Inverse topological pressure with applications to holomorphic dynamics of several complex variables

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Abstract

The notion of topological pressure for continuous maps has proved to be an extremely rich and beautiful subject, with many applications, for example to give estimates and formulae for the Hausdorff dimension of dynamically defined sets.

In this paper we define a few notions of inverse topological pressure ($\tilde{P}^-, P^-, P_-$) which in the case of endomorphisms take into consideration consecutive preimages of points (prehistories) instead of forward iterates. This inverse topological pressure has some properties similar to the regular (forward) pressure but, in general if the map is not a homeomorphism, they do not coincide.

In fact, there are several ways to define inverse topological pressure; for instance we show that the Bowen type definition coincides with the one using spanning sets.

Then we consider the case of a holomorphic map $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$ which is Axiom A and such that its critical set does not intersect a particular basic set Λ. Such maps were first studied by Fornaess-Sibony ([5]).

We will prove that, under a technical condition, the Hausdorff dimension of the intersection between the local stable manifold and the basic set is equal to t^s, i.e $HD(W^s_\delta(x) \cap \Lambda) = t^s$, for all points x belonging to Λ. Here t^s represents the unique zero of the function $t \to P^-(t\phi^s)$, with P^- denoting the inverse topological pressure and $\phi^s(x) = \log |Df|_{E^s_x}|, x \in \Lambda$. In general $HD(W^s_\delta(x) \cap \Lambda)$ will be estimated above by t^s and below by t^s_-, where t^s_- is the unique zero of the map $t \to P_-(t\phi^s)$.

As a corollary we obtain that, if the stable dimension is non-zero, then Λ must be a non-Jordan curve, and also, if $f|_\Lambda$ happens to be a homeomorphism (like in the examples from [14]), then the stable dimension cannot be zero.

Keywords: Inverse topological pressure, preimage entropy, stable Hausdorff dimension, holomorphic Axiom A maps

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1 Introduction

The notions of entropy and topological pressure have found many interesting applications in dynamical systems. In particular Ruelle ([18]) proved (see [2] for the first result of this type) that, for a hyperbolic rational map f on $\mathbb{P}^1\mathbb{C}$, the Hausdorff dimension of its Julia set is equal to the unique zero of the pressure function of $\phi(x) := -\log|Df(x)|$. This can be used further to give estimates on the Hausdorff dimension of the Julia set for some rational maps.

Notation: In the sequel we will denote the Hausdorff dimension of a set A by $HD(A)$; also, $\mathbb{P}^k$ will denote the k-dimensional complex projective space.

Relations between Lyapunov exponents, Hausdorff dimension and entropy were also given by Manning in [9]. Equalities between stable/unstable Hausdorff dimension and the zero of the contraction/expansion in the stable/unstable direction have been given for complex Henon mappings by Verjovsky-Wu [20]). They proved the following theorem (shown in the case of plane horseshoes by Manning and McCluskey in [10]):

Theorem (Verjovsky-Wu). *For any hyperbolic Henon map g on $\mathbb{C}^2$ and $x \in J$, (with J denoting the Julia set of g), the Hausdorff dimension t^s of $W^s_\varepsilon(x) \cap J$ is given by Bowen's formula:*

$$P_{g|J}(t\Phi^s) = 0,$$

where $0 < t^s < 2$ is independent of $x \in J$, and $\Phi^s(x) := \log|Dg|_{E^s_x}|$, with E^s_x the stable tangent space at x.

A similar equality is true for the unstable dimension $HD(W^u_\varepsilon(x) \cap J)$.

In [11] we noticed that a similar equality is not valid for the stable dimension of a holomorphic endomorphism of $\mathbb{P}^2$ which is hyperbolic on its nonwandering set $\Omega(f)$. However we still have one inequality:

Theorem ([11]). *If f is Axiom A, holomorphic map on $\mathbb{P}^2$, and Λ is a basic set of $\Omega(f)$, then $HD(W^s_\varepsilon(x) \cap \Lambda) \le t^s_0$, with t^s_0 the unique zero of the pressure function $t \to P(t\phi^s)$, but in general the inequality is strict.*

Counterexamples to the equality are given in [11]. In [12] we gave also a similar estimate for the Hausdorff dimension of the set K^- of points with "bounded inverse iterates"; K^- represents actually the union of unstable sets of points from the saddle part S_1 of the nonwandering set. For s-hyperbolic maps on $\mathbb{P}^2$, K^- was shown to have empty interior [12].

The problem of estimating $HD(W^s_\varepsilon(x) \cap \Lambda)$ for a hyperbolic holomorphic map f on $\mathbb{P}^2$ was studied in [13] as well. In that paper we obtained an upper bound using the number of preimages that a point from Λ can have in Λ.

Theorem (Mihailescu-Urbanski). *Assume f is an Axiom A, holomorphic map of degree $d \ge 2$ on $\mathbb{P}^2$, and Λ is one of the basic sets with unstable index equal to 1. Suppose $C_f \cap \Lambda = \emptyset$ ($C_f =$*

critical set of f) and also that $f|_\Lambda\colon\ \Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least $d' \leq d$ preimages in Λ. Then $HD(W^s_\varepsilon(x) \cap \Lambda) \leq \frac{\log d' - h_{top}(f|_\Lambda)}{\log \alpha + \log \gamma}$, where $\alpha := \frac{\sup_{x\in\Lambda} |Df|_{E^s_x}|}{\inf_{y\in\Lambda} |Df|_{E^s_y}|}$ and $\gamma := \sup_{x\in\Lambda} |Df|_{E^s_x}|$, as long as $\alpha \leq \gamma^{-1}$.

The number of preimages of a point, belonging to Λ, is not constant and is not stable under perturbation. In [14] we gave a large class of perturbations of the map $f(z,w) = (z^2 + c, w^2)$, $|c|$ small, which are homeomorphisms on their respective basic sets close to the basic set $\{p_0(c)\} \times S^1$ of the initial map, where $p_0(c)$ is a fixed attracting point of $z^2 + c$.

We would like now to introduce a notion of inverse topological pressure which is better suited to the stable Hausdorff dimension problem. This inverse topological pressure has some properties similar to the regular (forward) pressure, but in general they do not coincide if the map is not a homeomorphism. (compare also to the notions of inverse entropy studied in [6], [15], etc.).

In the sequel, let us introduce several notions that will be used throughout the paper. We will start with the topological entropy defined in the usual manner. The general setting is that of (X, d), a compact metric space, and $f\colon\ X \to X$ a continuous map. For n, a positive integer, $d_n(x,y) := \max\{d(f^i x, f^i y),\ i = 0, ..., n-1\}$ is a metric on X inducing the same topology as the metric d.

Definition 1. A subset $E \subset X$ is called (n,ε)-separated (for some $\varepsilon > 0$) if for all $x, y \in E$, $x \neq y$, we have $d_n(x,y) \geq \varepsilon$.

Definition 2. The *topological pressure* of f is the functional $P_f\colon\ \mathcal{C}(X,\mathbb{R}) \to \overline{\mathbb{R}}$ defined on the space of continuous functions by:

$$P_f(\varphi) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup \left\{ \sum_{x\in E} \exp\left(\sum_{i=0}^{n-1} \varphi(f^i x) \right),\ E \subset X,\ (n,\varepsilon)\text{-separated set.} \right\}.$$

Definition 3. When considering $\varphi \equiv 0$ in Definition 2, we obtain the notion of *topological entropy* of f.

There exists an interesting relationship between Borel invariant measures and P_f, contained in the following:

Theorem (Variational Principle). *In the above setting, $P_f(\varphi) = \sup_\mu\{h_\mu(f) + \int \varphi\, d\mu\}$, where the supremum is taken over all f-invariant Borel probability measures μ, and $h_\mu(f)$ = measure-theoretic entropy of μ.*

For the definition of $h_\mu(f)$ and proofs of all these facts, as we mentioned, a good reference is [21].

The topological pressure has several useful properties:

Theorem (Properties of Pressure). *If $f\colon\ X \to X$ is a continuous transformation, and $\varphi, \psi \in \mathcal{C}(X,\mathbb{R})$, then:*

1) $\varphi \leq \psi \Rightarrow P_f(\varphi) \leq P_f(\psi)$

2) $P_f(\cdot)$ *is either finitely valued or constantly* ∞

3) P_f *is convex*

4) for a strictly negative function φ*, the mapping* $t \to P_f(t\varphi)$ *is strictly decreasing if* $P(0) < \infty$.

5) P_f *is a topological conjugacy invariant.*

The need appears however for a notion of topological pressure on non-compact sets. This was done beautifully in a paper by Pesin-Pitskel [16]. However, in the case of defining the inverse topological pressure, there is no apriori backward sum similar to the forward sum $\phi(x) + \phi(fx) + \phi(f^2x) + ... + \phi(f^n x)$ that was taken in the definition of $P(\phi)$ in the usual case.

One can define a notion of inverse pressure using the supremum over all prehistories, or the infimum, or by restricting to a certain set of points and prehistories.

We will give in the following section a definition for the inverse pressure, P^-, which is good from the point of view of its similarity to the Pesin-Pitskel notion and, more importantly, since it will give an upper estimate for the stable dimension.

