Geometry and Ergodic Theory of Conformal Nonrecurrent Dynamics

Mariusz Urbański

Department of Mathematics, University of North Texas,
Denton, TX 76203-5116, USA
e-mail: urbanski@unt.edu

Abstract: Let h be the Hausdorff dimension of the Julia set of a rational function T with no non-periodic recurrent critical points and let m be the only h-conformal measure for T. We prove the existence of a σ-finite T-invariant measure μ equivalent with m. The measure μ is then proved to be ergodic and conservative and we study the set of those points whose all open neighborhoods have infinite measure μ. Developing the concept of the inverse jump transformation we show that the packing and Hausdorff dimensions of conformal measure are equal to h. We also provide some sufficient conditions for Hausdorff and box dimensions of the Julia set to be equal.

§**1. Introduction.** In this paper we continue our study on rational functions with no recurrent critical points in the Julia set which in the sequel will be abbreviated as NCP maps. We prove the existence of a σ-finite T-invariant measure μ equivalent with m. The measure μ is the proved to be ergodic and conservative and we study the set of those points whose all open neighborhoods have infinite measure μ. We provide some sufficient conditions for points of the Julia set to be of finite or infinite condensation. The results obtained incline us to formulate conjecture that in contrast to parabolic maps (see [ADU]) the cumulated influence of critical and parabolic points may result in a bigger set of of points of infinite condensation than just the set of parabolic points. We also show that such cumulated effect gives rise to a weaker sufficient requirement for parabolic points to be of infinite condensation. In other words critical points increase the chances of various points (particularly parabolic points) to be of infinite condensation.

In the second part of our paper we prove that the packing dimension of conformal measure is equal to the Hausdorff dimension of the Julia set. We develope here the argument used in [DU3] to show equality of box and Hausdorff dimensions of Julia sets of parabolic rational functions. Since the direct analog of the jump transformation does not seem to be usefull in the presence of critical points, we introduce the procedure of grouping inverse images simulating inverse branches of the jump transformation of parabolic mappings. As a byproduct we prove that the diameters of of sufficiently small balls under this procedure converge to zero exponentially fast which can be treated as an extention of the result of Carleson, Johns and Yoccoz (see [CJY]) to the case of nonrecurrent dynamics with parabolic points. We would also like to add here that in the lack of parabolic points the equality of box and Hausdorff dimensions has been shown by F. Przytycki (see [Pr]) for Collect-Eckmann Rational maps.

Throughout the whole paper we essentially respect the notation and terminology used in [Ur] (with minor changes indicated in Preliminaries, II). The paper is organized as follows.

Section 1 provides some general results from geometric measure theory and analytic functions and Section 2 provides some basic technical properties of NCP maps. In Section 4 we prove the existence of a unique σ-finite T-invariant measure μ absolutely continuous with respect to the conformal measure and we show that this measure is conservative and ergodic. The next section, Section 5 is is devoted to study points of infinite condensation of μ and Section 6, its natural continuation deals with parabolic points. Finally Section 7 which to some extent forms a separate unit contains results about dimensions of Julia sets.

§**2. Preliminaries, I.** In this section we collect some general facts not involving dynamics. To begin with given a subset A of a compact metric space (X, d), a countable family $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of open balls centered at points of A is said to be a packing of A if and only if for any pair $i \neq j$, $d(x_i, x_j) \geq r_i + r_j$. Given a nondecreasing function $g : (0, \varepsilon) \to (0, \infty)$ for some $\varepsilon > 0$, the g-dimensional outer Hausdorff measure $\mathrm{H}_g(A)$ of the set A is defined as

$$\mathrm{H}_g(A) = \sup_{\varepsilon > 0} \inf \Big\{ \sum_{i=1}^{\infty} g\big(\mathrm{diam}(A_i)\big) \Big\},$$

where infimum is taken over all countable covers $\{A_i : i \geq 1\}$ of A by arbitrary sets whose diameters do not exceed ε. If g is of the form x^t instead of writing H_{x^t} we write H_t and speak about t-dimensional outer Hausdorff measure. In this case one will get comparable numbers (in the sense that ratios are bounded away from zero and infinity) if instead of covering A by arbitrary sets one considers only open balls centered at points of A.

The g–dimensional outer packing measure $\Pi_g(A)$ of the set A is defined as

$$\Pi_g(A) = \inf_{\cup A_i = A} \Big\{ \sum_i \Pi_g^*(A_i) \Big\}$$

(A_i are arbitrary subsets of A), where Π_g^*, the g-packing premeasure is given by $\Pi_g^*(A) = \inf_{\varepsilon > 0} \sup\big\{\sum_{i=1}^{\infty} g(r_i)\big\}$. Here the supremum is taken over all packings $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of the set A by open balls centered at points of A with radii which do not exceed ε. Similarly as in the case of Hausdorff measures if g is of the form x^t instead of writing Π_{x^t} we write Π_t and speak about t-dimensional outer packing measure. These two outer measures H_g and Π_g define countable additive measures on Borel σ-algebra of X. For additional properties of packing measures and a comprehensive discussion of this and related notions the reader is referred to the paper [TT], [Mat] and [PU] books.

The definitions of the Hausdorff dimension $\mathrm{HD}(A)$ of A and packing dimension $\mathrm{PD}(A)$ are the following

$$\mathrm{HD}(A) = \inf\{t : \mathrm{H}_t(A) = 0\} = \sup\{t : \mathrm{H}_t(A) = \infty\}$$

and

$$\mathrm{PD}(A) = \inf\{t : \Pi_t(A) = 0\} = \sup\{t : \Pi_t(A) = \infty\}.$$

Recall also that given a Borel probability measure μ on X one defines its Hausdorff and packing dimensions respectively as

$$\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(Y) : \mu(Y) = 1\}$$

and

$$\mathrm{PD}(\mu) = \inf\{\mathrm{PD}(Y) : \mu(Y) = 1\}.$$

In this paper we shall also deal with the box dimension of the Julia set. In order to recall it we first define lower box dimension $\underline{\mathrm{BD}}(A)$ and upper box dimension $\overline{\mathrm{BD}}(A)$ respectively by the following two formulas.

$$\underline{\mathrm{BD}}(A) = \liminf_{r\to 0} \frac{\log N(A,r)}{-\log r} \quad \text{and} \quad \overline{\mathrm{BD}}(A) = \limsup_{r\to 0} \frac{\log N(A,r)}{-\log r},$$

where $N(A,r)$ is the minimal number of balls with radii $\leq r$ needed to cover A. If the lower and upper box dimensions coincide, then their common value $\mathrm{BD}(A) = \underline{\mathrm{BD}}(A) = \overline{\mathrm{BD}}(A)$ is called the box dimension of A. In the literature all box dimensions are also called, perhaps more properly, box counting dimensions; we will keep here the shorter name. We recall (see [Fa] for example) that $\mathrm{HD}(A) \leq \mathrm{PD}(A) \leq \overline{\mathrm{BD}}(A)$ and $\mathrm{HD}(A) \leq \underline{\mathrm{BD}}(A) = \underline{\mathrm{BD}}(\overline{A}) \leq \overline{\mathrm{BD}}(A) = \overline{\mathrm{BD}}(\overline{A})$.

Writing $A \preceq B$ we mean that there exists a universal constant $C \geq 1$ such that $A \leq CB$. The relation $A \asymp B$ means that $A \preceq B$ and $B \preceq A$. A probability measure μ on a metric space X is said to be

(a) upper geometric at a point $x \in X$ if and only if $\mu(B(x,r)) \preceq r^{\mathrm{HD}(X)}$ $(r > 0)$.

(b) geometrically strong at x if and only if $\mu(B(x,r)) \asymp r^{\mathrm{HD}(X)}$ $(0 < r \preceq 1)$.

(c) uniformly geometrically strong at x if and only if $\mu(B(z, \sigma\mathrm{dist}(x,z)) \asymp \mathrm{dist}^{\mathrm{HD}(X)}(x,z)$ for every $x \in X$, every $0 < r < 1$ and every σ ranging over a compact set of $(0,\infty)$.

Not that if μ is uniformly geometrically strong at a point of X, then μ is geometrically strong at this point. Note also that in [Ur] a different terminology has been used (see p. 394 there) like points which are upper estimable, lower estimable and strongly lower estimable.

Passing to conformal maps we consider the situation where $H : U_1 \to U_2$ is an analytic map of open subsets U_1, U_2 of the Riemann sphere $\overline{\mathbb{C}}$. We say that given t, Borel probability measures m_1 and m_2 on $\overline{\mathbb{C}}$ form a t-conformal pair of measures if and only if

$$m_2(H(A)) = \int_A |H'|^t \, dm_1$$

for every Borel subset A of U_1 such that $H|_A$ is one-to-one. A Borel probability measure is said to be t-conformal if and only if the measures $m_1 = m$ and $m_2 = m$ form a t-conformal pair of measures. We note that unless otherwise indicated we always work with spherical metric in $\overline{\mathbb{C}}$. In the sequel we will frequently use the following

Koebe Distortion Theorem (KDT). Given a number $s > 0$ there exists a function $k : [0,1) \to [1,\infty)$ such that for any $z \in \overline{\mathbb{C}}$, $r > 0, t \in [0,1)$ and any univalent analytic function $H : B(z,r) \to \overline{\mathbb{C}}$ such that the complement $\overline{\mathbb{C}} \setminus H(B(z,r))$ contains a ball of radius s we have

$$\sup\{|H'(x)| : x \in B(z,tr)\} \le k(t) \inf\{|H'(x)| : x \in B(z,tr)\}.$$

We put $K = k(1/2)$ and invoking this theorem we will usually refer simply to KDT. We shall also use the following more geometric version of the Koebe Distortion Theorem involving moduli of annuli.

Koebe Distortion Theorem, II (KDT(II)). Given a number $s > 0$ there exists a function $w : (0,+\infty) \to [1,\infty)$ such that for any two open topological disks $Q_1 \subset Q_2$ with $\mathrm{Mod}(Q_2 \setminus Q_1) \ge w$ and any univalent analytic function $H : Q_2 \to \overline{\mathbb{C}}$ such that the complement $\overline{\mathbb{C}} \setminus H(Q_2)$ contains a ball of radius s we have

$$\sup\{|H'(x)| : x \in Q_1\} \le w(t) \inf\{|H'(x)| : x \in Q_1\}.$$

If c is a critical point of an analytic map H, by $q(c)$ we denote its multiplicity. Then there exists $A = A(H,c) \ge 1$ such that

$$A^{-1}|z-c|^{q(c)} \le |H(z) - H(c)| \le A|z-c|^{q(c)}$$

and

$$A^{-1}|z-c|^{q(c)-1} \le |H'(z)| \le A|z-c|^{q(c)-1}$$

for all z sufficiently close to c. If there is no confusion we will write simply q instead of $q(c)$. We shall now collect some basic properties of conformal pairs.

Lemma 2.1. Suppose that c is a critical point of an analytic mapping $H : D \to \mathbb{C}$. Suppose that m_1 and m_2 form a t-conformal pair of measures for H. If $B \subset \mathbb{C}$ is sufficiently close to c and $m_2(G) \preceq \mathrm{diam}^t(G)$ for all Borel subsets G of $H(B)$, then

$$m_1(F) \preceq \mathrm{diam}^t(F)$$

for all Borel subsets F of B.

Proof. (Comp. Lemma 3.4, p. 398, [Ur]) Let L be the constant giving inequality $m_2(G) \preceq \mathrm{diam}^t(G)$. For $F \subset B$ and $n \ge 0$ set $\delta = \mathrm{diam}(F)$ and $R_n = F \cap \{z : \delta 2^{-(n+1)} < |z-c| \le \delta 2^{-n}\}$. Then $H(R_n) \subset \{z \in H(F) : A^{-1}\delta^q 2^{-q(n+1)} < |z - H(c)| \le \delta^q 2^{-qn}\}$, whence

$$\begin{aligned} L(2A\delta^q 2^{-qn})^t \ge m_2(H(R_n)) &= q^{-1}\int_{R_n} |H'(z)|^t \, dm_1(z) \\ &\ge q^{-1}A^{-1}\delta^{(q-1)t}2^{-(n+1)(q-1)t} m_1(R_n). \end{aligned}$$

Thus $m_1(R_n) \le qAL(2A)^t 2^{(q-1)t} 2^{-tn(q+1-q)} = qAL(2A)^t 2^{(q-1)t}\delta^t 2^{-tn}$ and therefore

$$m_1(F) = m_1\Big(\bigcup_{n\ge 0} R_n\Big) = \sum_{n\ge 0} m_1(R_n) \le qAL(2A)^t 2^{(q-1)t}\delta^t \sum_{n\ge 0} 2^{-tn}$$
$$= qAL(2A)^t 2^{(q-1)t}\frac{1}{1-2^{-t}}\mathrm{diam}^t(F).$$

The proof is finished. ♣

For A, B, any two subsets of a metric space put

$$\mathrm{dist}(A,B) = \inf\{\mathrm{dist}(a,b) : a \in A, b \in B\}$$

and

$$\mathrm{Dist}(A,B) = \sup\{\mathrm{dist}(a,b) : a \in A, b \in B\}.$$

If c is a critical point of an analytic mapping $H : D \to \overline{\mathbb{C}}$ and A is contained in a sufficiently small neighborhood of $H(c)$, then by $H_c^{-1}(A)$ we denote the subset of $H^{-1}(A)$ contained in a sufficiently small neighborhood of c.

We shall prove the following.

