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Gibbs States On The Symbolic Space Over An Infinite Alphabet

R. Daniel Mauldin[1], and Mariusz Urbański[1]

Department of Mathematics, University of North Texas,
Denton, TX 76203-1430, USA
e-mail:mauldin@unt.edu
Web: http://www.math.unt.edu/∼mauldin
e-mail:urbanski@unt.edu
Web: http://www.math.unt.edu/∼urbanski

Abstract. We consider subshifts of finite type on the symbolic space generated by incidence matrices over a countably infinite alphabet. We extend the definition of topological pressure to this context and, as our main result, we construct a new class of Gibbs states of Hölder continuous potentials on these symbol spaces. We establish some basic stochastic properties of these Gibbs states: exponential decay of correlations, central limit theorem and an a.s. invariance principle. This is accomplished via detailed studies of the associated Perron-Frobenius operator and its conjugate operator.

§0. Introduction. Preliminary notation. This paper has emerged as a natural consequence of our interests in geometrical and dynamical properties of the limit sets of conformal graph directed Markov systems, a generalization of infinite conformal iterated function systems systematically studied in [MU1], [MU2] and subsequent papers. Although our paper is self-contained, it could also be considered as the first step to developing the theory of conformal graph directed Markov systems. The central point of this paper, the existence of Gibbs states (and eigenmeasures of the operator conjugate to the Perron-Frobenius operator) for the shift map on the symbolic space generated by an infinite alphabet, and a Hölder continuous potential, is contained in Section 2. This is accomplished by producing Gibbs states for symbolic subspaces generated by finitely many elements of the alphabet and then demonstrating their tightness. In the first section we generalize the concept of topological pressure to the context of symbolic space over an infinite alphabet and we provide there several variational principles. Section 3 is devoted to systematic studies of Gibbs states and their relations with equilibrium states. In Section 4 we introduce the Perron-Frobenius operator and its conjugate. We deal here with their properties and we

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establish basic relations between Gibbs states and eigenmeasures of the conjugate Perron-Frobenius operator. In Section 5, following a classical approach, we investigate the Perron-Frobenius operator from the point of view of the Ionescu-Tulcea and Marinescu inequality. This is the key point to establish in the next section, Section 6, the stochastic properties of the Gibbs states. We obtain the weak-Bernoulli property, exponential decay of correlations, a central limit theorem, and an a. s. invariance principle. In the last section, Section 7, we compare our approach with that of O. Sarig.

Let us now present the general notation used throughout the whole paper. The letter I always means a countably infinite set, frequently called the alphabet, and sometimes identified with the set of positive integers $I\!N$. Any function $A : I \times I \to \{0,1\}$ is called an incidence matrix. The set

$$E^\infty = \{\omega \in I^\infty : A_{\omega_i\omega_{i+1}} = 1 \ \text{ for all } i \geq 1\},$$

the space of all A-admissible infinite sequences with terms in I is frequently called the symbol space (or the shift space) generated by the matrix A. By E^* we mean the set of all finite A-admissible sequences (words) and for every $n \geq 1$, E^n denotes the set of all A-admissible words of length n. Given $\omega \in E^\infty \cup E^*$, the symbol $|\omega|$ represents the number of letters forming ω. If $n \leq |\omega|$, then $\omega|_n = \omega_1\omega_2\ldots\omega_n$ is the restriction of ω to its first n letters. Finally $\sigma : I^\infty \to I^\infty$ is the (one-sided) shift map (cutting off the first coordinate), i.e. $\sigma(\{\omega_n\}_{n\geq 1}) = \{\omega_n\}_{n\geq 2}$. Notice that $\sigma(E^\infty) \subset E^\infty$ and therefore we can consider the map $\sigma : E^\infty \to E^\infty$, called in the sequel the subshift of finite type generated by the matrix A. We finally take $\beta > 0$ and consider the metric d_β on the space $I^\infty$by setting

$$d_\beta(\omega,\tau) = \mathrm{e}^{-\beta(|\omega\wedge\tau|-1)}$$

where $\omega \wedge \tau$ is the maximal initial common block of ω and τ (we use the convention $\mathrm{e}^{-\infty} = 0$. We consider the same metric on the space E^∞.

§**1. Topological pressure and variational principles.** We call the incidence matrix A irreducible if for all $i,j \in I$ there exists a path $\omega \in E^*$ such that $\omega_1 = i$ and $\omega_{|\omega|} = j$. We call it primitive if there exists $p \geq 1$ such that all the entries of A^p are positive, or in other words, for all $i,j \in I$ there exists a path $\omega \in E^p$ such that $\omega_1 = i$ and $\omega_{|\omega|} = j$. The matrix A is said to be finitely irreducible if there exists a finite set $\Lambda \subset E^*$ such that for all $i,j \in I$ there exists a path $\omega \in \Lambda$ for which $i\omega j \in E^*$ and finally A is said to be finitely primitive if there exists a finite set $\Lambda \subset E^*$ consisting of words of the same length such that for all $i,j \in E$ there exists a path $\omega \in \Lambda$ for which $i\omega j \in E^*$. Notice that a finitely irreducible matrix does not have to be primitive nor conversely. Notice also that the set Λ (associated either with a finitely irreducible or finitely primitive matrix) can be taken to be empty provided E^∞ is the full shift. Given a set $F \subset I$, we put

$$E_F^\infty = F^I \cap E^\infty = \{\omega \in F^I : A_{\omega_i\omega_{i+1}} = 1 \ \text{ for all } i \geq 1\}.$$

Given additionally a function $f : E_F^\infty \to I\!R$ we define the topological pressure of f with respect to the shift map $\sigma : E_F^\infty \to E_F^\infty$ to be

$$\mathrm{P}_F(f) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega\in F^n\cap E^n} \exp(\sup_{\tau\in[\omega\cap F]} \sum_{j=0}^{n-1} f(\sigma^j(\tau)) \right), \tag{1.1}$$

where $[\omega\cap F]=\{\tau\in E_F^\infty:\tau|_{|\omega|}=\omega\}$. If $F=I$ we simply write $[\omega]$ for $[\omega\cap F]$. Since the sequence $n\mapsto\log Z_n(F,f)$ is subadditive, where

$$Z_n(F,f)=\sum_{\omega\in F^n}\exp(\sup_{\tau\in[\omega\cap F]}\sum_{j=0}^{n-1}f(\sigma^j(\tau)),$$

the limit in (1.1) exists. If $F=I$, we suppress the subscript F and write simply $\mathrm{P}(f)$ for $\mathrm{P}_I(f)$ and $Z_n(f)$ for $Z_n(I,f)$.

There are several things concerning the pressure function which may differ radically from the case when the alphabet is finite. However, there is a reasonably wide class of functions introduced in [MU3] for which the pressure function is fairly well behaved.

Definition 1.1. (see [MU3]) A function $f:E^\infty\to I\!R$ is acceptable provided it is uniformly continuous with respect to the metric d_β for some $\beta>0$ and

$$\mathrm{osc}(f):=\sup_{i\in I}\{\sup(f|_{[i]})-\inf(f|_{[i]})\}<\infty.$$

We shall prove the following.

Theorem 1.2. If $f:E^\infty\to I\!R$ is acceptable and A is finitely irreducible, then

$$\mathrm{P}(f)=\sup\{P_F(f)\},$$

where the supremum is taken over all finite subsets F of I.

Proof. The inequality $\mathrm{P}(f)\ge\sup\{P_F(f)\}$ is obvious. To prove the converse suppose first that $\mathrm{P}(f)<\infty$. Put $p=\max\{|\omega|:\omega\in\Lambda\}$ and $T=\min\left\{\inf\sum_{j=0}^{|\omega|-1}f\circ\sigma^j|_{[\omega]}:\omega\in\Lambda\right\}$, where $[\omega]=\{\tau\in E^\infty:\tau|_{|\omega|}=\omega\}$. Fix $\varepsilon>0$. By the acceptability of f, there exists $l\ge1$ such that $|f(\omega)-f(\tau)|<\varepsilon$, if $\omega|_l=\tau|_l$ and $M=\mathrm{osc}(f)<\infty$. Now, fix $k\ge l$. By subadditivity, $\frac{1}{n}\log Z_n(f)\ge\mathrm{P}(f)$. For each $F\subset I$ and $m\in N$, set

$$\Gamma_m(F,f)=\sum_{\omega\in F^m\cap E^m}\exp(\sup_{\tau\in[\omega]}\sum_{j=0}^{m-1}f(\sigma^j(\tau)),$$

Then there exists a finite set $F\subset I$ such that

$$\frac{1}{k}\log\Gamma_k(F,f)\ge\mathrm{P}(f)-\varepsilon. \tag{1.2}$$

We may assume that F contains Λ. Put

$$\overline{f}=\sum_{j=0}^{k-1}f\circ\sigma^j.$$

Now, for every element $\tau = \tau_1, \tau_2, \ldots, \tau_n \in F^k \cap E^k \times \ldots \times F^k \cap E^k$ (n factors) one can choose elements $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Lambda$ such that $\overline{\tau} = \tau_1\alpha_1\tau_2\alpha_2 \ldots \tau_{n-1}\alpha_{n-1}\tau_n \in E^*$. Then for every $n \geq 1$

$$\begin{aligned}
\sum_{i=kn}^{kn+p(n-1)} Z_i(F,f) &\geq \sum_{\tau\in(F^k\cap E^k)^n} \exp\left(\sup_{[\overline{\tau}\cap F]} \sum_{j=0}^{k-1} f\circ\sigma^j\right) \\
&\geq \sum_{\tau\in(F^k\cap E^k)^n} \exp\left(\inf_{[\overline{\tau}]} \sum_{j=0}^{k-1} f\circ\sigma^j\right) \\
&\geq \sum_{\tau\in(F^k\cap E^k)^n} \exp\left(\sum_{i=1}^{n} \inf \overline{f}|_{[\tau_i]} + T(n-1)\right) \\
&= \exp(T(n-1)) \sum_{\tau\in(F^k\cap E^k)^n} \exp\sum_{i=1}^{n} \inf \overline{f}|_{[\tau_i]} \\
&\geq \exp(T(n-1)) \sum_{\tau\in(F^k\cap E^k)^n} \exp\left(\sum_{i=1}^{n} (\sup \overline{f}|_{[\tau_i]} - (k-l)\varepsilon - Ml)\right) \\
&= \exp(T(n-1) - (k-l)\varepsilon n - Mln) \sum_{\tau\in(F^k\cap E^k)^n} \exp\sum_{i=1}^{n} \sup \overline{f}|_{[\tau_i]} \\
&= \mathrm{e}^{-T} \exp\big(n(T-(k-l)\varepsilon - Ml)\big) \left(\sum_{\tau\in(F^k\cap E^k)} \exp(\sup f|_{[\tau]})\right)^n .
\end{aligned}$$

Hence, there exists $kn \leq i_n \leq (k+p)n$ such that

$$Z_{i_n}(F,f) \geq \frac{1}{pn}\mathrm{e}^{-T} \exp\big(n(T-(k-l)\varepsilon - Ml)\big)\Gamma_k(F,f)^n$$

and therefore, using (1.2), we obtain

$$\mathrm{P}_F(f) = \lim_{n\to\infty} \frac{1}{i_n} \log Z_{i_n}(F,f) \geq \frac{-|T|}{k} - \varepsilon + \frac{l\varepsilon}{k+p} - \frac{Ml}{k} + \mathrm{P}(f) - \varepsilon \geq \mathrm{P}(f) - 4\varepsilon$$

provided that k is large enough. Thus, letting $\varepsilon \searrow 0$, the theorem follows. The case $\mathrm{P}(f) = \infty$ can be treated similarly. ■

We say a σ-invariant Borel probability measure $\tilde{\mu}$ on E^∞ is finitely supported provided there exists a finite set $F \subset I$ such that $\tilde{\mu}(E_F^\infty) = 1$. The well-known variational principle (see [Wa], comp. [PU]) tells us that for every finite set $F \subset I$

$$\mathrm{P}_F(f) = \sup\{\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f\,d\tilde{\mu}\},$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures $\tilde{\mu}$ with $\tilde{\mu}(F^\infty) = 1$. Applying Theorem 1.2, we therefore get the following.

Theorem 1.3.(1st variational principle) If A is finitely irreducible and if $f : E^\infty \to I\!\!R$ is acceptable, then

$$\mathrm{P}(f) = \sup\{\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu}\},$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures $\tilde{\mu}$ which are finitely supported.

We would like to note that Theorem 1.2 and Theorem 1.3 have been proved in [Sa] as Theorem 2 for locally Hölder continuous potentials. We would like however to add that in [Sa] the shift map is only assumed to be topologically mixing whereas we need finite irreducibility. Let us also add that theorem similar to our Theorem 1.2 appeared in [Za] as Theorem 1.3. In [Za] however the author assumes that the potential has a continuous extension on a compactification of the coding space, hence, inparticular, it must be bounded whereas we allow unbounded potentials.

Now, given $n \geq 1$ let α_0^{n-1} be the standard partition of E^∞ into cylinders of length n:

$$\alpha_0^{n-1} = \{[\omega] : |\omega| = n\}.$$

We put

$$S_n f = \sum_{j=0}^{n-1} f \circ \sigma^j.$$

If $n = 1$ we write also α for α_0^0. Our next theorem is the following.

Theorem 1.4.(2nd variational principle) If $f : E^\infty \to I\!\!R$ is a continuous function and $\tilde{\mu}$ is a σ-invariant Borel probability measure on E^∞ such that $\int f d\tilde{\mu} > -\infty$, then

$$\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu} \leq \mathrm{P}(f).$$

In addition, if $\mathrm{P}(f) < \infty$, then $\mathrm{H}_{\tilde{\mu}}(\alpha) < \infty$.