Let us now also give the definition of inverse entropy, studied by Hurley [6], Nitecki-Przytycki [15], etc. Although the notion we will introduce will be in general different from this one, parallels between the two shall prove interesting. We will call a *branch* of length ℓ (or *prehistory* of length ℓ) in X, a sequence of preimages, $\beta = (z_0, z_{-1}, \dots, z_{-\ell})$, with $z_i \in X$, $-\ell \leq i \leq 0$, such that $f(z_{i-1}) = z_i$, $-\ell + 1 \leq i \leq 0$. For another branch $\beta' = (z'_0, \dots, z'_{-\ell})$ of same length, define their *branch distance* to be $d^b(\beta, \beta') = \max_{j=\overline{0,\ell}} d(z_{-j}, z'_{-j})$. The reader can notice the similarity between the branch distance and $d_n(\cdot,\cdot)$ introduced earlier. Like $d_n(\cdot,\cdot)$ for forward iterates, d^b measures the growth of inverse iterates. Using this, we now define a *branch metric* on X:

$$d^b_\ell(x, x') < \varepsilon,$$

if for every branch β of length ℓ with $z_0 = x$, there exists a branch β' of length ℓ with $z'_0 = x'$ such that $d^b(\beta, \beta') < \varepsilon$, and vice versa. Denote by $N_{\text{span}}(\varepsilon, d^b_\ell, X)$ the smallest cardinality of an ε-spanning set for X in the d^b_ℓ metric. Hence, if A is an ε-spanning set with $\#A = N_{\text{span}}(\varepsilon, d^b_\ell, X)$, then, $\forall x \in X$, $\exists y \in A$ with $d^b_\ell(x, y) < \varepsilon$. Let also $N_{\text{sep}}(\varepsilon, d^b_\ell, X)$ be the largest cardinality of an ε-separated set for X. So, if A is ε-separated, then for all $x, y \in A$, $x \neq y$, $d^b_\ell(x, y) > \varepsilon$. Like in the case of usual entropy, spanning and separated sets each approximate the preimage branch entropy.

Proposition ([15]). *For* $f\colon\ X \to X$ *continuous,* (X, d) *compact metric space, we have*

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log N_{sep}(\varepsilon, d^b_n, X) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log N_{\text{span}}(\varepsilon, d^b_n, X)$$

and the common value is called the preimage (branch) entropy*, denoted by* $h_i(f)$. $\square$

Proposition. *In the same setting as above, if* f *is a homeomorphism, then* $h_i(f) = h(f)$.

So, in the particular case of homeomorphisms, the two notions coincide; the proof is immediate. Let us recall now two cases when $h_i(f) = 0$.

a. Forward-expansive coverings

If X is a metric space, then a continuous map $f\colon\ X \to X$ is called *forward expansive* if there exists $\varepsilon_0 > 0$ such that for all $x, y \in X$ with $x \neq y$ there exists $m \geq 0$ with

$$d(f^m x, f^m y) \geq \varepsilon_0 > 0.$$

For example, f is forward-expansive on any invariant subset of a Riemannian manifold on which Df is expanding by a constant factor $\lambda > 1$. Recall that $f\colon\ X \to X$ is a *covering map* if for all $x \in X$ there exists a neighborhood U_x of x, such that $f^{-1}(U_x) = \bigcup\limits_i V^i$ with $\{V^i\}$ open disjoint sets, and such that $f\colon\ V^i \to U_x$ is homeomorphism.

Proposition ([15]). *If $f\colon\ X \to X$ is a* forward expansive covering *map, then $h_i(f) = 0$.* $\square$

b. Graph maps.

A *finite graph* is a compact metric space K with a distinguished finite set of points called *vertices*, whose complement has finitely many connected components, *edges*, homeomorphic to the open interval (0,1). We fix the metric on K by assigning length 1 to each edge and the distance between two points in K is the length of the shortest path connecting them.

Theorem (Nitecki-Przytycki, [15]). *Let K a finite graph and $f\colon\ K \to K$ continuous map. Then $h_i(f) = 0$.* $\square$

Corollary. *For any continuous self-map f of a closed interval $[a, b]$, or of the circle S^1, we have $h_i(f) = 0$.* $\square$

We end this section with a short discussion of possible definitions for inverse topological pressure and their advantages or disadvantages.

First, one may try to generalize the definition of preimage branch entropy $h_i(f)$ and obtain a notion of inverse pressure using the spanning sets in the metric d_n^b and then taking supremum over all the sums of the test function ϕ along prehistories of points in the spanning set.

This could be done in the spirit of [16].

However in this definition, if we concatenate prehistories of length $n_1, n_2, ..., n_m$, then there is no way one can obtain small sets in the $d^b_{n_1+n_2+...+n_m}$ metric. This, because the chosen prehistories form just a strict subset in the set of all $(n_1 + ... + n_m)$-prehistories of points in the tail.

The fact that prehistories do not concatenate makes the equality between the inverse pressure defined with spanning sets and the one using the outer measure construction (like in [1] or [16]) break down.

However we need the outer measure construction since it is better suited for the different diameters of the sets appearing in the definition of Hausdorff dimension.

In the next section we will address these questions and will introduce a notion of inverse topological pressure which is good from the point of view of stable dimension. By *stable dimension at* *a point* x we will understand the Hausdorff dimension of the intersection between the local stable manifold at x and the respective basic set Λ.

We will define P^- with a construction similar to [16] where we will take all possible prehistories covering a set. So, this time two points will be (n, ε)-close if they have some n-prehistories which are ε close at each level.

We shall define also another notion of inverse pressure, called P_- which uses bigger sets for the cover. The stable dimension will be contained between the zeros of $P_-(t\phi^s)$ and $P^-(t\phi^s)$.

2 Two definitions for inverse topological pressure

We shall start with the definition of $\tilde{P}^-$ and P^-. Let us fix $\varepsilon > 0$ small enough.

Let X a compact metric space, $Y \subset X$, and $f : X \to X$ a continuous map.

Denote by $\mathcal{C}_m(\varepsilon)$ the set of collections of length m of balls of radius ε centered at points of a certain prehistory, $C = \{U_0 = B(x_0, \varepsilon), ..., U_{m-1} = B(x_{m-1}, \varepsilon)\}$, where $f(x_{-1}) = x_0, ..., f(x_{-m+1}) = x_{-m+2}$ and of collections of length $k < m$ which are *terminal*, i.e $\{U_0 = B(x_0, \varepsilon), ..., U_{k-1} = B(x_{-k+1}, \varepsilon)\}$, with $f(x_{-i}) = x_{-i+1}, i = 1, ..., k-1$ and $f^{-1}(x_{-k+1}) = \emptyset$. We denote by $n(C)$ the number of elements of C. It is clear that terminal branches with $k < m$ can be taken only if the map f is not surjective. In most of our applications the map f will be surjective on X, however. Now let $C = \{U_0, ..., U_{k-1}\} \in \mathcal{C}_m(\varepsilon), k = n(C)$, and define

$$X(C) := \{y \in U_0, \exists y_{-1} \in f^{-1}(y) \cap U_1, \exists y_{-2} \in f^{-1}(y_{-1}) \cap U_2, ...\}.$$

For a real continuous function ϕ, on X, define also $S_k^- \phi(C) := sup\{\phi(y) + ... + \phi(y_{-k+1}), y \in X(C)$ and the prehistory $y, ...y_{-k+1}$ as in the definition of $X(C)\}$, where $k = n(C)$.

Remark: Let us denote by $\delta_\phi(\varepsilon)$ the maximum oscillation of the function ϕ on a ball of radius ε in X, i.e $\delta_\phi(\varepsilon) := \sup\{|\phi(x) - \phi(y)|\}$, where the supremum is taken over all pairs $x, y \in X$ for which there exists $z \in X$ such that $x, y \in B(z, \varepsilon)$.

Then, for any $y, y' \in X(C)$, $|\phi(y)+\phi(y_{-1})+...+\phi(y_{-n(C)+1})-\phi(y')-\phi(y'_{-1})-...-\phi(y'_{-n(C)+1})| \leq n(C)\delta_\phi(\varepsilon)$.

So, up to a difference of at most $n(C)\delta_\phi(\varepsilon)$, it does not matter which point in $X(C)$ we take to calculate $S_{n(C)}^- \phi(C)$. Let $\mathcal{C}(\varepsilon) = \bigcup_{m=1}^\infty \mathcal{C}_m(\varepsilon)$.

For an arbitrary function $\phi \in \mathcal{C}(X, \mathbb{R})$, a positive integer N, and a real number λ let:

$M(\lambda, \phi, Y, N, \varepsilon) := \inf\{\sum_{C \in \Gamma} \exp(-\lambda n(C) + S_{n(C)}^- \phi(C))\}$, where the infimum is taken over all $\Gamma \subset \mathcal{C}(\varepsilon)$ such that $Y \subset \bigcup_{C \in \Gamma} X(C), n(C) \geq N$, or $n(C) < N$ and C is terminal $\}$.

When N increases, the pool of possible candidates Γ appearing in the definition of $M(\lambda, \phi, Y, N, \varepsilon)$ decreases. Hence, there exists the limit $\lim_{N \to \infty} M(\lambda, \phi, Y, N, \varepsilon) =: m(\lambda, \phi, Y, \varepsilon)$. The notation $m(\lambda, \phi, Y, \varepsilon)$ emphasizes the nature of the construction in the spirit of Hausdorff outer measure. Now let $\tilde{P}^-(\phi, Y, \varepsilon) := \inf\{\lambda, m(\lambda, \phi, Y, \varepsilon) = 0\}$.

Remark: Obviously, $\tilde{P}^-$ depends on the map f; in general we will not record this when no confusion can arise, however if we want to emphasize the dependence on f we will write $\tilde{P}_f^-$.

Proposition 1. *Given a continuous function $f : X \to X$ as above, and $Y \subset X$, the limit $\lim_{\varepsilon \to 0} \tilde{P}^-(\phi, Y, \varepsilon)$ exists and is called the* ***inverse topological pressure*** *of ϕ on Y, and denoted by $\tilde{P}^-(\phi, Y)$. When we want to emphasize the dependence on f we will write $\tilde{P}_f^-(\phi, Y)$.*

Proof. Assume $0 < \varepsilon' < \varepsilon$ and take a collection $\Gamma \in \mathcal{C}(\varepsilon')$ covering Y Then taking the balls of same centers as the ones in $C' \in \Gamma$ and radius ε, we will obtain another cover of Y, this time from $\mathcal{C}(\varepsilon)$ and whose elements are denoted by C. As in the Remark above, if $\delta_\phi(\varepsilon)$ denotes the maximum oscillation of ϕ on a ball of radius ε in X, we get $S^-_{n(C)}\phi(C) \le S^-_{n(C)}\phi(C') + n(C)\delta_\phi(\varepsilon)$.

