Estimates for the stable dimension for holomorphic maps

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Abstract

We study the Hausdorff dimension of the intersection between stable manifolds and basic sets for an Axiom A holomorphic endomorphism on $\mathbb{P}^2$. For a map which is at least d'-to-1 on a basic set Λ, we are improving the upper estimate given in [11] by taking into consideration the number of preimages, and thus proving for non-invertible maps results parallel to those of Manning [10] and Verjovsky-Wu [16] from the case of diffeomorphisms.

Also a lower estimate for $HD(W^s_\varepsilon(x) \cap \Lambda)$ is given by using a concept of preimage entropy modeled after Bowen and similar to another one introduced by Hurley [6]. This preimage entropy plays the role of the entropy of f^{-1} from the case of homeomorphisms, but in general, if the map f is an endomorphism like in our situation, the preimage entropy does not coincide with $h(f|_\Lambda)$.

We also show that, given c with $|c|$ small, the perturbation $(z^2 + c + \varepsilon w, w^2)$ of $(z^2 + c, w^2)$ is injective on its basic set Λ_ε close to $\Lambda := p_0(c) \times S^1$ (where $p_0(c)$ is the fixed attracting point for $z^2 + c$) and also its stable dimension is strictly positive.

An interesting consequence of our results is the fact that $HD(W^s_\varepsilon(x) \cap \Lambda_g)$ does not depend continuously on g, (for $x \in \Lambda_g$), which is opposite to the situation of diffeomorphisms.

We study also the stable dimension for a large class of quadratic endomorphisms and show that it is either zero, or can be estimated easily using the derivative on the respective basic set.

Keywords: Hausdorff dimension, stable manifolds, preimage entropy, holomorphic endomorphisms, topological pressure.

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1 Introduction

We are considering Axiom A holomorphic functions $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ on the 2-dimensional complex projective space. For definitions of hyperbolicity and Axiom A we refer to [5], [7], [15] .

Let Λ be a basic set of the saddle part S_1 of the nonwandering set Ω.

If $x \in \Lambda$ and one denotes by $\phi^s(x) := \log |Df|_{E^s_x}|$, where E^s_x is the stable tangent subspace at x, then we have the result of [11]:

Proposition 1. *Let f a holomorphic Axiom A endomorphism of $\mathbb{P}^2$ such that $C_f \cap S_1 = \emptyset$, where C_f = critical set of f. Then $HD(W^s_\varepsilon(x) \cap \Lambda) \leq t^s$, where t^s is the only zero of the function $t \to P(t \cdot \phi^s)$. The estimate does not depend on $x \in \Lambda$.* □.

In the case of diffeomorphisms we have $HD(W^s_\varepsilon(x) \cap \Lambda) = t^s$ as was proved by Manning [10].

In this article, we denote the nonwandering set of f by Ω and we consider the partition of Ω as $S_0 \cup S_1 \cup S_2$, where the dimension of the unstable spaces over S_i is equal to i, $i \in \{0, 1, 2\}$. The function $P(\cdot)$ referred to in Proposition 1 is the topological pressure [17]. We will now define topological pressure and give some of its properties. Our purpose in doing this is two-fold: first as a convenience for the reader, second because later we shall define a related concept, that of preimage entropy. Parallels between these two notions shall prove very useful.

We will give now a brief description of selected topics from thermodynamic formalism ([17] for a more complete exposition).

The general setting is that of (X, d), a compact metric space, and $f\colon X \to X$ a continuous map. For n, a positive integer, $d_n(x, y) := \max\{d(f^i x, f^i y),\ i = 0, ..., n-1\}$ is a metric on X inducing the same topology as the metric d.

Definition 1. A subset $E \subset X$ is called (n, ε)-separated (for some $\varepsilon > 0$) if for all $x, y \in E$, $x \neq y$, we have $d_n(x, y) \geq \varepsilon$.

Definition 2. The *topological pressure* of f is the functional $P_f\colon \mathcal{C}(X, \mathbb{R}) \to \overline{\mathbb{R}}$ defined on the space of continuous functions by:

$$P_f(\varphi) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \exp \left(\sum_{i=0}^{n-1} \varphi(f^i x) \right), E \subset X, (n, \varepsilon)\text{-separated set.} \right\}.$$

Definition 3. When considering $\varphi \equiv 0$ in Definition 2, we obtain the notion of *topological entropy* of f.