Proof. If $\mathrm{P}(f) = +\infty$ there is nothing to prove. So, suppose that $\mathrm{P}(f) < \infty$. Then there exists $q \geq 1$ such that $Z_n(f) < \infty$ for every $n \geq q$. Also for every $n \geq 1$ we have

$$\sum_{|\omega|=n} \tilde{\mu}([\omega]) \sup(S_n f||\omega|) \geq \int S_n f d\tilde{\mu} = n \int f d\tilde{\mu} > -\infty.$$

Therefore, using concavity of the function $h(x) = -x\log x$, we obtain for every $n \geq q$

$$\begin{aligned}
\mathrm{H}_{\tilde{\mu}}(\alpha_0^{n-1}) + \int S_n f\, d\tilde{\mu} &\leq \sum_{|\omega|=n} \tilde{\mu}([\omega])(\sup S_n f|_{[\omega]} - \log \tilde{\mu}([\omega])) \\
&= Z_n(f) \sum_{|\omega|=n} Z_n(f)^{-1} h\big(\mathrm{e}^{\sup S_n f|_{[\omega]}} \tilde{\mu}([\omega])\big)\mathrm{e}^{-\sup S_n f|_{[\omega]}}\big) \\
&\leq Z_n(f) h\left(\sum_{|\omega|=n} Z_n(f)^{-1} \mathrm{e}^{\sup S_n f|_{[\omega]}} \tilde{\mu}([\omega])\mathrm{e}^{-\sup S_n f|_{[\omega]}}\right) \\
&= Z_n(f) h(Z_n(f)^{-1}) \\
&= \log\big(\sum_{|\omega|=n} \exp\big(\sup S_n f|_{[\omega]}\big)\big) = \log Z_n(f)
\end{aligned}$$

Hence $\mathrm{H}_{\tilde{\mu}}(\alpha_0^{n-1}) \leq \log Z_n(f) + n\int(-f)d\tilde{\mu} < \infty$ for every $n \geq q$. Thus $\mathrm{H}_{\tilde{\mu}}(\alpha) < \infty$. Since in addition α_0^{q-1} is a generator, we therefore get

$$\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f\, d\tilde{\mu} \leq \liminf_{n\to\infty} \left(\frac{1}{n}\left(\mathrm{H}_{\tilde{\mu}}(\alpha_0^{n-1}) + \int S_n f\, d\tilde{\mu})\right)\right) \leq \lim_{n\to\infty} \frac{1}{n} \log Z_n(f) = \mathrm{P}(f).$$

The proof is complete. ■

We would like to remark that under some additional assumptions, implying in particular that the potential f has a continuous extension on a compactification E^∞, this theorem follows from Theorem 1 in [PP].

As an immediate consequence of Theorem 1.3 and Theorem 1.4, we get the following.

Theorem 1.5.(3rd variational principle) Suppose that the incidence matrix is A is finitely irreducible. If $f : E^\infty \to I\!R$ is acceptable, then

$$\mathrm{P}(f) = \sup\{\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f\, d\tilde{\mu}\},$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures $\tilde{\mu}$ such that $\int f\, d\tilde{\mu} > -\infty$.

We would like to notice that the same theorem under weaker assumptions on the shift map and stronger assumptions on the potential f has been proved in [GS] and [Sa].

We end this section with the following.

Proposition 1.6. If the incidence matrix is finitely primitive and the function f is acceptable, then $\mathrm{P}(f) < \infty$ if and only if $Z_1(f) < \infty$.

Proof. Let $q \geq 1$ and $\Lambda \subset E^q$ be the objects resulting from finite primitivness of the incydency matrix A. Let

$$M = \min\left\{\inf_{[\alpha]}\left\{\sum_{j=0}^{q-1} f\circ\sigma^j\right\} : \alpha\in\Lambda\right\}.$$

For $n \geq 1$ and $\omega \in I^n$ let $\overline{\omega} = \omega_1\alpha_1\omega_2\alpha_2\ldots\omega_{n-1}\alpha_{n-1}\omega_n \in E^{n+q(n_1)}$, where all $\alpha_1,\ldots,\alpha_n$ are appropriately taken from Λ. Since f is acceptable, we therefore get

$$\begin{aligned}
Z_{n+q(n-1)} &= \sum_{\omega\in E^{n+q(n_1)}} \exp\left(\sup_{[\omega]}\left\{\sum_{j=0}^{q-1} f\circ\sigma^j\right\}\right) \geq \sum_{\omega\in I^n}\exp\left(\sup_{[\overline{\omega}]}\left\{\sum_{j=0}^{n+q(n-1)} f\circ\sigma^j\right\}\right)\\
&\geq \exp\left(\sum_{j=1}^{n}\inf\bigl(f|_{[\omega_j]}\bigr) + M(n-1)\right)\\
&\geq e^{M(n-1)}\exp\left(\sum_{j=1}^{n}\sup\bigl(f|_{[\omega_j]}\bigr) - \operatorname{osc}(f)n\right)\\
&= \exp\bigl(-M + (M-\operatorname{osc}(f))n\bigr)\left(\sum_{e\in I}\exp\bigl(\sup\bigl(f|_{[\omega_j]}\bigr)\bigr)\right)^n\\
&= \exp\bigl(-M + (M-\operatorname{osc}(f))n\bigr)Z_1(f)^n.
\end{aligned}$$

Thus $\mathrm{P}(f) \geq M - \operatorname{osc}(f) + \log Z_1(f)$. Hence, if $\mathrm{P}(f) < \infty$, then also $Z_1(f) < \infty$. The opposit implication is obvious since $Z_n(f) \leq Z_1(f)^n$. The proof is complete. ■

§2. The existence of eigenmeasures of the conjugate Perron-Frobenius operator and of Gibbs states. Here we prove the main result of our paper. It concerns the existence of eigenmeasures of the conjugate Perron-Frobenius operator and Gibbs states. If $f : E^\infty \to I\!R$ is a continuous function, then a Borel probability measure $\tilde{m}$ on E^∞ is called a Gibbs state (comp. [Bo], [HMU], [PU], [Ru], [Wa] and [Ur]) for f if there exist constants $Q \geq 1$ and $\mathrm{P}_{\tilde{m}}$ such that for every $\omega \in E^*$ and every $\tau \in [\omega]$

$$Q^{-1} \leq \frac{\tilde{m}([\omega])}{\exp\bigl(S_{|\omega|}f(\tau) - \mathrm{P}_{\tilde{m}}|\omega|\bigr)} \leq Q. \tag{2.1}$$

If addtionally $\tilde{m}$ is shift-invariant, it is then called an invariant Gibbs state.

Remark 2.1. Notice that the number $S_{|\omega|}f(\tau)$ in (2.1) can be replaced by $\sup(S_{|\omega|}f|_{[\omega]})$ or by $\inf(S_{|\omega|}f|_{[\omega]})$.

For the sake of completeness we provide a short direct proof of the following folklore result (see [Bo]) which wwill be needed in the sequel.

Proposition 2.2.
(a) For every Gibbs state $\tilde{m}$, $\mathrm{P}_{\tilde{m}} = \mathrm{P}(f)$.
(b) Any two Gibbs states of the function f are equivalent with Radon-Nikodym derivatives bounded away from zero and infinity.

Proof. We shall first prove (a). Towards this end fix $n \geq 1$ and, using Remark 2.1, sum up (2.1) over all words $\omega \in E^n$. Since $\sum_{|\omega|=n} \tilde{m}([\omega]) = 1$, we therefore get

$$Q^{-1}\mathrm{e}^{-\mathrm{P}_{\tilde{m}} n} \sum_{|\omega|=n} \exp\bigl(\sup S_n f|_{[\omega]}\bigr) \leq 1 \leq Q\mathrm{e}^{-\mathrm{P}_{\tilde{m}} n} \sum_{|\omega|=n} \exp\bigl(\sup S_n f|_{[\omega]}\bigr).$$

Applying logarithms to all three terms of this formula, dividing all the terms by n and taking the limit as $n \to \infty$, we obtain $-\mathrm{P}_{\tilde{m}} + \mathrm{P}(f) \leq 0 \leq -\mathrm{P}_{\tilde{m}} + \mathrm{P}(f)$, which means that $\mathrm{P}_{\tilde{m}} = \mathrm{P}(f)$. The proof of of item (a) is thus complete.

In order to prove part (b) suppose that m and ν are two Gibbs states of the function f. Notice now that part (a) implies the existence of a constant $T \geq 1$ such that

$$T^{-1} \leq \frac{\nu([\omega])}{m([\omega])} \leq T$$

for all words $\omega \in E^*$. A straightforward reasoning gives now that ν and m are equivalent and $T^{-1} \leq \frac{d\nu}{dm} \leq T$. The proof is complete. ■

We say that a function $f : E^\infty \to \mathbb{R}$ is Hölder continuous with an exponent $\alpha > 0$ if

$$V_\alpha(f) = \inf\{L : |f(\omega) - f(\tau)| \leq L d_\alpha(\omega, \tau),\ \omega_1 = \tau_1\} < \infty.$$

Note that each Hölder continuous function is acceptable.

Lemma 2.3. If $g : E^\infty \to \mathbb{C}$ and $V_\alpha(g) < \infty$, then for all $\omega\tau \in E^\infty$ with $\omega_1 = \tau_1$, all $n \geq 1$, and all $\rho \in E^n$ with $A_{\rho_n\omega_1} = A_{\tau_n\omega_1} = 1$ we have

$$|S_n g(\rho\omega) - S_n g(\rho\tau)| \leq \frac{V_\alpha(g)}{\mathrm{e}^\alpha - 1} d_\alpha(\omega, \tau).$$

Proof. We have

$$\begin{aligned}|S_n g(\rho\omega) - S_n g(\rho\tau)| &\leq \sum_{i=0}^{n-1} |g(\sigma^i(\rho\omega)) - g(\sigma^i(\tau\omega))| \leq \sum_{i=0}^{n-1} V_\alpha(g) d_\alpha(\sigma^i(\rho\omega), \sigma^i(\tau\omega)) \\ &\leq V_\alpha(g)\sum_{i=0}^{n-1} \mathrm{e}^{-\alpha(n-i)} d_\alpha(\omega,\tau) \leq V_\alpha(g)\frac{\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}} d_\alpha(\omega,\tau) \leq \frac{V_\alpha(g)}{\mathrm{e}^\alpha - 1} d_\alpha(\omega,\tau).\end{aligned}$$

The proof is complete. ■

We set

$$T(g) = \exp\left(\frac{V_\alpha(g)}{\mathrm{e}^\alpha - 1}\right).$$

From now throughout this section $f : E^\infty \to I\!R$ is assumed to be a Hölder continuous function with an exponent $\beta > 0$ and is assumed to satisfy the following summability requirement

$$\sum_{e\in I} \exp(\sup(f|_{[e]})) < \infty. \tag{2.2}$$

This requirement allows us to define the Perron-Frobenius operator $\mathcal{L}_f : C_b(E^\infty) \to C_b(E^\infty)$, acting on the space of bounded continuous functions $C_b(E^\infty)$, as follows

$$\mathcal{L}_f(g)(\omega) = \sum_{e\in I: A_{e\omega_1}=1} \exp(f(e\omega)g(e\omega).$$

Then $||\mathcal{L}_f||_0 \le \sum_{e\in I} \exp(\sup(f|_{[e]})) < \infty$ and for every $n \ge 1$

$$\mathcal{L}_f^n(g)(\omega) = \sum_{\tau\in E^n: A_{\tau_n\omega_1}=1} \exp\big(S_n f(\tau\omega)\big)g(\tau\omega).$$

The conjugate operator $\mathcal{L}_f^*$ acting on the space $C_b^*(E^\infty)$ is defined as follows.

$$\mathcal{L}_f^*(\mu)(g) = \mu(\mathcal{L}_f(g)).$$

From now throughout this section we assume that there exists a Borel probability measure $\tilde m$ which is an eigenmeasure of the conjugate operator $\mathcal{L}_f^* : C_b^*(E^\infty) \to C_b^*(E^\infty)$. The corresponding eigenvalue is denoted by λ. Since $\mathcal{L}_f$ is a positive operator, $\lambda \ge 0$. Obviously $\mathcal{L}_f^{*n}(\tilde m) = \lambda^n \tilde m$. The integral version of this equality takes on the following form

$$\int \sum_{\tau\in E^n: A_{\tau_n\omega_1}=1} \exp\big(S_n f(\tau\omega)\big)g(\tau\omega)d\tilde m(\omega) = \lambda^n \int g d\tilde m \tag{2.3}$$

for every function $g \in C_b^(E^\infty)$. In fact this equality extends to the space of all bounded Borel functions on E^∞. In particular, taking $\omega \in E^*$, say $\omega \in E^n$, a Borel set $A \subset E^\infty$ such that $A_{\omega_n\tau_1} = 1$ for every $\tau \in A$, and $g = 1\!\!1_{[\omega A]}$, we obtain from (2.3)

$$\begin{aligned}
\lambda^n \tilde m([\omega A]) &= \int \sum_{\tau\in E^n: A_{\tau_n\rho_1}=1} \exp\big(S_n f(\tau\rho)\big)1\!\!1_{[\omega A]}(\tau\rho)d\tilde m(\rho) \\
&= \int_{\rho\in A: A_{\omega_n\rho_1}=1} \exp\big(S_n f(\omega\rho)\big)d\tilde m(\rho) \\
&= \int_A \exp\big(S_n f(\omega\rho)\big)d\tilde m(\rho) \qquad (2.4)
\end{aligned}$$

Remark 2.4. Suppose now that (2.4) holds. Representing then any Borel set $B \subset E^\infty$ as a union $\bigcup_{\omega\in E^n}[\omega B_\omega]$, where $B_\omega = \{\alpha \in E^\infty : A_{\omega_n\alpha_1} = 1 \text{ and } \omega\alpha \in B\}$, a straightforward calculation based on (2.4) demonstrates that (2.3) is satisfied for the characteristic function $1\!\!1_B$ of the set B. By the standard approximation argument (2.3) is therefore satisfied for all $\tilde{m}$-integrable functions g. As the final conclusion we obtain that $\tilde{m}$ is an eigenmeasure of the conjugate operator $\mathcal{L}_f^*$ if and only if formula (2.4) is satisfied.

An alternative proof of the following theorem is included in the proof of Theorem 8 in [Sa] under the weaker assumption that the shift map has the big images property.

Theorem 2.5. If the incidence matrix is finitely irreducible, then the eigenmeasure $\tilde{m}$ is a Gibbs state for f. In addition $\lambda = \mathrm{e}^{\mathrm{P}(f)}$.