Therefore, $\tilde{P}^-(\phi, Y, \varepsilon) - \delta_\phi(\varepsilon) \le \liminf_{\varepsilon' \to 0} \tilde{P}^-(\phi, Y, \varepsilon')$. This shows that the limit in the proposition does exist.

□

Observation:

We will also denote by $\tilde{P}^-(\phi)$, $M(\lambda, \phi, N, \varepsilon)$, and $m(\lambda, \phi, \varepsilon)$, respectively, the quantities $\tilde{P}^-(\phi, X)$, $M(\lambda, \phi, X, N, \varepsilon)$ and $m(\lambda, \phi, X, \varepsilon)$, when no confusion arises.

A few properties of $\tilde{P}^-$ are easy to prove :

Proposition 2. *(1) If $Y_1 \subset Y_2 \subset X$, then $\tilde{P}^-(\phi, Y_1) \le \tilde{P}^-(\phi, Y_2)$.*

(2) If $Y = \cup_m Y_m$, then $\tilde{P}^-(\phi, Y) = \sup_i \tilde{P}^-(\phi, Y_i)$.

(3) If f is a homeomorphism of X, then $\tilde{P}^-(\phi, X) = P(\phi)$, so the inverse pressure of a function ϕ coincides with the usual (forward) one in the case of homeomorphisms.

4) $\tilde{P}^-(\phi, Y)$ is invariant to topological conjugacy.

We can also define another notion of inverse pressure using only collections C of the same length or which are terminal. This time we define it directly on the whole space X. Take $P_m^-(\phi, \varepsilon) := \inf\{\sum_{C \in \Gamma} exp(S^-_{n(C)}\phi(C)), \Gamma \in \mathcal{C}_m(\varepsilon), \Gamma \text{ covers } X\}$. Then, set $P^-(\phi, \varepsilon) := \overline{\lim}_{m \to \infty} \frac{1}{m} \log P_m^-(\phi, \varepsilon)$. Similarly as for $\tilde{P}^-$, we can prove that $P^-(\phi) := \lim_{\varepsilon \to 0} P^-(\phi, \varepsilon)$ does exist. Again we write $P^-(\phi)$ when the mapping f is fixed and there can be no confusion; if we want to record also the dependence on f we will use the notation $P_f^-(\phi)$.

Theorem 1. *If $f : X \to X$ is surjective, then $P^-(\phi) = \tilde{P}^-(\phi)$, for any continuous function $\phi \in \mathcal{C}(X, \mathbb{R})$.*

Proof. Let us note in the begining of our proof that, since f is surjective, we do not need to worry about terminal branches in $\mathcal{C}_m(\varepsilon)$ of length $n(C) < m$.

First we show that $\tilde{P}^-(\phi) \le P^-(\phi)$. Let us take $\lambda > P^-(\phi, \varepsilon)$. For every $0 < \eta < \lambda - P^-(\phi, \varepsilon)$ and all N large enough, we have

$$m(\lambda, \phi, \varepsilon) \le M(\lambda, \phi, N, \varepsilon) + \eta. \tag{1}$$

Since there exists $m > N$ such that $|P^-(\phi, \varepsilon) - \frac{1}{m} \log P_m^-(\phi, \varepsilon)| \le \eta$, we obtain

$$\begin{aligned} M(\lambda, \phi, N, \varepsilon) &\le \inf\{\sum_{C \in \Gamma} \exp(-\lambda m + S_m^-\phi(C)), \Gamma \in \mathcal{C}_m(\varepsilon) \text{ covers } X\} \\ &\le e^{-\lambda m} P_m^-(\phi, \varepsilon) \le \exp((-\lambda + P^-(\phi, \varepsilon) + \eta)m) \end{aligned}$$

as long as $m \geq N$. Combining this and (1) we get

$$m(\lambda, \phi, \varepsilon) \leq \exp((-\lambda + P^-(\phi, \varepsilon) + \eta)m) + \eta$$

and, since $-\lambda + P^-(\phi, \varepsilon) + \eta < 0$, letting $m \to \infty$, we get $m(\lambda, \phi, \varepsilon) \leq \eta$. Letting in turn $\eta \to 0$, we get $m(\lambda, \phi, \varepsilon) \leq 0$ which implies that $\tilde{P}^-(\phi, \varepsilon) \leq \lambda$. Since λ was chosen arbitrarily larger than $P^-(\phi, \varepsilon)$, the required inequality thus follows.

Let us show now the opposite inequality: $\tilde{P}^-(\phi) \geq P^-(\phi)$. Firstly, we study the concatenation of two prehistories. This is the main advantage of this definition for inverse pressure, i.e the possibility of joining two different prehistories to form another one, of length equal to the sum of lengths of its components.

Let $\Gamma_m \subset \mathcal{C}_m(\varepsilon), \Gamma_n \subset \mathcal{C}_n(\varepsilon)$, each covering X. Since X is compact, we may assume that both Γ_m and Γ_n have finite number of chains C. Take now $C \in \Gamma_m, C' \in \Gamma_n$. Assume that $C = \{U_0, ..., U_{-m+1}\}, C' = \{U'_0, ..., U'_{-n+1}\}$. Let us define $X(CC') := \{x \in X(C) : \ x_{-m+1} \in X(C')\}$. If $X(CC') \neq \emptyset$, then if y, z are points in $X(CC')$, z will have a preimage $z_{-1} \in U_1$, and y will have a preimage $y_{-1} \in U_1$, hence $d(y_{-1}, z_{-1}) < 2\varepsilon$. Similarly, in a prehistory attached to $X(C)$, y_{-m+1} has a prehistory y_m belonging to U'_1; so does z_{-m+1}, hence $d(y_{-m}, z_{-m}) < 2\varepsilon$. Repeating the reasoning, we get that, if $C'' := \{B(y_0, 2\varepsilon), ..., B(y_{-m-n+1}, 2\varepsilon)\} \in \mathcal{C}_{m+n}(2\varepsilon)$, then $X(CC') \subset X(C'')$. Therefore

$$S^-_{m+n}\phi(C'') \leq S^-_m(C) + S^-_n(C') + (m+n)\delta_\phi(2\varepsilon). \tag{2}$$

We now return to the problem of showing the inequality $\tilde{P}^-(\phi) \geq P^-(\phi)$. In order to prove it fix $\lambda > \tilde{P}^-(\phi)$. Then $m(\lambda, \phi, \varepsilon) = 0$ and therefore $\lim_{N\to\infty} M(\lambda, \phi, N, \varepsilon) = 0$. If N is large enough, then $M(\lambda, \phi, N, \varepsilon) < \frac{1}{2}$. Thus, there exists a covering $\Gamma \subset \mathcal{C}(\varepsilon)$ such that

$$\sum_{C\in\Gamma} \exp(-\lambda n(C) + S^-_{n(C)}\phi(C)) < \frac{1}{2}$$

Since Γ covers X which is compact, we can assume that Γ is finite. By raising the sum above to power s and then adding over s, we obtain that

$$\sum_{s\geq 0} \sum_{j_1,..,j_s} exp(-\lambda(n(C_{j_1}) + ... + n(C_{j_s})) + S^-_{n(C_{j_1})}\phi(C_{j_1}) + .. + S^-_{n(C_{j_s})}\phi(C_{j_s})) < \tag{3}$$

$$< M := \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^s < \infty. \tag{4}$$

Similarly as above, we can associate to $C_{j_1}, ..., C_{j_s}$ a chain denoted by $C_{j_1}...C_{j_s}$ obtained by concatenation. If C_j is chosen arbitrarily in Γ, and Γ covers X, then the set of all such $C_{j_1}...C_{j_s}$ gives a collection denoted $\Gamma^{j_1..j_s} \subset \mathcal{C}(2\varepsilon)$. By the same argument as in (2), we obtain the following:

$$S^-_{n_{j_1}+...+n_{j_s}}\phi(C_{j_1}..C_{j_s}) \leq S^-_{n_{j_1}}\phi(C_{j_1}) + ... + S^-_{n_{j_s}}\phi(C_{j_s}) + (n_{j_1} + ... + n_{j_s}) \cdot \delta_\phi(2\varepsilon) \tag{5}$$

If $\Gamma = \{C_1, ..., C_q\}$, put $N_0 := \max_{1 \leq i \leq q} n(C_i)$. Also denote the maximum oscillation $\delta_\phi(2\varepsilon)$ by $\delta(2\varepsilon)$ when no confusion arises. For any given $n \geq 0$, the sets $\{X(C_{j_1}...C_{j_s}), n \leq n_{j_1} + ... + n_{j_s} \leq n + N_0\}$ cover X. Denote the collection of these chains $\overline{C_{j_1}...C_{j_s}}$, by Γ_n. Note that for every chain $\overline{C_{j_1}...C_{j_s}} \in \Gamma_n$, we have

$$S_n^- \phi(\overline{C_{j_1}..C_{j_s}}) \leq S^-_{n_{j_1}+...+n_{j_s}} \phi(C_{j_1}..C_{j_s}) + N_0 ||\phi||_\infty.$$

Hence, applying (3) and (5), we get

$$\begin{aligned}
&\sum_{C \in \Gamma_n} \exp(- \lambda n + S_n^- \phi(C_{j_1}..C_{j_s}) - n\delta(2\varepsilon)) \leq \\
&\qquad \leq \exp(N_0 ||\phi||_\infty) \sum_{C \in \Gamma_n} \exp(-\lambda n + S^-_{n_{j_1}+...+n_{j_s}} \phi(C_{j_1}..C_{j_s}) - n\delta(2\varepsilon)) \\
&\qquad \leq \exp(N_0(||\phi||_\infty + |\lambda|)) \sum_{C \in \Gamma_n} \exp(-\lambda(n_{j_1} + .. + n_{j_s}) + S^-_{n_{j_1}} \phi(C_{j_1}) + ... + S^-_{n_{j_s}} \phi(C_{j_s}) + \\
&\qquad (n_{j_1} + ..n_{j_s})\delta(2\varepsilon) - n\delta(2\varepsilon)) \\
&\qquad \leq \exp(N_0(||\phi||_\infty + |\lambda| + \delta(2\varepsilon))) \cdot M < \infty
\end{aligned}$$

This proves that

$$\inf_{\Gamma' \subset \mathcal{C}(2\varepsilon)} \sum \exp(S_n^- \phi) \leq \exp(N_0(||\phi||_\infty + \delta(2\varepsilon) + |\lambda|)) \cdot M \exp((\lambda + \delta(2\varepsilon))n),$$

where M is a constant independent of n. In conclusion $\lambda + \delta(2\varepsilon) \geq P^-(\phi, 2\varepsilon)$. But ε can be taken arbitrarily small and λ was taken arbitarily larger than $\tilde{P}^-(\phi)$. Hence $\tilde{P}^-(\phi) \geq \lim_{\varepsilon \to 0} P^-(\phi, 2\varepsilon) = P^-(\phi)$. This finishes the proof of the required equality $P^-(\phi) = \tilde{P}^-(\phi)$. $\square$

Remark: If f is not surjective, it is not true in general in the above proof that, if C' is terminal with $n(C') < n$, then CC' gives also a terminal branch C'' with $n(C'') < m + n$.

