COMPLEX CONTINUED FRACTIONS WITH RESTRICTED ENTRIES.

PAWEL HANUS AND MARIUSZ URBAŃSKI

Department of Mathematics, University of North Texas,
Denton, TX 76203-5118.
E-mail: pgh0001@jove.acs.unt.edu, urbanski@unt.edu

1. Introduction

Let (X, ρ) be a compact metric space, and let I be a countable set with at least two elements. Define $S = \phi_i : X \to X \ : \ i \in I$, a collection of injective contractions from X to X for which there exists $0 < s < 1$ such that

$$\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y),$$

for every $i \in I$ and for every pair of points $x, y \in X$. Any such collection is called an iterated function system (abbreviated as i.f.s.). Set $I^* = \bigcup_{m \geq 1} I^m$ and for $\omega \in I^m, m \geq 1$ define

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_m}.$$

If $\omega \in I^* \cup I^\infty$ and $m \geq 1$ does not exceed the length of ω, we denote by $\omega|_m$ the word $\omega_1 \omega_2 \ldots \omega_m$.
In general, the main object of our interest is the limit set J associated to the system $S = \{\phi_i : X \to X \ : \ i \in I\}$, that is the set defined as

$$J = \bigcup_{\omega \in I^\infty} \bigcap_{m=1}^{\infty} \phi_{\omega|_m}(X). \tag{1}$$

An iterated function system, S, is said to satisfy Open Set Condition (OSC) if there exists a nonempty open set $U \subset X$ such that $\phi_i(U) \subset U$ for all $i \in I$ and also $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I, \ i \neq j$. From now on assume that X is a subset of a d-dimensional Euclidean

Date: April 1998.

space. An iterated function system is said to be conformal if the following conditions are satisfied.

• X is compact and connected, $U = \mathrm{Int}_{\mathbb{R}^d}(X)$, $\phi_i(U) \subset U$, $\phi_i(U) \cap \phi_j(U) = \emptyset$ for $i \neq j$.

• There exist $\alpha, l > 0$ such that for every $x \in \partial X$ there exists an open cone, $\mathrm{Con}(x, u_x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, direction vector u_x, central angle of Lebesgue measure α, and altitude l. This is so called cone property.

• There exists an open connected set $X \subset V \subset \mathbb{R}^d$ such that $\phi_i : V \to V$ for all i and every map ϕ_i can be extended to a $C^{1+\epsilon}$ diffeomorphism on V, and the extended map is conformal on V.

• There exists $K \geq 1$ such that $|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$ for every $\omega \in I^*$ and every pair of points $x, y \in V$. This is so called Bounded Distortion Property (BDP).

Let $X(\infty)$ be the set of limit points of all sequences $x_i \in \phi_i(X),\ i \in I'$ where I' ranges over all infinite subsets of I.

The topological pressure function, P, for iterated function systems is defined as follows. For every $t \geq 0$ consider the series

$$\psi_1(t) = \sum_{i \in I} ||\phi'_i||^t,$$

and more generally define for every integer $n \geq 1$

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi'_\omega||^t.$$

Now set

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \psi_n(t). \tag{2}$$

As it was shown in [MU1] there are two disjoint classes of conformal iterated function systems, regular and irregular. A system is regular if there exists $t \geq 0$ such that $P(t) = 0$. Otherwise the system is irregular.

Moreover the following property holds.

Theorem 1. $\dim_H(J) = \sup\{\dim_H(J_F) :\ F \subset I \textit{ finite}\} = \inf\{t \geq 0 : P(t) \leq 0\}$. *If a system is regular and* $P(t) = 0$ *then* $t = \dim_H(J)$.

A Borel probability measure m is said to be t-conformal if $m(J) = 1$ and for every Borel set A and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi'_i|^t \, dm \tag{3}$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0, \tag{4}$$

for every pair $i, j \in I,\ i \neq j$. Lemma 3.13. in [MU1] states that the conformal iterated function system is regular if and only if there exists a t-conformal measure (t is such that $P(t) = 0$), and then $t = \dim_H(J)$.