There exists an interesting relationship between Borel invariant measures and P_f, contained in the following:

Theorem (Variational Principle). *In the above setting, $P_f(\varphi) = \sup_\mu \{h_\mu(f) + \int \varphi \, d\mu\}$, where the supremum is taken over all f-invariant Borel probability measures μ, and $h_\mu(f)$ = measure-theoretic entropy of μ.*

For the (long) definition of $h_\mu(f)$ and proofs of all these facts, as we mentioned, a good reference is [17].

Theorem (Properties of Pressure). *If $f\colon X \to X$ is a continuous transformation, and $\varphi, \psi \in \mathcal{C}(X, \mathbb{R})$, then:*

1) $\varphi \le \psi \Rightarrow P_f(\varphi) \le P_f(\psi)$

2) $P_f(\cdot)$ is either finitely valued or constantly ∞

3) P_f is convex

4) for a strictly negative function φ, the mapping $t \to P_f(t\varphi)$ is strictly decreasing if $P(0) < \infty$.

5) P_f is a topological conjugacy invariant.

In our situation $X = \Lambda$ (Λ a basic set of S_1); then, because $\phi^s < 0$, it follows that $t^s =$ zero of $P(t\phi^s)$ is uniquely defined, and $t^s \ge 0$ since $P(0) = h(f|_\Lambda) \ge 0$, $P(t\phi^s) < 0$ for large enough t, and $t \to P(t\phi^s)$ is strictly decreasing.

Define now the inverse limit of X, $\widehat{X} := \{(x_n)_{n\le 0},\ f(x_{n-1}) = x_n,\ n \le 0,\ x_n \in X\}$, and call an element $\hat{x} = (x_n)_{n\le 0}$ of $\widehat{X}$ a prehistory of x_0. The metric on $\widehat{X}$ is the usual metric, see for example [12].

The map f extends to a homeomorphism $\hat{f}$ of $\widehat{X}$ given by $\hat{f}((x_n)_{n\le 0}) = (x_{n+1})_{n\le 0}$, where $x_1 = f(x_0)$. In this case, denoting by $p(\hat{x}) = x_0$ the canonical projection, $p\colon \widehat{X} \to X$, the following diagram commutes:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\hat{f}} & \widehat{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X \end{array}$$

It is well-known that, in this case, $h_f = h_{\hat{f}}$.

We now come back to $HD(W^s_\varepsilon(x) \cap \Lambda)$. In [11] the first author studied the maps $f(z, w) = (z^2 + c, w^2 + d)$, $c \ne 0$, $d \ne 0$ and showed that if c is chosen such that $|1 - \sqrt{1-4c}| = \frac{4}{5}$, then $P(2\phi^s) > 0$ implies $t^s > 2$.

So, the estimate $HD(W^s_\varepsilon(x) \cap \Lambda) \le t^s$ is not sharp because $W^s_\varepsilon(x) \cap \Lambda$ consists of only one point. A different estimate for a certain class of maps will be proved in §2 to take this fact into consideration.

Theorem 1. *Assume f is Axiom A, holomorphic endomorphism on $\mathbb{P}^2$ of degree d, and Λ is one of the basic sets with unstable index 1. Suppose that $C_f \cap \Lambda = \emptyset$ and that $f|_\Lambda\colon \Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least d' preimages in Λ, $d' \le d$. Then $HD(W^s_\varepsilon(x) \cap \Lambda) \le \frac{\log d' - h_{top}(f|_\Lambda)}{\log \alpha + \log \gamma}$, as long as $\alpha\gamma \le 1$ where $\alpha := \frac{\sup_{x\in\Lambda} |Df|_{E^s_x}|}{\inf_{y\in\Lambda} |Df|_{E^s_y}|}$, and $\gamma := \sup_{x\in\Lambda} |Df|_{E^s x}|$. The estimate is independent on $x \in \Lambda$.* $\square$

This result complements the theorem from [12] stating that the interior of the union of all unstable manifolds corresponding to points in S_1, is empty.