Proof. It immediately follows from (2.4) and Lemma 2.3 that for every $\omega \in E^*$ and every $\tau \in [\omega]$

$$\tilde{m}([\omega]) \le \lambda^{-n} T(f) \exp\big(S_n f(\tau)\big) = T(f) \exp\big(S_n f(\tau) - n \log \lambda\big),$$

where $n = |\omega|$. On the other hand, let Λ be given by finite irreducibility of A. For every $\alpha \in \Lambda$ let

$$E_a = \{\tau \in E^\infty : \omega\alpha\tau \in E^\infty\}.$$

By the definition of Λ, $\bigcup_{\alpha\in\Lambda} E_a = E^\infty$. Hence, there exists $\gamma \in \Lambda$ such that $\tilde{m}(E_\gamma) \ge (\#\Lambda)^{-1}$. Writing $p = |\gamma|$ we therefore have

$$\begin{aligned}
\tilde{m}([\omega]) \ge \tilde{m}([\omega\gamma]) &= \lambda^{-(n+p)} \int_{\rho\in E^\infty : A_{\gamma_p\rho_1}=1} \exp\big(S_{n+p} f(\omega\gamma\rho)\big) d\tilde{m}(\rho) \\
&= \lambda^{-(n+p)} \int_{\rho\in E^\infty : A_{\gamma_p\rho_1}=1} \exp\big(S_n f(\omega\gamma\rho)\big) \exp\big(S_p f(\gamma\rho)\big) d\tilde{m}(\rho) \\
&\ge \lambda^{-n} \exp\big(\min\{\inf(S_{|\alpha|} f|_{[\alpha]}) : \alpha \in \Lambda\} - p \log \lambda\big) \int_{\rho\in E^\infty : A_{\gamma_p\rho_1}=1} \exp\big(S_n f(\omega\gamma\rho)\big) d\tilde{m}(\rho) \\
&= C\lambda^{-n} \int_{E_\gamma} \exp\big(S_n f(\omega\gamma\rho)\big) d\tilde{m}(\rho) \ge CT(f)^{-1} \lambda^{-n} m(E_\gamma) \exp\big(S_n f(\tau)\big) \\
&\ge CT(f)^{-1} (\#\Lambda)^{-1} \exp\big(S_n f(\tau) - n \log \lambda\big),
\end{aligned}$$

where $C = \exp\big(\min\{\inf(S_{|\alpha|} f|_{[\alpha]}) : \alpha \in \Lambda\} - p \log \lambda\big)$. Thus $\tilde{m}$ is a Gibbs state for f. The equality $\lambda = \mathrm{e}^{\mathrm{P}(f)}$ follows now immediately from Proposition 2.2. The proof is complete. ■

In order to simplify notation we will skip in the rest of this section the subscript f. We begin our "existence" considerations with the following result whose first proof can be found in [Bo].

Lemma 2.6. If the alphabet I is finite and the incidence matrix is irreducible, then there exists an eigenmeasure $\tilde{m}$ of the conjugate operator $\mathcal{L}_f^*$.

Proof. By our assumption $\mathcal{L}_f$ is a strictly positive operator (in the sens that it maps strictly positive functions into strictly positive functions). In particular the following formula

$$\nu \mapsto \frac{\mathcal{L}_f^*(\nu)}{\mathcal{L}_f^*(\nu)(1\!\!1)}$$

defines a continuous map of the space of Borel probability measures on E^∞ into itself. Since E^∞ is a compact metric space, the Schauder-Tichonov theorem applies, and as its consequence, we conclude that the map defined above has a fixed point, say $\tilde{m}$. Then $\mathcal{L}_f^*(\tilde{m}) = \lambda \tilde{m}$, where $\lambda = \mathcal{L}_f^*(\tilde{m})(1\!\!1)$. The proof is complete. ■

In Lemma 2.8, actually the main result of this paper, we will need a simple fact about irreducible matrices. We will provide its short proof for the sake of completeness. It is more natural and convenient to formulate it in the language of oriented graphs. Let us recall that an oriented grapph is said to be strongly connected if and only if its incidence matrix is irreducible. In other words, it means that every two vertices can be joined by a path of admissible edges.

Lemma 2.7. If $\Gamma =< E, V >$ is a strongly connected oriented graph, then there exists a sequence of strongly connected subgraphs $< E_n, V_n >$ of Γ such that all the vertices $V_n \subset V$ and all the edges E_n are finite, $\{V_n\}_{n=1}^\infty$ is an increasing sequence of vertices, $\{E_n\}_{n=1}^\infty$ is an increasing sequnce of edges, $\bigcup_{n=1}^\infty V_n = V$ and $\bigcup_{n=1}^\infty E_n = E$.

Proof. Indeed, let $V = \{v_n : n \geq 1\}$ be a sequence of all vertices of Γ. and let $E = \{e_n : n \geq 1\}$ be a sequence of edges of Γ. We will proceed inductively to construct the sequences $\{V_n\}_{n=1}^\infty$ and $\{E_n\}_{n=1}^\infty$. In order to construct $< E_1, V_1 >$ let α be a path joining v_1 and v_2 ($i(\alpha) = v_1$, $t(\alpha) = v_2$) and let β be a path joining v_2 and v_1 ($i(\beta) = v_2$, $t(\beta) = v_1$). These paths exist since Γ is strongly connected. We define $V_1 \subset V$ to be the set of all vertices of paths α and β and $E_1 \subset E$ to be the set of all edges from α and β enlarged by e_1 if this edge is among all the edges joining the vertices of V_1. Obviously $< E_1, V_1 >$ is strongly connected and the first step of inductive procedure is complete. Suppose now that a strongly connected graph $< E_n, V_n >$ has been constructed. If $v_{n+1} \in V_n$, we set $V_{n+1} = V_n$ and E_{n+1} is then defined to be the union of E_n and all the edges from $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ that are among all the edges joining the vertices of V_n. If $v_{n+1} \notin V_n$, let α_n be a path joining v_n and v_{n+1} and let β_n be a path joining v_{n+1} and v_n. We define V_{n+1} to be the union of V_n and the set of all vertices of α_n and β_n. E_{n+1} is then defined to be the union of E_n, all the edges building the paths α_n and β_n and all the edges from $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ that are among all the edges joining the vertices of V_{n+1}. Since $< E_n, V_n >$ was strongly connected, so is $< E_{n+1}, V_{n+1} >$. The inductive procedure is complete. It immediately follows from the construction that $V_n \subset V_{n+1}$, $E_n \subset E_{n+1}$. $\bigcup_{n=1}^\infty V_n = V$ and $\bigcup_{n=1}^\infty E_n = E$. We are done. ■

Our main result is the following.

Lemma 2.8. Suppose that $f : E^\infty \to I\!\!R$ is a Holder continuous function such that $\sum_{e \in I} \exp(\sup(f|_{[e]})) < \infty$, and the incidence matrix is irreducible. Then there exists a

Borel probability eigenmeasure $\tilde{m}$ of the conjugate operator $\mathcal{L}_f^*$.

Proof. Without loosing generality we may assume that $I = I\!N$. Since the incidence matrix is irreducible, it follows from Lemma 2.7 that we can reorder the set $I\!N$ such that there exists an increasing to infinity sequence $\{l_n\}_{n\geq 1}$ for every $n \geq 1$ the matrix $A|_{\{1,\ldots,l_n\}\times\{1,\ldots,l_n\}}$ is irreducible. Then, in view of Lemma 2.6, there exists an eigenmeasure $\tilde{m}_n$ of the operator $\mathcal{L}_n^*$, conjugate to Perron-Frobenius operator

$$\mathcal{L}_n : C(E_{l_n}^\infty) \to C(E_{l_n}^\infty)$$

associated to the function $f|_{E_{l_n}^\infty}$, where, for any $q \geq 1$,

$$E_q^\infty = E^\infty \cap \{1,\ldots,q\}^\infty = \{(e_k)_{k\geq 1} : 1 \leq e_k \leq q \text{ and } A_{e_k e_{k+1}} = 1 \text{ for all } k \geq 1\}.$$

Occasionally we will also treat $\mathcal{L}_n$ as acting on $C(E^\infty)$ and $\mathcal{L}_n^*$ as acting on $C^*(E^\infty)$. Our first aim is to show that the sequence $\{\tilde{m}_n\}_{n\geq 1}$ is tight where $\tilde{m}_n$, $n \geq 1$, are treated her as Borel probability measures on E^∞. Let $\mathrm{P}_n = \mathrm{P}(\sigma|_{E_{l_n}^\infty}, f_{E_{l_n}^\infty})$. Obviously $\mathrm{P}_n \geq \mathrm{P}_1$ for all $n \geq 1$. For every $k \geq 1$ let $\pi_k : E^\infty \to I\!N$ be the projection onto k-th coordinate, i.e. $\pi(\{(e_u)_{u\geq 1}\}) = e_k$. By Theorem 2.5, $\mathrm{e}^{\mathrm{P}_n}$ is the eigenvalue of $\mathcal{L}_n^*$ corresponding to the eigenmeasure $\tilde{m}_n$. Therefore, we obtain for every $n \geq 1$, every $k \geq 1$, and every $e \in I\!N$ that

$$\begin{aligned}
\tilde{m}_n(\pi_k^{-1}(e)) &= \sum_{\omega\in E_{l_n}^k:\omega_k=e} \tilde{m}_n([\omega]) \leq \sum_{\omega\in E_{l_n}^k:\omega_k=e} \exp\big(\sup(S_k f|_{[\omega]}) - \mathrm{P}_n k\big) \\
&\leq \mathrm{e}^{-\mathrm{P}_n k} \sum_{\omega\in E_{l_n}^k:\omega_k=e} \exp\big(\sup(S_{k-1} f|_{[\omega]}) + \sup(f|_{[e]})\big) \\
&\leq \mathrm{e}^{-\mathrm{P}_1 k} \left(\sum_{i\in I\!N} \mathrm{e}^{\sup(f|_{[i]})}\right)^{k-1} \mathrm{e}^{\sup(f|_{[e]})}
\end{aligned}$$

Therefore

$$\tilde{m}_n(\pi_k^{-1}([e+1,\infty))) \leq \mathrm{e}^{-\mathrm{P}_1 k} \left(\sum_{i\in I\!N} \mathrm{e}^{\sup(f|_{[i]})}\right)^{k-1} \sum_{j>e} \mathrm{e}^{\sup(f|_{[j]})}.$$

Fix now $\varepsilon > 0$ and for every $k \geq 1$ choose an integer $n_k \geq 1$ such that

$$\mathrm{e}^{-\mathrm{P}_1 k} \left(\sum_{i\in I\!N} \mathrm{e}^{\sup(f|_{[i]})}\right)^{k-1} \sum_{j>n_k} \mathrm{e}^{\sup(f|_{[j]})} \leq \frac{\varepsilon}{2^k}.$$

Then, for every $n \geq 1$ and every $k \geq 1$, $\tilde{m}_n(\pi_k^{-1}([n_k+1,\infty))) \leq \frac{\varepsilon}{2^k}$. Hence

$$\tilde{m}_n\left(E^\infty \cap \prod_{k\geq 1}[1,n_k]\right) \geq 1 - \sum_{k\geq 1} \tilde{m}_n(\pi_k^{-1}([n_k+1,\infty))) \geq 1 - \sum_{k\geq 1} \frac{\varepsilon}{2^k} = 1 - \varepsilon.$$

Since $E^\infty \cap \prod_{k\geq 1}[1, n_k]$ is a compact subset of E^∞, the tightness of the sequence $\{\tilde{m}_n\}_{n\geq 1}$ is therefore proved. Thus, in view of Prochorov's theorem there exists $\tilde{m}$, a weak-limit point of the sequence $\{\tilde{m}_n\}_{n\geq 1}$. Let now $\mathcal{L}_{0,n} = \mathrm{e}^{-\mathrm{P}_n}\mathcal{L}_n$ and $\mathcal{L}_0 = \mathrm{e}^{-\mathrm{P}(f)}\mathcal{L}$ be the corresponding normalized operators. Fix $g \in C_b(E^\infty)$ and $\varepsilon > 0$. Let us now consider an integer $n \geq 1$ so large that the following requirements are satisfied.

$$\sum_{i>n} ||g||_0 \exp\big(\sup(f|_{[i]}) - \mathrm{P}(f)\big) \leq \frac{e}{6}, \tag{x.1}$$

$$\sum_{i\geq 1} ||g||_0 \exp\big(\sup(f|_{[i]})\big)\mathrm{e}^{-\mathrm{P}_1}(\mathrm{P}(f) - \mathrm{P}_n) \leq \frac{e}{6}, \tag{2.6}$$

$$|\tilde{m}_n(g) - \tilde{m}(g)| \leq \frac{e}{3}, \tag{2.7}$$

and

$$\left|\int \mathcal{L}_0(g)d\tilde{m} - \int \mathcal{L}_0(g)d\tilde{m}_n\right| \leq \frac{e}{3}. \tag{2.8}$$

It is possible to make condition (2.6) satisfied since, due to Theorem 1.2, $\lim_{n\to\infty} \mathrm{P}_n = \mathrm{P}(f)$. Let $g_n = g|_{E^\infty_{I_n}}$. The first two observations are the following.

$$\begin{aligned}
\mathcal{L}^*_{0,n}\tilde{m}_n(g) &= \int_{E^\infty} \sum_{i\leq n: A_{i\omega_n}=1} g(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde{m}_n(\omega) \\
&= \int_{E^\infty_{I_n}} \sum_{i\leq n: A_{i\omega_n}=1} g(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde{m}_n(\omega) \\
&= \int_{E^\infty_{I_n}} \sum_{i\leq n: A_{i\omega_n}=1} g_n(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde{m}_n(\omega) \\
&= \mathcal{L}^*_{0,n}\tilde{m}_n(g_n) = \tilde{m}_n(g_n) \qquad (2.9)
\end{aligned}$$

and

$$\tilde{m}_n(g_n) - \tilde{m}_n(g) = \int_{E^\infty_{I_n}} (g_n - g)d\tilde{m}_n = \int_{E^\infty_{I_n}} 0 d\tilde{m}_n = 0. \tag{2.10}$$

Using the triangle inequality we get the following.

$$\begin{aligned}
\mathcal{L}^*_0\tilde{m}(g) - \tilde{m}(g)| \leq |\mathcal{L}^*_0\tilde{m}(g) - \mathcal{L}^*_0\tilde{m}_n(g)| + |\mathcal{L}^*_0\tilde{m}_n(g) - \mathcal{L}^*_{0,n}\tilde{m}_n(g)| + \\
+ |\mathcal{L}^*_{0,n}\tilde{m}_n(g) - \tilde{m}_n(g_n)| + |\tilde{m}_n(g_n) - \tilde{m}_n(g)| + |\tilde{m}_n(g) - \tilde{m}(g)| \qquad (2.11)
\end{aligned}$$

Let us now look at the second summand. Applying (2.6) and (2.5) we get

$$\begin{aligned}
&|\mathcal{L}_0^*\tilde m_n(g) - \mathcal{L}_{0,n}^*\tilde m_n(g)| = \\
&= \bigg|\int_{E^\infty} \sum_{i\le n: A_{i\omega_n}=1} g(i\omega)\big(\exp(f(i\omega)-\mathrm{P}(f) - \exp(f(i\omega)-\mathrm{P}_n)\big)d\tilde m_n(\omega) \\
&+ \int_{E^\infty} \sum_{i> n: A_{i\omega_n}=1} g(i\omega)\exp(f(i\omega)-\mathrm{P}(f))d\tilde m_n(\omega)\bigg| \\
&\le \sum_{i\le n} ||g||_0 \mathrm{e}^{f(i\omega)}\mathrm{e}^{-\mathrm{P}_n}(\mathrm{P}(f)-\mathrm{P}_n) + \sum_{i>n} ||g||_0 \exp\big(\sup(f|_{[i]}-\mathrm{P}(f)\big) \\
&\le \sum_{i\ge 1} ||g||_0 \exp\big(\sup(f|_{[i]})\big)\mathrm{e}^{-\mathrm{P}_1}(\mathrm{P}(f)-\mathrm{P}_n) + \frac{\varepsilon}{6} \\
&\le \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}. \qquad (2.12)
\end{aligned}$$

Combining now in turn (2.8), (2.12), (2.9), (2.10) and (2.7) we get from (2.11) that

$$|\mathcal{L}_0^*\tilde m(g) - \tilde m(g)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Letting $\varepsilon \searrow 0$ we therefore get $\mathcal{L}_0^*\tilde m(g) = \tilde m(g)$ or $\mathcal{L}_f^*\tilde m(g) = \mathrm{e}^{\mathrm{P}(f)}\tilde m(g)$. Hence $\mathcal{L}_f^*\tilde m = \mathrm{e}^{\mathrm{P}(f)}\tilde m$ and the proof is complete. ■

As an immediate consequence of this theorem and Theorem 2.5, we get the following.