2. Results

In this paper we focus our attention on a special example of an infinite conformal iterated function system. Namely, let X be a closed disc on a complex plane centered at the point $1/2$ with radius $1/2$, and let $V = B(1/2, 3/4)$. Set $I = \{n + ki\ :\ n \in \mathbb{N},\ k \geq 1 \text{ fixed}\}$, and for every index $n + ki \in I$ define $\phi_{n+ki} : V \to V$ as follows

$$\phi_{n+ki}(t) = \frac{1}{n + ki + t}.$$

One can easily verify that for every positive integer n, $\phi_{n+ki}(X) \subseteq X$ and $\phi_{n+ki}(V) \subseteq V$, or even more precisely

$$\phi_{n+ki}(B(1/2, 1/2)) = \frac{1}{B(n + 1/2 + ki, 1/2)}. \tag{5}$$

Moreover we have that $\phi'_{n+ki}(t) = -(n+ki+t)^{-2}$, and hence $||\phi'_{n+ki}|| = |n + ki|^{-2} < (1 + k^2)^{-1} < 1$. That gives us the universal contractive constant from the definition of i.f.s. It is also easy to check that our system is conformal (all four conditions from the definition are trivially satisfied). As announced at the beginning of the paper we want to turn our attention to the limit set J_k associated with the system. In particular we want to investigate the Hausdorff dimension, and then the h-dimensional Hausdorff and packing measures of this set, where $h = \dim_H(J_k)$.

The similar systems were introduced and studied in [GM] and [MU1]. In particular it was shown in [MU1] that the limit set J related to the system where there is no restriction for an index k ($k \in \mathbb{Z}$ arbitrary) has the following properties:

$$1.2484 < h = \dim_H(J) < 1.9,$$

$$\mathcal{H}^h(J) = 0,$$

$$0 < \mathcal{P}^h(J) < \infty,$$

where $\mathcal{H}^h$ and $\mathcal{P}^h$ denote h-dimensional Hausdorff and packing measures respectively.

We start our investigation with the following lemma.

Lemma 2. $\lim_{k\to\infty} \dim_H(J_k) = 1/2$.

Proof. According to what we said earlier $\dim_H(J_k) = \inf\{t \geq 0 : P(t) \leq 0\}$. One can prove using the chain rule that the sequence $\psi_n(t)$ is subadditive, that is

$$K^{-t}\psi_n(t)\psi_m(t) \leq \psi_{n+m}(t) \leq \psi_n(t)\psi_m(t) \leq \psi_1(t)^{n+m}. \tag{6}$$

Therefore the fact that $P(t) = \infty$ is equivalent to saying that $\psi_1(t) = \infty$. In our case

$$\psi_1(t) = \sum_{n=1}^{\infty} ||\phi'_{n+ki}||^t = \sum_{n=1}^{\infty} \frac{1}{|n+ki|^{2t}}.$$

In particular $\psi_1(1/2) = \sum_{n=1}^{\infty} |n+ki|^{-1} = \infty$. The last proves that the system is regular and $\dim_H(J_k) > 1/2$ for all k.
Fix $\epsilon > 0$, and choose k, depending on ϵ, so big that

$$\psi_1\left(\frac{1}{2}+\epsilon\right) = \sum_{n=1}^{\infty} \frac{1}{|n+ki|^{1+\epsilon/2}} < 1.$$

Then using the subadditive property (6) we obtain

$$\begin{aligned} P\left(\frac{1}{2}+\epsilon\right) &= \lim_{m\to\infty} \frac{1}{m} \log \psi_m\left(\frac{1}{2}+\epsilon\right) \leq \lim_{m\to\infty} \frac{1}{m} \log \psi_1\left(\frac{1}{2}+\epsilon\right)^m \\ &= \log \psi_1\left(\frac{1}{2}+\epsilon\right) < 0. \end{aligned}$$

We get that $\dim_H(J_k) < 1/2+\epsilon$, since our system is regular. When we let $\epsilon \searrow 0$, which implies $k \to \infty$, our proof will be finished. ♣

Remark. Notice that if $k = 0$ we have a system of real continued fractions $\phi_n(x) = (n+x)^{-1}$, for which the limit set J_0 is the unit interval without rational numbers. Obviously in this case the Hausdorff dimension of the limit set is equal to 1, and both 1-dimensional Hausdorff measure of J_0 and 1-dimensional packing measure of J_0 are 1.