Lemma 2.2. Suppose that c is a critical point of an analytic mapping $H : D \to \overline{\mathbb{C}}$. Suppose that m_1 and m_2 form a t-conformal pair of measures for H. If A is a Borel set sufficiently close to $H(c)$, then

$$m_2(A)\big(\mathrm{Dist}(H(c),A)\big)^{\frac{1-q}{q}t} \preceq m_1\big(H_c^{-1}(A)\big) \preceq m_2(A)\big(\mathrm{dist}(H(c),A)\big)^{\frac{1-q}{q}t},$$

or equivalently

$$\frac{m_2(A)}{\mathrm{Dist}^t(H(c),A)} \preceq \frac{m_1(H_c^{-1}(A))}{\mathrm{Dist}^t(c,H_c^{-1}(A))} \preceq \frac{m_1(H_c^{-1}(A))}{\mathrm{dist}^t(c,H_c^{-1}(A))} \preceq \frac{m_2(A)}{\mathrm{dist}^t(H(c),A)}$$

Proof. We have

$$m_2(A) = m_2(H(H_c^{-1}(A))) \ge \frac{1}{q}\int_{H_c^{-1}(A)} |H'|^t\, dm$$
$$\succeq m_1(H_c^{-1}(A))\big(\mathrm{dist}(c,H_c^{-1}A)\big)^{(q-1)t}$$
$$\asymp m_1(H_c^{-1}(A))\big(\mathrm{dist}(H(c),A)\big)^{\frac{q-1}{q}t}.$$

Hence $m_1(H_c^{-1}(A)) \preceq m_2(A)\big(\mathrm{dist}(H(c),A)\big)^{\frac{1-q}{q}t}$. A similar estimate gives $m_1(H_c^{-1}(A)) \succeq m_2(A)\big(\mathrm{Dist}(H(c),A)\big)^{\frac{1-q}{q}t}$ and we are done. ♣

Lemma 2.3. Suppose that c is a critical point of an analytic map $H : D \to \overline{\mathbb{C}}$. Suppose that m_1 and m_2 form a t-conformal pair of measures for H. If $m_1(B(c,r)) \preceq r^t$ for every $r > 0$ small enough, then

$$m_1(H_c^{-1}(A)) \preceq m_2(A)^{1/q(c)}$$

for every Borel set $A \subset D$ sufficiently close to $H(c)$.

Proof. We may assume that $A \subset B(H(c),1)$ and $H_c^{-1}(A) \subset B(c,1)$. By conformality of m_1 and m_2 we may also assume that $m_2(A) > 0$. Let $R(z,r_1,r_2) = \{y \in \mathbb{C} : r_1 < |y-z| < r_2\}$ and let

$$R = \inf\{r \geq 0 : m_2(A \cap R(H(c),r,1)) < r^t\}.$$

Then $0 < R < 1$, $m_2\big(A \cap R(H(c),R,1)\big) \leq R^t$ and $m_2\big(A \cap \overline{R(H(c),R,1)}\big) \geq R^t$. Therefore

$$\begin{aligned}
m_1(H_c^{-1}(A)) &= m_1\big(H_c^{-1}(A \cap \overline{B(H(c),R)}\big) + m_1\big(H_c^{-1}(A \cap R(H(c),R,1)\big) \\
&\leq m_1\big(H_c^{-1}(\overline{B(H(c),R)}\big) + m_1\big(H_c^{-1}(A \cap R(H(c),R,1)\big) \\
&\preceq m_1\big(B(c,AR^{1/q})\big) + q(R^{\frac{1}{q}-1})^t m_2\big(A \cap R(H(c),R,1\big) \\
&\preceq R^{t/q} + R^{\frac{t}{q}-t}R^t \asymp R^{t/q} \leq m_2\big(A \cap \overline{R(H(c),R,1)}\big)^{1/q} \\
&\preceq m_2(A)^{1/q}.
\end{aligned}$$

The proof is finished. ♣

Lemma 2.4. Suppose that H is a holomorphic univalent map on a neighborhood $U \subset \overline{\mathbb{C}}$ of a point x_0. Suppose that m_1 and m_2 form a t-conformal pair of measures for H. If $F \subset B(x_0,r) \subset B(x_0,2r) \subset U$ is a Borel set such that $m_2(F) \succeq \mathrm{diam}^\alpha(F)$, for some $\alpha \in \mathbb{R}$, then $m_1(H^{-1}(F)) \succeq \mathrm{diam}^\alpha(H^{-1}(F))$.

Proof. By conformality, KDT, and our assumptions

$$\begin{aligned}
m_1(H^{-1}(F)) &\asymp |H'(x_0)|^{-t} m_2(F) \succeq |H'(x_0)|^{-t}\mathrm{diam}^\alpha(F) \\
&= (|H'_{x_0}|^{-1}\mathrm{diam}(F))^\alpha |H'(x_0)|^{\alpha-t} \succeq (|H'(x_0)|^{-1}\mathrm{diam}(F))^\alpha \\
&\asymp \mathrm{diam}^\alpha(H^{-1}(F)).
\end{aligned}$$

The proof is finished. ♣

Lemma 2.5. Suppose that c is a critical point of an analytic map $H : D \to \mathbb{C}$. Suppose that m_1 and m_2 form a t-conformal pair of measures for H. If F is a Borel set sufficiently close to $H(c)$ and $m_2(F) \succeq \mathrm{Dist}^\alpha(F,H(c))$ for some $\alpha > 0$, then

$$m_1(H_c^{-1}(F)) \succeq \mathrm{Dist}(H_c^{-1}(F),c)^{t+q(c)(\alpha-t)},$$

In particular if F contains $H(c)$, then dist can be replaced everywhere by diam.

Proof. By conformality and by our assumptions we can estimate as follows

$$\begin{aligned}
\mathrm{Dist}^{q\alpha}(H_c^{-1}(F),c) \asymp \mathrm{Dist}^\alpha(F,H(c)) \leq m_2(F) &\preceq \int_{H_c^{-1}(F)} |H'(z)|^t \, dm_1(z) \\
&\preceq \int_{H_c^{-1}(F)} |z-c|^{(q-1)t} \, dm_1(z) \\
&\preceq \mathrm{Dist}(H_c^{-1}(F),c)^{(q-1)t} m_1(H_c^{-1}(F)).
\end{aligned}$$

Hence $m_1(H_c^{-1}(F)) \succeq \mathrm{Dist}(H_c^{-1}(F), c)^{(q\alpha-(q-1)t} = \mathrm{Dist}(H_c^{-1}(F), c)^{t+q(\alpha-t)}$. The proof is finished. ♣

§**3. Preliminaries, II.** Let us now briefly discuss some aspects of rational functions with no recurrent critical points in the Julia set. In the sequel they will be reffered to as NCP maps. In this paper we will essentially follow the notation and terminology of [Ur]. If F is a subset of $\overline{\mathbb{C}}$ then by $O_+(F)$ we denote the forward orbit of F under T, that is

$$O_+(F) = \bigcup_{n\geq 0} T^n(F).$$

By $\mathrm{Crit}(T)$ we denote the set of all critical points of T lying in the Julia set $J(T)$, by $\Omega(T)$ the set of all parabolic periodic points of T (rationally indifferent periodic points), by $\mathrm{Sing}(T)$ the union $\bigcup_{n\geq 0} T^{-n}\big(\Omega(T)\cup \mathrm{Crit}(J(T))\big)$, and by

$$\mathrm{PC}(T) = \overline{O_+(T(\mathrm{Crit}(T)))}$$

the postcritical set of T. Let h denote the Hausdorff dimension of $J(T)$ and let m be the (up to equivalency of measures) h-conformal measure for $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ supported on $J(T)$. Observe that for every $n \geq 1$, $J(T^n) = J(T)$ and by the chain rule m is h-conformal for T^n. Also observe that if ν is a T-invariant measure, it is also T^n-invariant and that $c\sum_{j=0}^{n-1} \mu\circ T^{-j}$ is T-invariant for every constant $c > 0$ and every T^n-invariant measure μ. Therefore as long as we are concerned with geometry of the Julia set and invariant measures for T, passing to a sufficiently high iterate we may assume that $\Omega(T)$, which is finite, consists of fixed points only and $T'(\omega) = 1$ for all $\omega \in \Omega(T)$.

Let now $Q \subset \overline{\mathbb{C}}$ be a connected set that is not disjoint from $J(T)$. We say that a connected component Q_n of $T^{-n}(Q)$, $n \geq 0$, has the weak property $(*)$ if either Q or $T^{n-1}(Q_n)$ is disjoint from $T_\omega^{-1}(B(\Omega,\theta))$, where θ is at least as small as required in [Ur, (5.2), p.402].

Let $\beta > 0$ and $\gamma > 0$ be at least so small as required in [Ur, p.402]. We say that a connected component Q_n of $T^{-n}(Q)$, $n \geq 0$, has the property $(*)$ if there exists $z \in J(T)$ such that $T^n(Q_n) \subset B(z,\gamma)$ and the component $\hat{Q}_n$ of $T^{-n}B(z,2\gamma)$ containing Q_n satisfies weak property $(*)$. The family of all those components will be denoted by $\mathrm{Comp}_0^n(Q)$.

Let $\kappa = \big(\Pi_{c\in\mathrm{Crit}(T)} q(c)\big)^{-1}$. We shall prove the following.

Lemma 3.1. Suppose that $z \in J(T)$, $Q^{(2)} = B(z,2\gamma)$ and $Q^{(1)} = B(z,\gamma)$. Suppose that $n \geq 0$ is a non-negative integer, $Q_n^{(1)}$ and $Q_n^{(2)}$ are connected components of $T^{-n}(Q^{(1)})$ and $T^{-n}(Q^{(2)})$ respectively, such that $Q_n^{(1)} \subset Q_n^{(2)}$. If $Q_n^{(2)}$ satisfies weak property $(*)$, then $\mathrm{Mod}(Q_n^{(2)} \setminus Q_n^{(1)}) \geq \kappa \log 2/\#\mathrm{Crit}(T)$.

Proof. In view of [Ur, Lemma 5.1] there exists a geometric annulus $R \subset Q^{(2)} \setminus Q^{(1)}$ centered at z of modulus $\log 2/\#\mathrm{Crit}(T)$ such that $R \cap T^n(Q_n^{(2)} \cap \mathrm{Crit}(T^n)) = \emptyset$. Since covering maps increase moduli of annuli at most by factors equal to their degrees, we conclude that

$$\mathrm{Mod}(Q_n^{(2)} \setminus Q_n^{(1)}) \geq \mathrm{Mod}(R_n) \geq \log 2/\#\mathrm{Crit}(T)/\Pi_{c\in\mathrm{Crit}(T)}q(c) = \frac{\kappa \log 2}{\#\mathrm{Crit}(T)},$$

where $R_n \subset Q_n^{(2)}$ is the component of $T^{-n}(Q^{(2)})$ enclosing $Q_n^{(1)}$. ♣

As an immediate consequence of this lemma and KDT(II) we get the following.

Lemma 3.2. Suppose that $Q \subset \overline{\mathbb{C}}$ is a connected set and $Q_n \in \mathrm{Comp}_0^n(Q)$. If $0 \leq k \leq n$ is an integer such that $T^k : \hat{Q}_n \to T^k(\hat{Q}_n)$ is univalent, then

$$\frac{|(T^k)'(y)|}{|(T^k)'(x)|} \leq \mathrm{const}$$

for all $x, y \in Q_n$, where const is a number depending only on $\#\mathrm{Crit}(T)$ and κ.

The statements to the following two lemmas the reader can find in [Pr].

Lemma 3.3. Suppose that $Q^{(1)} \subset Q^{(2)}$ are two connected sets and $Q^{(2)} \in \mathrm{Comp}_0^n(Q^{(2)})$. If $Q_n^{(1)}$ is a connected component of $T^{-n}(Q^{(1)})$ contained in $Q_n^{(2)}$, then

$$\frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \succeq \frac{\mathrm{diam}(Q^{(1)})}{\mathrm{diam}(Q^{(2)})}.$$

Proof. We shall prove this lemma by induction on n. For $n = 0$ we have even equality. Assume therefore that it is true for some $n \geq 0$. Let $Q_{n+1}^{(2)} \in \mathrm{Comp}_0^{n+1}(Q^{(2)})$. Consider two cases, First suppose that $\hat{Q}_{n+1}^{(2)}$ contains no critical point of T. Let $0 \leq k \leq n$ be the largest integer such that $\hat{Q}_k^{(2)} = T^{n+1-k}(\hat{Q}_{n+1}^{(2)})$ contains a critical point of T. If no such a number exists set $k = 0$. Then by Lemma 3.2 and by inductive assumption

$$\frac{\mathrm{diam}(Q_{n+1}^{(1)})}{\mathrm{diam}(Q_{n+1}^{(2)})} \asymp \frac{\mathrm{diam}(Q_k^{(1)})}{\mathrm{diam}(Q_k^{(2)})} \succeq \frac{\mathrm{diam}(Q^{(1)})}{\mathrm{diam}(Q^{(2)})}.$$

and we are done in this case. So, suppose that $\hat{Q}_{n+1}^{(2)}$ contains some critical point, say c, and denote its order by q. Set $Q_n^{(i)} = T(Q_{n+1}^{(i)})$, $i = 1, 2$. As $Q_{n+1}^{(i)}$ are both connected we therefore get

$$\begin{aligned}\mathrm{diam}(Q_n^{(i)}) &\asymp \mathrm{diam}(Q_{n+1}^{(i)}) \sup\{|T'(x)| : x \in Q_{n+1}^{(i)}\} \\ &\asymp \mathrm{diam}(Q_{n+1}^{(i)}) \mathrm{Dist}(c, Q_{n+1}^{(i)})^{q-1}.\end{aligned}$$

Hence

$$\frac{\mathrm{diam}(Q_{n+1}^{(1)})}{\mathrm{diam}(Q_{n+1}^{(2)})} \asymp \frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \cdot \frac{\mathrm{Dist}(c, Q_n^{(2)})^{q-1}}{\mathrm{Dist}(c, Q_n^{(1)})^{q-1}} \geq \frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})}.$$