Corollary 2.9. Suppose that $f : E^\infty \to I\!\!R$ is a Holder continuous function such that $\sum_{e\in I}\exp(\sup(f|_{[e]})) < \infty$ and the incidence matrix is finitely irreducible. Then there exists a Gibbs state for f.

As an immediate consequence of Theorem 2.8, Theorem 4.3, Theorem 2.5, and Theorem 3.2, we get the following.

Corollary 2.10. Suppose that $f : E^\infty \to I\!\!R$ is a Holder continuous function such that $\sum_{e\in I}\exp(\sup(f|_{[e]})) < \infty$ and the incidence matrix is finitely primitive. Then

(a) There exists a unique eigenmeasure $\tilde m_f$ of the conjugate Perron-Frobenius operator $\mathcal{L}_f^*$ and the corresponding eigenvalue is equal to $\mathrm{e}^{\mathrm{P}(f)}$.
(b) The eigenmeasure $\tilde m_f$ is a Gibbs state for f.
(c) The function $f : E^\infty \to I\!\!R$ has a unique σ-invariant Gibbs state $\tilde\mu_f$, this Gibbs state is completely ergodic and the stochastic laws presented in Section 6 are satisfied.

The character of the following chapters is somewhat different. We mainly examine properties of Perron-Frobenius operators, Gibbs states and eigenmeasures of the conjugate Perron-Frobenius operators assuming the existence of these measures and imposing some additional requirements, frequently weaker than those needed in Section 2 for the proof of the existence of such measures.

§3. Properties of Gibbs states and equilibrium states. As an immediate consequence of (2.1) and Remark 2.1 we get the following.

Proposition 3.1. Any uniformly continuous function $f : E^\infty \to I\!R$ that has a Gibbs state is acceptable.

Given $\omega \in E^*$ and $n \geq 1$ let

$$E_n^\omega = \{\tau \in E^n : A_{\tau_n \omega_1} = 1\} \text{ and } E_*^\omega = \{\tau \in E^* : A_{\tau_{|\tau|}\omega_1} = 1\}.$$

We shall prove the following result which is well-known (see for ex. Lemma 2.1 in [ADU]) except the right-hand side inequality in formula (2.1).

Theorem 3.2. If an acceptable function f has a Gibbs state and the incidence matrix A is finitely primitive, then f has a unique invariant Gibbs state. Moreover, this invariant Gibbs state is exact.

Proof. Let $\tilde{m}$ be a Gibbs state for f. Since the matrix A is finitely primitive, $\inf\{\tilde{m}(\sigma([i])) : i \in I\} > 0$ and it therefore follows from Lemma 2.1 in [ADU] that there exists a σ-invariant Borel probability measure $\tilde{\mu}$ absolutely continuous with respect to $\tilde{m}$ and even more, the left-hand side inequality in formula (2.1) holds. Let now the finite set Λ and the integer $q \geq 0$ be given by finite primitiveness of the incidence matrix A. By acceptability of f,

$$T = \min\{\inf(S_q f|_{[\alpha]}) : \alpha \in \Lambda\} > -\infty.$$

Fixing $\omega \in E^*$, using (2.1), Remark 2.1 and Proposition 2.2(a) we get for everyl $n \geq q$

$$
\begin{aligned}
\tilde{m}(\sigma^{-n}([\omega])) &= \sum_{\tau\in E_n^\omega} \tilde{m}([\tau\omega]) \geq \sum_{\alpha\in\Lambda\cap E_q^{(\omega_1)}} \sum_{\tau\in E_{n-q}^\alpha} \tilde{m}([\tau\alpha\omega]) \\
&\geq \sum_{\alpha\in\Lambda\cap E_q^{(\omega_1)}} \sum_{\tau\in E_{n-q}^\alpha} Q^{-1} \exp\bigl(\inf(S_{n+|\omega|}f|_{[\tau\alpha\omega]}) - \mathrm{P}(f)(n+|\omega|)\bigr) \\
&\geq Q^{-1} \sum_{\alpha\in\Lambda\cap E_q^{(\omega_1)}} \sum_{\tau\in E_{n-q}^\alpha} \exp\bigl(\inf(S_{n-q}f|_{[\tau]}) - \mathrm{P}(f)(n-q)+ \\
&+ \inf(S_q f|_{[\alpha]}) - q\mathrm{P}(f) + \inf(S_{|\omega|}f|_{[\omega]}) - \mathrm{P}(f)|\omega|\bigr) \\
&\geq Q^{-1}\mathrm{e}^T\mathrm{e}^{-q\mathrm{P}(f)} \exp\bigl(\inf(S_{|\omega|}f|_{[\omega]}) - \mathrm{P}(f)|\omega|\bigr)\times \\
&\times \sum_{\alpha\in\Lambda\cap E_q^{(\omega_1)}} \sum_{\tau\in E_{n-q}^\alpha} \exp\bigl(\inf(S_{n-q}f|_{[\tau]}) - \mathrm{P}(f)(n-q)\bigr) \\
&\geq Q^{-2}\exp(T-\mathrm{P}(f)q)\tilde{m}([\omega]) \sum_{\alpha\in\Lambda\cap E_q^{(\omega_1)}} \sum_{\tau\in E_{n-q}^\alpha} \exp\bigl(\inf(S_{n-q}f|_{[\tau]}) - \mathrm{P}(f)(n-q)\bigr) \\
&\geq Q^{-2}\exp(T-\mathrm{P}(f)q)\tilde{m}([\omega])Q^{-1} \sum_{|\tau=n-q} \tilde{m}([\tau]) = Q^{-3}\exp(T-\mathrm{P}(f)q)\tilde{m}([\omega])
\end{aligned}
$$

Since $\tilde{\mu}([\omega]) \geq \liminf_{n\to\infty} \tilde{m}(\sigma^{-n}([\omega]))$, we therefore conclude that $\tilde{\mu}([\omega]) \geq Q^{-3}\exp(T - \mathrm{P}(f)q)\tilde{m}([\omega])$ and consequently $\tilde{\mu}$ is a Gibbs state. Exactness of such a measure is well-known (see Theorem 3.2 in [ADU] for ex.) and uniqueness follows immediately from ergodicity of any invariant Gibbs state and Proposition 2.2(b). The proof is complete. ■

We say that two functions $f, g : E^\infty \to I\!R$ are cohomologous in a class $\mathcal{H}$ if there exists a function $u : E^\infty \to I\!R$ in the class $\mathcal{H}$ such that

$$
g - f = u - u\circ\sigma.
$$

We shall provide now a list of necessary and sufficient conditions for two Hölder continuous functions to have the same invariant Gibbs states. The proof is analogous to the proof of Theorem 1.28 in [Bo] (see also [HMU])

Theorem 3.3. Suppose that $f, g : E^\infty \to I\!R$ are two Hölder continuous functions that have invariant Gibbs states $\tilde{\mu}_f$ and $\tilde{\mu}_g$ respectively. Suppose also that the incidence matrix A is finitely irreducible. Then the following conditions are equivalent:

(1) $\tilde{\mu}_f = \tilde{\mu}_g$.

(2) There exists a constant R such that if $\sigma^n(\omega) = \omega$, then

$$
S_n f(\omega) - S_n g(\omega) = nR
$$

(3) The difference $f - g$ is cohomologous to a constant in the class of bounded Hölder continuous functions.

(4) The difference $g-f$ is cohomologous to a constant in the class of bounded continuous functions.

(5) There exist constants S and T such that for every $\omega \in E^\infty$ and every $n \geq 1$

$$|S_nf(\omega) - S_ng(\omega) - Sn| \leq T.$$

If these conditions are satisfied then $R = S = \mathrm{P}(f) - \mathrm{P}(g)$.

Proof. (1) $\Rightarrow$ (2). It follows from (2.1) that

$$Q^{-2} \leq \frac{\exp\bigl(S_kf(\omega) - \mathrm{P}(f)k\bigr)}{\exp\bigl(S_kg(\omega) - \mathrm{P}(g)k\bigr)} \leq Q^2$$

for every $\omega \in E^\infty$ and every $k \geq 1$. Suppose now that $\sigma^n(\omega) = \omega$. Then for every $k = ln$, $l \geq 1$,

$$Q^{-4} \leq \exp\bigl(l(S_nf(\omega) - S_ng(\omega)) - (\mathrm{P}(f) - \mathrm{P}(g))n\bigr) \leq Q^4.$$

Hence, there exists a constant $T \geq 0$ such that

$$l|S_nf(\omega) - S_ng(\omega) - (\mathrm{P}(f) - \mathrm{P}(g))n| \leq T$$

and therefore, letting $l \nearrow \infty$, we conclude that $S_nf(\omega) - S_ng(\omega) = (\mathrm{P}(f) - \mathrm{P}(g))n$. Thus, putting $R = \mathrm{P}(f) - \mathrm{P}(g)$ completes the proof of the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Define

$$\eta = f - g - R.$$

Since the incidence matrix A is irreducible, there exists a point $\tau \in E^\infty$ transitive for the shift map $\sigma : E^\infty \to E^\infty$. Put

$$\Gamma = \{\sigma^k(\tau) : k \geq 1\}$$

and define the function $u : \Gamma \to I\!R$ by setting

$$u(\sigma^k(\tau)) = \sum_{j=0}^{k-1} \eta(\sigma^j(\tau)).$$

Note that the function u is well-defined since all points $\sigma^k(\tau)$, $k \geq 1$, are mutually distinct. Taking the minimum of exponents we may assume that both functions f and g are Hölder continuous with the same order β. Let Λ be the set coming from finite irreducibility of the incidence matrix A. Let $|\Lambda| = \sup\{|\alpha| : \alpha \in \Lambda\}$ and $S = \sup\{|S_{|\alpha|}\eta| : \alpha \in \Lambda\}$. Fix $k \geq 1$

and consider periodic point $\omega = (\tau|_k\alpha)^\infty$. Then by our assumption

$$\begin{aligned}
|u(\sigma^k(\tau)| = &\Bigg|\sum_{j=0}^{k-1}\big(\eta(\sigma^j(\tau)) - (f(\sigma^j(\omega)) - g(\sigma^j(\omega))\big) + Rk+ \\
&+ \sum_{j=0}^{|\alpha|-1}\big(g(\sigma^{k+j}\omega) - f(\sigma^{k+j}\omega)\big) + R|\alpha|\Bigg| \\
&= \left|\sum_{j=0}^{k-1}\big((f(\sigma^j(\tau)) - f(\sigma^j(\omega))) - (g(\sigma^j(\tau)) - g(\sigma^j(\omega))) - S_{|\alpha|}\eta(\sigma^k(\omega)\big)\right| \\
&\le \sum_{j=0}^{k-1}|f(\sigma^j(\tau)) - f(\sigma^j(\omega))| + \sum_{j=0}^{k-1}|g(\sigma^j(\tau)) - g(\sigma^j(\omega))| \\
&\le \sum_{j=0}^{k-1}V_\beta(f)\mathrm{e}^{-\beta(k-j)} + |S_{|\alpha|}\eta(\sigma^k(\omega)|\sum_{j=0}^{k-1}V_\beta(g)\mathrm{e}^{-\beta(k-j)} + S
\end{aligned}$$

$$\le (V_\beta(f) + V_\beta(g))\frac{\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}} + S < \infty. \tag{3.1}$$

Assume now $\sigma^k(\tau)|_r = \sigma^l(\tau)|_r$ for some $k < l$ and some $r \ge 1$. Let $\omega = \tau|_k(\sigma^k(\tau)|_{l-k})^\infty$. By our assumption $\sum_{j=k}^{l-1}\eta(\sigma^j(\omega)) = 0$. Hence,

$$\begin{aligned}
|u(\sigma^l(\tau)) - u(\sigma^k(\tau))| &= \left|\sum_{j=k}^{l-1}\eta(\sigma^j(\tau))\right| = \left|\sum_{j=k}^{l-1}\eta(\sigma^j(\tau)) - \eta(\sigma^j(\omega))\right| \\
&\le \sum_{j=k}^{l-1}\big(|f(\sigma^j(\tau)) - f(\sigma^j(\omega))| + |g(\sigma^j(\tau)) - g(\sigma^j(\omega))|\big) \\
&\le \sum_{j=k}^{l-1}(V_\beta(f) + V_\beta(g))\mathrm{e}^{-\beta(r+l-j-1)}
\end{aligned}$$

$$\le \mathrm{e}^{-\beta r}(V_\beta(f) + V_\beta(g))\sum_{j=0}^{\infty}\mathrm{e}^{-\beta j} = \frac{V_\beta(f) + V_\beta(g)}{1-\mathrm{e}^{-\beta}}\mathrm{e}^{-\beta r} \tag{3.2}$$

In particular it follows from (3.2) that u is uniformly continuous on Γ. Since Γ is a dense subset of E^∞ we therefore conclude that u has a unique continuous extension on E^∞. Moreover, it follows from (3.1) and (3.2) that u is bounded and Hölder continuous. The proof of the implication (2) $\Rightarrow$ (3) is therefore complete.