Theorem 3. *Let k be so that $1/2 < h = \dim_H(J_k) < 1$. Then $\mathcal{H}^h(J_k) = 0$.*

Proof. Let m be the conformal measure associated to our conformal iterated function system. The idea of the proof is based on the following fact (Lemma 4.9 in [MU1]).
If S is a regular c.i.f.s. and there exists a sequence of points $z_j \in X(\infty)$ and the sequence of positive reals r_j, $j \geq 1$, such that $r_j \to 0$ and

$$\limsup_{j\to\infty} \frac{m(B(z_j, r_j))}{r_j^h} = \infty,$$

then $\mathcal{H}^h(J_k) = 0$.
In our case $z_j = 0$ for every j, since 0 is the only point in $X(\infty)$. Hence

it is sufficient to show that

$$\overline{\lim_{r\to 0}}\frac{m(B(0,r))}{r^h}=\infty. \tag{7}$$

Notice that

$$\begin{aligned}
m(B(0,r)) &\geq \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} m(\phi_{n+ki}(X)) \\
&= \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} \int_X |\phi_{n+ki}|^h\,dm \\
&\geq \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} K^{-h}||\phi'_{n+ki}||^h \\
&= K^{-h}\sum_{\phi_{n+ki}(X)\subseteq B(0,r)} |n+ki|^{-2h} \\
&\asymp K^{-h}\sum_{\phi_{n+ki}(X)\subseteq B(0,r)} n^{-2h},
\end{aligned}$$

where by $A\asymp B$ we mean that there exists some constant $C\geq 1$ such that $C^{-1}\leq A/B\leq C$.
We have to find out for how many n, $B(0,r)\cap\phi_{n+ki}(X)\neq\emptyset$ or for which n, $\phi_{n+ki}(X)\subset B(0,r)$.

$$\begin{aligned}
y\in B(0,r)\cap\phi_{n+ki}(X) &\Leftrightarrow |y|<r \;\text{ and }\; y\in\frac{1}{B(n+1/2+ki,1/2)} \\
&\Leftrightarrow \frac{1}{|y|}>\frac{1}{r} \;\text{ and }\; \frac{1}{y}\in B(n+1/2+ki,1/2)
\end{aligned}$$

Hence $B(0,r)\cap\phi_{n+ki}(X)\neq\emptyset$ if and only if $B(n+1/2+ki,1/2)$ contains a complex number of modulus bigger than $1/r$.
Notice that if z is a point in $\overline{B(n+1/2+ki,1/2)}$ of maximal possible modulus, then

$$|z|=\sqrt{n^2+n+\frac{1}{4}+k^2}+\frac{1}{2}.$$

Therefore $B(0,r)\cap\phi_{n+ki}(X)\neq\emptyset$ only if the following condition holds

$$\sqrt{n^2+n+k^2+\frac{1}{4}}+\frac{1}{2}>\frac{1}{r}.$$

Let $A=\{n|\ \sqrt{n^2+n+k^2+1/4}+1/2>1/r\}$, and let $n(r)$ be the minimal element of A. One can see that $n(r)\asymp 1/r$ by the minimality of $n(r)$. Using the integral test we are able to evaluate the limit introduced at the beginning of the proof.

$$\overline{\lim_{r\to 0}}\frac{m(B(0,r))}{r^h} \geq \overline{\lim_{r\to 0}}\frac{K^{-h}\sum_{n\geq n(r)+1} n^{-2h}}{r^h} \asymp \overline{\lim_{r\to 0}}\frac{K^{-h}\int_{n(r)}^{\infty} x^{-2h}\,dx}{r^h}$$
$$= \overline{\lim_{r\to 0}}\frac{n(r)^{1-2h}}{(2h-1)K^h r^h} \asymp \overline{\lim_{r\to 0}}\frac{r^{2h-1}}{(2h-1)K^h r^h}$$
$$= \overline{\lim_{r\to 0}}\frac{r^{h-1}}{(2h-1)K^h} = \infty$$

We conclude that $\mathcal{H}^h(J_k) = 0$, which completes the proof. ♣

Now we turn our attention to the h-dimensional packing measure of the limit set J_k. We begin with a simple lemma.