The proof is finished. ♣

Lemma 3.4. Suppose that $Q^{(1)} \subset Q^{(2)}$ are two connected sets and $Q^{(2)} \in \mathrm{Comp}_0^n(Q^{(2)})$. If $Q_n^{(1)}$ is a connected component of $T^{-n}(Q^{(1)})$ contained in $Q_n^{(2)}$, then

$$\frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \preceq \frac{\mathrm{diam}^{\kappa}(Q^{(1)})}{\mathrm{diam}^{\kappa}(Q^{(2)})}.$$

Proof. We shall prove this lemma by induction on n. For $n = 0$ we have even equality. Assume therefore that it is true for some $n \geq 0$. Let $Q_{n+1}^{(2)} \in \mathrm{Comp}_0^{n+1}(Q^{(2)})$. Consider two cases, First suppose that $\hat{Q}_{n+1}^{(2)}$ contains no critical point of T. Let $0 \leq k \leq n$ be the largest integer such that $\hat{Q}_k^{(2)} = T^{n+1-k}(\hat{Q}_{n+1}^{(2)})$ contains a critical point of T. If no such a number exists set $k = 0$. Then by Lemma 3.2 and by inductive assumption

$$\frac{\mathrm{diam}(Q_{n+1}^{(1)})}{\mathrm{diam}(Q_{n+1}^{(2)})} \asymp \frac{\mathrm{diam}(Q_k^{(1)})}{\mathrm{diam}(Q_k^{(2)})} \preceq \frac{\mathrm{diam}^{\kappa}(Q^{(1)})}{\mathrm{diam}^{\kappa}(Q^{(2)})}.$$

and we are done in this case. So, suppose that $\hat{Q}_{n+1}^{(2)}$ contains some critical point, say c, and denote its order by q. Set $Q_n^{(i)} = T(Q_{n+1}^{(i)})$, $i = 1, 2$. As $Q_{n+1}^{(i)}$ are both connected we therefore get

$$\begin{aligned}\mathrm{diam}(Q_n^{(i)} &\asymp \mathrm{diam}(Q_{n+1}^{(i)}) \sup\{|T'(x)| : x \in Q_{n+1}^{(i)}\} \\ &\asymp \mathrm{diam}(Q_{n+1}^{(i)}) \mathrm{Dist}(c, Q_{n+1}^{(i)})^{q-1}.\end{aligned}$$

Hence

$$\frac{\mathrm{diam}(Q_{n+1}^{(1)})}{\mathrm{diam}(Q_{n+1}^{(2)})} \asymp \frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \cdot \frac{\mathrm{Dist}(c, Q_n^{(2)})^{q-1}}{\mathrm{Dist}(c, Q_n^{(1)})^{q-1}} \asymp \frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \cdot \frac{\mathrm{Dist}(T(c), Q_n^{(2)})^{\frac{q-1}{q}}}{\mathrm{Dist}(T(c), Q_n^{(2)})^{\frac{q-1}{q}}}$$

Now, if $\mathrm{Dist}(T(c), Q_n^{(2)}) \leq 2\mathrm{diam}(Q_n^{(2)})$, then by inductive assumption

$$\begin{aligned}\frac{\mathrm{diam}(Q_{n+1}^{(1)})}{\mathrm{diam}(Q_{n+1}^{(2)})} &\preceq \frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \cdot 2^{q-1} \frac{\mathrm{diam}^{\frac{q-1}{q}}(Q_n^{(2)})}{\mathrm{diam}^{\frac{q-1}{q}}(Q_n^{(1)})} \\ &\asymp \frac{\mathrm{diam}^{\frac{1}{q}}(Q_n^{(1)})}{\mathrm{diam}^{\frac{1}{q}}(Q_n^{(2)})} \preceq \frac{\mathrm{diam}^{\frac{1}{q}\frac{1}{q|n}}(Q^{(1)})}{\mathrm{diam}^{\frac{1}{q}\frac{1}{q|n}}(Q^{(2)})} \\ &= \frac{\mathrm{diam}^{\frac{1}{q|n+1}}(Q^{(1)})}{\mathrm{diam}^{\frac{1}{q|n+1}}(Q^{(2)})}\end{aligned}$$

and we are done. Otherwise $\mathrm{Dist}(T(c), Q_n^{(2)}) \geq 2\mathrm{diam}(Q_n^{(2)})$. Since $\mathrm{Dist}(T(c), Q_n^{(2)}) \leq \mathrm{dist}(T(c), Q_n^{(2)}) + \mathrm{diam}(Q_n^{(2)})$ we thus conclude that $\mathrm{diam}(Q_n^{(2)}) \leq \mathrm{dist}(T(c), Q_n^{(2)})$. Hence $\mathrm{Dist}(T(c), Q_n^{(2)}) \leq 2\mathrm{dist}(T(c), Q_n^{(2)})$. But then using inductive assumption

$$\begin{aligned}\frac{\mathrm{diam}(Q_{n+1}^{(1)})}{\mathrm{diam}(Q_{n+1}^{(2)})} &\preceq \frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \cdot 2^{q-1} \frac{\mathrm{dist}(T(c), Q_n^{(2)})^{\frac{q-1}{q}}}{\mathrm{dist}(T(c), Q_n^{(2)})^{\frac{q-1}{q}}} \\ &\preceq 2^{q-1} \frac{\mathrm{diam}(Q_n^{(1)})}{\mathrm{diam}(Q_n^{(2)})} \preceq \frac{\mathrm{diam}^{\frac{1}{q|n}}(Q^{(1)})}{\mathrm{diam}^{\frac{1}{q|n}}(Q^{(2)})} \\ &\preceq \frac{\mathrm{diam}^{\frac{1}{q|n+1}}(Q^{(1)})}{\mathrm{diam}^{\frac{1}{q|n+1}}(Q^{(2)})}.\end{aligned}$$

The proof is finished. ♣

§**4. Invariant measure - existence, ergodicity, and conservativity.** In this section we prove the following two results.

Theorem 4.1. If $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an NCP map and h is the Hausdorff dimension of the Julia set $J(T)$, then there exists exactly one h-conformal measure for T. Moreover m is conservative and ergodic.

Theorem 4.2. Up to a multilicative constant there exists exactly one T-invariant σ-finite measure μ absolutely continuous with respect to m. Moreover μ is equivalent with m, conservative and ergodic.

Proof of Theorem 4.1. First notice that one proves here similarly as in [DU1] that that any two h-conformal measures are equivalent on the set $J(T) \setminus \mathrm{Sing}(T)$. Since after normalizations the continuous and atomic parts of any h-conformal measure would form h-conformal measures (mutually singular of course) themselves, it follows from this observation and [Ur, Theorem 7.6] that any two h-conformal measures are equivalent and nonatomic.

Let us now prove that any h-conformal measure is ergodic. Indeed, suppose to the contrary that m is an h-conformal measure and $T^{-1}(G) = G$ for some Borel set $G \subset J(T)$ with $0 < m(G) < 1$. But then the two conditional measures m_G and $m_{J(T)\setminus G}$

$$m_G(B) = \frac{m(B \cap G)}{m(G)}, \quad m_{J(T)\setminus G}(B) = \frac{m(B \cap J(T) \setminus G)}{m(J(T) \setminus G)}$$

would be h-conformal and singular. Contradiction.

If now m and ν are two arbitrary h-conformal measures, then by a simple computation based on the definition of conformal measures we see that the Radon-Nikodyn derivative $\phi = d\nu/dm$ is constant on grand orbits of T. Therefore by ergodicity of m we conclude that ϕ is constant m almost ewerywhere. As both m and ν are probability measures it implies that $\phi = 1$ a.e., hence $\nu = m$.

It is left to show that m is conservative and in order to demonstrate it we improve slightly the reasoning contained in the very first part of Section 7 of [Ur]. First notice that if $m(G \triangle B) = 0$, then also $m(T(G) \triangle T(B)) = 0$. This follows immediately from the formula defining conformality and possibility of partitioning $J(T)$ into finitely many sets on each of them T is one-to-one.

Now we shall prove that every forward invariant ($T(F) \subset F$) subset F of $J(T)$ is either of measure 0 or 1. Indeed, suppose to the contrary that $0 < m(F) < 1$. Since m is nonatomic it suffices to show that

$$m(F \setminus \mathrm{Sing}(T)) = 0.$$

Denote by Z the set of all points $z \in F \setminus \mathrm{Sing}(T)$ such that

$$\lim_{r \to 0} \frac{m(B(z,r) \cap (F \setminus \mathrm{Sing}(T)))}{m(B(z,r))} = 1 \tag{4.1}$$

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [Fe]), $m(Z) = m(Y)$. Suppose now that $m(Z) > 0$ and fix $z \in Y$. Let x, $\eta(z)$, and the sequence $\{n_j\}$ be the objects produced by [Ur, Prop. 6.1] which reads the following.

If $z \in J(T) \setminus \mathrm{Sing}(T)$, then there exist a positive number $\eta(z)$, an increasing sequence of positive integers $\{n_j\}_{j \geq 1}$, and a point $x \in \omega(z) \setminus (\Omega(T) \cup \omega(\mathrm{Crit}(z)))$ such that $\lim_{j \to \infty} T^{n_j}(z) = x$ and for every $j \geq 0$

$$\mathrm{Comp}(z, T^{n_j}(z), T^{n_j}, \eta(z)) \cap \mathrm{Crit}(T^{n_j}) = \emptyset.$$

Since $T : J(T) \to J(T)$ is topologically exact, there exists $n \geq 0$ such that $T^n(J(T) \cap B(x, \eta(z)/4)) = J(T)$, whence $m(T^n(B(x, \eta(z)/4)) = 1$. Since also $T^n(F) \subset F$, $m(F) < 1$, and

$$m\big(T^n(F \cap B(x, \eta(z)/4)) \triangle T^n(B(x, \eta(z)/4)\big) = m\big((F \cap B(x, \eta(z)/4)) \triangle B(x, \eta(z)/4)\big)$$

we therefore get

$$\begin{aligned} m\big(B(x, \eta(z)/4) \setminus F\big) &= m\big((F \cap B(x, \eta(z)/4)) \triangle B(x, \eta(z)/4)\big) = 1 - m\big((T^n(F \cap B(x, \eta(z)/4))\big) \\ &\geq 1 - m(T^n(F)) \geq 1 - m(F) > 0. \end{aligned}$$

Hence for every $j \geq 1$, $m\big(B(T^{n_j}(z), \eta(z)/2) \setminus F\big) \geq m\big(B(x, \eta(z)/4) \setminus F\big) > 0$. Therefore, as $T^{-1}(J(T) \setminus F) \subset J(T) \setminus F$, the standard application of Koebe Distortion Theorem shows that

$$\limsup_{r \to 0} \frac{m(B(z,r) \setminus F)}{m(B(z,r))} > 0$$

which contradicts (4.1). Thus either $m(F) = 0$ or $m(F) = 1$.

Now conservativity is straightforward. One needs to prove that for every Borel set $B \subset J(T)$ with $m(B) > 0$ one has $m(G) = 0$, where

$$G = \{x \in J(T) : \sum_{n \geq 0} \chi_B(T^n(x)) < +\infty\}.$$

Indeed, suppose that $m(G) > 0$ and for all $n \geq 0$ let

$$G_n = \{x \in J(T) : \sum_{n\geq 0} \chi_B(T^n(x)) = 0\} = \{x \in J(T) : T^k(x) \notin B \ \text{ for all } \ k \geq n\}.$$

Since $G = \bigcup_{n\geq 0} G_n$, there exists $k \geq 0$ such that $m(G_k) > 0$. Since all the sets G_n are forward invariant we conclude that $m(G_k) = 1$. But on the other hand all the sets $T^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_k. This contradiction finishes the proof. ♣

The idea of the proof of Theorem 4.2 is to apply a general sufficient condition for the existence of σ-finite absolutely continuous invariant measure proven in [Mar]. In order to formulate this condition suppose that X is a σ-compact metric space, m is a Borel probability measure on X, positive on open sets, and that a measurable map $T : X \to X$ is given with respect to which measure m is quasi-invariant, i.e. $m \circ T^{-1} \ll m$. Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure m. We also assume that $m(X \setminus \bigcup_{n\geq 0} A_n) = 0$, and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$m(T^{-k}(A_m) \cap A_n) > 0,$$

then the partition α is called irreducible. Martnes' result comprising Proposition 2.6 and Theorem 2.9 of [Mar] reads the following.

Theorem 4.3. Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure m. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n

$$K_n^{-1} \frac{m(A)}{m(A_n)} \leq \frac{m(T^{-k}(A))}{m(T^{-k}(A_n))} \leq K_n \frac{m(A)}{m(A_n)}, \tag{4.2}$$

then T has a σ-finite T-ivariant measure μ absolutely continuous with respect to m. Additionally μ is equivalent with m, conservative and ergodic, and unique up to a multiplicative constant.

Proof of Theorem 4.2. Since in the sequel we will not only need Theorem 4.2 but a bit more, the way in which the invariant measure claimed in Theorem 4.3 is produced we shall also describe this procedure briefly. Following Martens one fixes an element $A \in \alpha = \{A_n : n \geq 0\}$ and then one considers the following sequences of measures

$$S_k(m) = \sum_{i=0}^{k-1} m \circ T^{-i} \quad \text{and} \quad Q_k(m) = \frac{S_k(m)}{S_k(m)(A)}.$$

It is proven in [Mar] that each weak limit μ of the sequence $Q_k(m)$ has the properties required in Theorem 4.3, where a sequence $\{\nu_k : k \geq 1\}$ of measures on X is said to converge weakly if for all $n \geq 1$ the measures ν_k converge weakly on all compact subsets of A_n. In fact it turns out that the sequence $Q_k(m)$ converges and

$$\mu(F) = \lim_{n \to \infty} Q_k(m)(F)$$

for every Borel set $F \subset X$. Of course $\mu(A) \leq 1 < \infty$. Making use of (4.1) one proves (see [Mar, Lemma 2.4]) the following.