Now, the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (1). It follows from (5) and (2.1) that for every $\omega \in E^*$, say $\omega \in E^n$

$$Q^{-2}\mathrm{e}^{-T}\exp\big((S + \mathrm{P}(g) - \mathrm{P}(f))n\big) \le \frac{\tilde\mu_f([\omega])}{\tilde\mu_g([\omega])} \le Q^2\mathrm{e}^T\exp\big((S + \mathrm{P}(g) - \mathrm{P}(f))n\big). \tag{3.3}$$

Suppose that $S \neq \mathrm{P}(f) - \mathrm{P}(g)$. Without loosing generality we may assume that $S < \mathrm{P}(f) - \mathrm{P}(g)$. But then it would follow from (3.3) that that for every $n \geq 1$

$$1 = \tilde{\mu}_f(E^\infty) = \sum_{|\omega|=n} \tilde{\mu}_f([\omega]) \leq Q^4 \mathrm{e}^T \exp\big((S + \mathrm{P}(g) - \mathrm{P}(f))n\big)$$

which gives contradiction for $n \geq 1$ large enough. Hence $S = \mathrm{P}(f) - \mathrm{P}(g)$. But then (3.3) implies that the measures $\tilde{\mu}_f$ and $\tilde{\mu}_g$ are equivalent. Since, in view of Theorem 3.2 these measures are ergodic, they must conincide. The proof of the implication (5) $\Rightarrow$ (1) and simultaneously of the whole Theorem 3.3 is complete. ■

We call a σ-invariant probability measure $\tilde{\mu}$ an equilibrium state of the potential f if $\int -f d\mu < +\infty$ and

$$\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu} = \mathrm{P}(f).$$

We end this section with the following two results.

Lemma 3.4. Suppose that the incidence matrix A is finitely primitive and that a continuous function $f : E^\infty \to I\!\!R$ has a Gibbs state. Denote by $\tilde{\mu}_f$ its unique invariant Gibbs state (see Theorem 3.2). Then the following three conditions are equivalent:

(a) $\int_{E^\infty} -f d\tilde{\mu}_f < \infty$.

(b) $\sum_{i \in I} \inf(-f|_{[i]}) \exp(\inf f|_{[i]}) < \infty$.

(c) $\mathrm{H}_{\tilde{\mu}_f}(\alpha) < \infty$, where $\alpha = \{[i] : i \in I\}$ is the partition of E^∞ into initial cylinders of length 1.

Proof. *(a)* $\Rightarrow$ *(b)*. Suppose that $\int -f d\tilde{\mu}_f < \infty$. This means that $\sum_{i \in I} \int_{[i]} -f d\tilde{\mu}_f < \infty$ and consequently

$$\begin{aligned}\infty > \sum_{i \in I} \inf(-f|_{[i]})\tilde{\mu}_f([i]) &\geq Q^{-1} \sum_{i \in I} \inf(-f|_{[i]}) \exp\big(\inf f|_{[i]} - \mathrm{P}(f)\big) \\ &= Q^{-1}\mathrm{e}^{-\mathrm{P}(f)} \sum_{i \in I} \inf(-f|_{[i]}) \exp\big(\inf f|_{[i]}\big).\end{aligned}$$

(b) $\Rightarrow$ *(c)*. Assume that $\sum_{i \in I} \inf(-f|_{[i]}) \exp\big(\inf(f|_{[i]})\big) < \infty$. We shall show that $\mathrm{H}_{\tilde{\mu}_f}(\alpha) < \infty$. By definition,

$$\mathrm{H}_{\tilde{\mu}_f}(\alpha) = \sum_{i \in I} -\tilde{\mu}_f([i]) \log \tilde{\mu}_f([i]) \leq \sum_{i \in I} -\tilde{\mu}_f([i])\big(\inf(f|_{[i]}) - \mathrm{P}(f) - \log Q\big).$$

Since $\sum_{i \in I} \tilde{\mu}_f([i])(\mathrm{P}(f) + \log Q\big) < \infty$, it suffices to show that $\sum_{i \in I} -\tilde{\mu}_f([i]) \inf(f|_{[i]}) < \infty$. And indeed,

$$\sum_{i \in I} -\tilde{\mu}_f([i]) \inf(f|_{[i]}) = \sum_{i \in I} \tilde{\mu}_f([i]) \sup(-f|_{[i]}) \leq \sum_{i \in I} \tilde{\mu}_f([i])\big(\inf(-f|_{[i]}) + \mathrm{osc}(f)\big).$$

Since $\sum_{i\in I}\tilde{\mu}_f([i])\mathrm{osc}(f)=\mathrm{osc}(f)$, it is enough to show that

$$\sum_{i\in I}\tilde{\mu}_f([i])\inf(-f|_{[i]})<\infty.$$

Since $\tilde{\mu}_f$ is a probability measure, $\lim_{i\to\infty}\tilde{\mu}_f([i])=0$. Therefore, it follows from (2.1) that $\lim_{i\to\infty}\bigl(\sup(f|_{[i]})-\mathrm{P}(f)\bigr)=-\infty$. Thus, for all i sufficiently large, say $i\ge k$, $\sup(f|_{[i]})<0$. Hence, for all $i\ge k$, $\inf(-f|_{[i]})=-\sup(f|_{[i]})>0$. So, using (2.1) again, we get

$$\begin{aligned}\sum_{i\ge k}\tilde{\mu}_f([i])\inf(-f|_{[i]}) &\le \sum_{i\ge k} Q\exp\bigl(\inf(f|_{[i]})-\mathrm{P}(f)\bigr)\inf(-f|_{[i]})\\ &= Q\mathrm{e}^{-\mathrm{P}(f)}\sum_{i\ge k}\exp\bigl(\inf(f|_{[i]})\bigr)\inf(-f|_{[i]})\end{aligned}$$

which is finite due to our asssumption. Finally $\sum_{i\in I}\tilde{\mu}_f([i])\inf(-f|_{[i]})$ is finite.

$(c)\Rightarrow(a)$. Suppose that $\mathrm{H}_{\tilde{\mu}_f}(\alpha)<\infty$. We need to show that $\int -f\,d\tilde{\mu}_f<\infty$. We have

$$\infty>\mathrm{H}_{\tilde{\mu}_f}(\alpha)=\sum_{i\in I}-\tilde{\mu}_f([i])\log\bigl(\tilde{\mu}_f([i])\bigr)\le\sum_{i\in I}-\tilde{\mu}_f([i])\bigl(\inf(f|_{[i]})-\mathrm{P}(f)-\log Q\bigr).$$

Hence, $\sum_{i\in I}-\tilde{\mu}_f([i])\inf(f|_{[i]})<\infty$ and therefore

$$\int -f\,d\tilde{\mu}_f=\sum_{i\in I}\int_{[i]}-f\,d\tilde{\mu}_f\le\sum_{i\in I}\sup(-f|_{[i]})\tilde{\mu}_f([i])=\sum_{i\in I}-\inf(f|_{[i]})\tilde{\mu}_f([i])<\infty.$$

The proof is complete. ■

Theorem 3.5. Suppose that the incidence matrix A is finitely primitive. Suppose that $f:E^\infty\to I\!\!R$ is a Hölder continuous bounded function that has a Gibbs state and that $\int -f\,d\tilde{\mu}_f<\infty$, where $\tilde{\mu}_f$ is the unique invariant Gibbs state for the potential f (see Theorem 3.2). Then $\tilde{\mu}_f$ is the unique equilibrium state for the potential f.

Proof. In order to show that $\tilde{\mu}_F$ is an equilibrium state of the potential f consider $\alpha=\{[i]:i\in I\}$, the partition of E^∞ into initial cylinders of length one. By Lemma 3.4, $\mathrm{H}_{\tilde{\mu}_f}(\alpha)<\infty$. Applying the Breiman-Shanon-McMillan theorem, Birkhoff's ergodic theorem, and (2.1), we get for $\tilde{\mu}_F$-a.e. $\omega\in E^\infty$

$$\begin{aligned}\mathrm{h}_{\tilde{\mu}_f}(\sigma)\ge\mathrm{h}_{\tilde{\mu}_f}(\sigma,\alpha)&=\lim_{n\to\infty}-\frac{1}{n}\log\tilde{\mu}_f([\omega|_n])\\ &\ge\lim_{n\to\infty}-\frac{1}{n}\bigl(\log Q+S_nf(\omega)-\mathrm{P}(f)n\bigr)\\ &=\lim_{n\to\infty}\frac{-1}{n}S_nf(\omega)+\mathrm{P}(f)=\int -f\,d\tilde{\mu}_f+\mathrm{P}(f)\end{aligned}$$

which, in view of Theorem 1.3, implies that $\tilde{\mu}_f$ is an equilibrium state for the potential f.

In order to prove uniqueness of equilibrium states we follow the reasoning taken from the proof of Theorem 1 in [DKU]. So, suppose that $\tilde\nu \neq \tilde\mu_f$ is an equilibrium state for the potential $f : E^\infty \to I\!R$. Applying the ergodic decomposition theorem, we may assume that $\tilde\nu$ is ergodic. Then, using (2.1), we can write for every $n \geq 1$ as follows.

$$
\begin{aligned}
0 &= n(\mathrm{h}_{\tilde\nu}(\sigma) + \int (f - \mathrm{P}(f))d\tilde\nu) \leq \mathrm{H}_{\tilde\nu}(\alpha_n) + \int (S_n f - \mathrm{P}(f)n)d\tilde\nu \\
&= -\sum_{|\omega|=n} \tilde\nu([\omega]) \left(\log \tilde\nu([\omega]) - \frac{1}{\tilde\nu([\omega])} \int_{[\omega]} (S_n f - \mathrm{P}(f)n)d\tilde\nu \right) \\
&\leq -\sum_{|\omega|=n} \tilde\nu([\omega]) \left(\log \tilde\nu([\omega]) - (S_n f(\tau_\omega) - \mathrm{P}(f)n)\right) \text{ for a suitable } \tau_\omega \in [\omega] \\
&= -\sum_{|\omega|=n} \tilde\nu([\omega]) \big(\log \tilde\nu([\omega]) \exp\big(\mathrm{P}(f)n - S_n f(\tau_\omega)\big)\big) \\
&\leq -\sum_{|\omega|=n} \tilde\nu([\omega]) \big(\log \tilde\mu_f([\omega]) Q^{-1} \tilde\mu([\omega])\big) \\
&= \log Q - \sum_{|\omega|=n} \tilde\nu([\omega]) \log \left(\frac{\tilde\nu([\omega])}{\tilde\mu_f([\omega])} \right).
\end{aligned}
$$

Therefore, in order to conclude the proof, it suffices to show that

$$
\lim_{n\to\infty} \left(-\sum_{|\omega|=n} \tilde\nu([\omega]) \log \left(\frac{\tilde\nu([\omega])}{\tilde\mu_f([\omega])} \right) \right) = -\infty.
$$

Since both measures $\tilde\nu$ and $\tilde\mu_f$ are ergodic and $\tilde\nu \neq \tilde\mu_f$, the measures $\tilde\nu$ and $\tilde\mu_f$ must be mutually singular. In particular

$$
\lim_{n\to\infty} \tilde\nu \left(\left\{ \omega \in E^\infty : \frac{\tilde\nu([\omega|_n])}{\tilde\mu_f([\omega|_n])} \leq S \right\} \right) = 0
$$

for every $S > 0$. For every $j \in \boldsymbol{Z}$ and every $n \geq 1$ define now

$$
F_{n,j} = \left\{ \omega \in E^\infty : \mathrm{e}^{-j} \leq \frac{\tilde\nu([\omega|_n])}{\tilde\mu_f([\omega|_n])} < \mathrm{e}^{-j+1} \right\}.
$$

Then

$$
\tilde\nu(F_{n,j}) = \int_{F_{n,j}} \frac{\tilde\nu([\omega|_n])}{\tilde\mu_f([\omega|_n])} d\tilde\mu_f(\omega) \leq \mathrm{e}^{-j+1} \tilde\mu_f(F_{n,j}) \leq \mathrm{e}^{-j+1}
$$

and we have for each $k = -1, -2, -3, \ldots$

$$
\begin{aligned}
-\sum_{|\omega|=n} \tilde{\nu}([\omega]) \log\left(\frac{\tilde{\nu}([\omega])}{\tilde{\mu}_f([\omega])}\right) &= -\int \log\left(\frac{\tilde{\nu}([\omega])}{\tilde{\mu}_f([\omega])}\right) d\tilde{\mu}_f(\omega) \\
&\le \sum_{j\in \boldsymbol{Z}} \tilde{\nu}(F_{n,j}) \le k \sum_{j\le k} \tilde{\nu}(F_{n,j}) + \sum_{j\ge 1} j\mathrm{e}^{-j+1} \\
&= k\tilde{\nu}\left(\left\{\omega \in E^\infty : \frac{\tilde{\nu}([\omega|_n])}{\tilde{\mu}_f([\omega|_n])} \ge \mathrm{e}^{-k}\right\}\right) + \sum_{j\ge 1} j\mathrm{e}^{-j+1} \\
&\longrightarrow k + \sum_{j\ge 1} j\mathrm{e}^{-j+1} \text{ as } n \to \infty.
\end{aligned}
$$

The proof is complete. ■

§4. Properties of the Perron-Frobenius operator.

Let

$$\mathcal{L}_0 = \mathrm{e}^{\mathrm{P}(f)} \mathcal{L}_f.$$

The first result concerning the Perron-Frobenius operator is the following.