However, even if f is not surjective the proof above still gives the following.

Proposition 3. *For any continuous map $f : X \to X$, $\tilde{P}^-(\phi) \leq P^-(\phi), \forall \phi \in \mathcal{C}(X, \mathbb{R})$.*

The following properties of inverse pressure are similar to those of the usual (forward) topological pressure; we denote by P^- the functional P^-_f in the following:

Proposition 4. *If $f : X \to X$ is a continuous map of a compact metric space, and if $\phi, \psi \in \mathcal{C}(X, \mathbb{R})$, we have:*

(a) $\tilde{P}^-(\phi + \alpha) = \tilde{P}^-(\phi) + \alpha$, for a real constant α.

(b) if $\phi \leq \psi$, then $\tilde{P}^-(\phi) \leq \tilde{P}^-(\psi)$. Hence if we denote by $\tilde{h}^-(f) := \tilde{P}^-_f(0)$, then $\tilde{h}^-(f) + inf\, \phi \leq \tilde{P}^-(\phi) \leq \tilde{h}^-(f) + sup\ \phi$.

(c) $\tilde{P}^-(\cdot)$ is either finitely valued or constantly ∞.

(d) $|\tilde{P}^-(\phi) - \tilde{P}^-(\psi)| \leq ||\phi - \psi||$ as long as $\tilde{P}^-(\cdot)$ is finite.

(e) $\tilde{P}^-(\phi + \psi \circ f - \psi) = \tilde{P}^-(\phi)$.

(f) for a strictly negative function ϕ, the mapping $t \to \tilde{P}^-(t\phi)$ is strictly decreasing if $\tilde{P}^-(0) < \infty$.

Proof. The items (a) and (d) follow immediately from the definition. The first part of (b) is clear from the definition of inverse pressure. The second part follows from the first part combined with (a).

In order to prove (c) notice that from (b) we have

$$h^-(f) + \inf\phi \leq P^-(\phi) \leq h^-(f) + \sup\phi.$$

Hence, if there exists $\phi \in \mathcal{C}(X, \mathbb{R})$, such that $P^-(\phi) = \infty$, then, since ϕ is bounded on X, it follows that $h^-(f) = \infty$, hence $P^-(\psi)$ is infinite for any $\psi \in \mathcal{C}(X, \mathbb{R})$. The item (e) follows from the fact that

$$\exp(S_n^-(\phi + \psi \circ f - \psi)(\beta)) = \exp(S_n^-(\phi)(\beta)) \cdot \exp(\psi(fx) - \psi(x_{-n+1})),$$

where $\beta = (x_{-n+1}, ..., x_0 = x)$ is a prehistory of x and in general we define $S_n^-\phi(\beta) := \phi(x_{-n+1}) + ... + \phi(x)$. Now we use the fact that, for two prehistories β, γ corresponding to two points x, y in a set of the form $X(C)$, (i.e assume β, γ follow C), $|S_n^-\phi(\beta) - S_n^-\phi(\gamma)| \leq n\delta_\phi(\varepsilon)$, with $\delta_\phi(\varepsilon)$ the maximum oscillation of ϕ on a set of radius ε in X. Then by taking the limit over n approaching infinity, one obtains the equality in the statement.

The item (f) follows easily from (a) and (c).

□

The notions of inverse pressure introduced, will give in particular two inverse entropies, $\tilde{h}^- := \tilde{P}^-(0), h^- := P^-(0)$ and from Proposition 4, $\tilde{h}^- \leq h^-$.

Proposition 5. $0 \leq \tilde{h}^- \leq h^- \leq h_i$.

Proof. The proof follows from the definitions. If two points are (n, ε)-close in the d_n^b metric, then obviously they will be (n, ε) close also from the point of view of entropy h^-. Hence we need more (n, ε) spanning sets to cover X for h_i, than we need for h^-.

□

We shall give now an example showing that in general $h^- \neq h_i$.

Example with $h^- \neq h_i$. The example is basically one of a smooth map with infinite h_i given in [15]. In the notation of [15] we will need just the following properties:

(i) $X := B(\frac{1}{2}) \cup S(1)$, where $S(r) := \{z \in \mathbb{C}, |z| = r\}$

(ii) $f : X \to X$, and $f|_{S(1)}$ is the map $z \to z^2$ and also $f(B(\frac{1}{2})) \subset S(1)$

(iii) $B(\frac{1}{2}) \subset S(\frac{1}{2})$.

(iv) the metric on X is the one induced from the real plane.

(v) $h_i(f) = \infty$; this is proved in [15].

We now calculate $h^-(f)$. For the points in $S(1)$ we consider only prehistories whose elements are all in $S(1)$. So, the number of (n, ε) spanning sets necessary to cover $S(1)$ in the definition of h^- is smaller than the number of spanning sets in the d_n^b metric used for $h_i(f|_{S(1)})$. Then, since $f(B(\frac{1}{2})) \subset S(1)$, and hence the points from $B(\frac{1}{2})$ have no preimages in X, it follows that we can cover $B(\frac{1}{2})$ only with $(0,\varepsilon)$ balls centered at points of $B(\frac{1}{2})$, if $\varepsilon < \frac{1}{2}$. Indeed the only terminal branches of points from $B(\frac{1}{2})$ are the 0-branches.

But the number of such balls is independent of n, hence using also the fact from Section 1, that $h_i(f|_{S(1)}) = 0$, we get $h^-(f) = 0$.

So, $h^-(f) \neq h_i(f)$, since from property (v) above, $h_i(f) = \infty$ $\square$

Let us introduce now another notion of inverse topological pressure, this time using inverse spanning sets.

We will start with a continuous *surjective* map $f : X \to X$, where X is a compact metric space. The model we have in mind is that of a holomorphic map of algebraic degree $d \geq 2$ on $\mathbb{P}^2$ which is Axiom A and of a basic set Λ in the nonwandering set of f. Since $f : X \to X$ is a surjective map, any point $x \in X$ will have n-prehistories for any positive integer n. Given an n-prehistory $\beta := (x_0, ..., x_{-n+1})$ of x, we say that C is a branch **modeled after** β if $C = \{B(x_0, \varepsilon), ..., B(x_{-n+1}, \varepsilon)\}$.

Definition 4. We shall call (n,ε)-inverse ball centered at x, the set $\underset{C}{\cup} X(C)$, where C ranges over all branches modeled after the n-prehistories of x. It will be denoted by $B_n^-(x,\varepsilon)$.

Obviously if $x \in B_n^-(y,\varepsilon)$, then $y \in B_n^-(x,\varepsilon)$ as well.

Similar to the first definition of inverse topological pressure, and given $f : X \to X$ surjective, we introduce for an arbitrary function $\phi \in \mathcal{C}(X,\mathbb{R})$, a real number λ, a positive integer N, and a subset Y of X, the following quantity:

$$M_-(\lambda, \phi, Y, N, \varepsilon) := \inf\{\sum_F \exp(-\lambda n_x + S_{n_x,-}\phi(x)), \text{where } Y \subset \underset{x\in F}{\cup} B_{n_x}^-(x,\varepsilon), n_x \geq N, \forall x \in F\},$$

where $S_{n,-}\phi(x) := \inf\{\phi(x) + \phi(x_{-1}) + ... + \phi(x_{-n+1}), (x, x_{-1}, ..., x_{-n+1}) \text{an n-prehistory of } x\}$.

If N increases, we have less sets in the infimum above, therefore the limit $\lim\limits_{N\to\infty} M_-(\lambda,\phi,Y,N,\varepsilon)$ exists. We shall denote this limit by $m_-(\lambda,\phi,Y,\varepsilon)$ in order to keep a similar notation as for P^-.

We take also $P_-(\phi,Y,\varepsilon) := \inf\{\lambda, m_-(\lambda,\phi,Y,\varepsilon) = 0\}$. Identically as for P^- one can prove that:

Proposition 6. *The limit* $\lim\limits_{\varepsilon\to 0} P_-(\phi,Y,\varepsilon)$ *exists; it will be denoted by* $P_-(\phi,Y)$ *and will be called the inverse lower topological pressure of* ϕ *on* Y *relative to the map* f.

Remark:

The inverse lower pressure P_- depends on f, although we did not record this dependence in order not to burden notation. When it will be necessary to record the dependence on f we will write $P_{-,f}$.

The following properties of P_- are similar to those of P^- from Proposition 2. Notice also that since f was assumed surjective on X, P^- and $\tilde{P}^-$ coincide.

Proposition 7. *If* $f : X \to X$ *is a continuous map of a compact metric space, and if* $\phi, \psi \in \mathcal{C}(X,\mathbb{R})$*, then*

(a) $P_-(\phi + \alpha) = P_-(\phi) + \alpha$ *for a constant* α.