In general, if $f|_\Lambda\colon\ \Lambda \to \Lambda$ is d'-to-1, then $h(f|_\Lambda) \geq \log d'$ (this is proved similarly to the Misiurewicz-Przytycki theorem, [7]). And, if $\alpha\gamma \leq 1$, $\log\alpha + \log\gamma \leq 0$, so $\frac{\log d' - h(f|_\Lambda)}{\log\alpha + \log\gamma} \geq 0$. For the example given above, $f(z,w) = (z^2+c, w^2+d)$, $h(f|_\Lambda) = \log 2$, $d' = 2$, hence the above estimate is sharp.

In the case of diffeomorphisms, $HD(W^s_\varepsilon(x) \cap \Lambda) = t^s$ ([16]). However in our case, that proof will not work mainly because the estimates using $h_\mu(f)$ break down due to the non-invertibility of f. The main problem is that the entropy was defined by considering the forward iterates of f, and, if f is not injective, $h(f)$ does not shed any light upon the growth of preimages. To compensate this, we will use in Theorem 2 a notion of *preimage (branch) entropy.*

Let us call a *branch* of length ℓ (or *prehistory* of length ℓ) in X, a sequence of preimages, $\beta = (z_0, z_{-1}, \dots, z_{-\ell})$, with $z_i \in X$, $-\ell \leq i \leq 0$, such that $f(z_{i-1}) = z_i$, $-\ell+1 \leq i \leq 0$. Given another branch $\beta' = (z'_0, \dots, z'_{-\ell})$ of same length, define their *branch distance* to be $d^b(\beta,\beta') = \max\limits_{j=\overline{0,\ell}} d(z_{-j}, z'_{-j})$. The reader can notice the similarity between the branch distance and $d_n(\cdot,\cdot)$ introduced earlier. Like $d_n(\cdot,\cdot)$ for forward iterates, d^b measures the growth of inverse iterates. Using this, we now define a *branch metric* on X:

$$d^b_\ell(x,x') < \varepsilon,$$

if for every branch β of length ℓ with $z_0 = x$, there exists a branch β' of length ℓ with $z'_0 = x'$ such that $d^b(\beta,\beta') < \varepsilon$, and vice versa. Denote by $N_{\text{span}}(\varepsilon, d^b_\ell, X)$ the smallest cardinality of an ε-spanning set for X in the d^b_ℓ metric. Hence, if A is an ε-spanning set with $\#A = N_{\text{span}}(\varepsilon, d^b_\ell, X)$, then, $\forall x \in X$, $\exists y \in A$ with $d^b_\ell(x,y) < \varepsilon$.

Let also $N_{\text{sep}}(\varepsilon, d^b_\ell, X)$ be the largest cardinality of an ε-separated set for X. So, if A is ε-separated, then for all $x,y \in A$, $x \neq y$, $d^b_\ell(x,y) > \varepsilon$.

The following proposition gives the definition of the preimage entropy $h_i(f)$ and two ways to calculate it.

Proposition ([13]). *For $f\colon\ X \to X$ continuous, (X,d) compact metric space, we have*

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log N_{sep}(\varepsilon, d^b_n, X) = \lim_{\varepsilon\to 0} \underline{\lim_{n\to\infty}} \frac{1}{n} \log N_{\text{span}}(\varepsilon, d^b_n, X)$$

and the common value is called the preimage (branch) entropy*, denoted by $h_i(f)$.* □

In general there is no relation between $h_i(f)$ and $h(f)$. Besides $h_i(f)$, there are several other notions of inverse entropy, a local one called $h_m(f)$ and a global one called preimage relation entropy, and denoted by $h_r(f)$, but we will not use them directly in our proofs.

Their definitions and properties can be found in the articles by Langevin and Walczak [9] and Hurley [6]. Hurley [6] proved the following.