Theorem 4.1. If a function $f : E^\infty \to I\!R$ has a Gibbs state, then for every $n \ge 1$ and every $\omega \in I^n$

$$\mathcal{L}_0^n(1\!\!1)(\omega) \le Q.$$

Proof. Let ν be a Gibbs measure for f. In view of Lemma 2.3 and the definition of Gibbs states we get

$$\mathcal{L}_0^n(1\!\!1)(\omega) = \sum_{\tau \in \sigma^{-n}(\omega)} \exp\bigl(S_n f(\tau) - \mathrm{P}(f)n\bigr) \le \sum_{\tau \in \sigma^{-n}(\omega)} Q\nu([\tau|_n]) \le Q\nu(\sigma^{-n}([\omega])) \le Q.$$

The proof is complete. ■

We would like to emphasize that in in Theorem 4.1 we assumed only the existence of a Gibbs state and not an eigenmeasure of the conjugate Perron-Frobenius operator. We shall now prove the following.

Theorem 4.2. If the incidence matrix is finitely primitive, then there exists a constant $R > 0$ such that

$$\mathcal{L}_0^n(1\!\!1)(\omega) \ge R$$

for all $n \ge 1$ and all $\omega \in E^\infty$.

Proof. It follows from (2.3) and Theorem 2.5 that $\int \mathcal{L}_0^n(1\!\!1) d\tilde{m} = 1$ for all $n \ge 1$. Hence for every $n \ge 1$ there exists $\omega(n) \in E^\infty$ such that $\mathcal{L}_0^n(1\!\!1)(\omega(n)) \ge 1$. Let now $\Lambda \subset E^q$ be the set given by finite primitivness of the incidence matrix A. Since Λ is finite and f is Hölder continuous, we have

$$N = \min\{\exp\bigl(\inf(S_q f|_{[\alpha]}) - q\mathrm{P}(f)\bigr) : \alpha \in \Lambda\} > 0.$$

Applying now Lemma 2.3, we get for every $n \geq q+1$ and every $\tau \in E^\infty$ the following

$$\begin{aligned}
&\mathcal{L}_0^n(1\!\!1)(\tau) = \\
&= \sum_{\omega \in E^n : A_{\omega_n \tau_1} = 1} \exp\bigl(S_n f(\omega\tau) - \mathrm{P}(f)n\bigr) \\
&\geq \sum_{\omega \in E^{n-q}} \exp\bigl(S_n f(\omega\alpha(\omega)\tau) - \mathrm{P}(f)n\bigr) \\
&\geq \sum_{\omega \in E^{n-q} : A_{\omega_{n-q}\omega(n-q)_1} = 1} \exp\bigl(S_n f(\omega\alpha(\omega)\tau) - \mathrm{P}(f)n\bigr) \\
&\geq \sum_{\omega \in E^{n-q} : A_{\omega_{n-q}\omega(n-q)_1} = 1} \exp\bigl(S_{n-q} f(\omega\alpha(\omega)\tau) - \mathrm{P}(f)(n-q)\bigr) \exp\bigl(S_q f(\alpha(\omega)\tau) - \mathrm{P}(f)q\bigr) \\
&\geq N \sum_{\omega \in E^{n-q} : A_{\omega_{n-q}\omega(n-q)_1} = 1} \exp\bigl(S_{n-q} f(\omega\alpha(\omega)\tau) - \mathrm{P}(f)(n-q)\bigr) \\
&\geq NT(f)^{-1} \sum_{\omega \in E^{n-q} : A_{\omega_{n-q}\omega(n-q)_1} = 1} \exp\bigl(S_{n-q} f(\omega\omega(n-q)) - \mathrm{P}(f)(n-q)\bigr) \\
&= NT(f)^{-1} \mathcal{L}_0^{n-q}(1\!\!1)(\omega(n-q)) \geq NT(f)^{-1},
\end{aligned}$$

where $\alpha(\omega)$ is an element of Λ such that $A_{\omega_n \alpha(\omega)_1} = A_{\alpha(\omega)_q \tau_1} = 1$. Since by finite primitivness of A, $\sup_{j \in I}\{\inf\{i \in I : A_{ij} = 1\}\} < \infty$, we deduce that $\min_{n \leq q}\{\inf(\mathcal{L}_0^n(1\!\!1))\} > 0$. Combining this and the last display we conclude the proof. ■

Theorem 4.3. If the incidence matrix is finitely primitive, then there exists at most one Borel probability fixed point of the conjugate operator $\mathcal{L}_0^*$.

Proof. Suppose that $\tilde{m}$ and $\tilde{m}_1$ are such two fixed points. In view of Proposition 2.2(b) and Theorem 2.5 the measures $\tilde{m}$ and $\tilde{m}_1$ are equivalent. Put $\rho = \frac{d\tilde{m}_1}{d\tilde{m}}$. Fix temporarily $\omega \in E^*$, say $\omega \in E^n$. It then follows from (3.1) and Theorem 2.5 that

$$\begin{aligned}
&\tilde{m}([\omega]) = \\
&= \int_{\tau \in E^\infty : A_{\omega_n \tau_1} = 1} \exp\bigl(S_n f(\omega\tau) - \mathrm{P}(f)n\bigr) d\tilde{m}(\tau) \\
&= \int_{\tau \in E^\infty : A_{\omega_n \tau_1} = 1} \exp\bigl(S_n f(\sigma(\omega\tau))\bigr) - \mathrm{P}(f)(n-1)\bigr) \exp\bigl(f(\omega\tau) - \mathrm{P}(f)\bigr) d\tilde{m}(\tau) \\
&= \int_{\tau \in E^\infty : A_{(\sigma(\omega))_{n-1}\tau_1} = 1} \exp\bigl(S_n f(\sigma(\omega\tau))\bigr) - \mathrm{P}(f)(n-1)\bigr) \exp\bigl(f(\omega\tau) - \mathrm{P}(f)\bigr) d\tilde{m}(\tau).
\end{aligned}$$

Hence

$$\inf\bigl(\exp\bigl(f|_{[\omega]} - \mathrm{P}(f)\bigr)\bigr)\tilde{m}([\sigma\omega]) \le \tilde{m}([\omega]) \le \sup\bigl(\exp\bigl(f|_{[\omega]} - \mathrm{P}(f)\bigr)\bigr)\tilde{m}([\sigma\omega]).$$

Since $f : E^\infty \to I\!\!R$ is Hölder continuous, we therefore conclude that for every $\omega \in E^\infty$

$$\lim_{n\to\infty} \frac{\tilde{m}\bigl([\omega|_n]\bigr)}{\tilde{m}\bigl([\sigma(\omega)|_{n-1}]\bigr)} = \exp\bigl(f(\omega) - \mathrm{P}(f)\bigr) \tag{4.1}$$

and the same formula is true with $\tilde{m}$ replaced by $\tilde{m}_1$. In view of Theorem 2.5 and Theorem 3.2 there exists a set of points $\omega \in E^\infty$ with $\tilde{m}$ measure 1 for which the Radon-Nikodym derivatives $\rho(\omega)$ and $\rho(\sigma(\omega))$ are both defined. Let $\omega \in E^\infty$ be such a point. Then using (4.1) and its version for $\tilde{m}_1$ we obtain

$$\begin{aligned}\rho(\omega) = \lim_{n\to\infty}\left(\frac{\tilde{m}_1\bigl([\omega|_n]\bigr)}{\tilde{m}\bigl([\omega|_n]\bigr)}\right) &= \lim_{n\to\infty}\left(\frac{\tilde{m}_1\bigl([\omega|_n]\bigr)}{\tilde{m}_1\bigl([\sigma(\omega)|_{n-1}]\bigr)}\cdot\frac{\tilde{m}_1\bigl([\sigma(\omega)|_{n-1}]\bigr)}{\tilde{m}\bigl([\sigma(\omega)|_{n-1}]\bigr)}\cdot\frac{\tilde{m}\bigl([\sigma(\omega)|_{n-1}]\bigr)}{\tilde{m}\bigl([\omega|_n]\bigr)}\right)\\ &= \exp\bigl(f(\omega) - \mathrm{P}(f)\bigr)\rho(\sigma(\omega))\exp\bigl(\mathrm{P}(f) - f(\omega)\bigr) = \rho(\sigma(\omega)).\end{aligned}$$

But since, in view of Theorem 3.2, $\sigma : E^\infty \to E^\infty$ is ergodic with respect to a σ-invariant measure equivalent with $\tilde{m}$, we conclude that ρ is $\tilde{m}$-almost ewerywhere constant. Since $\tilde{m}_1$ and $\tilde{m}$ are both probabilistic, $\tilde{m}_1 = \tilde{m}$. The proof is complete. ■

§**5. Ionescu-Tulcea and Marinescu inequality.** Alternative proofs of most results of this section can be found in Chapter 4 of [Ar] and in [AD] (for ex. Proposition 1.4 in [AD] states the same as our Lemma 5.1). In particular the reader should notice that Gibbs-Markov maps considered in [AD] are a generalization of our subshifts with finitely irreducible incidence matrix and Hölder continuous potentials.

Let

$$\mathcal{H}_0 = \{g : I^\infty \to C\!\!\!\!I : g \text{ is bounded and continuous}\}$$

and for every $\alpha > 0$ let

$$\mathcal{H}_\alpha = \{g \in \mathcal{H}_0 : V_\alpha(g) < \infty\}.$$

The set $\mathcal{H}_\alpha$ becomes a Banach space when endowed with the norm

$$||g||_\alpha = ||g||_0 + V_\alpha(g).$$

The main technical result of this section, called Ionescu-Tulcea and Marinescu inequality, is the following.

Lemma 5.1. Suppose that a Hölder continuous function $f : E^\infty \to I\!\!R$, say with an exponent $\beta > 0$, satisfying (2.2) has a Gibbs state. Then the normalized operator $\mathcal{L}_0 : \mathcal{H}_0 \to \mathcal{H}_0$ preserves the space $\mathcal{H}_\beta$ and moreover there exists a constant $C > 0$, such that for every $n \ge 1$ and every $g \in \mathcal{H}_\beta$

$$||\mathcal{L}_0^n(g)||_\beta \le Q\mathrm{e}^{-\beta n}||g||_\beta + C||g||_0.$$

Proof. Let $\tau, \rho \in E^\infty$, $\tau|_k = \rho|_k$ and $\tau_{k+1} \neq \rho_{k+1}$ for some $k \geq 1$. Then for every $n \geq 1$

$$
\begin{aligned}
\mathcal{L}_0^n(g)(\rho) - \mathcal{L}_0^n(g)(\tau) &= \sum_{\omega \in E^n : A_{\omega_n \rho_1} = 1} \exp(S_n f(\omega\rho) - \mathrm{P}(f)) g(\omega\rho) - \\
&\quad - \sum_{\omega \in E^n : A_{\omega_n \rho_1} = 1} \exp(S_n f(\omega\tau) - \mathrm{P}(f)) g(\omega\tau) \\
&= \sum_{\omega \in E^n : A_{\omega_n \rho_1} = 1} \exp(S_n f(\omega\rho) - \mathrm{P}(f)) \big(g(\omega\rho) - g(\omega\tau)\big) + \\
&\quad + \sum_{\omega \in E^n : A_{\omega_n \rho_1} = 1} g(\omega\tau) \big(\exp(S_n f(\omega\rho) - \mathrm{P}(f)) - \exp(S_n f(\omega\tau) - \mathrm{P}(f))\big) \qquad (5.1)
\end{aligned}
$$

But $|g(\omega\rho) - g(\omega\tau)| \leq V_\beta(g) \mathrm{e}^{-\beta(n+k)}$, and therefore, employing Theorem 4.1 we obtain

$$
\begin{aligned}
\sum_{\omega \in E^n : A_{\omega_n \rho_1} = 1} \exp(S_n f(\omega\rho) - \mathrm{P}(f)n) |g(\omega\rho) - g(\omega\tau)| &\leq \mathcal{L}_0^n(1\!\!1)(\rho) V_\beta(g) \mathrm{e}^{-\beta(n+k)} \leq \\
\leq Q V_\beta(g) \mathrm{e}^{-\beta(n+k)} \leq \mathrm{e}^{-\beta n} Q ||g||_\beta d_\beta(\rho, \tau) \qquad (5.2)
\end{aligned}
$$

Now notice that there exists a constant $M \geq 1$ such that $|1 - \mathrm{e}^x| \leq M|x|$ for all x with $|x| \leq \log(T(f))$. Since by Lemma 2.3, $|S_n f(\omega\rho) - S_n f(\omega\tau)| \leq d_\beta(\rho, \tau) \log(T(f)) \leq \log(T(f))$, we can estimate as follows.

$$
\begin{aligned}
|\exp(S_n f(\omega\rho) - \mathrm{P}(f)n) - \exp(S_n f(\omega\tau) - \mathrm{P}(f)n)| &= \\
&= \exp(S_n f(\omega\rho) - \mathrm{P}(f)n) |1 - \exp(S_n f(\omega\tau) - S_n f(\omega\rho))| \\
&\leq M \exp(S_n f(\omega\rho) - \mathrm{P}(f)n) |S_n f(\omega\rho) - S_n f(\omega\tau)| \\
&\leq M \exp(S_n f(\omega\rho) - \mathrm{P}(f)n) \log(T(f)) d_\beta(\rho, \tau) \\
&= M \log(T(f)) \exp(S_n f(\omega\rho) - \mathrm{P}(f)n) d_\beta(\rho, \tau)
\end{aligned}
$$

Hence, using Theorem 4 again, we get

$$
\begin{aligned}
\sum_{\omega \in E^n : A_{\omega_n \rho_1} = 1} |g(\omega\tau)| |\exp(S_n f(\omega\rho) - \mathrm{P}(f)n) - \exp(S_n f(\omega\tau) - \mathrm{P}(f)n)| \\
\leq ||g||_0 M \log(T(f)) d_\beta(\rho, \tau) \sum_{\omega \in E^n : A_{\omega_n \rho_1} = 1} \exp(S_n f(\omega\rho) - \mathrm{P}(f)n) \\
= ||g||_0 M \log(T(f)) d_\beta(\rho, \tau) \mathcal{L}_0^n(1\!\!1)(\rho) \\
\leq MQ \log(T(f)) ||g||_0 d_\beta(\rho, \tau)
\end{aligned}
$$

Combining this inequality, (5.2) and (5.1), we get

$$
|\mathcal{L}_0^n(g)(\rho) - \mathcal{L}_0^n(g)(\tau)| \leq \mathrm{e}^{-\beta n} Q ||g||_\beta d_\beta(\rho, \tau) + MQ \log(T(f)) ||g||_0 d_\beta(\rho, \tau).
$$

Combining in turn this and Theorem 4.1 we get

$$
|\mathcal{L}_0^n(g)||_\beta \leq Q \mathrm{e}^{-\beta n} ||g||_\beta + Q(M \log(T(f)) + 1) ||g||_\beta.
$$

The proof is finished. ■

Remark 5.2. We would like to remark that in fact in the proof of Lemma 5.1 we used only "weaker" property of Gibbs states, namely the right-hand side inequality of (2.1).