Lemma 4.4. For every $n \geq 0$ we have $0 < \mu(A_n) < \infty$.

Since ergodicity and conservativity of m is given by Theorem 4.1, we only need to construct an irreducible partition α with property (4.2). Indeed, set $Y = J(T) \setminus (\mathrm{PC}(T) \cup \Omega)$. For every $y \in Y$ consider a ball $B(y, r(y))$ such that $r(y) > 0$, $m(\partial B(y, r(y))) = 0$, and $r(y) < (1/2)\mathrm{dist}(y, \mathrm{PC}(T) \cup \Omega)$. The balls $B(y, r(y))$, $y \in Y$, cover Y and since Y is a metric separable space, one can choose a countable cover, say $\{\tilde{A}_n : n \geq 0\}$, from them. We may additionally require that the family $\{\tilde{A}_n : n \geq 0\}$ is locally finite that is that each point $x \in Y$ has an open neighborhood intersecting only finitely many balls $\tilde{A}_n$, $n \geq 0$. We now define the family $\alpha = \{A_n : n \geq 0\}$ inductively setting

$$A_0 = \tilde{A}_0 \quad \text{and} \quad A_{n+1} = \tilde{A}_{n+1} \setminus \bigcup_{k=1}^{n} \overline{\tilde{A}_n}$$

(and throwing away empty sets). Obviously α is a disjoint family and

$$\bigcup_{n \geq 1} A_n \supset J(T) \setminus (\mathrm{PC}(T) \cup \Omega) \setminus \bigcup_{n \geq 0} \partial \tilde{A}_n$$

whence in view of [Ur, Corollary 7.2], $m\big(\bigcup_{n \geq 0} A_n\big) = 1$. The distortion condition (4.2) follows now from KDT with all constants $K_n = K$, and irreducibility of partition α follows from openess of the sets A_n and topological exactness of the map $T : J(T) \to J(T)$. ♣

§**5. Invariant measure - points of infinite condensation.** We say that the T-invariant measure μ produced in Theorem 4.3 is of finite condensation at $x \in J(T)$ if and only if there exists an open neighborhood V of x such that $\mu(V) < \infty$. Otherwise μ is said to be of infinite condensation at x. We respectively say that x is a point of finite or infinite condensation of μ. We begin with the following obvious

Lemma 5.1. If x is a point of infinite condensation of μ, then each point of the closure $\overline{\{T^n(x) : n \geq 0\}}$ is also of infinite condensation of μ.

Lemma 5.2. The set of points of infinite condensation of measure μ is contained in $\mathrm{PC}(T) \cup \Omega$.

Proof. If $z \notin \mathrm{PC}(T) \cup \Omega$, then by local finiteness of the family $\{\tilde{A}_n : n \geq 0\}$ there exist an open neighborhood V of z and an integer $k \geq 0$ such that $m\big(V \setminus \bigcup_{j=0}^k A_j\big) = 0$. Hence, in view of Lemma 4.4 and Theorem 4.2 ($\mu \ll m$) we get $\mu(A) \leq \sum_{j=0}^k \mu(A_j) < \infty$. The proof is finished. ♣

We say that $z \in J(T) \setminus \Omega$ is geometrically good if

$$m(B_n) \preceq \mathrm{diam}^h(B_n) \tag{5.1}$$

for every set B of sufficiently small diameter containing x, every $n \geq 0$ and every connected component B_n of $T^{-n}(B)$. The direction of the inequality above makes that checking geometrical goodness one can assume the sets B to be balls centered at x. The most general sufficient condition for finite condensation is the following.

Lemma 5.3. If $z \in J(T) \setminus \Omega$ is geometrically good, then z is a point of finite condensation of measure μ.

Proof. Since $z \notin \Omega$, taking θ and γ sufficiently small, there exists a disk $B = B(z,r)$ such that all the connected components of all the sets $T^{-n}(B)$, $n \geq 0$, satisfy property $(*)$. Since $m(B) > 0$ and $m(\bigcup_{n\geq 0} A_n) = 1$, there exists $i \geq 0$ such that $m(B \cap A_i) > 0$. Since $B \cap A_i$ is an open set having nonempty intersection with $J(T)$, there exists an open ball $F \subset B \cap A_i$ having nonempty intersection with $J(T)$. Of course $m(F) > 0$. For every $n \geq 0$ let B_n be a connected component of $T^{-n}(B)$ and let $F_n \subset B_n$ be some connected component of $T^{-n}(F)$ contained in B_n. Using KDT and the fact that z is geometrically good, we get

$$m(F_n) \asymp \mathrm{diam}^h(F_n) = \left(\frac{\mathrm{diam}(F_n)}{\mathrm{diam}(B_n)}\right)^h \mathrm{diam}^h(B_n) \succeq m(B_n)\left(\frac{\mathrm{diam}(F_n)}{\mathrm{diam}(B_n)}\right)^h$$

Applying now Lemma 3.3 to the connected sets F and B we obtain

$$m(F_n) \succeq m(B_n)\left(\frac{\mathrm{diam}(F)}{\mathrm{diam}(B)}\right)^h.$$

Thus $S_n(m)(B) \preceq S_n(m)(F) \preceq S_n(m)(A_i)$. Hence, using Lemma 4.4, we get $\mu(B) \preceq \mu(A_i) < \infty$ and therefore z is a point of finite condensation of μ. ♣

In order to make use of this lemma we need to provide sufficient conditions for points to be geometrically good. This is done below.

Lemma 5.4. If $z \in J(T) \setminus \Omega$ and μ is upper geometric at z, then z is geometrically good.

Proof. The proof of this lemma follows by a straightforward inductive argument incorporating KDT, Lemma 2.1, finiteness of the set of critical points of T, [Ur, Lemma 5.1], and [Ur, Lemma 2.13] (or equivalently [Ur,(5.4)]). ♣

Lemma 5.5. If $z \in J(T)$ and $\omega(z) \cap \Omega = \emptyset$, then the conformal measure m considered on $J(T)$ is strongly geometric at z.

Proof. For every $\theta > 0$ let

$$A(\theta) = \{z \in J(T) : \mathrm{dist}(O_+(z), \Omega) \geq \theta\}$$

Since the set of critical points of T is finite there is $\theta_0 > 0$ such that

$$\{c \in \mathrm{Crit}(T) : \omega(c) \cap \Omega = \emptyset\} \subset A(\theta_0) \tag{5.2}$$

We shall prove first that for every $0 < \theta < \theta_0$ measure m is uniformly strongly geometric at every point of $A(\theta)$. We will proceed inductively with respect to the stratification of $J(T)$ and with respect to the order of critical points. In order to take care of the base of induction consider the set $J_0(T) \cap A(\theta)$. This set is disjoint from $\mathrm{Crit}(T)$. Taking $\varepsilon > 0$ sufficiently small, in fact $\varepsilon < \lambda(\theta\tau)^{-1}\rho/2$ and any radius $0 < r < \tau\theta||T'||^{-1}\lambda^{-1}$, we conclude from [Ur, Prop. 6.4(b)] that if $z \in J_0(T) \cap A(\theta)$, then $s = s(\lambda, \varepsilon, r, z) = u = u(\lambda, r, z)$, where u is the number produced in [Ur, Prop. 6.3] and s is produced in [Ur, Prop. 6.4]. Combining now [Ur, Prop. 6.3(d)] along with [Ur, Prop. 6.4(c)] and results of sections 2 and 3 of [Ur], we deduce that m is uniformly strongly geometric at z. The inductive step consists of two parts. First suppose that for some $0 \leq i \leq p-1$ the conformal measure m is uniformly strongly geometric at every point of the set $J_i(T) \cap A(\theta)$. We shall note that m is uniformly strongly geometric at every point of the set $Cr_{i+1}(T) \cap A(\theta)$. Indeed, it immediately follows from the second inclusion of [Ur, Lemma 5.9] followed by [Ur, Lemma 3.4] and [Ur, Lemma 3.5]. In the second step suppose that for some $0 \leq i \leq p$, m is strongly geometric at all points of $S_i(T) \cap A(\theta)$. Since $\mathrm{Crit}(T)$ is finite there exists $\zeta > 0$ such that if $c \in \mathrm{Crit}(T)$ and $\omega(c) \cap \Omega \neq \emptyset$, then $O_+(y) \cap B(\Omega, \theta) \neq \emptyset$ for all $y \in B(c, \zeta)$. Fix $\varepsilon > 0$ so small that $\varepsilon\lambda^{-1}\theta\tau < \min\{\rho/2, \zeta\}$. Consider any point $z \in J_i(T) \cap A(\theta)$ and any radius $r > 0$. If $s(\lambda, \varepsilon, r, z) = u(\lambda, r, z)$, then by [Ur, Prop. 6.3(d)] (notice that $T^u(z) \in A(\theta) \subset J(T) \setminus B(\Omega, \theta)$), [Ur, Prop. 6.4(c)] and results of sections 2 and 3 of [Ur], we conclude that m is uniformly strongly geometric for z and r. If $s < u$, then by [Ur, Prop. 6.4(b)] there exists $c \in \mathrm{Crit}(T)$ such that $|T^s(z) - c| \leq \varepsilon r|(T^s)'(z)|$. By the choice of ε we have $\varepsilon r|(T^s)'(z)| \leq \varepsilon\lambda^{-1}\theta\tau \leq \rho/2$. Hence $|T^s(z) - c| \leq \rho/2$ and by the definition of $J_i(T)$ we conclude that $c \in S_i(T)$. Also $\varepsilon r|(T^s)'(z)| \leq \varepsilon\lambda^{-1}\theta\tau < \zeta$. Thus $\omega(c) \cap \Omega = \emptyset$ since otherwise we would get $O_+(T^s(z)) \cap B(\Omega, \theta) \neq \emptyset$ contradicting the assumption $z \in A(\theta)$. Hence by (5.2), $c \in A(\theta_0) \subset A(\theta)$. So, we have shown that $c \in S_i(T) \cap A(\theta)$ and therefore m is uniformly strongly geometric for z and r because of our inductive assumption, [Ur, Prop. 6.4(c)] and results of sections 2 and 3 of [Ur]. Thus the proof of uniform strong geometricity of m in $A(\theta)$ has been completed. Observing that $\{z : \omega(z) \cap \Omega = \emptyset\} = \bigcup_{0<\theta<\theta_0} A(\theta)$, completes the proof. ♣

Corollary 5.6. If $\Omega = \emptyset$, then the conformal measure m is strongly geometric. (Recall that strong geometricity implies uniform strong geometricity)

Proof. That m is strongly geometric follows immediately from Lemma 5.5. The stronger result that m is uniformly strongly geometric follows from the fact that in this case $J(T) = A(\theta)$ for every $\theta > 0$ and from the proof of Lemma 5.5. ♣

Proposition 5.7. If z is a point of infinite condensation of μ, then $z \in \omega(\mathrm{Crit}(T)) \cup \Omega$ and $\omega(z) \cap \Omega \neq \emptyset$. Additionally the intersection $\omega(z) \cap \Omega$ consists of points of infinite condensation of μ.

Proof. The last part of this proposition follows immediately from Lemma 5.1. If $z \notin \omega(\mathrm{Crit}(T)) \cup \Omega$ then there exists $n \geq 0$ such that $T^{-n}(z) \cap (\mathrm{PC}(T) \cup \Omega) = \emptyset$ and therefore by Lemma 5.2 there exists $\zeta > 0$ such that $\mu(B(T^{-n}(z), \zeta)) < \infty$. Now there exists $r > 0$ such that $T^{-n}(B(z,r)) \subset B(T^{-n}(z), \zeta)$, hence $\mu(B(z,r)) = \mu(T^{-n}(B(z,r)) \leq \mu(B(T^{-n}(z), \zeta)) < \infty$. Thus z is a point of finite condensation of μ. So, it is left to show that $\omega(z) \cap \Omega \neq \emptyset$. This is obvious (actually vacuous) if $z \in \Omega$ and we may assume that $z \notin \Omega$. But then the formula $\omega(z) \cap \Omega \neq \emptyset$ is an immediate consequence of Lemma 5.5, Lemma 5.4, and Lemma 5.3. The proof is completed. ♣

Corollary 5.8. If $\Omega = \emptyset$, then there exists a T-invariant probability measure μ equivalent to m.

Corollary 5.9. If $\omega(\mathrm{Crit}(T)) \cap \Omega = \emptyset$, then the set of points of infinite condensation of μ is contained in Ω.

Theorem 5.10. If $h = \mathrm{HD}(J(T)) \geq 1$, then the set of points of infinite condensation of μ is contained in the set of parabolic points.