(b) if $\phi \leq \psi$, then $P_-(\phi) \leq P_-(\psi)$. If $h_-(f) := P_-(0, f)$, then $h_-(f) + \inf \phi \leq P_-(\phi) \leq h_-(f) + \sup \phi$.

(c) $P_-(\cdot)$ is either finite valued or constantly ∞.

(d) $|P_-(\phi) - P_-(\psi)| \leq ||\phi - \psi||$ when $P_-(\cdot)$ is finite.

(e) $P_-(\phi + \psi \circ f - \psi) = P_-(\phi)$.

(f) if $\phi < 0$ on X and $h_-(f) < \infty$, then the map $t \to P_-(t\phi)$ is strictly decreasing and has a unique zero.

(g) if f is a homeomorphism on X, then $P^-(\phi) = P_-(\phi)$, $\forall \phi \in \mathcal{C}(X, \mathbb{R})$. Hence P_-, P^-, P coincide for homeomorphisms.

The decreasing part in consequence (f) above follows from (a) and (b); the uniqueness of the zero follows from the fact that $P_-(0) = h_-(f) \geq 0$ and $P_-(t\phi) < 0$ if t is large enough.

The name "inverse lower pressure" is justified by the following Proposition:

Proposition 8. *For a continuous surjective map $f : X \to X$, and a continuous arbitrary function $\phi \in \mathcal{C}(X, \mathbb{R})$, we have $P^-(\phi) \geq P_-(\phi)$.*

Proof. Let us take a covering of X with sets $X(C)$, where C belongs to a finite set Γ. For each $C \in \Gamma$, assume that C corresponds to an $n(C)$-prehistory of a point $x(C)$. Let us denote the set of all points $x(C)$ obtained in this fashion (when $Cin\Gamma$), by F. We could have several C's from Γ corresponding to the same $x \in F$. If this happens then we take $n(x)$ to be the smallest $n(C)$ among all the C's giving x. Then $B^-_{n(x)}(x, \varepsilon)$ contains all the sets of the form $X(C)$ for all the C's in Γ with $x(C) = x$. This implies that $X = \underset{x \in F}{\cup} B^-_{n(x)}(x, \varepsilon)$. Also, it is clear that, if $n(C) \geq N$ for all $C \in \Gamma$, then also $n(x) \geq N$, for all $x \in F$. On the other hand, if for a branch C it happens that $x(C) = x$ and $n(C) = n(x)$, then from definitions it follows that $S^-_{n(C)}\phi(C) \geq S_{n(x),-}\phi(x)$. If for the branch C, it happens that $x(C) = x$, but $n(C) > n(x)$, then we do not even consider the corresponding term in the sum from the definition of $M_-(\lambda, \phi, X, N, \varepsilon)$. Hence in the sum from the definition of $M_-(\lambda, \phi, X, N, \varepsilon)$ we have less sets than in $M(\lambda, \phi, X, N, \varepsilon)$, and for the ones that appear in both sums, we have $S^-_n\phi(C) \geq S_{n,-}\phi(x(C))$. Therefore $M_-(\lambda, \phi, X, N, \varepsilon) \leq M(\lambda, \phi, X, N, \varepsilon), \forall \lambda, N, \varepsilon$. In conclusion we get $P_-(\phi) \leq P^-(\phi)$.

□

3 Stable dimension

This section will present the main application of the previously introduced notions of inverse topological pressure. We will study the availability of a Bowen type relation for the Hausdorff dimension of the intersection between a local stable manifold and a given basic set for a holomorphic Axiom A map f of $\mathbb{P}^2$. We shall call this Hausdorff dimension *the stable dimension* for short. Precise definitions will be given below.

It was observed in [11] that the stable dimension is in general just smaller than the zero t_0^s of the function $t \to P(t\phi^s)$, where $P(\cdot)$ denotes the usual (forward) pressure. In that article there are also given examples with strict inequality. This is due to the fact that $P(\cdot)$ takes into consideration only forward orbits and hence we cannot estimate from below the diameters of the sets in a covering using derivatives of the form $||Df^n|_{E^s}||$.

In [13] we also showed that the gap between t_0^s and the stable dimension can be explained partially by the number of preimages that a point in Λ has in Λ.

Here we will prove some estimates between the unique zero of the inverse pressure P^-, t^s, the unique zero of the inverse lower pressure P_-, t_-^s, and the stable dimension. This will imply that if the stable dimension is non-zero, then the basic set cannot be a Jordan curve. Let us first introduce some notation.

Our setting throughout this section is that of a holomorphic mapping on the complex projective plane $f : \mathbb{P}^2 \to \mathbb{P}^2$.

Set

$$\Omega(f) := \{x \in \mathbb{P}^2, \forall (r > 0) \exists (n \geq 1) \ s.t \ f^n(B(x,r)) \cap B(x,r) \neq \emptyset\},$$

the non-wandering set of f. The space of prehistories in Ω is denoted by $\hat{\Omega} := \{\hat{x} := (x_n)_{n \leq 0}, f(x_{n-1}) = x_n, x_n \in \Omega, \forall n \leq 0\}$.

From now on f is assumed hyperbolic. In particular there exists a continuous splitting of the tangent bundle over $\hat{\Omega}$ into subspaces which are invariated by Df, $T_{\hat{\Omega}}(\hat{x}) = E^s_{x_0} \oplus E^u_{\hat{x}}$ and constants $c \geq 0$ and $\lambda > 1$ such that

$$||Df^n(v)|| \leq c\lambda^{-n} \cdot ||v||, v \in E^s_{x_0}$$

and

$$||Df^n(v')|| \geq c^{-1}\lambda^n ||v'||, v' \in E^u_{\hat{x}}, \forall n \geq 0.$$

Up to a change of metric it can be proved that in the above inequalities one can take $c = 1$. It can be shown that the above splitting will give birth to local stable and unstable manifolds respectively denoted by

$$W^s_\varepsilon(x) := \{y \in \mathbb{P}^2, d(f^n x, f^n y) \leq \varepsilon, \forall n \geq 0\}$$

and

$$W^u_\varepsilon(\hat{x}) := \{y \in \mathbb{P}^2, y \text{ has a prehistory } \hat{y} = (y_n)_{n \leq 0}, d(x_n, y_n) \leq \varepsilon, \forall n \leq 0\}.$$

$W^s_\varepsilon(x)$ and $W^u_\varepsilon(\hat{x})$ are complex disks. More information about this subject and proofs can be found in [5], [19], [7].

We now assume that f is **Axiom A**, meaning that there exists a hyperbolic splitting of the tangent bundle as above and that periodic points are dense in $\Omega(f)$.

In this case $\Omega(f)$ will decompose as a union of finitely many invariant sets Ω_i, called **basic sets**. A good general reference for Axiom A in the case of endomorphisms is for example Ruelle's book [19]. For Axiom A maps it also makes sense to define the **No-Cycle Property** which says that there can be no cycles among the basic sets for the ordering $\Omega_i \succ \Omega_j$ iff $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \emptyset$, where W^u, W^s are the global unstable/stable sets (as defined for example in [5]).

In the following we will be interested only in basic sets of saddle type, i.e which have both stable and unstable directions (complex dimensions dim $E^s_x = 1$ and dim $E^u_{\hat{x}} = 1$) and will denote in general such a set by Λ. By **stable dimension** at the point x from Λ, we will understand $HD(W^s_\varepsilon(x) \cap \Lambda)$, for some $\varepsilon > 0$.

It follows from the definition of local manifolds that $W^s_\varepsilon(x)$ depends only on its base point x, whereas $W^u_\varepsilon(\hat{x})$ can depend on the whole prehistory $\hat{x} \in \hat{\Omega}$.

Denote also by C_f the critical set of f. This is an analytic variety in $\mathbb{P}^2$. We fix a basic set of saddle type, Λ, and will assume in the sequel that $C_f \cap \Lambda = \emptyset$.

Any holomorphic map on $\mathbb{P}^2$ is of the form $[z : w : t] \to [P_1(z,w,t) : P_2(z,w,t) : P_3(z,w,t)]$, with P_i homogeneous polynomials in (z,w,t) of the same degree. This common degree is called the *algebraic degree* of f. We will assume that this algebraic degree is larger or equal than 2.

Proposition 9. *If f is an Axiom A holomorphic map on $\mathbb{P}^2$ of algebraic degree $d \geq 2$, and Λ is a basic set of saddle type, then $\tilde{P}^-_{f|_\Lambda}(\phi) = P^-_{f|_\Lambda}(\phi)$, for any $\phi \in \mathcal{C}(\Lambda, \mathbb{R})$.*

Proof. Since f is surjective on Λ, we can apply Theorem 1 on the compact space $X = \Lambda$ and we are done.

□

Proposition 10. *(a) Consider an Axiom A map f as above, f holomorphic on $\mathbb{P}^2$, and Λ one of the basic sets of f, such that $\Lambda \cap \mathcal{C}_f = \emptyset$. Assume also that Λ can be written as a union of finitely (countably) many compact, pathwise connected and simply connected subsets $(V_i)_i$, and that f has no cycles among its basic sets. Perturb the map f to a holomorphic map g on $\mathbb{P}^2$, such that the corresponding basic set of g close to Λ is Λ_g. Then Λ_g can also be written as a (possibly uncountable) union of compact, pathwise connected and simply connected subsets which are homeomorphic images of the sets $V_i, i > 0$.*

(b) In the above setting, if $f|_\Lambda$ is a homeomorphism, and g is a close pertubation of f, then $g|_{\Lambda_g}$ is also a homeomorphism.

Proof. Let us take V a subset of Λ which is pathwise connected, compact and simply connected.

Since f is Axiom A and with the No Cycle condition, one can apply the Stability Theorem ([17]) for a close perturbation g of f, to obtain that:

i) g is also Axiom A and there exists a basic set Λ_g of g, such that Λ_g is close to Λ and

ii) there exists a homeomorphism $h : \widehat{\Lambda} \to \widehat{\Lambda_g}$ commuting with the liftings $\hat{f}, \hat{g}$, i.e $h \circ \hat{f} = \hat{g} \circ h$, or equivalently $\hat{g}^{-1} \circ h = h \circ \hat{f}^{-1}$.