If the unit ball in $\mathcal{H}_\beta$ were compact as a subset of the Banach space $\mathcal{H}_0$ with the supremum norm $||\cdot||_0$, we could use now the famous Ionescu-Tulcea and Marinescu Theorem (see [ITM]) to establish some useful spectral properties of the Perron-Frobenius operator $\mathcal{L}_0$. But this ball is compact only in the topology of uniform convegence on compact subsets of E^∞ and we need to prove these properties directly ommitting the Ionescu-Tulcea and Marinescu Theorem theorem. Beginning with Lemma 5.5 we develop the approach from [PU].

Theorem 5.3. Suppose that a Hölder continuous function $f : E^\infty \to I\!\!R$, say with an exponent $\beta > 0$, satisfying (2.2) has a Gibbs state and the operator conjugate to the normalized Perron-Frobenius operator $\mathcal{L}_0$ has a Borel probability fixed point $\tilde{m}$. Then the operator $\mathcal{L}_0 : \mathcal{H}_\beta \to \mathcal{H}_\beta$ has a fixed point $\psi \le Q$ such that $\int \psi d\tilde{m} = 1$. If, in addition, the incydence matrix A is finitely primitive then $\psi \ge R$, where R is the constant produced in Theorem 4.2.

Proof. In view of Lemma 5.1, $||\mathcal{L}_0^n(1\!\!1)||_\beta \le Q + C$ for every $n \ge 0$. Hence

$$\left\| \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}_0^j(1\!\!1) \right\|_\beta \le Q + C \tag{5.3}$$

for every $n \ge 1$. Therefore, by the Ascoli-Arzela theorem there exists an increasing to infinity sequence of positive integers $\{n_k\}_{k\ge1}$ such that the sequence $\left\{\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}_0^j(1\!\!1)\right\}_{k\ge1}$ converges on compact sets of E^∞, say to $\psi : E^\infty \to I\!\!R$. Obviously $||\psi||_\beta \le Q + C$ and, in particular $\psi \in \mathcal{H}_\beta$. Since $\tilde{m}$ is a fixed point of the operator conjugate to $\mathcal{L}_0$, $\int \mathcal{L}_0^j(1\!\!1)dm = 1$ for every $j \ge 0$. Consequently $\int \frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_0^j(1\!\!1)dm = 1$ for every $n \ge 1$. Hence, applying Lebesgue's dominated convegence theorem along with Theorem 4.1, we conclude that $\int \psi d\tilde{m} = 1$ and $\psi \le Q$. Assuming in addition that the incydence matrix A is finitely primitive, using Theorem 4.2, we simultaneouslu get $\psi \ge R$. We are left to show that $\mathcal{L}_0(\psi) = \psi$. And indeed, using Theorem 4.1, we get for every $k \ge 1$ that

$$\left\| \mathcal{L}_0\left(\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}_0^j(1\!\!1)\right) - \frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}_0^j(1\!\!1) \right\| = \frac{1}{n_k}\left||\mathcal{L}_0^{n_k}(1\!\!1) - \mathcal{L}_0(1\!\!1)\right|_0 \le \frac{1}{n_k}Q.$$

Hence

$$\mathcal{L}_0\left(\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}_0^j(1\!\!1)\right) - \frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}_0^j(1\!\!1) \to 0 \tag{5.4}$$

uniformly. Therefore, in order to conclude the proof it sufficies to show that if a sequnce $\{g_k\}_{k=1}^\infty \subset \mathcal{H}_0$ is uniformly bounded and converges uniformly on compact sets of E^∞, say to a function g, then $\mathcal{L}_0(g_k)$, $k \ge 1$, converges uniformly on compact sets of E^∞ to $\mathcal{L}_0(g)$. And indeed, first notice that $||g||_0 \le B$, where B is an upper bound of the sequnce $\{g_k\}_{k=1}^\infty$. Fix now $\varepsilon > 0$. Since f has a Gibbs state, the series $M = \sum_{i\in I} \exp\bigl(\sup(f|_{[i]}) - \mathrm{P}\bigr)$ converges and there therefore exists a finite set $I_\varepsilon \subset I$ such that

$$\sum_{i\in I\setminus I_\varepsilon} 2B\exp\bigl(\sup(f|_{[i]}) - \mathrm{P}\bigr) < \frac{\varepsilon}{2}. \tag{5.5}$$

Fix now an arbitrary compact set $K \subset E^\infty$. Then for every $i \in I$, the set $iK = \{i\omega : \omega \in K \text{ and } A_{i\omega_1} = 1\}$ is also compact and so is the set $\bigcup_{i\in I_\varepsilon} iK$. Since $\{g_k\}_{k=1}^\infty$ converges uniformly on compact sets to g, there exists $q \ge 1$ such that for every $n \ge q$, $||(g_n - g)|_{\bigcup_{i\in I_\varepsilon} iK} \le \frac{\varepsilon}{2M}$. Applying this, Theorem 4.1 and (5.5), we get for every $n \ge q$ and every $\omega \in K$ that

$$\begin{aligned}
&|\mathcal{L}_0(g)(\omega) - \mathcal{L}_0(g_n)(\omega)| = \\
&= |\mathcal{L}_0(g - g_n)(\omega)| \\
&\le \sum_{i\in I_\varepsilon : A_{i\omega_1}=1} |g_n(i\omega) - g(i\omega)| \exp(f(i\omega) - \mathrm{P}) + \sum_{i\in I\setminus I_\varepsilon : A_{i\omega_1}=1} |g_n(i\omega) - g(i\omega)| \exp(f(i\omega) - \mathrm{P}) \\
&\le \sum_{i\in I_\varepsilon : A_{i\omega_1}=1} \frac{\varepsilon}{2M} \exp(f(i\omega) - \mathrm{P}) + \sum_{i\in I\setminus I_\varepsilon : A_{i\omega_1}=1} |g_n(i\omega) + g(i\omega)| \exp(f(i\omega) - \mathrm{P}) \\
&\le \frac{\varepsilon}{2M} \sum_{i\in I} \exp\bigl(\sup(f|_{[i]}) - \mathrm{P}\bigr) + 2B \sum_{i\in I\setminus I_\varepsilon : A_{i\omega_1}=1} \exp\bigl(\sup(f|_{[i]}) - \mathrm{P}\bigr) \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

The proof is complete. ■

From now on we assume in this section that the incidence matrix A is finitely primitive. Then $\psi \ge R$ and therefore the operator $\hat{\mathcal{L}} : \mathcal{H}_0 \to \mathcal{H}_0$, given by the formula

$$\hat{\mathcal{L}}(g) = \frac{1}{\psi}\mathcal{L}_0(g\psi) \tag{5.6}$$

is well-defined. It is straightforward to check that $\hat{\mathcal{L}}(\mathcal{H}_\beta) \subset \mathcal{H}_\beta$ (i.e. $1/\psi$ and the product of any two functions in $\mathcal{H}_\beta$ are again in $\mathcal{H}_\beta$. The basic properties of the operator $\hat{\mathcal{L}}$ following from Lemma 5.1, Theorem 4.2 and Theorem 5.3 are listed below.

Theorem 5.4. Writing

$$u_n(\omega) = \exp(S_n f(\omega) - \mathrm{P}n)\frac{\psi(\omega)}{\psi(\sigma^n(\omega))},\ n \ge 1,$$

we have for all $g \in \mathcal{H}_\beta$ and all $n \geq 1$

(a) $\hat{\mathcal{L}}^n(g)(\omega) = \frac{1}{\psi(\omega)}\mathcal{L}_0^n(g\psi)(\omega) = \sum_{\tau \in E^n : A_{\tau_n \omega_1}=1} u_n(\tau\omega)g(\tau\omega)$.

(b) $\hat{\mathcal{L}}^n(1\!\!1) = 1\!\!1$ and $||\hat{\mathcal{L}}^n||_0 = 1$.

(c) $M = \sup_{n\geq 1}\{||\hat{\mathcal{L}}^n||_\beta\} < \infty$.

(d) $\hat{\mathcal{L}}^*(\tilde{\mu}_f) = \tilde{\mu}_f$. In particular the closed subspaces $\mathcal{H}_0^0 = \{g \in \mathcal{H}_0 : \tilde{\mu}_f(g) = 0\}$ and $\mathcal{H}_\beta^0 = \{g \in \mathcal{H}_\beta : \tilde{\mu}_f(g) = 0\}$ are $\hat{\mathcal{L}}$-invariant.

(e) $\mathcal{H}_\beta = I\!\!R 1\!\!1 \bigoplus \mathcal{H}_\beta^0$ $(g = \tilde{\mu}_f(g)1\!\!1 + (g - \tilde{\mu}_f(g)1\!\!1))$.

Denote

$$\mathcal{H}_\beta^{0,1} = \{g \in \mathcal{H}_\beta^0 : ||g||_\beta \leq 1\}.$$

We shall prove the following.

Lemma 5.5. For every $n \geq 0$ define

$$b_n = \sup\{||\hat{\mathcal{L}}^n(g)||_\beta : g \in \mathcal{H}_\beta^{0,1}\}.$$

Then $\lim_{n\to\infty} b_n = 0$.

Proof. Define for every $n \geq 0$

$$a_n = \sup\{||\hat{\mathcal{L}}^n(g)||_0 : g \in \mathcal{H}_\beta^{0,1}\}.$$

It immediately follows from Theorem 5.4(b) that the sequence $\{a_n\}_{n\geq 1}$ is (weakly) decreasing. We shall show first that

$$\lim_{n\to\infty} a_n = 0.$$

Suppose on the contrary that $a = \lim_{n\to\infty} a_n > 0$. By Theorem 5.4(c), $\sup_{n\geq 0} b_n \leq M < \infty$. There therefore exists $\delta > 0$ such that if $d_\beta(\omega,\tau) \leq \delta$, then $|\hat{\mathcal{L}}^n(g)(\tau) - \hat{\mathcal{L}}^n(g)(\omega)| \leq a/2$ for all $g \in \mathcal{H}_\beta^{0,1}$ and all $n \geq 0$. Since A is finitely primitive, there exists $p \geq 0$ such that for every $\omega \in E^\infty$,

$$\sigma^p(B(\omega,\delta)) = E^\infty \tag{5.7}$$

Fix now $g \in \mathcal{H}_\beta^{0,1}$ and $n \geq 0$. Since $\int \hat{\mathcal{L}}^n g \, d\tilde{\mu}_f = 0$, there exists $\tau \in I^\infty$ such that $\hat{\mathcal{L}}^n(g)(\tau) \leq 0$. By (5.7) for every $\omega \in I^\infty$ there exists $\rho \in B(\tau,\delta) \cap \sigma^{-p}(\omega)$. Then $\hat{\mathcal{L}}^n(g)(\rho) \leq \hat{\mathcal{L}}^n(g)(\tau) + \frac{a}{2} \leq \frac{a}{2} \leq a_n - \frac{a}{2}$. Thus

$$\begin{aligned}
\hat{\mathcal{L}}^p(\hat{\mathcal{L}}^n g)(\omega) &= \hat{\mathcal{L}}^n g(\rho)u_p(\rho) + \sum_{\eta \in \sigma^{-p}(\omega)\setminus\{\rho\}} \hat{\mathcal{L}}^n g(\eta)u_p(\eta) \\
&\leq \left(a_n - \frac{a}{2}\right) u_p(\rho) + a_n \sum_{\eta \in \sigma^{-p}(\omega)\setminus\{\rho\}} \\
&= a_n - \frac{a}{2}u_p(\rho) \leq a_n - \frac{a}{2}\inf(u_p)
\end{aligned}$$

Similarly we get $\hat{\mathcal{L}}^p(\hat{\mathcal{L}}^n g)(\omega) \geq -a_n + \frac{a}{2}\inf(u_p)$ and in consequence, $||\hat{\mathcal{L}}^{p+n} g||_0 \leq a_n - \frac{a}{2}\inf(u_p)$ or

$$||\hat{\mathcal{L}}^n g||_0 \leq a_{n-p} - \frac{a}{2}\inf(u_p)$$

for every $n \geq p$. Taking the supremum over all $g \in \mathcal{H}_\beta^{0,1}$, we thus get $a_n \leq a_{n-p} - \frac{a}{2}\inf(u_p)$. So, $a = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} a_{n-p} - \frac{a}{2}\inf(u_p) = a - \frac{a}{2}\inf(u_p) < a$. This contradiction shows that $\lim_{n\to\infty} a_n = 0$.

Fix now $\varepsilon > 0$ and then an integer $v \geq 1$ so large that $a_v \leq \frac{\varepsilon}{2C}$ and $Q\mathrm{e}^{-\beta n}M \leq \varepsilon/2$ for all $n \geq v$. Then, in view of Lemma 5.1 for every $n \geq 2v$ and every $g \in \mathcal{H}_\beta^{0,1}$ we get

$$||\hat{\mathcal{L}}^n g||_\beta \leq ||\hat{\mathcal{L}}^{n-v}(\hat{\mathcal{L}}^v g)||_\beta \leq Q\mathrm{e}^{-\beta(n-v)}||\hat{\mathcal{L}}^v g||_\beta + C||\hat{\mathcal{L}}^v g||_0 \leq \frac{\varepsilon}{2}M + Ca_v \leq \varepsilon.$$

So, $b_n \leq \varepsilon$ and the proof is complete. ■

Theorem 5.6. Suppose that a Hölder continuous function $f : E^\infty \to I\!R$, say with an exponent $\beta > 0$, satisfies (2.2). Suppose also that the normalized conjugate Perron-Frobenius operator $\mathcal{L}_0^*$ has a Borel probability fixed point $\tilde{m}$. Assume that the incidence matrix A is finitely primitive. Then there exist constants $\overline{M} > 0$ and $0 < \gamma < 1$ such that for every $g \in \mathcal{H}_\beta$ and every $n \geq 0$

$$\text{(a)} \qquad ||\hat{\mathcal{L}}^n(g) - \int g\, d\tilde{\mu}_f||_\beta \leq \overline{M}\gamma^n ||g||_\beta$$

and

$$\text{(b)} \qquad ||\mathcal{L}_0^n(g) - \int g\, d\tilde{m}_f\psi||_\beta \leq \overline{M}\gamma^n ||g||_\beta$$

where $\hat{\mathcal{L}}$ is the operator defined by (5.6) and $\tilde{\mu}_f$ is the unique invariant Gibbs state of the potential f whose existence and uniqueness follow from Theorem 3.2 and Theorem 2.5.