Proof. The proof of [Ur, Lemma 7.12] shows that each point $z \in J(T)$ is upper geometric. Therefore the proof of Theorem 5.10 is completed by applying Lemma 5.4 and Lemma 5.3. ♣

§**6. Invariant measure - Parabolic points.** From what we have shown in the previous section, it is clear that in order to localize the points of infinite condensation of μ we have to egzamine the parabolic points. We shall now explore this topic in details. So, fix $\omega \in \Omega$. Then there exists a number $\zeta > 0$ such that each connected component C of $T^{-1}(B(\omega, \zeta))$ contains exactly one element of $T^{-1}(\omega)$. Moreover we require that if y is not a critical point of T, then the degree of $T|_C$ is equal to the order of y. Since Ω is finite we may fix $\zeta > 0$ good for all parabolic points. If now F is a subset of $B(\omega, \zeta)$ and $y \in T^{-1}(\omega)$, then by $T_y^{-1}(F)$ we denote the intersection $T^{-1}(F) \cap C_y$, where C_y is the unique connected component of $T^{-1}(B(\omega, \zeta))$ containing y. If y is not a critical point we also treat T_y^{-1} as the inverse map of T defined on $B(\omega, \zeta)$ and sending ω to y. With these conventions given $B \subset B(\omega, \zeta)$ and $n \geq 1$ we can write

$$S_n(m)(B) = m(T_\omega^{-(n-1)}(B)) + \sum_{y \in T^{-1}(\omega) \setminus \omega} \sum_{k=0}^{n-2} S_{n-(k+1)}(m)(T_y^{-1}(T_\omega^{-k}(B))) \tag{6.1}$$

On a sufficiently small open neighbourhood of ω the Taylor series expansion of T takes on the form

$$T(z) = z - a(z-\omega)^{p(\omega)+1} + a_2(z-\omega)^{p(\omega)+2} + a_3(z-\omega)^{p(\omega)+3} + \dots,$$

where $a \neq 0$ and $p = p(\omega)$ is a positive integer. In particular the number $p(\omega)$ is uniquely determined. Looking at (6.1) it is not difficult to predict that we will need the following lemma several times.

Lemma 6.1. For every $r > 0$ sufficiently small (say $r < \zeta$) there exists a constant $C(r) > 0$ such that for all $\omega \in \Omega$ and for all $k \geq 0$

$$m\big(T_\omega^{-k}(B(\omega,r)\cap J(T))\big) \asymp C(r)\big((k+1)^{-(\frac{1}{p(\omega)})}\big)^{h+p(\omega)(h-1)}.$$

The proof of this lemma follows immediately from the following two facts about local behaviour of T around ω which can be found in Section 4 of [Ur], comp. [ADU], [DU2], and [DU3]. The first fact is that $m(B(\omega,r)) \asymp r^{h+p(\omega)(h-1)}$ for all $r > 0$ sufficiently small and the second one is that $T_\omega^{-k}(B(\omega,r)\cap J(T)) \asymp B(\omega, M(r)(k+1)^{-(\frac{1}{p(\omega)})})$ for some constant $M(r)$ depending on r only.

Our first result about condensation of μ at parabolic points is the following.

Proposition 6.2. If $\omega \in \Omega \setminus \mathrm{PC}(T)$, then μ is of infinite condensation at ω if and only if $h \leq \frac{2p(\omega)}{p(\omega)+1}$.

Proof. Since $\omega \in \Omega\setminus \mathrm{PC}(T)$, in view of Lemma 5.2 for every $y \in T^{-1}(\omega)\setminus\{\omega\}$ there exists an open neighborhood B_y of y such that $\mu(B_y) < \infty$ and $B_y \subset \tilde{A}_j$ for some $j \geq 0$. Take B to be a ball in $J(T)$ (either closed or open) centered at ω and of radius $< \zeta$ so small that $T_y^{-1}(B) \subset B_y$ for all $y \in T^{-1}(\omega)\setminus\{\omega\}$. Then, using Lemma 6.1 and KDT we conclude that

$$\begin{aligned}
S_{n-(k+1)}(m)\big(T_y^{-1}(T_\omega^{-k}(B))\big) &\asymp S_{n-(k+1)}(m)(B_y)\frac{m\big(T_y^{-1}(T_\omega^{-k}(B))\big)}{m(B_y)}\\
&\asymp S_{n-(k+1)}(m)(B_y)(k+1)^{-\frac{h}{p}+(1-h)},
\end{aligned}$$

where $p = p(\omega)$. Thus, using (6.1), we can write

$$\begin{aligned}
Q_n(B) &= \frac{S_n(m)(B)}{S_n(m)(A)}\\
&\asymp \frac{n^{-\frac{h}{p}+(1-h)}}{S_n(m)(A)} +\\
&\quad + \sum_{y\in T^{-1}(\omega)\setminus\{\omega\}}\sum_{k=0}^{n-2}(k+1)^{-\frac{h}{p}+(1-h)}\frac{S_{n-(k+1)}(m)(B_y)}{S_{n-(k+1)}(m)(A)}\cdot\frac{S_{n-(k+1)}(m)(A)}{S_n(m)(A)}\\
&\asymp \sum_{k=1}^{n-1}k^{-\frac{h}{p}+(1-h)}\frac{S_{n-k}(m)(A)}{S_n(m)(A)}, \qquad (6.2)
\end{aligned}$$

where writing the second comparability sign we used the fact that $-\frac{h}{p}+1-h \leq 0$ (see [Ur, Theorem 7.14), that $\sum_{j\geq 0} m(T^{-j}(A)) = \infty$ required in Theorem 4.3, and Lemma 2.4 from [Mar]. Since $S_{n-k}(m)(A) \leq S_n(m)(A)$, from (6.2) we get $Q_n(m)(B) \preceq \sum_{k=1}^{n-1} k^{-\frac{h}{p}+(1-h)}$. Hence if $h > 2p/(p+1)$, then the series $\sum_{k=1}^{\infty} k^{-\frac{h}{p}+(1-h)}$ converges and the measures $Q_n(m)(B)$ are uniformly bounded from above. The last part of the proposition is therefore proven.

The second part, that is assuming $h \le 2p/(p+1)$, follows immediately combining formula (6.2), the fact that the series $\sum_{k=1}^\infty k^{-\frac{h}{p}+(1-h)}$ diverges, and the fact that for every k, $\lim_{n\to\infty} \frac{S_{n-k}(m)(A)}{S_n(m)(A)} = 1$. The proof is finished. ♣

Proposition 6.3. If $\omega \in \Omega$ and $h \le \frac{2p(\omega)}{p(\omega)+1}$, then μ has infinite condensation at ω.

Proof. With a little refinement the proof of this proposition is contained in the proof of Proposition 6.2. Indeed, since the set $\bigcup_{n\ge 0} T^{-n}\big(T^{-1}(\omega) \setminus \{\omega\}\big)$ is dense in $J(T)$ and since by [Ur,Lemma 5.2] the set $\mathrm{PC}(T)$ is nowhere-dense in $J(T)$, there exist $k \ge 1$ and $x \in T^{-k}(\omega) \setminus (\mathrm{PC}(T) \cup \Omega)$. Let $T_x^{-k} = (T_x^k)^{-1}$ have the same the same meaning, with T replaced by T^k, as the objects T_y^{-1} introduced before Lemma 6.1. Similarily as in the proof of Proposition 6.2, in view of Lemma 5.2 there exists an open neighborhood B_x of x such that $\mu(B_x) < \infty$ and $B_x \subset \tilde{A}_j$ for some $j \ge 0$. Take any ball $B \subset B(\omega, \zeta)$ in $J(T)$ centered at ω so small that $T_x^{-k}(B)$ is well defined and $T_x^{-k}(B) \subset B_x$. Then using Lemma 3.2, Lemma 2.2 with an observation that $\mathrm{Dist}(R(c), A)^{\frac{1-q}{t}t} \ge 1$, and Lemma 6.1, we can write for all $n \ge k$

$$
\begin{aligned}
S_n(m)(B) &\ge \sum_{j=0}^{n-k} S_{n-(j+k)}(m)\big(T_x^{-k}(T_\omega^{-j}(B))\big) \\
&\asymp \sum_{j=0}^{n-k} S_{n-(j+k)}(m)(B_x)\frac{m\big(T_x^{-k}(T_\omega^{-j}(B))\big)}{m(B_x)} \\
&\succeq \sum_{j=0}^{n-k} S_{n-(j+k)}(m)(B_x)(j+1)^{-\frac{h}{p}+(1-h)}.
\end{aligned}
$$

Thus for all $n \ge k$ we have

$$
\begin{aligned}
Q_n(B) = \frac{S_n(m)(B)}{S_n(m)(A)} &\succeq \sum_{j=0}^{n-k}(j+1)^{-\frac{h}{p}+(1-h)}\frac{S_{n-(j+k)}(m)(B_x)}{S_{n-(j+k)}(m)(A)} \cdot \frac{S_{n-(j+k)}(m)(A)}{S_n(m)(A)} \\
&\succeq \sum_{j=0}^{n-k}(j+1)^{-\frac{h}{p}+(1-h)}\frac{S_{n-(j+k)}(m)(A)}{S_n(m)(A)}.
\end{aligned}
$$

Now we complete the proof in exactly the same way as the proof of Proposition 6.2. ♣

Comparing Proposition 6.2 and Proposition 6.3 with [ADU,Theorem 9.8] one finds that the presence of critical points does not create barriers for parabolic points to be of infinite condensation. Just opposite, we shall now prove a result showing that in fact critical points may affect parabolic points ω lying in their ω-limit set so much that these latter ones become of infinite condensation even if $h > \frac{2p(\omega)}{p(\omega)+1}$ (without critical points, according to [ADU, Theorem 9.8] ω would be of finite condensation).

Theorem 6.4. If $c \in J(T)$ is a critical point of T of order q, $\omega = T(c) \in \Omega$, and $h \le \frac{2qp(\omega)}{p(\omega)+1}$, then ω is of infinite condensation of measure μ.

Proof. Let $p = p(\omega)$. First we shall show that $h > \frac{qp(\omega)}{p(\omega)+1}$ and

$$m(B(c,r)) \asymp r^{h+qp(h-1)}. \tag{6.3}$$

The proof of this formula is given by a simple "integration" based on [DU3, Lemma 4.8]. Indeed, given $x \in \mathbb{C}$ and $0 < r_1 < r_2$ let $R(x, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - x| \le r_2\}$. Using then [DU3, Lemma 4.8], for every sufficiently small $r > 0$ we can write

$$(r^q)^{h+p(h-1)} \asymp m\big(R(T(c), r^q, (2r)^q)\big) \asymp m\big(T(R(c, r, 2r))\big) \asymp m\big(R(c, r, 2r)\big) r^{(q-1)h}$$

and therefore $m\big(R(c, r, 2r)\big) \asymp r^{h+qp(h-1)}$. Thus

$$\begin{aligned} m(B(c,r)) &= \sum_{n=1}^{\infty} m\big(R(c, 2^{-n}r, 2^{-(n-1)}r)\big) = \sum_{n=1}^{\infty} r^{h+qp(h-1)} 2^{-n(h+qp(h-1)} \\ &= r^{h+qp(h-1)} \sum_{n=1}^{\infty} 2^{-n(h+qp(h-1)}. \end{aligned}$$

Since $m(B(c,r))$ is finite, first it follows from this formula that $h + qp(h-1) > 0$, or equivalently that $h > \frac{qp}{1+qp}$, and then that $m(B(c,r)) \asymp r^{h+qp(h-1)}$.

Recall now from [DU2] that for every $r > 0$ small enough the set $T_\omega^{-k}(J(T) \cap B(\omega, r))$ is comparable with the ball centered at ω of radius comparable with $(k+1)^{-1/p}$, where the comparability constants may depend on r but given $r > 0$ these are independent of k. Hence using (6.3) we conclude that for all sufficiently small balls B centered at ω

$$m\big(T_c^{-1}(T_\omega^{-k}(B))\big) \asymp (k+1)^{-\frac{1}{pq}(h+qp(h-1))}. \tag{6.4}$$

Now we proceed similarily as in the proof of Proposition 6.3. Since the set $\bigcup_{n \ge 2} T^{-n}(c)$ is dense in $J(T)$ and since by [Ur, Lemma 5.2] the set $\mathrm{PC}(T) \cup \Omega$ is nowhere dense in $J(T)$, there exists $k \ge 2$ and $x \in T^{-(k-1)}(c) \setminus (\mathrm{PC}(T) \cup \Omega)$. Similarly as in the proof of Proposition 6.3, in view of Lemma 5.2 there exists an open neighborhood B_x of x such that $\mu(B_x) < \infty$ and $B_x \subset \tilde{A}_j$ for some $j \ge 0$. With the obvious meaning of the operation $T_x^{-(k-1)}$ consider any ball $B \subset B(\omega, \zeta)$ so small that $T_c^{-1}(B)$ and $T_x^{-(k-1)}(T_c^{-1}(B))$ are well-defined and $T_x^{-k}(B) = T_x^{-(k-1)}(T_c^{-1}(B)) \subset B_x$. Then using Lemma 3.2, Lemma 2.2 with the observation that $\mathrm{Dist}(R(c), A)^{\frac{1-q}{q}t} \ge 1$, and (6.4), we get for all $n \ge k$

$$S_n(m)(B) \succeq \sum_{j=0}^{n-k} S_{n-(j+k)}(m)(B_x)(j+1)^{-\frac{h}{pq}+(1-h)}$$

and consequently

$$\begin{aligned} Q_n(B) &= \frac{S_n(m)(B)}{S_n(m)(A)} \\ &\ge \sum_{j=0}^{n-k} (j+1)^{-\frac{h}{pq}+(1-h)} \frac{S_{n-(j+k)}(m)(B_x)}{S_{n-(j+k)}(m)(A)} \frac{S_{n-(j+k)}(m)(A)}{S_n(m)(A)}. \end{aligned}$$

Now we complete the proof in exactly the same way as the proof of Proposition 6.2. ♣

Remark 6.5. The same proof carries through with an obvious minor modification if we assume only that $T^k(c) = \omega$ for some $k \geq 1$.

§7. **Dimensions of the Julia set.** The main purpose of this section is to prove Theorem 7.10 saying that Hausdorff and packing dimension of conformal measure of the Julia set of a rational function with no recurrent critical points are equal. We begin this section with the following general result.

Theorem 7.1. If $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational function of degree ≥ 2 then the upper box dimension of $J(T)$ and the packing dimension of $J(T)$ are equal.