The homeomorphism h is uniquely defined with the above commuting property. Let us notice also that h depends continuously on g; when we will want to emphasize this dependence on g, we shall write h_g. Hence from the continuity of the homeomorphism h_g with respect to g, it follows that $\pi \circ h_g$ converges towards the canonical projection $\pi : \widehat{\Lambda} \to \Lambda$, when $g \to f$.

(a) Now, since V is simply connected and Λ does not intersect the critical set $\mathcal{C}_f$ of f, we can define a branch of f^{-1} on V, call it f_*^{-1}, which takes its values in Λ. But in this case, $f_*^{-1}(V)$ will be also simply connected, and contained in the set Λ which does not intersect the critical set $\mathcal{C}_f$, hence we can define again a branch of f^{-1}, this time on $f_*^{-1}(V)$; in this way, we can define on V a sequence of inverse branches $f_*^{-n}, n > 0$. If x is a point in Λ, let us define $\sigma(x) := (x, x^*_{-1}, x^*_{-2}, ..., x^*_{-n}, ...)$, where in general $x^*_{-n} := f_*^{-n}(x)$, for all integers $n > 0$. So σ is a section over V of the canonical projection $\pi : \widehat{\Lambda} \to \Lambda$. Let $\tilde{V}$ denote the set $\sigma(V)$. $\sigma : V \to \tilde{V}$ is a homeomorphism. Let also $V_g := \pi(h(\tilde{V}))$. We shall prove that $\pi \circ h \circ \sigma : V \to V_g$ is a homeomorphism, which will imply the conclusion of the theorem. Let us assume that there exist points x, y in V such that $\pi(h(\hat{x})) = \pi(h(\hat{y}))$, where $\hat{x} = \sigma(x), \hat{y} = \sigma(y)$. In this case we can write:

$$h(\hat{x}) = (z_0, z_{-1}, ..., z_{-k+1}, z_{-k}, z_{-k-1}, ...),$$

$$h(\hat{y}) = (z_0, z_{-1}, ..., z_{-k+1}, z'_{-k}, z'_{-k-1}, ...)$$

where k is the first positive integer such that the k-preimages of z_0 are different, i.e $z_{-k} \neq z'_{-k}$ and $z_{-j} = z'_{-j}, j < k$. Also, from the definition of h, we have z_{-i}, z'_{-i} in Λ_g for all i.

Notice that from the construction of h, we get that $\hat{g}^{-1} \circ h = h \circ \hat{f}^{-1}$, and by induction, $\hat{g}^{-n} \circ h = h \circ \hat{f}^{-n}$, for all $n \geq 0$. But $\hat{g}^{-k} \circ h(\hat{x}) = (z_{-k}, z_{-k-1}, ...)$ and $\hat{g}^{-k} \circ h(\hat{y}) = (z'_{-k}, z'_{-k-1}, ...)$.

On the other hand,

$$\hat{f}^{-k}(\hat{x}) = (x_{-k}, x_{-k-1}, ...), \text{and} \hat{f}^{-k}(\hat{y}) = (y_{-k}, y_{-k-1}, ...)$$

x_{-k}, y_{-k} are the images of x, y by the k-th inverse iterate f_*^{-k} on V.

Since $\Lambda \cap \mathcal{C}_f = \emptyset$, it follows that, if the perturbation g is close enough to f, there exists a positive constant δ_0 depending only on f such that

$$d(\xi, \xi') > \delta_0,$$

for any $\xi, \xi' \in \Lambda_g, \xi \neq \xi'$ with $g(\xi) = g(\xi')$. This is true since Λ_g is close to Λ_f and $\mathcal{C}_g$ is close to Λ_f. Fix now $\eta > 0$ sufficiently small such that $\eta < \frac{\delta_0}{3}$ and

$$d(\xi, \xi') < 2\eta \text{ implies } d(f_*^{-1}(\xi), f_*^{-1}(\xi')) < \frac{\delta_0}{3}, \forall \xi, \xi' \in V$$

Consider now g so close to f such that $d(\pi \circ h_g(\hat{\xi}), \xi_0) < \eta$, for all $\hat{\xi} \in \widehat{\Lambda}$.

But in our case, $\hat{g}^{-j} \circ h_g(\hat{x}) = (z_{-j}, ...) = h_g \circ \hat{f}^{-j}(\hat{x}), \forall j > 0$. Hence from above, $d(x_{-j}, z_{-j}) < \eta, \forall j > 0$. Similarly, $d(y_{-j}, z'_{-j}) < \eta, \forall j > 0$.

This implies that x_{-k+1} is η-close to z_{-k+1} and y_{-k+1} is η-close to $z'_{-k+1} = z_{-k+1}$.

But $x_{-k} = f_*^{-1}(x_{-k+1}), y_{-k} = f_*^{-1}(y_{-k+1})$ and from above, $d(x_{-k+1}, y_{-k+1}) < 2\eta$. So, from the way η was chosen, we obtain

$$d(x_{-k}, y_{-k}) < \frac{\delta_0}{3}$$

But, from the way g was taken close to f,

$$d(\pi \circ h_g(\hat{f}^{-k}\hat{x}),\ x_{-k}) < \eta, \text{ and } d(\pi \circ h_g(\hat{f}^{-k}\hat{y}), y_{-k}) < \eta$$

So from the preceding two inequalities, we get

$$d(\pi \circ h_g(\hat{f}^{-k}\hat{x}), \pi \circ h_g(\hat{f}^{-k}\hat{y})) < \delta_0$$

But $\pi \circ h_g(\hat{f}^{-k}\hat{x}) = z_{-k}, \pi \circ h_g(\hat{f}^{-k}\hat{y}) = z'_{-k}$, and z_{-k}, z'_{-k} are different g-preimages in Λ_g of the same point z_{-k+1}, hence from the definition of δ_0, we should have $d(z_{-k}, z'_{-k}) > \delta_0$. We have thus obtained a contradiction.

Therefore the map $\pi \circ h_g \circ \sigma : V \to V_g$ is a homeomorphism, so V_g has all the topological properties of V, in particular it is compact, pathwise connected and simply connected.

Now, σ was determined by a fixed prehistory of an arbitrary point x from V. By taking all the possible prehistories of x, and the corresponding sections σ given by them, we will obtain for each such σ a homeomorphic image in Λ_g. In conclusion if we take all such sections σ for all the sets V_i which cover Λ, we will obtain a cover of Λ_g.

(b) For the proof of (b), we already have the (unique) sequence of inverse iterates defined on the whole Λ, since $f|_\Lambda$ is a homeomorphism. So we do not need anymore that Λ is written as a union of simply connected subsets.

Thus we have just one section σ as in (a) and hence Λ_g is homeomorphic to Λ.

□

As was already said, in this section we assume $C_f \cap \Lambda = \emptyset$. Therefore $|Df|E^s_x| \neq 0, \forall x \in \Lambda$. Hence, it makes sense to consider the function ϕ^s, from $\mathcal{C}(X, \mathbb{R})$, defined by $\phi^s(x) := \log|Df|_{E^s_x}|$. We now use the fact that, since ϕ^s is strictly negative on the space $X = \Lambda$, the mapping $t \to P^-(t\phi^s)$ is strictly decreasing (Proposition 3 and 4 (f)), and the fact that $P^-(0) = h^- \geq 0$. We assume also that $h^- < \infty$. Also, it is not difficult to see that $P^-(t\phi^s) < 0$ for t large enough. Hence this implies that there exists a unique $t^s \geq 0$ such that $P^-(t^s\phi^s) = 0$.

The same argument can be used to find a unique zero of $t \to P_-(t\phi^s)$.

Definition 5. In the above setting, the unique $t \geq 0$ such that $P^-(t\phi^s) = 0$ will be denoted by t^s and will be called the *zero of* $P^-(t\phi^s)$. The unique zero of $t \to P_-(t\phi^s)$ will be denoted by t^s_- and will be called the *zero of* $P_-(t\phi^s)$.

We are now ready to prove that, under a certain technical condition, t^s is equal to the Hausdorff dimension of the intersection between any local stable manifold and Λ, where Λ is a basic set with both stable and unstable directions (such basic sets are called *of saddle type*), and that in general, without the technical condition, $t^s_- \leq HD(W^s_\varepsilon(x) \cap \Lambda) \leq t^s$. Let us denote $\delta^s(x) = HD(W^s_\varepsilon(x) \cap \Lambda)$.

Theorem 2. *(a) Let f be a holomorphic Axiom A map of $\mathbb{P}^2$ and Λ a basic set of saddle type of the nonwandering set of f. Assume that $C_f \cap \Lambda = \emptyset$ and that Λ can be written as the union of countably many compact, pathwise connected and simply connected subsets.*

Then $t^s = \delta_s(x)$, for any point $x \in \Lambda$. In particular, $HD(W^s_\varepsilon(x) \cap \Lambda)$ does not depend on $x \in \Lambda$ in this case.

(b) In the setting from (a), if Λ is not necessarily the union of countably many compact, pathwise connected and simply connected sets, then $t^s \geq \delta^s(x) \geq \sup\limits_V t^s(V)$, where the supremum in the last

inequality is taken over all compact, pathwise connected and simply connected subsets V of Λ, and where $t^s(V)$ denotes the unique zero of the map $t \to P^-(t\phi^s, V)$.

Proof. In this proof we take $X = \Lambda$, $f : \Lambda \to \Lambda$ and x an arbitrary point in Λ. All the inverse pressures considered are relative to f as a surjective mapping on X. Therefore from Theorem 1, we know that $\tilde{P}^-(\cdot) = P^-(\cdot)$.

We will use interchangeably these two definitions of P^-.

