$$

where $L = T^{-1}K \cap Y$, and (17) follows since $x \in K$ was arbitrary. $\square$

With the estimates (18), (19) in hand we are now ready to prove the main result of this paper, giving sufficient conditions for a function f to have an essentially fixed point normal form, which is also a strong normal form. The hypotheses below are exactly the same as for Lemmas 5.12 and 5.13, but for convenience we state them in full.

Theorem 5.14. *Suppose the subshift of finite type Σ_A is primitive. Suppose $f : \Sigma_A \to I\!R$ has summable variations, and is bounded above. Suppose the inequality (13) holds for some primitive pair $(N, \mathbb{M})$ and some set Z such that $T(Z) = \Sigma_A$.*

Then f is essentially compact, and has a strong normal form $\tilde{f}$. Consequently

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset .$$

Proof. As noted previously, the fixed point equation $M_{f,\lambda}\varphi_\lambda = \varphi_\lambda$ is equivalent to

$$M_{f,\lambda}\varphi_\lambda^* = \varphi_\lambda^* + (1-\lambda)\inf \varphi_\lambda , \tag{20}$$

and we want to let $\lambda \nearrow 1$ in (20) in order to show that $M_f\varphi_1^* = \varphi_1^* + c(f)$. If $K \subset \Sigma_A$ is compact then

$$\begin{aligned} ||M_{f,\lambda}\varphi_\lambda^* - M_f\varphi_1^*||_{\infty,K} &\le ||M_{f,\lambda}\varphi_\lambda^* - M_f\varphi_\lambda^*||_{\infty,K} + ||M_f\varphi_\lambda^* - M_f\varphi_1^*||_{\infty,K} \\ &\le (1-\lambda)||\varphi_\lambda^*||_{\infty,L} + ||\varphi_1^* - \varphi_\lambda^*||_{\infty,L} , \end{aligned}$$

by (18) and (19), where $L = L(K)$ is a compact subset of Σ_A. Since $||\varphi_\lambda^*||_{\infty,L}$ is bounded independently of λ, and $\varphi_\lambda^* \to \varphi_1^*$ subsequentially in $C(\Sigma_A)$, we see that $M_{f,\lambda}\varphi_\lambda^* \to M_f\varphi_1^*$ subsequentially in $C(\Sigma_A)$ as $\lambda \nearrow 1$.

So $(M_{f,\lambda} - Id)(\varphi_\lambda^*) \to (M_f - Id)(\varphi_1^*)$ subsequentially in $C(\Sigma_A)$ as $\lambda \nearrow 1$. Since each $(M_{f,\lambda} - Id)(\varphi_\lambda^*)$ is a constant function, and constant functions form a closed subspace of $C(\Sigma_A)$, the function $(M_f - Id)(\varphi_1^*)$ is also a constant. By Lemma 3.4, the constant in question must be $c(f) = \lim_n \frac{1}{n}\sup_{x\in X} S_nf(x)$. Therefore

$$M_f\varphi_1^* = \varphi_1^* + c(f) , \tag{21}$$

and φ_1^* is an essentially fixed point for M_f.

If $Y = \cup_{i=1}^J [i]_A$, where $J \in I\!N$ is as in Lemma 5.12, then setting $\lambda = \mu = 1$ in (14), and combining with (21), gives

$$\varphi_1^*(x) + c(f) = M_f\varphi_1^*(x) = \max_{y\in T^{-1}(x)\cap Y} (f + \varphi_1^*)(y)$$

for all $x \in \Sigma_A$, which is precisely condition (c) of Definition 4.1. The fact that Y is a finite union of cylinder sets ensures that conditions (a) and (b) of Definition 4.1 are also satisfied (cf. Remark 5.10 (d)). Therefore f is essentially compact. Since Σ_A is a Polish space, Theorem 4.3 then implies that $\tilde{f} := f + \varphi_1^* - \varphi_1^* \circ T$ is a strong normal form for f, and that $\mathcal{M}_{\max}(f) = \left\{ m \in \mathcal{M} : \mathrm{supp}(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset$. □

In the case where the subshift of finite type Σ_A is the full shift, Theorem 5.14 implies:

Corollary 5.15. *Let $\Sigma = \mathbb{N}^{\mathbb{N}}$ be the full shift. Suppose $f : \Sigma \to \mathbb{R}$ is bounded above, has summable variations, and that there exists $I \in \mathbb{N}$ such that*

$$\sum_{j=1}^{\infty} var_j(f) < \inf f|_{[I]} - \sup f|_{[i]} \tag{22}$$

for all i sufficiently large.
Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset .$$

Proof. For the full shift we may choose $Z = [I]$ and $N = 0$. The inequality (22) then implies (13), and the result follows from Theorem 5.14. □

The inequality (13) asserts that the (finite) infimum of f on some set Z is sufficiently larger than its values "at infinity". An extreme example of this is when $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$.

Corollary 5.16. *Let Σ_A be primitive. Suppose f has summable variations, that there is a primitive alphabet $\mathbb{M}$ for which $\inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})} > -\infty$, and that $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$.*
Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset .$$

Proof. The lefthand side of (13) is finite, because f is bounded above, with summable variations, and $\inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})} > -\infty$. The set $Z = \Pi_{A,1}^{-1}(\mathbb{M})$ satisfies $T(Z) = \Sigma_A$, and $\inf f|_Z > -\infty$, so the righthand side of (13) tends to $+\infty$ as $i \to \infty$. Therefore (13) holds for all sufficiently large i, and the result follows from Theorem 5.14. □

The case where $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$ occurs in particular when the *summability condition*

$$\sum_{i \in \mathbb{N}} \exp(\sup f|_{[i]_A}) < \infty \tag{23}$$

holds. This condition plays a key role in the development of the thermodynamic formalism for countable state subshifts of finite type, allowing us to define the *Ruelle operator* $\mathcal{L}_f \varphi(x) = \sum_{Ty=x} e^{f(y)} \varphi(y)$. If Σ_A is *finitely* primitive and f has summable variations then (23) is equivalent (cf. [MU], Prop. 2.7) to the finiteness of the topological pressure

$$P(f) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{T^n y = y} \exp \left(\sup_{\omega \in [\Pi_{A,n}(y)]} \sum_{i=0}^{n-1} f(T^i \omega) \right) ,$$

and is a necessary condition for the existence of an invariant Gibbs measure for f. Any f satisfying the summability condition (23) is necessarily bounded above, and if $I\!N_A = \{i \in I\!N : [i]_A \neq \emptyset\}$ is infinite then f is unbounded below.

Corollary 5.17. *Let Σ_A be finitely primitive. Suppose f has summable variations and satisfies the summability condition (23).*
Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset\,.$$

Proof. If $\mathbb{M}$ is a finite primitive alphabet for Σ_A then $\inf f|_{\Pi_{A,1}^{-1}(\mathbb{M})} > -\infty$. The summability condition (23) implies that $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$, with the convention that $\sup f|_{[i]_A} = -\infty$ whenever $[i]_A$ is empty. The result now follows from Corollary 5.16. $\square$

In particular we deduce

Corollary 5.18. *Let Σ_A be finitely primitive. If f has summable variations and an invariant Gibbs measure then it also has a maximizing measure.*

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