Proof. First notice that since T is Lipshitz continuous and it is topologically exact, $\overline{\mathrm{BD}}(V) = \overline{\mathrm{BD}}(J(T))$ for every nonempty open subset V of J. Fix a number $t < \overline{\mathrm{BD}}(J(T))$ and consider an arbitrary Borel countable cover $\{X_n : n \geq 1\}$ of $J(T)$. Since $J(T)$ is compact, by the Baire category theorem there exists $n \geq 1$ such that $\overline{X_n}$ has nonempty interior. Since $t < \overline{\mathrm{BD}}(J(T)) = \overline{\mathrm{BD}}(X_n)$, we have $\Pi_t^*(X_n) = \infty$. Thus $\sum_{k\geq 1} \Pi_t^*(X_k) = \infty$ and consequently $\Pi_t(J(T)) = \infty$. This finishes the proof. ♣

Lemma 7.2. If $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an NCP map and $h = \mathrm{HD}(J(T)) \leq 1$, then $\overline{\mathrm{BD}}(J(T)) = \mathrm{HD}(J(T))$.

Proof. The proof is an immediate consequence of Theorem 7.1 and [Ur, Lemma 7.13, p.413].

Passing to more technical topics we begin with the following.

Lemma 7.3. If $Q \subset \overline{\mathbb{C}}$ is a connected set, then

$$\frac{m(Q_n)}{\mathrm{diam}^h(Q_n)} \succeq \frac{m(Q)}{\mathrm{diam}^h(Q)}$$

for every $n \geq 0$, and every connected component $Q_n \in \mathrm{Comp}_0^n(Q)$.

Proof. We shall prove the lemma by induction on n. For $n = 0$ the assertion is obvious. If $\hat{Q}_{n+1}$ contains no critical points of T then let $0 \leq k \leq n$ be the largest integer such that $\hat{Q}_k = T^{n+1-k}(\hat{Q}_{n+1})$ contains a critical point of T (set $k = 0$ if no such a number exists set). Choosing an arbitrary $y \in Q_{n+1}$ we obtain

$$\frac{m(Q_{n+1})}{\mathrm{diam}^h(Q_{n+1})} \asymp \frac{|(T^{n+1-k})'(y)|^{-h} m(Q_k)}{|(T^{n+1-k})'(y)|^{-h} \mathrm{diam}^h(Q_k)} \succeq \frac{m(Q)}{\mathrm{diam}^h(Q)}.$$

Now suppose that $\hat{Q}_{n+1}$ contains a critical point, say c. Write $Q_n = T(Q_{n+1})$. If $\mathrm{Dist}(T(c), Q_n) \leq 2\mathrm{dist}(T(c), Q_n)$, then $\mathrm{Dist}(T(c), Q_n) \asymp \mathrm{dist}(T(c), Q_n)$ and therefore

$$\frac{m(Q_{n+1})}{\mathrm{diam}^h(Q_{n+1})} \asymp \frac{\mathrm{Dist}^{(\frac{1}{q(c)}-1)h}(T(c), Q_n)}{\mathrm{Dist}^{(\frac{1}{q(c)}-1)h}(T(c), Q_n)} \cdot \frac{m(Q_n)}{\mathrm{diam}^h(Q_n)} \asymp \frac{m(Q_n)}{\mathrm{diam}^h(Q_n)} \succeq \frac{m(Q)}{\mathrm{diam}^h(Q)}$$

and we are done. So, consider the worst case when $\mathrm{Dist}(T(c), Q_n) \geq 2\mathrm{dist}(T(c), Q_n)$. Since $\mathrm{Dist}(T(c), Q_n) \leq \mathrm{dist}(T(c), Q_n) + \mathrm{diam}(Q_n)$ it implies that $\mathrm{dist}(T(c), Q_n) \leq \mathrm{diam}(Q_n)$ and consequently $\mathrm{Dist}(T(c), Q_n) \leq 2\mathrm{diam}(Q_n)$. Therefore, in view of the second version of Lemma 2.2,

$$\frac{m(Q_{n+1})}{\mathrm{diam}^h(Q_{n+1})} \succeq \frac{m(Q_{n+1})}{\mathrm{Dist}^h(c, Q_{n+1})} \succeq \frac{m(Q_n)}{\mathrm{Dist}(T(c), Q_n)} \geq \frac{1}{2^h}\frac{m(Q_n)}{\mathrm{diam}^h(Q_n)} \succeq \frac{m(Q_n)}{\mathrm{diam}^h(Q_n)}.$$

The proof is finished. ♣

Lemma 7.4. If $h \geq 1$, then there exists a constant $M > 0$ such that if $Q \subset \overline{\mathbb{C}}$ is a connected set, then

$$\frac{m\big(Q_n \cap T^{-n}(A)\big)}{m(Q_n)} \leq Mm(A)^\kappa$$

for every Borel set $A \subset Q$, every $n \geq 0$, and every connected component $Q_n \in \mathrm{Comp}_0^n(Q)$.

Proof. Denote by $q|_n$ the product of multiplicities of critical points of this map. We prove by induction on n that

$$\frac{m(Q_n \cap T^{-n}(A)}{m(Q_n)} \preceq m(A)^{1/q|_n}.$$

For $n = 0$ this is obvious. Fix some $n \geq 0$ and assume that it is true for all $0 \leq k \leq n$. Consider two cases. First suppose that $\hat{Q}_{n+1}$ contains no critical points of T. Let $0 \leq k \leq n$ be the largest integer such that $\hat{Q}_k = T^{n+1-k}(\hat{Q}_{n+1})$ contains a critical point of T. If no such a number exists set $k = 0$. Then, with an arbitrary point $y \in Q_{n+1}$, by Lemma 3.2, and the inductive assumption we have

$$\begin{aligned}\frac{m\big(Q_{n+1} \cap T^{-(n+1)}(A)\big)}{m(Q_{n+1})} &\asymp \frac{|(T^{n+1-k})'(y)|^{-h}m\big(Q_k) \cap T^{-k}(A)\big)}{|(T^{n+1-k})'(y)|^{-h}m(Q_k)} \\ &\preceq \frac{m(A)^{1/q|_n}}{m(Q)} \asymp m(A)^{1/q|_n}.\end{aligned}$$

Now suppose that $\hat{Q}_{n+1}$ contains a critical point, say c. Write $Q_n = T(Q_{n+1})$. If $\mathrm{Dist}(T(c), Q_n) \leq 2\mathrm{dist}(T(c), Q_n)$, then $\mathrm{Dist}(T(c), Q_n) \asymp 2\mathrm{dist}(T(c), Q_n)$ and therefore

$$\begin{aligned}\frac{m\big(Q_{n+1} \cap T^{-(n+1)}(A)\big)}{m(Q_{n+1})} &\asymp \frac{\mathrm{Dist}^{\frac{1}{q(c)}-1}(T(c), Q_n)}{\mathrm{Dist}^{\frac{1}{q(c)}-1}(T(c), Q_n)}\frac{m\big(Q_n \cap T^{-n}(A)\big)}{m(Q_n)} \\ &= \frac{m\big(Q_n \cap T^{-n}(A)\big)}{m(Q_n)} \preceq m(A)^{1/q|_n}\end{aligned}$$

and we are done in this case. So, suppose that $\mathrm{Dist}(T(c), Q_n) \geq 2\mathrm{dist}(T(c), Q_n)$. As $\mathrm{Dist}(T(c), Q_n) \leq \mathrm{dist}(T(c), Q_n) + \mathrm{diam}(Q_n)$ it implies that $\mathrm{dist}(T(c), Q_n) \leq \mathrm{diam}(Q_n)$ and consequently $\mathrm{Dist}(T(c), Q_n) \leq 2\mathrm{diam}(Q_n)$. Thus in view of Lemma 2.2 we get

$$m(Q_{n+1}) \succeq m(Q_n)\mathrm{Dist}(T(c), Q_n)^{\frac{1-q(c)}{q(c)}h} \succeq m(Q_n)\mathrm{diam}(Q_n)^{\frac{1-q(c)}{q(c)}h}.$$

Hence, using Lemma 2.3 and Lemma 7.3, we obtain

$$
\begin{aligned}
\frac{m\big(Q_{n+1}\cap T^{-(n+1)}(A)\big)}{m(Q_{n+1})} &\preceq m(A)^{(q(c)q|_n)^{-1}} \frac{m(Q_n)^{\frac{1}{q(c)}-1}}{\operatorname{diam}(Q_n)^{\frac{1-q(c)}{q(c)}h}} \\
&= m(A)^{(q(c)q|_n)^{-1}} \left(\frac{m(Q_n)}{\operatorname{diam}(Q_n)^h}\right)^{\frac{1-q(c)}{q(c)}} \\
&\preceq m(A)^{(q(c)q|_n)^{-1}} \left(\frac{m(Q)}{\operatorname{diam}(Q)^h}\right)^{\frac{1-q(c)}{q(c)}} \asymp m(A)^{(q(c)q|_n)^{-1}}.
\end{aligned}
$$

The proof is finished. ♣

Let $\theta > 0$ be so small that for every $\omega \in \Omega$ the local map T_ω^{-1} is well defined on $B(\omega, 3\theta)$,

$$T_\omega^{-1}(B(\omega,3\theta)\cap J(T)) \subset B(\omega,3\theta)$$

and

$$|(T_\omega^{-1})'(z)| < 1$$

for all $z \in B(\omega, 3\theta)$. Let $0 < \xi < \theta/||T'||^2$ be so small that the following properties are satisfied for every $\omega \in \Omega$.

All the sets

$$T_\omega^{-n}\big((B(\omega,2\theta)\setminus B(\omega,\theta/||T'||^2))\cap B(J(T),\xi)\big), n \geq 0, \tag{7.1}$$

are well defined and mutually disjoint.

$$\operatorname{dist}\big(\overline{\mathbb{C}}\setminus B(\omega,\theta), T_\omega^{-1}(J(T)\cap B(\omega,\theta+\xi))\big) > 2\xi, \tag{7.2}$$

$$\sup\{|(T_\omega^{-n})'(z)| : z \in (B(\omega,2\theta)\setminus B(\omega,\theta/||T'||^2))\cap B(J(T),\xi), n \geq 1\} \leq 1. \tag{7.3}$$

It follows immediately from (7.1), (7.2), and (7.3) that

$$T_\omega^{-j}(Q)\cap T_\omega^{-i}(Q') = \emptyset \tag{7.4}$$

for all integers $i, j \geq 0$ and all disjoint sets Q and Q' intersecting $J(T)\cap\partial B(\omega,\theta)$ and with diameters $\leq \xi$. Finally we require the constant $0 < \eta < \min\{\theta, \xi\}$ introduced in [Ur, Prop. 6.1] be at least so small that for every connected set Q with diameter $\leq \eta$, every $n \geq 0$ and every connected component Q_n of $T^{-n}(Q)$ satisfying the property $(*)$, $\operatorname{diam}(Q_n) < \xi$. Letting now Q be a connected set with diameter $\leq \xi$ and not disjoint from $J(T)$ we define the family $\operatorname{Comp}_*^1(Q)$ consisting of connected components of backward iterates of Q as follows.

(a) If $Q \cap B(\Omega, \theta) = \emptyset$,
then $\mathrm{Comp}_*^1(Q)$ is formed by the family of all connected components of $T^{-1}(Q)$.

(b) If $Q \cap T_\omega^{-1}(B(\Omega, \theta)) \neq \emptyset$ for some $\omega \in \Omega$,
then $\mathrm{Comp}_*^1(Q)$ is formed by the family of all connected components of $T^{-1}(Q)$ except for $T_\omega^{-1}(Q)$ which is well defined as $Q \subset B(\omega, 2\theta)$.

(c) Finally, if $Q \cap B(\Omega, \theta) \neq \emptyset$ for some $\omega \in \Omega$ but $Q \cap T_\omega^{-1}(B(\Omega, \theta)) = \emptyset$,
then $\mathrm{Comp}_*^1(Q)$ is formed by all the connected components of $T^{-1}(Q)$ and all the sets of the form $T_\omega^{-n}(Q)$ which are well defined due to the choice of ξ and since $\mathrm{diam}(Q) \leq \xi$.

To every element $P \in \mathrm{Comp}_*^1(Q)$ we ascribe the number $k_Q(P)$ to be n if $P = T_\omega^{-n}(Q)$ and to be 1 in all the other cases. Notice that always P is a connected component of $T^{-k_Q(P)}(Q)$ satisfying property $(*)$.

Notice that the family $\mathrm{Comp}_*^1(Q)$ consists again of connected sets and if we start with Q of diameter $\leq \eta$, we can proceed inductively defining the families $\mathrm{Comp}_*^n(Q)$ by setting

$$\mathrm{Comp}_*^{n+1}(Q) = \bigcup \{\mathrm{Comp}_*^1(Q_n^*) : Q_n^* \in \mathrm{Comp}_*^n(Q), n \geq 1\}$$

We shall prove the following.

Lemma 7.5. If $Q \subset \overline{\mathbb{C}}$ is a connected set not disjoint from $J(T)$ and $\mathrm{diam}(Q) \leq \eta$, then for every $n \geq 0$ the family $\mathrm{Comp}_*^n(Q)$ consists of mutually disjoint sets.