Proof of (a):

First let us show that $t^s \geq \delta^s(x)$, where we remind the notation $HD(W^s_\varepsilon(x) \cap \Lambda) =: \delta^s(x)$. So, take an arbitrary $t > t^s$. Then, $P^-(t\phi^s) < \beta$ for some $\beta < 0$. Hence $\lim\limits_{n\to\infty} \frac{1}{n} \log P^-_n(\phi, \varepsilon') < \beta < 0$, for $\varepsilon' > 0$ small enough. So, there exists a collection $\Gamma \subset \mathcal{C}_n(\varepsilon')$ such that $\sum\limits_{C\in\Gamma} \exp(S^-_n\phi(C)) < e^{n\beta}$, for n large enough. But since Γ covers Λ, i.e. since $\Lambda = \bigcup\limits_{C\in\Gamma} \Lambda(C)$, we then have $\Lambda(C) \cap W^s_\varepsilon(x) \subset f^{n-1}(W^s_\varepsilon(x_{-n+1}))$, where $x \in \Lambda$ and C is associated to an n-prehistory of x, $(x_0, x_{-1}, ..., x_{-n+1})$, (where $x_0 = x$).

Denote $W := W^s_\varepsilon(x) \cap \Lambda$. Now by using the Distortion Lemma from [11], and the fact that f is conformal on its stable sets, we obtain that $\text{diam}(\Lambda(C) \cap W) \leq A\varepsilon|Df^n|_{E^s_\xi}|$, where ξ is some point in $B(x_{-n+1}, \varepsilon')$ and A is a fixed constant independent of n. This implies that:

$$\sum_{C\in\Gamma} (\text{diam}\Lambda(C))^t \leq A^t\varepsilon^t \sum_{C\in\Gamma} \exp(S^-_n(t\phi^s)(C)) \cdot \exp(n\delta_\phi(\varepsilon')) \leq e^{n\beta} e^{n\delta_\phi(\varepsilon')}\varepsilon^t.$$

(In fact due to the bounded distortion property $\exp(n\delta_\phi(\varepsilon'))$ can even be replaced by a constant). But in the last inequality, $\beta < 0$ is independent of ε' and we can assume that $\beta > 2\delta_\phi(\varepsilon')$ which implies that for n large, $\sum\limits_{C\in\Gamma} (\text{diam}\Lambda(C))^t < 1$. Since $\text{diam}(\Lambda(C)) \to 0$ when $n \to \infty$, we obtain $\delta \leq t$. But since t was taken arbitrarily larger than t^s, we conclude that $t^s \geq \delta^s(x)$.

We now proceed to the proof of the other inequality, i.e $\delta^s(x) \geq t^s$. Take an arbitrary $t > \delta^s(x)$. We shall show that $t \geq t^s$, i.e. that $P^-(t\phi^s) \leq 0$.

Fix $\varepsilon' > 0$ and $\eta > 0$. Since Λ was assumed to be a basic set, the map $f|_\Lambda$ is transitive. Using this and the local product structure proof of Theorem 2 of [13], that there exists a positive integer m such that $f^{-m}(W^s_\varepsilon(x) \cap \Lambda) \cap \Lambda$ is $\varepsilon'/2$-dense in Λ and that any local unstable manifold, $W^u_{\varepsilon'/2}(\hat{z})$ intersects it, for all $\hat{z} \in \hat{\Lambda}$. Since f is a local homeomorphism near Λ (due to $C_f \cap \Lambda = \emptyset$), we have also that

$$HD(f^{-m}(W^s_\varepsilon(x) \cap \Lambda)) = HD(W^s_\varepsilon(x) \cap \Lambda)$$

We can use now the fact that Λ is written as the union of countably many compact, pathwise connected, simply connected subsets V_i, $i > 0$ integer.

From Proposition 2 we know that $P^-(\phi) = \sup_i P^-(\phi, V_i)$. If we denote the zero of the map $t \to P^-(t\phi^s, V_i)$ by $t^s(V_i)$, it is easy to see that $t^s = \sup_i t^s(V_i)$.

Therefore, if we prove that for all integers $i > 0$, $\delta^s(x) \geq t^s(V_i)$, then the inequality $\delta^s(x) \geq t^s$ will hold as well. So, in the sequel we restrict attention to such a simply connected subset V_i, which will be denoted for brevity by V.

As in the proof of Proposition 10, one can define on V a sequence of inverse iterates $f_*^{-n}, n > 0$. Therefore we have on V a well-defined distribution of unstable local manifolds given by the prehistories obtained from the inverse iterates f_*^{-n}.

For a point y from V denote by $\psi(y)$ the local unstable manifold $W^u_{\frac{\varepsilon'}{2}}(\hat{y})$, where $\hat{y}$ is the prehistory defined by taking $y_{-j} = f_*^{-j}(y), j > 0, y_0 = y$.

Consider now a point y from V and z belonging to $\psi(y)$. Then there exists a prehistory $(z_{-k})_{k\geq 0}$ of z such that $d(z_{-k}, y_{-k}) < \frac{\varepsilon'}{2}, \forall k \geq 0$.

We claim there is only one prehistory of z with this property. Indeed, if $(z'_{-k})_k$ were another such prehistory, we would have $d(z_{-k}, z'_{-k}) < \varepsilon', \forall k > 0$. Assuming that N is an integer such that $z_{-k} = z'_{-k}, 0 < k < N$ and $z_{-N} \neq z'_{-N}$, it follows that z_{-N} and z'_{-N} are two different preimages of the same point z_{-N+1}, and since $\mathcal{C}_f \cap \Lambda = \emptyset$, we cannot have $d(z_{-N}, z'_{-N}) < \varepsilon'$ if ε' has been chosen small enough. Therefore the prehistory $(z_{-k})_k$ is uniquely defined by being the prehistory of z which follows the prehistory of y corresponding to $\psi(y)$.

Hence we can extend the inverse iterates $f_*^{-k}, k > 0$ also to the unstable manifolds $\psi(y), y \in V$.

Since $t > \delta^s(x)$, there exists $\mathcal{U} = (U_i)_{i\in I}$, a cover of $f^{-m}(W^s_\varepsilon(x) \cap \Lambda)$ by open balls with *mesh* $(\mathcal{U}) < \eta/2$, $\eta << \varepsilon'$ and such that

$$\sum_i (\mathrm{diam}(U_i))^t \leq 1.$$

In the sequel we shall denote $|Df^k|_{E^s_x}|$ by $|Df^k_s(x)|$.

For every set U_i let us take $\tilde{U}_i := \bigcup_{y\in V, \psi(y)\cap U_i \neq \emptyset} \psi(y), i > 0$. Obviously $\bigcup_i \tilde{U}_i$ covers the entire set V. From the discussion above, the inverse iterates f_*^{-n} are well defined on the sets $\tilde{U}_i$.

For each i, take n_i the first positive integer such that $\mathrm{diam} f_*^{-n}(\tilde{U}_i) \leq \varepsilon', n < n_i$ and $\mathrm{diam} f_*^{-n_i}(\tilde{U}_i) > \varepsilon'$. If $z \in \psi(y), y \in V$, then $f^{-n}(z)_*$ becomes closer and closer to $f_*^{-n}(y)$ when n increases, in any case closer than $\frac{\varepsilon'}{2}$. Hence $d(f^{-n}(\tilde{U}_i), f^{-n}(U_i)) < \frac{\varepsilon'}{2}, \forall n > 0$.

Notice however that the set $U_i \cap \tilde{U}_i$ is contained in a small open analytic disk inside $f^{-m}(W^s_\varepsilon(x))$. Also, the stable tangent space makes sense at all points in $f^{-m}(W^s_\varepsilon(x))$, not only at points from Λ.

Let us denote $U_i^* := U_i \cap \tilde{U}_i$.

By inflating slightly U_i^* to an open set D_i contained in $f^{-m}(W^s_\varepsilon(x))$, we can still define the inverse iterates $f_*^{-k}, k \leq n_i$ on D_i. This is possible due to the fact that, first, $\mathrm{diam} f_*^{-1}(U_i^*) < \varepsilon'$ and the distance between different preimages of an open analytic disk neighbourhood of U_i inside $f^{-m}(W^s_\varepsilon(x))$, is bounded below by a positive constant α independent of i (since $\mathcal{C}_f \cap \Lambda = \emptyset$ and $\tilde{U}_i$ is ε'-close to Λ). One can arrange for ε' to be less than $\frac{\alpha}{4}$. Hence $f_*^{-1}(U_i^*)$ is contained in just one analytic disk D_i^{-1}, preimage of an analytic disk D_i containing U_i^* in $f^{-m}(W^s_\varepsilon(x))$ (since the distance between different components of $f^{-1}(D_i)$ would be, as we saw, larger than $\alpha > 2\mathrm{diam} f_*^{-1}(U_i^*)$).

Afterwards, we apply the same argument again to preimages of D_i^{-1}. Inductively we obtain that $f^{-n_i}(U_i^*)$ is contained in an analytic disk, denoted by $D_i^{-n_i}$ where $D_i^{-n_i} \subset f^{-m-n_i}(W^s_\varepsilon(x))$. So the tangent space to this analytic disk is in the stable direction, induced from $W^s_\varepsilon(x)$.

Hence there exists a point $\xi_i \in D_i^{-n_i}$ (ξ_i is actually in the convex cover of $f_*^{-n_i}(U_i^*)$ in the analytic disk $D_i^{-n_i}$) such that $d(f^k(\xi_i), f_*^{k-n_i}(U_i^*)) < \frac{\varepsilon'}{4}, 0 \le k \le n_i$, n_i large, (because f^k contracts distances on the stable disk $D_i^{-n_i}$ and, from the Mean Value Inequality on this analytic disk,

$$\varepsilon' < \mathrm{diam} f_*^{-n_i}(U_i) \le 2\mathrm{diam} U_i \cdot |Df_s^{n_i}(\xi_i)|^{-1}$$

But $d(f^k(\xi_i), f_*^{k-n_i}(U_i^*)) < \frac{\varepsilon'}{4}, k \le n_i$, and $\mathrm{diam}(f_*^{-k}(U_i)) < \varepsilon'$, therefore we can take a point $y_i \in V \cap \tilde{U}_i$ such that $d(f^k(\xi_i), f^k(y_i)) < 2\varepsilon'$. So the above inequality becomes:

$$(\varepsilon')^t < (\mathrm{diam} f_*^{-n_i}(U_i))^t \le 2^t (\mathrm{diam} U_i)^t \cdot |Df_s^{n_i}(y_i)|)^{-t} \exp(tn_i \cdot \delta(2\varepsilon')),$$

where $\delta(2\varepsilon')$ is the maximum oscillation of $t\phi^s$ on a ball of radius $2\varepsilon'$.