Lemma 4. *If $f(z) = 1/z$ is a function defined on a complex plane and $C(x,r)$ is the circle centered at x and of radius r, then*

$$f(C(x,r)) = C\Big(\frac{\overline{x}}{|x|^2 - r^2}, \frac{r}{||x|^2 - r^2|}\Big). \tag{8}$$

Proof. Recall from the theory of analytic functions that for $\lambda > 0$, $\lambda \neq 1$ the equation

$$\left|\frac{z-p}{z-q}\right| = \lambda$$

represents the circle with respect to which the points p and q are symmetric, see [P]. Moreover the center and the radius of this circle are given by the formulas

$$x = \frac{p - \lambda^2 q}{1-\lambda^2}, \quad r = \lambda\frac{|p-q|}{|1-\lambda^2|}.$$

Fix $C(x,r)$. Let $p = x + r/2$, $q = x + 2r$; one can see that p and q are symmetric with respect to the circle $C(x,r)$. Using point $x + r$ which lies on this circle we obtain that $\lambda = 1/2$.
The image of our circle under function f is the circle

$$\frac{|w - 1/p|}{|w - 1/q|} = \lambda\left|\frac{q}{p}\right|.$$

Notice that the points $p' = 1/p = 2/(2x+r)$ and $q' = 1/q = 1/(x+2r)$ are symmetric with respect to the new circle. Let $\lambda' = \lambda|q/p|$. Then

$$\lambda' = \lambda\frac{|x+2r|}{|x+r/2|} = \frac{1}{2}\frac{|x+2r|}{|x+r/2|} = \frac{|x+2r|}{|2x+r|}.$$

Let $C(x', r') = f(C(x, r))$. Then

$$x' = \frac{p' - \lambda'^2 q'}{1 - \lambda'^2} = \frac{\frac{2}{2x+r} - \left|\frac{x+2r}{2x+r}\right|^2 \frac{1}{x+2r}}{1 - \left|\frac{x+2r}{2x+r}\right|^2} = \frac{\frac{2}{2x+r} - \frac{\overline{x+2r}}{(2x+r)\overline{(2x+r)}}}{\frac{(2x+r)\overline{(2x+r)} - (x+2r)\overline{(x+2r)}}{(2x+r)\overline{(2x+r)}}}$$

$$= \frac{2\overline{(2x+r)} - \overline{x+2r}}{(2x+r)\overline{(2x+r)} - (x+2r)\overline{(x+2r)}}$$

$$= \frac{4\overline{x} + 2r - \overline{x} - 2r}{4|x|^2 + 2xr + 2\overline{x}r + r^2 - |x|^2 - 2xr - 2\overline{x}r - 4r^2}$$

$$= \frac{\overline{x}}{|x|^2 - r^2},$$

and also

$$r' = \lambda' \frac{|p' - q'|}{|1 - \lambda'^2|} = \left|\frac{x+2r}{2x+r}\right| \frac{\left|\frac{2}{2x+r} - \frac{1}{x+2r}\right|}{\left|1 - \left|\frac{x+2r}{2x+r}\right|^2\right|}$$

$$= \frac{\left|\frac{x+2r}{2x+r}\right| \frac{|2x+4r-2x-r|}{|(2x+r)(x+2r)|}}{\left|\frac{|2x+r|^2 - |x+2r|^2}{|2x+r|^2}\right|} = \frac{\frac{3r}{|2x+r|^2}}{\frac{|3|x|^2 - 3r^2|}{|2x+r|^2}} = \frac{r}{||x|^2 - r^2|},$$

which finishes the proof. ♣

Theorem 5. *Let k be so that $1/2 < h < 1$. Then $0 < \mathcal{P}^h(J_k) < \infty$.*