Proof. We need to prove that if Q is of diameter $\leq \eta$, then $\mathrm{Comp}_*^1(Q)$ consists of mutually disjoint sets and that if Q and Q' are two disjoint disks, then all the elements of $\mathrm{Comp}_*^1(Q)$ are mutually disjoint from all the elements of $\mathrm{Comp}_*^1(Q')$. Then the obvious induction works. So, let us deal first with the family $\mathrm{Comp}_*^1(Q)$ alone. Indeed, if either $Q \cap B(\Omega, \theta) = \emptyset$ or $Q \cap T_\omega^{-1}(B(\Omega, \theta)) \neq \emptyset$, the the disjointness of the family $\mathrm{Comp}_*^1(Q)$ is obvious. Thus we can suppose that $Q \cap B(\Omega, \theta) \neq \emptyset$ for some $\omega \in \Omega$ but $Q \cap T_\omega^{-1}(B(\Omega, \theta)) = \emptyset$. Then all the connected components of $T^{-1}(Q)$ are mutually disjoint and all of them except for $T_\omega^{-1}(Q)$ are disjoint from $B(\Omega, 2\theta)$. Since $\bigcup_{n \geq 0} T_\omega^{-n}(Q) \subset B(\omega, \theta)$, it is enough to know that the sets $T_\omega^{-n}(Q)$, $n \geq 0$, are mutually disjoint which in turn follows from the definition of ξ. Consider now Q and Q', two connected sets of diameter $\leq \xi$. The only problem with disjointness may occur if for at least one of these sets, say for example for Q, we have $Q \cap B(\Omega, \theta) \neq \emptyset$ for some $\omega \in \Omega$ but $Q \cap T_\omega^{-1}(B(\Omega, \theta)) = \emptyset$. Then also $Q \subset B(\omega, \theta + \xi)$ and $\bigcup_{n \geq 1} T_\omega^{-n}(Q) \subset B(\omega, \theta)$. So, if $Q' \cap T_\omega^{-1}(B(\Omega, \theta)) \neq \emptyset$, then $\mathrm{Comp}_*^1(Q') \cap B(\Omega, 2\theta) = \emptyset$ and we are done. If $Q' \cap B(\Omega, \theta) = \emptyset$, then the only difficulty may appear if there exists a connected component of $T^{-1}(Q')$ intersecting $B(\Omega, \theta)$. But then this component equals $T_\omega^{-1}(Q')$ and as Q' and Q are disjoint it folows from (7.4) that $\bigcup_{n \geq 1} T_\omega^{-n}(Q) \cap T_\omega^{-1}(Q') = \emptyset$ and we are also done. Thus it is left to consider the case when $Q' \cap B(\Omega, \theta) \neq \emptyset$ but $Q' \cap T_\omega^{-1}(B(\Omega, \theta)) = \emptyset$. Then again the connected components of $T^{-1}(Q)$ and $T^{-1}(Q')$ are mutually disjoint and all of them are disjoint from $B(\Omega, \theta)$ except for $T_\omega^{-1}(Q)$ and $T_\omega^{-1}(Q')$. So, we only need to know that the unions $\bigcup_{n \geq 1} T_\omega^{-n}(Q)$ and $\bigcup_{n \geq 1} T_\omega^{-n}(Q')$ are disjoint which follows immediately from (7.4). The proof is finished. ♣

Combining Lemma 7.4 and Lemma 7.5 we get the following.

Lemma 7.6. If $h \geq 1$, then there exists a constant $M > 0$ such that for every topological disk Q of diameter $\leq \eta$ and every Borel set $A \subset Q$ we have

$$\mu_n^*(A) := \sum_{Q_n \in \mathrm{Comp}_*^n(Q)} m\big(Q_n \cap T^{-n}(A)\big) \leq M m(A)^\kappa.$$

We shall now prove the lemma which adopts to our setting the results of Carleson, Jones, and Yoccoz from [CJY] about exponential decrease of diameters of connected components of backward images of small balls.

Lemma 7.7. There exist constants $C > 0$ and $\rho > 0$ such that if Q is a geometric disk centered at $J(T)$ and of radius $\leq \eta/2$ (η as above sufficiently small), then for every $n \geq 0$

$$\sup\{\mathrm{diam}(Q_n^*) : Q_n^* \in \mathrm{Comp}_*^n(Q)\} \leq C\mathrm{e}^{-\rho n}.$$

Proof. In view of Lemma 1.1 of [Pr]

$$\lim_{n\to\infty} \sup\{\mathrm{diam}(P_n)\} = 0,$$

where the supremum is taken over all geometric disks centered at $J(T)$ of radii $\leq \eta/2$ and over all connected components $P_n \in \mathrm{Comp}_0^n(P)$. In particular there exists a number $q \geq 1$ such that $\mathrm{diam}(P_j) \leq \eta/4$ for every $j \geq q$ and every geometric disk of radius $\leq \eta/2$. Fix now a chain sequence $Q_n^* \in \mathrm{Comp}_*^n(Q)$, that is a sequence satisfying $Q_{n+1}^* \in \mathrm{Comp}_*^1(Q_n)$ for all $n \geq 0$. For every $n \geq 0$ choose in Q_n^* one element $y_n^* \in Q_n^* \cap T^{-k_{Q,n}(Q_n^*)}(Q)$, where $k_{Q,n}(Q_n^*) = \sum_{j=0}^{n-1} k_{Q_j^*}(Q_{j+1}^*)$. In the sequel, if it does not cause confusion we will write k_n for $k_{Q,n}(Q_n^*)$. If $s \geq 1$ and $0 \leq n \leq s/q$ we define the following four sets.

$$\begin{aligned}
B_{s,n} &= \mathrm{Comp}\big(y_s^*, T^{-(k_s - k_{qn})}(B(y_{qn}^*, \eta/2))\big) \\
V_n^* &= \mathrm{Comp}\big(y_{qn}^*, T^{-(k_{qn} - k_{q(n-1)})}(B(y_{q(n-1)}^*, \eta/2))\big) \\
V_{s,n} &= \mathrm{Comp}\big(y_s^*, T^{-(k_s - k_{qn})}(V_n^*)\big) \\
A_{s,n} &= B_{s,n} \setminus V_{s,n}
\end{aligned}$$

Observe now that the following two formulas hold

$$B_{s,n} \subset B_{s,n+1} \tag{7.5}$$

and

$$A_{s,n+1} \cap B_{s,n} = \emptyset \tag{7.6}$$

Indeed, since $T^{k_{q(n+1)}-k_{qn}}(T^{k_s-k_{q(n+1)}}(B_{s,n}) \subset B((y^*_{qn}, \eta/2)$ and since $T^{k_s-k_{q(n+1)}}(B_{s,n})$ is a connected set containing $y^*_{q(n+1)}$, we conclude that

$$T^{k_s-k_{q(n+1)}}(B_{s,n}) \subset V^*_{n+1}. \tag{7.7}$$

Since additionally, by the definition of q, $V^*_{n+1} \subset B(y^*_{q(n+1)}, \eta/4)$ (pay attension: V^*_n satisfies property $(*)$!) we obtain

$$B_{s,n} \subset \mathrm{Comp}\big(y^*_s, T^{-(k_s-k_{q(n+1)})}(B(y^*_{q(n+1)}, \eta/2))\big) = B_{s,n+1}.$$

So, (7.5) is proved. In order to prove (7.6) notice that

$$T^{k_s-k_{q(n+1)}}(A_{s,n+1}) \subset B(y^*_{q(n-1)}, \eta/2) \setminus V^*_{n+1}$$

which combined with (7.7) completes the argument.

Now by Lemma 3.1 each set $A_{s,n}$ contains an annulus of modulus $\geq \kappa \log 2/\#(\mathrm{Crit}(T))$ surronding y^*_s. Since additionally $B_{s,0} = Q^*_s$ and $B_{s,[s/q]} \subset B(y^*_s, \varepsilon)$, it follows from (7.5), (7.6) and a result of [McM] that the hyperbolic diameter of Q^*_s in the hyperbolic disk $B(y^*_s, \varepsilon)$ is bounded from above by $\exp\big(-\frac{\kappa \log 2}{\#(\mathrm{Crit}(T)} \cdot [\frac{s}{q}]\big)$ multiplied by a universal constant. Since y^*_s, the center of the disk $B(y^*_s, \varepsilon)$, is in Q^*_s, for large s the hyperbolic diameter of Q^*_s and its euclidean diameter are comparable. This completes the proof of Lemma 7.7. ♣

Lemma 7.8. If $x \in J(T)$ and $k = k(x,r) \leq u(x,r)$ is the least integer such that $T^j(x) \in B(\Omega, d)$ for all $j = k, k+1, \dots, u$, then

$$m\big(B(T^k(x), r|(T^k)'(x)|\big) \succeq (r|(T^k)'(x)|)^{h+p(\omega)(h-1)}.$$

Proof. The proof of this lemma is an immediate consequence of Lemma 2.4 combined with the following facts. Firstly, $m\big(B(T^u(x), r|(T^u)'(x)|\big) \succeq (r|(T^u)'(x)|)^{h+p(\omega)(h-1)}$. Secondly T^{u-k} maps univalently a neighbourhood of $T^k(x)$ of size $r|(T^k)'(x)|$ onto a neighbourhood of $T^u(x)$ of size $r|(T^u)'(x)|$, and thirdly $|(T^{u-k})'(x)| \geq 1$ and $h + p(\omega)(h-1) \geq h$. ♣

Now we can prove the following.

Lemma 7.9. There exists a constant $\Gamma > 0$ such that $m(B(x,r)) \succeq r^\Gamma$ for all $x \in J(T)$ and all $0 < r \leq 1$.

Proof. We proceed inductively with respect to the sets $Cr_0, J_0, Cr_1, J_1, \dots, Cr_p, J_p$. More precisely, we shall construct inductively a non-decreasing sequence of numbers $\Gamma_{0,1} \leq \Gamma_{0,2} \leq \Gamma_{1,1} \leq \Gamma_{1,2} \leq \dots \leq \Gamma_{p,1} \leq \Gamma_{p,2}$ such that $m(B(x,r)) \succeq r^{\Gamma_{i,1}}$ for all $x \in Cr_0 \cup J_0 \cup \dots \cup J_{i-1} \cup Cr_i = J_{i-1} \cup S_i(T)$ and $m(B(x,r)) \succeq r^{\Gamma_{i,2}}$ for all $x \in Cr_0 \cup J_0 \cup \dots \cup Cr_i \cup J_i = J_i \cup S_i(T)$ $(0 \leq r \leq 1)$. Indeed, since $Cr_0 = \emptyset$, any positive number $\Gamma_{0,1}$ works and we may declare $\Gamma_{0,1} = h + P(h-1)$, where $P = \max\{p(\omega) : \omega \in \Omega\}$. Suppose now that for some $0 \leq i \leq p$ all the numbers $\Gamma_{0,1} \leq \Gamma_{0,2} \leq \dots \leq \Gamma_{i,1}$ have been already defined and $m(B(x,r)) \succeq r^{\Gamma_{i,1}}$ for all $x \in J_{i-1} \cup S_i(T)$ with $0 \leq r \leq 1$. We shall check that there exists a number $\Gamma_{i,2} \geq \Gamma_{i,1}$ working for the set $J_i \cup S_i(T)$. Indeed, fixing $\varepsilon > 0$ appearing in [Ur, Prop.6.4] so small that $\varepsilon\theta\tau\lambda^{-1} < \rho$, we conclude from this proposition and from the definition of J_i that if either u is not defined or $s < u$ (and then also $s < k$), then the critical point produced in this proposition belongs to $S_i(T)$. Therefore applying either our inductive assumption (in the former case) or Lemma 7.8 (in the latter case) we conclude that in either case (note also that $\Gamma_{i,1} \geq h + P(h-1)$)

$$m(B(T^d(x), r|(T^d)'(x)|)) \succeq (r|(T^d)'(x)|)^{\Gamma_{i,1}}, \tag{7.8}$$

where $d = \min\{s, k\}$. Consider now Q_d, the connected component of $T^{-d}(Q)$ containing x where $Q = B(T^d(x), \eta/2)$. Since all the connected components of all the inverse images of Q_d satisfy the property $(*)$, applying Lemma C.7 along with [Ur, Prop.6.4(c)] and KDT, we get

$$\frac{r}{(r|(T^d)'(x)|)^\kappa} \preceq \mathrm{diam}(Q_d)\mathrm{diam}^\kappa(Q) \leq \eta^{-\kappa} \preceq 1$$

and consequently $|(T^d)'(x)| \succeq r^{\frac{1-\kappa}{\kappa}}$. Thus applying (7.6) and using [Ur, Prop.6.4] again, we get

$$\begin{aligned} m(B(x,r)) &\asymp |(T^d)'(x)|^{-h} m(B(T^d(x), r|(T^d)'(x)|) \\ &\succeq |(T^d)'(x)|^{-h} r^{\Gamma_{i,1}} |(T^d)'(x)|^{\Gamma_{i,1}} = r^{\Gamma_{i,1}} |(T^d)'(x)|^{\Gamma_{i,1}-h} \\ &\succeq r^{\Gamma_{i,1}+(\Gamma_{i,1}-h)(1-\kappa)/\kappa} \end{aligned}$$

and setting $\Gamma_{i,2} = \Gamma_{i,1} + (\Gamma_{i,1} - h)(1-\kappa)/\kappa = h + (\Gamma_{i,1} - h)/\kappa$ finishes this part of the inductive step.

Now suppose that for some $0 \leq i \leq p-1$ all the numbers $\Gamma_{0,1} \leq \Gamma_{0,2} \leq \ldots \leq \Gamma_{i,1} \leq \Gamma_{i,2}$ have been already defined and $m(B(x,r)) \succeq r^{\Gamma_{i,2}}$ for all $x \in J_i \cup S_i(T)$. It then immediately follows from [Ur, Lemma 5.9] and Lemma 2.5 that $m(B(c,r)) \succeq r^{h+q(c)(\Gamma_{i,2}-h)}$ for every critical point $c \in Cr_{i+1}(T)$. So, as $\Gamma_{i+1,1}$ one can take $\max\{\Gamma_{i,2}, h + (\Gamma_{i,2} - h)\max\{q(c) : c \in \mathrm{Crit}(T)\}\} = h + (\Gamma_{i,2} - h)/\kappa$. Thus the inductive proof is finished and simultaneously the proof of Lemma 7.9 taking $\Gamma = \Gamma_{p,2}$. ♣

Theorem 7.10. If $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an NCP map and m is the only h-conformal measure, then $\mathrm{PD}(m) = \mathrm{HD}(J(T))$.