Proposition. *For X compact metric space, $f\colon\ X \to X$ continuous,*

$$0 \leq h_m(f) \leq h(f) \leq h_m(f) + h_i(f) \leq h_m(f) + h_r(f).$$

If f is a homeomorphism, then $h_m(f) = 0$ and $h(f) = h_i(f) = h_r(f)$. $\square$

Let us recall now two cases when $h_i(f) = 0$.

a. Forward-expansive coverings

If X is a metric space, then a continuous map $f\colon X \to X$ is called *forward expansive* if $\exists \varepsilon_0 > 0$ such that whenever $x, y \in X$, $x \neq y$, $\exists m \geq 0$ with

$$d(f^m x, f^m y) \geq \varepsilon_0 > 0.$$

For example, f is forward-expansive on any invariant subset of a Riemannian manifold on which Df is expanding by a constant factor $\lambda > 1$. Denker and Urbanski [4] (see also Coven and Reddy [3]) showed that, if X is a compact metric space, then a forward-expansive map is in fact *expanding*, i.e. for some metric d on X (equivalent to the original one), there exists $b > 0$, $\lambda > 1$ constants such that $d(x, y) < b$ implies $d(f(x), f(y)) \geq \lambda d(x, y)$. We are also saying that $f\colon X \to X$ is a *covering map* if $(\forall) x \in X, \exists$ a neighborhood U_x of x, such that $f^{-1}(U_x) = \bigcup\limits_i V^i$ with $\{V^i\}$ open disjoint sets, and such that $f\colon V^i \to U_x$ is homeomorphism.

If f is forward-expansive on X compact metric space, then we can use Denker-Urbanski [4] characterization in connection to $h_i(f)$ because $h_i(f)$ is a topological concept which does not depend on the metric ([13]). This is the idea of the proof for the following:

Proposition ([13]). *If $f\colon X \to X$ is a* forward expansive covering *map, then $h_i(f) = 0$.* $\square$

b. A second situation when $h_i(f) = 0$ is that of graph maps.

A *finite graph* is a compact metric space K with a distinguished finite set of points called *vertices*, whose complement has finitely many connected components, *edges*, homeomorphic to the open interval (0,1). We fix the metric on K by assigning length 1 to each edge and the distance between two points in K is the length of the shortest path connecting them.

Theorem (Nitecki-Przytycki, [13]). *Let K a finite graph and $f\colon K \to K$ continuous map. Then $h_i(f) = 0$.* $\square$

Corollary. *For any continuous self-map f of a closed interval $[a, b]$, or of the circle S^1, we have $h_i(f) = 0$.* $\square$

We will start by defining another concept of preimage entropy, modeled after Bowen's covering type entropy ([1]) which does not even require X to be compact, or even a metric space. Suppose $\mathcal{A}$ is an open cover of X. For a subset $E \subset X$, define $n_{\mathcal{A}}^{-}(E)$ as the largest integer s.t. for all $0 \leq k < n_{\mathcal{A}}^{-}(E)$, and all $x, y \in E$, and every prehistory of x of length k, $\hat{x} = (x, x_{-1}, \dots, x_{-k})$, $\exists \hat{y} = (y, y_{-1}, \dots, y_{-k})$ prehistory of y of length k, and $\exists k$ sets from $\mathcal{A}, E^0, \dots, E^k$, s.t. $(x, y) \in E^0 \times E^0$, $(x_{-1}, y_{-1}) \in E^1 \times E^1, \dots, (x_{-k}, y_{-k}) \in E^k \times E^k$. If $\mathcal{B} = \{B_1, B_2, \dots\}$ has union covering X, set

$$\mathcal{D}_{\mathcal{A}}(\mathcal{B}, \lambda) := \sum_{i=1}^{\infty} \exp(-\lambda n_{\mathcal{A}}^{-}(B_i)).$$

Like in the definition of Hausdorff measures, let

$$m^-_{\mathcal{A},\lambda}(X) := \lim_{\varepsilon\to 0} \inf\{\mathcal{D}_{\mathcal{A}}(\mathcal{B},\lambda); \mathcal{B} = \{B_1, B_2, \dots\}, X \subset \bigcup_{i=1}^{\infty} B_i, \text{ and } \exp(-n^-_{\mathcal{A}}(B_i)) < \varepsilon, (\forall)i\}$$

Then, let $h^-_{\mathcal{A}}(f) := \inf\{\lambda, m^-_{\mathcal{A},\lambda}(X) = 0\}$, and $h^-(f) := \sup\limits_{\mathcal{A}} h^-_{\mathcal{A}}(f)$. It is easy to check that $h^-_{\mathcal{A}} \ge h^-_{\mathcal{A}'}$, when $\mathcal{A}, \mathcal{A}'$ are two open covers and $\mathcal{A}$ refines $\mathcal{A}'$, so $h^-(f)$ is the $\limsup h^-_{\mathcal{A}}(f)$ when $\mathrm{mesh}(A) \to 0$.