Proof. It is a general fact that if the limit set J of a c.i.f.s. has nonempty intersection with an interior of the set X, then the packing measure of this set is always positive. Hence we only need to show that the packing measure of J_k is finite.
Following Theorem 2.5 from [MU2] we must prove that there exist three constants $L > 0$, $\xi > 0$, and $\gamma \geq 1$ and there exists a finite set F such that for all $n \in I \setminus F$ and for all r with $\gamma \mathrm{diam}(\phi_{n+ki}(X)) < r \leq \xi$ there is some $x \in \phi_{n+ki}(X)$ such that

$$\frac{m(B(x,r))}{r^h} \geq L. \tag{9}$$

According to the result proven in the lemma above the diameter of the ball $\phi_{p+ki}(X)$ is

$$\mathrm{diam}(\phi_{p+ki}(X)) = 2\frac{1/2}{||p + 1/2 + ki|^2 - 1/4|} \asymp \frac{1}{p^2}, \tag{10}$$

for p large enough. Let Γ_p denote an arc, which is an image of the half-line $pt + ki$, $t \geq 1$ under function f. One can see that two the most distant points of Γ_p lying in $\phi_{p+ki}(X)$ are $(p+1+ki)^{-1}$ and $(p+ki)^{-1}$. Therefore we can choose an x to be the point in $\Gamma_p \cap \phi_{p+ki}(X)$ with $|x| = 1/(p+1)$. Additionally set $\xi = 1$ and $\gamma = 8$. To prove the theorem we only have to find $L > 0$ such (9) holds. To do it we have to consider two separate cases.

Case 1. Suppose that $|x| \le r$. In this case the ball $B(x,r)$ contains infinitely many balls of the form $\phi_{n+ki}(X)$, in fact it contains all the balls for n greater than some n_0.

On the arc Γ_p choose a point y such that $\rho(x,y) = r$. Let l denote the length of a part of Γ_p from 0 to y. There exists a unique $1 \le m \le p$ so that

$$\frac{m}{p+1} \le r \le \frac{m+1}{p+1}.$$

A simple geometry gives us that

$$\begin{aligned} |y| &> \frac{1}{\pi} l > \frac{1}{\pi}\Big(\frac{1}{p+1} + r\Big) > \frac{1}{\pi}\frac{m+1}{p+1} \\ &= \frac{1}{\frac{\pi(p+1)}{m+1}} \ge \frac{1}{8\lfloor p/m \rfloor} \end{aligned}$$

The above computation tells us that

$$\frac{1}{B(8\lfloor p/m \rfloor + 1/2 + ki, 1/2)} \cap B(x,r) \ne \emptyset, \tag{11}$$

so in other words we can assume that $n_0 = 8\lfloor p/m \rfloor$.

We have that

$$\begin{aligned} m(B(x,r)) &\ge \sum_{n=8\lfloor \frac{p}{m} \rfloor + 1}^{\infty} n^{-2h} \asymp \int_{8\lfloor \frac{p}{m} \rfloor}^{\infty} x^{-2h}\, dx \\ &= \frac{1}{2h-1}\frac{1}{(8\lfloor \frac{p}{m} \rfloor)^{2h-1}} = \frac{1}{8^{2h-1}(2h-1)}\frac{1}{\lfloor \frac{p}{m} \rfloor^{2h-1}} \\ &\ge \frac{1}{8^{2h-1}(2h-1)}\frac{1}{\lfloor \frac{p}{r(p+1)} \rfloor^{2h-1}} > \frac{1}{8^{2h-1}(2h-1)}\frac{1}{(\frac{1}{r})^{2h-1}} \\ &= \frac{1}{8^{2h-1}(2h-1)} r^{2h-1} \ge \frac{1}{8^{2h-1}(2h-1)} r^{h}. \end{aligned}$$

Case 2. Assume that $r < |x|$. In this case the ball $B(x,r)$ contains only finitely many balls of the form $\phi_{n+ki}(X)$. It certainly contains the pth ball, so all we have to do is to find the maximum index l so that $\phi_{l+ki}(X) \subset B(x,r)$.

Then there exists a unique $l \ge p$ such that

$$\sum_{n=p}^{l} \operatorname{diam}\Big(\frac{1}{B(n+\frac{1}{2}+ki, \frac{1}{2})}\Big) \le r \le \sum_{n=p}^{l+1} \operatorname{diam}\Big(\frac{1}{B(n+\frac{1}{2}+ki, \frac{1}{2})}\Big)$$

The formula (10) for the diameter of $\phi_{n+ki}(X)$ immediately gives that for n large enough

$$\frac{1}{(n+1)^2} < \operatorname{diam}\Big(\frac{1}{B(n+1/2+ki,1/2)}\Big) < \frac{1}{n^2} \tag{12}$$