Proof. If $h \leq 1$, the result is covered by Lemma 7.2. So, suppose that $h \geq 1$. Fix an $\varepsilon > 0$. Given $r > 0$ consider the set

$$X(r) = \{z : (r|(T^s)'(z)|)^{\Gamma-h} \leq r^\varepsilon\}.$$

Our aim is to get a good estimate from above on the measure $m(X(r))$. So, fix $x \in X(r)$. Let again $d = \min\{s(x,r), k(x,r)\}$, let $Q = B(T^d(x), \eta/2)$, and let Q_d be the connected component of $T^{-d}(Q)$ containing x. Since $|(T^{s-d})'(x)| \geq 1$, it follows from the definition of $X(r)$ that

$$|(T^d)'(x)| \leq r^{\frac{\varepsilon-(\Gamma-h)}{\Gamma-h}} \leq r^{-1} \tag{7.9}$$

Observe that by the definition of Q and by the definition of families $\mathrm{Comp}_*^n(Q)$, there exists $0 \le d^* \le d$ such that $Q_d \in \mathrm{Comp}_*^{d^*}(Q)$ and it follows from Lemma 7.7 and Lemma C.7 that

$$\mathrm{e}^{-\rho d^*} \ge \frac{\mathrm{diam}(Q_d)}{\mathrm{diam}^\kappa(Q)} \ge \frac{r}{(r|(T^d)'(x)|)^\kappa} = r^{1-\kappa}|(T^d)'(x)|^{-\kappa}.$$

Combining this and (7.9) we get $\mathrm{e}^{-\rho}d^* \ge r^{1-\kappa}r^\kappa = r$, hence

$$d^* \le -\frac{1}{\rho}\log r \tag{7.10}$$

Recall now that if $s < u$, then $s = d$ and $T^d(x) \in B(c, \mathrm{const}.r|(T^d)'(x)|)$ for some critical point c of T, and if $s = u$, then $T^u(x) \in B(\Omega, \mathrm{const}.r|(T^u)'(x)|)$. But $T^d(x) = T^k(x)$ is closer to Ω than $T^u(x)$ and therefore $T^d(x) \in B(\Omega, \mathrm{const}.r|(T^u)'(x)|)$. Since $B(x,r) \subset Q_d \cap T^{-d}(B(T^d(x), \mathrm{const}.r|(T^d)'(x)|))$, we thus get

$$\bigcup_{x\in X(r)} B(x,r) \subset \bigcup_{j=0}^{-\frac{1}{\rho}\log r} \bigcup_{Q_j\in \mathrm{Comp}_*^j(Q)} \big(Q_j \cap T^{-j}(B(\Omega\cup \mathrm{Crit}(T), \mathrm{const}.r^{\frac{\varepsilon}{\Gamma-h}}))\big).$$

Since $\Omega \cup \mathrm{Crit}(T)$ is a finite set, it follows from this inclusion, Lemma 7.6, inequality $m(B(x,r)) \preceq r^h$, and (7.8) that

$$m\Big(\bigcup_{x\in X(r)} B(x,r)\Big) \preceq -\frac{1}{\rho}\log r\big(r^{\frac{\varepsilon\Gamma}{\Gamma-h}}\big) \preceq r^\nu$$

for any fixed $0 < \nu < \frac{\varepsilon\Gamma}{\Gamma-h}$ and every $r > 0$ sufficiently small (depending on ν), say $r \le r_\nu$. Hence for any integer n so large that $2^{-n} \le r_\nu$, we have

$$m\Big(\bigcup_{j\ge n} X(2^{-j})\Big) \le \sum_{j\ge n} m(X(2^{-j})) \preceq \sum_{j\ge n} 2^{-j\nu} = \frac{2^{-n\nu}}{1-2^{-\nu}}$$

and consequently for every $k \ge 1$ there exists $N_k \ge 1$ such that $m\big(\bigcup_{j\ge N_k} X(2^{-j})\big) \le 1/k$. Thus denoting by Y_k the complement of $\bigcup_{j\ge N_k} X(2^{-j})$ in $J(T)$, we get $m(Y_k) \ge 1 - \frac{1}{k}$. We shall show that $\overline{\mathrm{BD}}(Y_k) \le h+\varepsilon$. Indeed, let $B(y_1, 2^{-j}), B(y_2, 2^{-j}), \dots B(y_p, 2^{-j})$ be an arbitrary packing of Y_k by open balls centered at points of Y_k and with radii 2^{-j}, $j \ge N_k$. Since $Y_k \subset J(T) \setminus X(2^{-j})$, we have $(2^{-j}|(T^{d_i})'(y_i)|)^{\Gamma-h} \ge 2^{-j\varepsilon}$ for every $i = 1, 2, \dots, p$ and $d_i = d(y_i, 2^{-j})$. Hence by [Ur, Proposition 6.4(c)] and Lemma 7.9 we get

$$\begin{aligned} m(B(y_i, 2^{-j})) &\asymp |(T^{d_i})'(y_i)|^{-h} m\big(B(T^{d_i}, 2^{-j}|(T^{d_i})'(y_i)|)\big) \\ &\succeq |(T^{d_i})'(y_i)|^{-h}(2^{-j}|(T^{d_i})'(y_i)|)^\Gamma \\ &= (2^{-j}|(T^{d_i})'(y_i)|)^{\Gamma-h}(2^{-j})^h \ge (2^{-j})^{h+\varepsilon}. \end{aligned}$$

Hence, using disjointness of balls $B(y_1, 2^{-j}), B(y_2, 2^{-j}), \dots B(y_p, 2^{-j})$, we get $p(2^{-j})^{h+\varepsilon} \le \sum_{i=1}^p m(B(y_i, 2^{-j})) \le 1$ and consequently $\frac{\log p}{-\log(2^{-j})} \le h+\varepsilon$. Letting $j \to \infty$ this shows that $\overline{\mathrm{BD}}(Y_k) \le h+\varepsilon$ and, in particular, $\mathrm{PD}(Y_k) \le h+\varepsilon$. Finally setting $Y = \bigcup_{n\ge1} Y(1/n)$, we end up with the formula $m(Y) = 1$ and $\mathrm{PD}(Y) \le h$. That $\mathrm{PD}(Y) \ge h$ follows from the fact that m is abosolutely continuous with respect to Π_h proven in [Ur]. The proof of Theorem 7.10 is finished. ♣

Let us now prove two sufficient conditions for the equality $\overline{\mathrm{BD}}(J(T)) = \mathrm{HD}(J(T))$ to hold.

Theorem 7.11. If $m(B(x,r)) \succeq r^\theta$ for some some $\theta < \frac{h}{1-\kappa}$ and all $x \in \Omega \cup \mathrm{Crit}(T)$, then $\overline{\mathrm{BD}}(J(T)) = \mathrm{HD}(J(T))$.

Proof. As the case $h \le 1$ is covered by Lemma 7.2 we assume $h \ge 1$. Fix an $\varepsilon > 0$. For $r > 0$ and $q \ge 1$ set

$$X(r,q) = \{x \in X(r) : \mathrm{e}^{-(q+1)} \le r|(T^d)'(x)| \le \mathrm{e}^{-q}\},$$

where $X(r)$ and d are the same as in the proof of Theorem 7.10. The same arguments as in the proof of Theorem 7.10 give for every $q \ge 1$

$$\bigcup_{x\in X(r,q)} B(x,r) \subset \bigcup_{j=0}^{r^{-\varepsilon}} \bigcup_{Q_j\in \mathrm{Comp}_*^j(Q)} \big(Q_j \cap T^{-j}(B(S,\mathrm{e}^{-q})\big),$$

where $S = \Omega \cup \mathrm{Crit}(T)$. Hence, applying Lemma 7.6 we get

$$m\big(\bigcup_{x\in X(r,q)} B(x,r)\big) \le r^{-\varepsilon} m(B(S,\mathrm{e}^{-q}))^\kappa. \tag{7.11}$$

Let now $P \subset J(T)$ be an arbitrary r-packing of $J(T)$. By (7.12) for every $q \ge 1$ we have

$$m\big(\bigcup_{x\in P\cap X(r,q)} B(x,r)\big) \le r^{-\varepsilon} m(B(S,\mathrm{e}^{-q}))^\kappa.$$

On the other hand by Lemma 7.9

$$\begin{aligned}
m\big(\bigcup_{x\in P\cap X(r,q)} B(x,r)\big) &= \sum_{x\in P\cap X(r,q)} m(B(x,r)) \\
&\asymp \sum_{x\in P\cap X(r,q)} |(T^d)'(x)|^{-h} m(B(T^d(x), r|(T^d)'(x)|) \\
&\asymp \sum_{x\in P\cap X(r,q)} |(T^d)'(x)|^{-h} m(B(S,\mathrm{e}^{-q})) \\
&\asymp \sum_{x\in P\cap X(r,q)} \left(\frac{r|(T^d)'(x)|}{r}\right)^{-h} m(B(S,\mathrm{e}^{-q})) \\
&\succeq \#(P\cap X(r,q)) r^h \mathrm{e}^{qh} m(B(S,\mathrm{e}^{-q})).
\end{aligned}$$

So, combining this estimate and (7.11) we get

$$\begin{aligned}
\#(P\cap X(r,q)) &\le r^{-h}\mathrm{e}^{-qh}m(B(S,\mathrm{e}^{-q}))^{\kappa-1}r^{-\varepsilon}\\
&= r^{-(h+\varepsilon)}\mathrm{e}^{-qh}m(B(S,\mathrm{e}^{-q}))^{\kappa-1}\\
&\preceq r^{-(h+\varepsilon)}\mathrm{e}^{-q(h-\theta(1-\kappa)}
\end{aligned}$$

Therefore, as $X(r)=\bigcup_{q\ge 1}X(r,q)$, and $h-\theta(1-\kappa>0$, $\#(P\cap X(r))\preceq r^{-(h+\varepsilon)}$. On the other hand if $x\in P\setminus X(r)$, then by [Ur, Proposition 6.4(c)] and Lemma 7.9 we get

$$\begin{aligned}
m(B(x,r)) &\asymp |(T^d)'(x)|^{-h}m\big(B(T^d(x),r|(T^d)'(x)|)\big)\\
&\succeq |(T^d)'(x)|^{-h}(r|(T^d)'(x)|)^{\Gamma}\\
&= (r|(T^d)'(x)|)^{\Gamma-h}r^h \ge r^{h+\varepsilon}.
\end{aligned}$$

Consequently $\#P\preceq r^{-(h+\varepsilon)}$. Therefore letting now $\varepsilon\to 0$ finishes the proof. ♣

Corollary 7.12. If $P=\max\{p(\omega):\omega\in\Omega\}$ and $P(1-\kappa)(h-1)<h\kappa^{2\#(\mathrm{Crit}(T))}$, then $\overline{\mathrm{BD}}(J(T))=\mathrm{HD}(J(T))$.

Proof. It follows from the proof of Lemma 7.9 that $\Gamma\le h+P(h-1)/\kappa^{2\#(\mathrm{Crit}(T))-1}$. Hence, in view of Lemma 7.9, $\overline{\mathrm{BD}}(J(T))=\mathrm{HD}(J(T))$ and we are done. ♣

Let us now state as a corollary a result proved in [DU3].

Corollary 7.13. If $J(T)$ contains no critical point ($\kappa=1$), then $\overline{\mathrm{BD}}(J(T))=\mathrm{HD}(J(T))$.

Let us also formulate the following.

Corollary 7.14. If $P=1$ and there is only one critical point in $J(T)$, which is moreover quadratic, then $\overline{\mathrm{BD}}(J(T))=\mathrm{HD}(J(T))$.

We would like to end up the paper with the remark that proceeding similarily as in the proof of Theorem 7.10 (reversing inequalities) one can prove the following.

Theorem 7.15 If m is the h-conformal measure for $T:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$, then $\mathrm{HD}(m)=\mathrm{HD}(J(T))$.

References

[ADU] J. Aaronson, M. Denker, M. Urbański, Ergodic theory for Markov fibred systems and parabolic rational maps, Transactions A.M.S. 337 (1993), 495-548.

[CJY] L. Carleson, P. W. Jones, J.-Ch. Yoccoz, Julia and John, Bol. Soc. Bras. Mat. 25(1994), 1-30.

[DU1] M. Denker, M. Urbański, Hausdorff and conformal measures on Julia sets with a rationally indifferent periodic point, J. London Math. Soc. 43 (1991), 107-118.

[DU2] M. Denker, M. Urbański, Geometric measures for parabolic rational maps, Ergod. Th. and Dynam. Sys. 12 (1992), 53-66.

[DU3] M. Denker, M. Urbański, The Capacity of parabolic Julia sets, Math. Zeitsch. 211 (1992), 73 – 86.

[Fa] K. J. Falconer, The Geometry of fractal sets, Cambridge University Press, 1985.

[Fe] H.F. Federer, Geometric measure Theory, Springer - Verlag, New York Inc. 1969.

[Man] R. Mane, On a lemma of Fatou, Bol. Soc. Bras. Mat. 24(1993), 1-12.

[Mar] M. Martens, The existence of σ-finite invariant measures, Applications to real one-dimensional dynamics, Preprint.

[Mat] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces,

[McM] C. McMullen, Complex dynamics and renormalization, Annals of Math. Studies 135, Princeton University Press, 1994.

[Pr] F. Przytycki, Iterations of holomorphic Collect-Eckmann maps: conformal measures, Preprint 1994.

[PU] F. Przytycki, M. Urbański, Fractals in the complex plane - ergodic theory methods, to appear.

[TT] S.J. Taylor, C. Tricot, Packing measure, and its evaluation for a Brownian path, Trans. A.M.S. 288 (1985), 679 - 699.

[Ur] M. Urbański, Rational functions with no recurrent critical points, to appear Ergod. Th. & Dynam. Sys. 14 (1994), 391-414.