This definition is inspired by Bowen's definition of topological entropy for non-compact spaces in [1]; we mention also that the notion from [1] was employed succesfully to give estimates on Hausdorff dimension in [10].

Noticing that the same idea of taking prehistories of points and defining "closeness" with regard to them appears in both $h^-(f)$ and $h_i(f)$, one can use the methods of Bowen ([1], pg. 126) to show that:

Proposition 2. *If $f\colon X \to X$ continuous map on a compact metric space X, then $0 \le h^-(f) \le h_i(f)$. Also, if f is a homeomorphism on X, then $h^-(f) = h(f^{-1}) = h(f)$.* □

In general it is not clear whether the equality $h^-(f) = h_i(f)$ holds.

Remark:

Let us note that Proposition 2 and the discussion above about h_i, imply that $h^-(f) = 0$ for forward expansive maps and graph maps.

We are ready now to state Theorem 2 which gives a lower estimate for $HD(W^s_\varepsilon(x) \cap \Lambda)$.

Theorem 2. *If Λ is a basic set for an Axiom A holomorphic endomorphism $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ and $C_f \cap \Lambda = \emptyset$, then for any $x \in \Lambda$,*

$$HD(W^s_\varepsilon(x) \cap \Lambda) \ge \left| \log \inf_{y\in\Lambda} |Df|_{E^s_y}| \right|^{-1} \cdot h^-(f|_\Lambda) \ge 0 \qquad \square$$

Remark. For $f(z,w) = (z^2 + c, w^2 + d), c, d$ small and non-zero, and $\Lambda = \{z_0\} \times J_{(w^2+d)}$, with $J_{(w^2+d)}$ the Julia set of $w \to w^2 + d$, and z_0 an attracting fixed point for $z^2 + c$, we already know that $W^s_\varepsilon(x) \cap \Lambda$ consists of a single point for $x \in \Lambda$, and therefore $HD(W^s_\varepsilon(x) \cap \Lambda) = 0$. This is in accordance with the fact from the Corollary, that $h_i(f|_\Lambda) = 0$ if Λ is a circle (or homeomorphic with a circle), hence also $h^-(f|_\Lambda) = 0$. □

We also show that, given c with $|c|$ small, the perturbation $g(z,w) = (z^2 + c + \varepsilon w, w^2)$ of $(z^2 + c, w^2)$ is injective on its basic set Λ_g close to $\Lambda := p_0(c) \times S^1$ (where $p_0(c)$ is the fixed attracting point for $z^2 + c$).

Then using Theorem 2 and the observation that for any homeomorphism g we have $h^-(g) = h(g) = log 2$, we will get also that its stable dimension is greater than a positive number independent of g.

Since the perturbation can be taken arbitrarily close to the map $(z,w) \to (z^2 + c, w^2)$, it will follow in Section 4 that $HD(W^s_\varepsilon(x) \cap \Lambda_g$ does not depend continuously on the map g, when $x \in \Lambda_g$, which is a contradiction to the case of Henon maps, [16].

In Section 4 we study a large class of quadratic maps on $\mathbb{P}^2$ and prove that, if the basic set Λ is connected, then one can control the number of preimages and hence, by using Theorems 1 and 2, the stable dimension.

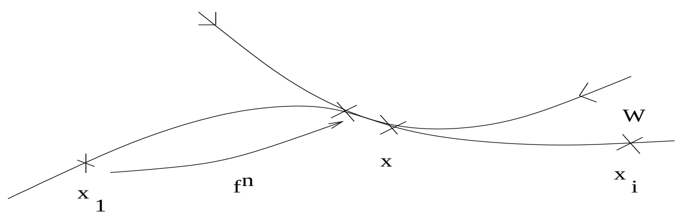
All this will be made precise in the sequel.