Hence, using the integral test we obtain

$$\frac{1}{p+1}-\frac{1}{l+2}=\int_{p+1}^{l+2} x^{-2}\,dx \le \sum_{n=p+1}^{l+1}\frac{1}{n^2} < r$$
$$< \sum_{n=p}^{l+1}\frac{1}{n^2} \le \int_{p-1}^{l+1} x^{-2}\,dx = \frac{1}{p-1}-\frac{1}{l+1}.$$

The last gives us that

$$\frac{1}{p+1}-\frac{1}{l+2} \le r \le \frac{1}{p-1}-\frac{1}{l+1} \tag{13}$$

Applying the Mean Value Theorem we get

$$\begin{aligned}
m(B(x,r)) &\ge \sum_{n=p}^{l} n^{-2h} \asymp \int_p^l x^{-2h}\,dx \\
&= \frac{1}{2h-1}\Big(\Big(\frac{1}{p}\Big)^{2h-1}-\Big(\frac{1}{l}\Big)^{2h-1}\Big) \\
&\ge \frac{1}{2h-1}\Big(\frac{1}{p}-\frac{1}{l}\Big)(2h-1)\Big(\frac{1}{z}\Big)^{2h-2} \\
&= \Big(\frac{1}{p}-\frac{1}{l}\Big)\Big(\frac{1}{z^2}\Big)^{h-1},
\end{aligned}$$

for some $p \le z \le l$. Recall that $r \ge 8/p^2$, which implies $z^{-2} \le r/8$. Hence, using (13)

$$\begin{aligned}
&m(B(x,r)) \ge \Big(\frac{1}{p}-\frac{1}{l}\Big)\Big(\frac{r}{8}\Big)^{h-1} \\
&=\Big[\Big(\frac{1}{p}-\frac{1}{p-1}\Big)+\Big(\frac{1}{p-1}-\frac{1}{l+1}\Big)+\Big(\frac{1}{l+1}-\frac{1}{l}\Big)\Big]8^{1-h}r^{h-1} \\
&\ge\Big[-\frac{1}{p(p-1)}+r-\frac{1}{l(l+1)}\Big]8^{1-h}r^{h-1} \\
&\ge\Big[r-\frac{2}{p(p-1)}\Big]8^{1-h}r^{h-1} \\
&\ge\Big[r-\frac{4}{p^2}\Big]8^{1-h}r^{h-1} \ge \Big[r-\frac{r}{2}\Big]8^{1-h}r^{h-1} = \frac{1}{2}8^{1-h}r^{h}.
\end{aligned}$$

Choosing L to be $\min\{8^{1-2h}(2h-1)^{-1},\ 2^{-1}8^{1-h}\}$ completes the proof of the theorem. ♣

3. Additional Remarks

According to Theorem 1 and since the Hausdorff dimension of the limit set of the system $\{\phi_{n+ki} : n \geq 1, k \in \mathbb{Z}\}$ is greater than 1, we can add to the family $\{\phi_{n+ki} : n \geq 1, k \text{ fixed}\}$ finitely many mappings from $\{\phi_{n+ki} : n \geq 1, k \in \mathbb{Z}\}$ to obtain the systems whose limit sets J have Hausdorff dimensions greater than 1. Then, it turns out that employing similar methods as those used in the proofs of Theorems 3 and 5, $0 < \mathcal{H}^h(J) < \infty$ and $\mathcal{P}^h(J) < \infty$. Let us remark that this fact distinguishes these systems from the full family $\{\phi_{n+ki} : n \geq 1, k \in \mathbb{Z}\}$ and the families investigated in this paper since for them the packing measure is finite and positive whereas here the Hausdorff measure is positive and finite.

References

[GM] R.J.Gardner, R.D.Mauldin, On the Hausdorff dimension of a set of complex continued fractions, Illinois J. of Math. 27 (1983), 334-344.

[P] H.A.Priestley, Introduction to Complex Analysis, Clarendon Press Oxford (1985)

[MU1] R.D.Mauldin, M.Urbański, Dimensions and measures in infinite iterated function systems, Proceedings London Math. Soc., (3) 73 (1996), 105-154.

[MU2] R.D.Mauldin, M.Urbański, The geometry of continued fractions, to appear Trans. of A.M.S.