We may in addition require $mesh(\mathcal{U})$ to be so small that $n_i \ge N, \forall i \in I$, N arbitrarily large.

We shall now denote by C_i, $i \in I$, the ordered collection of balls,

$\{B(f^{n_i}(y_i), 2\varepsilon'), B(f^{n_i-1}(y_i), 2\varepsilon'), ..., B(f(y_i), 2\varepsilon')\}$, with y_i found above.

We can then estimate as follows:

$$1 \ge \sum_{i\in I} (\mathrm{diam}(U_i))^t \ge C(\varepsilon')^t \cdot \sum_{i \in I} |Df_s^{n_i}(y_i)|^t \exp(-n_i \cdot 2\delta(2\varepsilon')t) \ge C(\varepsilon')^t \sum_{i\in I} \exp(S_{n_i}^-(t\phi^s)(C_i) - 2n_i\delta(2\varepsilon')t), \tag{6}$$

where C is a positive universal constant.

We shall show now that $V \subset \bigcup_i \Lambda(C_i)$. This follows immediately from the fact that $V = \bigcup_i \tilde{U}_i$ and since $\tilde{U}_i \subset \Lambda(C_i)$. (since $\tilde{U}_i$ is just a union of local unstable manifolds of the form $\psi(y)$).

Using the definition of $M(\cdot,\cdot,\cdot,\cdot)$, and inequality (6), we therefore get that for any $\lambda > 3\delta(2\varepsilon')$

$$\begin{aligned} M(\lambda, t\phi^s, N, \varepsilon') &\le \sum_{i\in I} \exp(-\lambda n(C_i) + S_{n_i}^-(t\phi^s)(C_i)) \le \\ &\le e^{-\delta(2\varepsilon')N} \sum_{i \in I} \exp(S_{n_i}^-(t\phi^s)(C_i) - n_i\delta(2\varepsilon')) \le C^{-1}(\varepsilon')^{-t} e^{-\delta(2\varepsilon')N} \end{aligned}$$

Hence, taking the limit if $N \to \infty$, we obtain $m(\lambda, t\phi^s, \varepsilon') = 0$, hence $\tilde{P}^-(t\phi^s, V, \varepsilon') \le 3\delta(2\varepsilon')$. Then by taking ε' approaching zero, it follows that $\tilde{P}^-(t\phi^s, V) \le 0$ and therefore $t \ge t^s(V)$. Consequently $\delta^s(x) \ge \sup_i t^s(V_i)$, and $\delta^s(x) \ge t^s$. In conclusion the proof of (a) is finished.

Proof of (b):

In this case the proof follows in the same way as for (a). The inequality $t^s \ge \delta^s(x)$ does not use the fact that Λ can be written as a countable union of simply connected sets V_i, hence it still holds in case (b). In the end of the proof of (a), since the simply connected set V is arbitrary, we get that $\delta^s(x) \ge \sup_V t^s(V)$, which represents the lower estimate in the statement.

□

Another lower estimate for the stable dimension can be given using the inverse lower pressure P_- and the unique zero t_-^s of the map $t \to P_-(t\phi^s)$:

Theorem 3. *Let f be a holomorphic Axiom A map of $\mathbb{P}^2$, and Λ a basic set of saddle type for f such that $\mathcal{C}_f \cap \Lambda = \emptyset$. Then $HD(W^s_\varepsilon(x) \cap \Lambda) \geq t^s_-$, for any point x in Λ.*

Proof. The proof here will follow similarly as in the proof of Theorem 2. We denote $\delta^s(x) := HD(W^s_\varepsilon(x) \cap \Lambda)$ and then take $t > \delta^s(x)$ arbitrary.

Therefore there will exist $\mathcal{U} = (U_i)_{i \in I}$, a cover of $f^{-m}(W^s_\varepsilon(x) \cap \Lambda)$ with $mesh(\mathcal{U}) < \eta << \varepsilon'$, such that $\sum_i (\text{diam}(U_i))^t \leq 1$; m and ε' have been taken as in the proof of part (a) of Theorem 2.

Now, for each U_i let us take n_i the largest integer such that all components of $f^{-k}(U_i)$ have diameter less than ε' small, and $f^{-n_i}(U_i)$ has a component $U_i^{n_i}$ of diameter larger or equal than ε'.

But, from the Mean Value Inequality there will exist then a point ξ_i in $U_i^{n_i}$ such that

$$\text{diam}(U_i) \geq \varepsilon' |Df_s^{n_i}(\xi_i)| \geq \varepsilon' \exp(S_{n_i,-}\phi^s(f^{n_i}(\xi_i))), i \in I.$$

The point $f^{n_i}(\xi_i)$ belongs to U_i, for all $i \in I$. Since all unstable manifolds of points in Λ, of size $\frac{\varepsilon'}{2}$ intersect the set $f^{-m}(W^s_\varepsilon(x) \cap \Lambda)$, it follows that Λ is the union of the inverse balls $B^-_{n_i}(f^{n_i}(\xi_i), 2\varepsilon'), i \in I$. For each U_i we have only one inverse ball $B^-_{n_i}(f^{n_i}(\xi_i), 2\varepsilon')$ and $(\text{diam}(U_i))^t \geq \exp(S_{n_i,-}t\phi^s(f^{n_i}(\xi_i)))$, so

$$1 \geq \sum_i (\text{diam}(U_i))^t \geq \sum_i \exp(S_{n_i,-}t\phi^s(f^{n_i}(\xi_i)))$$

But from the fact that the sets $B^-_{n_i}(f^{n_i}(\xi_i), 2\varepsilon'), i \in I$, cover Λ, it follows that $P_-(t\phi^s) \leq 0$, hence $t \geq t^s_-$. Since t was chosen arbitrarily bigger than $\delta^s(x)$, we get that $\delta^s(x) \geq t^s_-$, for all x in Λ.

$\square$

Corollary 1. *In the above setting, if f is an Axiom A holomorphic map on $\mathbb{P}^2$ and Λ a basic set of f, then, if there exists a point $x \in \Lambda$ such that $HD(W^s_\varepsilon(x) \cap \Lambda) \neq 0$, it follows that Λ cannot be a Jordan curve.*

Proof. Let us assume on the contrary, that there exists a basic set Λ such that $HD(W) \neq 0$ and Λ is a Jordan curve, where again we denoted $W := W^s_\varepsilon(x) \cap \Lambda$. But we proved in Proposition 5 that $0 \leq h^- \leq h_i$. Now we use the Theorem of Nitecki-Przytycki saying that if f is a continuous map on a Jordan curve Λ, then $h_i(f|_\Lambda) = 0$. (as notations, $h^- = h^-(f), h_i(f) = h_i$ in the current situation).

Therefore, $h^- = P^-(0) = 0$, so $t^s = 0$; contradiction with the fact that $t^s \geq HD(W^s_\varepsilon(x) \cap \Lambda) > 0$.

$\square$

Corollary 2. *In the above setting, if $h(f|_\Lambda) \neq 0$ and $f|_\Lambda$ is a homeomorphism, then $HD(W^s_\varepsilon(x) \cap \Lambda) \neq 0, \forall x \in \Lambda$.*

Proof. Indeed, if $f|_\Lambda$ is a homeomorphism, then, from the properties of lower inverse pressure (Proposition 7 g)), we have that $h(f|_\Lambda) = h^-(f|_\Lambda) = h_-(f|_\Lambda) \neq 0$. This implies that $P_-(0) \neq 0$, hence $t^s_- \neq 0$, so Theorem 3 now shows that $HD(W^s_\varepsilon(x) \cap \Lambda) \neq 0$.

$\square$

Examples:

In [14] we gave a large class of examples of perturbations of the map $(z, w) \to (z^2 + c, w^2)$, on $\mathbb{P}^2$, with basic sets on which they are homeomorphisms. These are maps of the form $f_{\varepsilon'}(z, w) = (z^2 + a\varepsilon' z + b\varepsilon' w + c + d\varepsilon' zw + e\varepsilon' w^2, w^2)$, where $|c| \neq 0$, small, and $|c| < c(a, b, d, e), b \neq 0, \varepsilon' < \varepsilon'(a, b, c, d, e)$. $\Lambda_{\varepsilon'}$ denotes the basic set of $f_{\varepsilon'}$ close to the circle $\{p_0(c)\} \times S^1$, where $p_0(c)$ is a fixed attracting point of the map $z \to z^2 + c$. It is shown in [14] that $f_{\varepsilon'}|_{\Lambda_{\varepsilon'}}$ is a homeomorphism.

Being perturbations of $f(z, w) = (z^2 + c, w^2)$ and since their considered basic set is close to $p_0(c) \times S^1$, it will follow from the Stability Theorem ([17]) that there exists a homeomorphism $h : \hat{\Lambda} \to \hat{\Lambda}_{\varepsilon'}$ commuting with $\hat{f}$ and $\hat{f}_{\varepsilon'}$. Hence $h(f) = h(\hat{f}) = h(\hat{f}_{\varepsilon'}) = h(f_{\varepsilon'}) = h^-(f_{\varepsilon'}) = \log 2$. Therefore, for this type of maps, Corollary 2 implies that the stable dimension is non-zero.

Also, let us notice that, according to Proposition 10 b), if g is itself a small perturbation of $f_{\varepsilon'}$, (for an ε' fixed, small) then $g|_{\Lambda_g} : \Lambda_g \to \Lambda_g$ is a homeomorphism too, so we can apply again Corollary 2 to get that the stable dimension on Λ_g is non-zero and that Λ_g is not a graph.

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