Proof. Lemma 5.5 says that $\lim_{n\to\infty} ||\hat{\mathcal{L}}|_{\mathcal{H}_\beta^0}^n||_\beta = 0$. There thus exists $q \geq 1$ such that $\hat{\mathcal{L}}|_{\mathcal{H}_\beta^0}^q||_\beta \leq (1/2)$. So, by an immediate induction, $||\hat{\mathcal{L}}|_{\mathcal{H}_\beta^0}^{qn}||_\beta \leq (1/2)^n$. Consider now an arbitrary $n \geq 0$ and write $n = pq+r$, $0 \leq r \leq q-1$. Then, using in addition Theorem 5.4(c), we get for every $\zeta \in \mathcal{H}_\beta^0$

$$\begin{aligned} ||\hat{\mathcal{L}}^n\zeta||_\beta &= ||\hat{\mathcal{L}}^{pq}(\hat{\mathcal{L}}^r\zeta)||_\beta \leq (1/2)^p||\hat{\mathcal{L}}^r\zeta||_\beta \leq M(1/2)^p = M(1/2)^{\frac{n-r}{q}} \\ &\leq M(1/2)^{\frac{n-q+1}{q}} = M(1/2)^{\frac{1-q}{q}}(1/2)^{\frac{n}{q}} \end{aligned}$$

and therefore for every $n \geq 0$, $||\hat{\mathcal{L}}|_{\mathcal{H}_\beta^0}^q||_\beta \leq M(1/2)^{\frac{1-q}{q}}\gamma^n$, where $\gamma = (1/2)^{1/q} < 1$. If now $g \in \mathcal{H}_\beta$, then $g - \tilde{\mu}_f(g) \in \mathcal{H}_\beta^0$ and $||g - \tilde{\mu}_f(g)||_\beta \leq ||g||_\beta + ||\tilde{\mu}_f(g)||_\beta \leq 2||g||_\beta$. Thus, for every $n \geq 0$

$$||\hat{\mathcal{L}}^n(g - \tilde{\mu}_f(g))||_\beta \leq 2M\gamma^n||g||_\beta$$

and the proof of Theorem 5.6(a) is complete. Part (b) is an immediate consequence of part (a). ■

The next proposition, the last result of this section, explains the real dynamical meaning of the fixed points of the normalized Perron-Frobenius operator $\mathcal{L}_0$.

Proposition 5.7. Assume that the operator conjugate to the normalized Perron-Frobenius operator $\mathcal{L}_0 = \mathrm{e}^{-\mathrm{P}(f)}\mathcal{L}_f$ has a Borel probability fixed point $\tilde{m}$. Let

$$\mathrm{Fix}(\mathcal{L}_0) = \{g \in L_1(\tilde{m}) : \mathcal{L}_0(g) = g, \int g d\tilde{m} = 1, \text{ and } g \geq 0\}$$

and

$$\mathrm{AI}(\tilde{m}) = \{g \in L_1(\tilde{m}) : g\tilde{m} \circ \sigma^{-1} = g\tilde{m}, \int g d\tilde{m} = 1, \text{ and } g \geq 0\}$$

Then $\mathrm{Fix}(\mathcal{L}_0) = \mathrm{AI}(\tilde{m})$.

Proof. It follows from (4.1) that for every $i \in I$ and every $\omega \in E^\infty$ with $A_{i\omega_1} = 1$, we have

$$\frac{d\tilde{m} \circ i}{d\tilde{m}}(\omega) = \exp\bigl(f(i\omega) - \mathrm{P}(f)\bigr),$$

where we treat $i : \{\omega \in E^\infty : A_{i\omega_1} = 1\} \to E^\infty$ as the map defined by the formula $i(\omega) = i\omega$. Therefore, the Perron-Frobenius operator $\mathcal{L}_0$ sends the density of a measure $\tilde{\mu}$ absolutely continuous with respect to $\tilde{m}$ to the density of the measure $\tilde{\mu} \circ \sigma^{-1}$. Hence, the proposition follows. ■

§**5. Stochastic laws.** In this section we closely follow §3 of [DU1]. Let Γ be a finite or countable measurable partition of a probability space $(Y, \mathcal{F}, \nu)$ and let $S : Y \to Y$ be a measure preserving transformation. For $0 \leq a \leq b \leq \infty$, set $\Gamma_a^b = \bigvee_{a \leq l \leq b} S^{-l}\Gamma$. The measure ν is said to be absolutely regular with respect to the filtration defined by Γ, if there exists a sequence $\beta(n) \searrow 0$ such that

$$\int_Y \sup_a \sup_{A \in \Gamma_{a+n}^\infty} |\nu(A|\Gamma_0^a) - \nu(A)| d\nu \leq \beta(n).$$

The numbers $\beta(n)$, $(n \geq 1)$, are called coefficients of absolute regularity. Let α be the partition of I^∞ into initial cylinders of length 1. Using Theorem 5.6, and proceeding exactly as in the proof of [Ry, §3 of Theorem 5] we derive the following (with the notation of previous sections).

Theorem 6.1. The measure $\tilde{\mu}_f$ is absolutely regular with respect to the filtration defined by the partition α. The coefficients of absolute regularity decrease to 0 at an exponential rate.

Theorem 6.1 says in particular that the dynamical system $(\sigma, \tilde{\mu}_f)$ is weak-Bernoulli (see [Or]). As an immediate consequence of this theorem and the results proved in [Or] we get the following.

Theorem 6.2. The natural extension of the dynamical system $(\sigma, \tilde{\mu}_f)$ is isomorphic with some Bernoulli shift.

It follows from this theorem that the theory of absolutely regular processes applies ([IL], [PS]). We sketch this application briefly. We say that a measurable function $g : I^\infty \to I\!R$ belongs to the space $L^*(\sigma)$ if there exist constants $\alpha, \gamma, M > 0$ such that $\int \|g\|_0^{2+\alpha} \, d\tilde{\mu}_f < \infty$ and

$$\int \|g - E_{\tilde{\mu}_f}\big(g|(\alpha)^n\big)\|_0^{2+\alpha} \, d\tilde{\mu}_f \le M n^{-2-\gamma}$$

for all $n \ge 1$, where $E_{\tilde{\mu}_f}\big(g|(\alpha)^{n-1}\big)$ denotes the conditional expectation of g with respect to the partition $(\alpha)^{n-1}$ and the measure $\tilde{\mu}_f$. $L^*(\sigma)$ is a linear space. It follows from Theorem 5.1, [IL] and [PS] that with $\tilde{\mu}_f(g) = \int g \, d\tilde{\mu}_f$ the series

$$\sigma^2 = \sigma^2(g) = \int_{I^\infty} (g - \tilde{\mu}_f(g))^2 \, d\tilde{\mu}_f + 2\sum_{n=1}^{\infty} \int_{I^\infty} (g - \tilde{\mu}_f(g))(g \circ \sigma^n - \tilde{\mu}_f(g)) \, d\tilde{\mu}_f$$

is absolutely convergent and non–negative. The reader should not be confused by two different meanings of the symbol σ: the number defined above and the shift map. Then the process $(g \circ \sigma^n : n \ge 1)$ exhibits an exponential decay of corellations and if $\sigma^2 > 0$, it satisfies the central limit theorem. More precisely, we have the following.

Theorem 6.3. If $u, v \in L^*(\sigma)$ then there are constants $C, \theta > 0$ such that for every $n \ge 1$ we have

$$\int (g - Eu)\big((g - Ev) \circ \sigma^n\big) \, d\mu_f \le C\mathrm{e}^{-\theta n},$$

where $Eu = \int u \, d\tilde{\mu}_f$ and $Ev = \int v \, d\tilde{\mu}_f$.

Theorem 6.4. If $g \in L^*(\sigma)$ and $\sigma^2(g) > 0$, then for all r

$$\tilde{\mu}_f\left(\left\{\omega \in E^\infty : \frac{\sum_{j=0}^{n-1} g \circ \sigma^j - nEg}{\sqrt{n}} < r\right\}\right) \to \frac{1}{\sigma(g)\sqrt{2\pi}} \int_{-\infty}^{r} e^{-t^2/2\sigma(g)^2} \, dt.$$

The most fruitful in geometric applications is a.s. invariance principle and therefore we would like to devote it more time. This principle means that one can redefine the process $(g \circ \sigma^n : n \ge 1)$ on some probability space on which there is defined a standard Brownian motion $(B(t) : t \ge 0)$ such that for some $\lambda > 0$

$$\sum_{0 \le j \le t} \big[g \circ \sigma^j - \tilde{\mu}_f(g)\big] - B(\sigma^2 t) = O(t^{\frac{1}{2} - \lambda}) \quad \tilde{\mu}_f \text{ a.e.}$$

Let $h : [1, \infty) \longrightarrow I\!R$ be a positive non–decreasing function. The function h is said to belong to the lower class if

$$\int_1^\infty \frac{h(t)}{t} \exp\big(-\frac{1}{2} h(t)^2\big) dt < \infty$$

and to the upper class if

$$\int_1^\infty \frac{h(t)}{t} \exp\big(-\frac{1}{2} h(t)^2\big) dt = \infty.$$

Well-known results for Brownian motion imply (see Theorem A in [PS]) the following.

Theorem 6.5. If $g \in L^*(\sigma)$ and $\sigma^2(g) > 0$ then

$$\tilde{\mu}_f\bigg(\{\omega \in I^\infty : \sum_{j=0}^{n-1} \big(g(\sigma^j(\omega)) - \tilde{\mu}_f(g)\big) > \sigma(g) h(n) \sqrt{n} \text{ for infinitely many } n \geq 1\} \bigg)$$
$$= \begin{cases} 0 & \text{if } h \text{ belongs to the lower class,} \\ 1 & \text{if } h \text{ belongs to the upper class.} \end{cases}$$

Our last goal in this section is to provide a sufficient condition for a function ψ to belong to the space $L^*(\sigma)$.

Lemma 6.6. Each Hölder continuous function which has some finite moment greater than 2 belongs to $L^*(\sigma)$.

Proof. It suffices to show that any Hölder continuous function $\psi : \Sigma \to I\!R$ satisfies the requirement $\int \|\psi - E_{\tilde{\mu}_f}\big(\psi|(\alpha)^n\big)\|_0^3 \, d\tilde{\mu}_f \leq M n^{-2-\gamma}$ which will finish the proof. So, given $n \geq 1$ suppose that $\omega, \tau \in A$ for some $A \in \alpha_0^{n-1}$. In particular $\omega|_n = \tau|_n$. Hence $|\psi(\omega) - \psi(\tau)| \leq V_\beta(\psi) \mathrm{e}^{-\beta n}$ which means that $\psi(\tau) - V_\beta(\psi)\mathrm{e}^{-\beta n} \leq \psi(\omega) \leq \psi(\tau) + V_\beta(\psi)\mathrm{e}^{-\beta n}$. Integrating these inequalities against the measure $\tilde{\mu}_f$ and keeping ω fixed, we obtain

$$\int_A \psi d\tilde{\mu}_f - V_\beta(\psi)\mathrm{e}^{-\beta n} \tilde{\mu}_f(A) \leq \psi(\omega)\tilde{\mu}_f(A) \leq \int_A \psi d\tilde{\mu}_f + V_\beta(\psi)\mathrm{e}^{-\beta n}\tilde{\mu}_f(A).$$

Dividing these inequalities by $\tilde{\mu}_f(A)$ we deduce that

$$\left| \psi(\omega) - \frac{1}{\tilde{\mu}_f(A)} \int_A \psi d\tilde{\mu}_f \right| \leq V_\beta(\psi)\mathrm{e}^{-\beta n}.$$

Thus $\int \|\psi(\omega) - E_{\tilde{\mu}_f}\big(\psi|(\alpha)^n\big)\|_0^3 \, d\tilde{\mu}_f \leq V_\beta(\psi)^3 \mathrm{e}^{-3\beta n}$ and we are done. ■

§7. **A comparison with Sarig's approach.** In [Sa] O. Sarig has proposed a different definition of pressure for the class of locally Hölder continuous functions closely generalizing

Gurevich's definition (se [Gu]) of topological entropy of subshifts with an infinite alphabet. Sarig's approach is the following. Fix $i \in I$ and define

$$Z_n(f,i) = \sum_{\{\omega \in E^\infty : \sigma^n(\omega)=\omega, \omega_1=i\}} \exp(S_n f(\omega)).$$

It can be proven that that the limit $\lim_{n\to\infty} \frac{1}{n} \log Z_n(f,i)$ exists and is independent of i. This number is just the topological pressure of f introduced by Sarig in [Sa]. We will denote it by $\mathrm{P}_o(f)$. As we already mentiond working always with locally Hölder continuous functions Sarig has proved in [Sa] Theorems 1.3 and 1.5 with $\mathrm{P}(f)$ replaced by $\mathrm{P}_o(f)$. He has also proved Theorem 1.2 and Theorem 1.4 without any assumptions on the incidence matrix but with the pressure $\mathrm{P}(f)$ replaced by $\mathrm{P}_o(f)$. Consequently, we always have $\mathrm{P}_o(f) \le \mathrm{P}(f)$ and $\mathrm{P}_0(f) = \mathrm{P}(f)$ if the icidence matrix is finitely irreducible (in general this equality fails). We would like to add that O. Sarig has provided a simple short argument for equality $\mathrm{P}_o(f) = \mathrm{P}(f)$ in the case when $E^\infty = I^\infty$, i.e. when E^∞ is the full shift. This argument can be found for example in [HU]. We would like to add the remark that although Sarig's pressure $\mathrm{P}_o(f)$ behaves better as a theoretical notion (the varaitional pronciple is satisfied if both pressures differ), the more traditional definition of pressure like $\mathrm{P}(f)$ fits better to our future geometrical applications. Our last comment is that the existence of Gibbs states constructed in Section 6 does not follow from the sufficient conditions provided in [Sa].

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