2 Upper Estimate for $HD(W^s_\varepsilon(x)\cap\Lambda)$

Theorem 1 *Assume f is an Axiom A, holomorphic map of degree $d \geq 2$ on $\mathbb{P}^2$, and Λ is one of the basic sets with unstable index equal to 1. Suppose $C_f \cap \Lambda = \emptyset$ (C_f= critical set of f) and also that $f|_\Lambda\colon\ \Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least $d' \leq d$ preimages in Λ. Then $HD(W^s_\varepsilon(x)\cap\Lambda) \leq \frac{\log d' - h_{top}(f|_\Lambda)}{\log\alpha+\log\gamma}$, where $\alpha := \frac{\sup\limits_{x\in\Lambda}|Df|_{E^s_x}|}{\inf\limits_{y\in\Lambda}|Df|_{E^s_y}|}$ and $\gamma := \sup\limits_{x\in\Lambda}|Df|_{E^s_x}|$, as long as $\alpha \leq \gamma^{-1}$.*

Observation: In the case of diffeomorphisms, $HD(W^s_\varepsilon(x)\cap\Lambda) = t^s$, where t^s is the unique zero of the stable function $t \to P(t\phi^s)$, $\phi^s(x) := \log|Df|_{E^s_x}|$. Also, from the Variational Principle, it follows that there exists a probability measure μ_s such that $t^s = \frac{h\mu_s}{\lambda_s}$, λ_s being the Lyapunov exponent of μ_s. In the case of endomorphisms, it was showed in [11] that $HD(W^s_\varepsilon(x)\cap\Lambda) \leq t^s$, but in general, the inequality is strict. For example for the case of a map $f(z,w) = (P(z),Q(w))$, with P,Q hyperbolic on their Julia sets, $HD(W^s_\varepsilon(z,w)\cap\Lambda) = 0$, for $(z,w) \in \Lambda = \{(z_0,w_0)\} \times J_Q$, where Q_0 is an attracting periodic point for P. So, the formula in the Theorem explains that the gap between $HD(W^s_\varepsilon(x)\cap\Lambda)$ and t^s is due to the number of preimages.

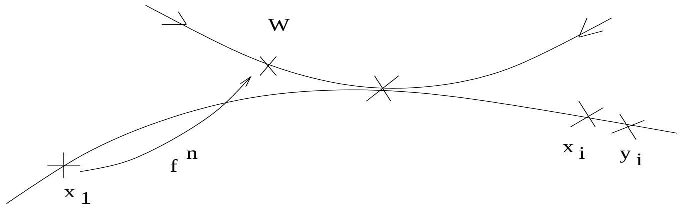
Proof. Denote $W := W^s_\varepsilon(x)\cap\Lambda$. Let E be an (n,δ)-separated set of maximal cardinality inside $f^{-n}(W)\cap\Lambda$ with $\delta << \varepsilon$ to be determined in the course of the proof.



We let now a point $x_1 \in E$, $f^n(x_1) \in W$, and $f^n(x_1)$ has at least d'^n different preimages belonging to Λ.

$$f^{-n}(f^n x_1)\cap\Lambda = \{x_1, x_2, \dots, x_{n(x_1)}\},$$

and denote by x_{i_1} the point of this set such that $\|Df^n|_{E^s_{x_{i_1}}}\| = \max\{\|Df^n|_{E^s_{x_i}}\|, i = 1,...,n(x_1)\}$. Then there exists a universal constant A such that $A\cdot\|Df^n|_{E^s_{x_{i_1}}}\| \geq \operatorname{diam} f^n(W^s_\delta(x_i))$, $i = 1,...,n(x_1)$, where this inequality follows from Lemma 1 of [11].



Since E is a maximal (n,δ)-separated set, there exist points $y_i \in E$ such that $y_1 = x_1$ and $x_i \in W^s_\delta(y_i)$, $i = 1, ..., n(x_1) \Leftrightarrow y_i \in W^s_\delta(x_i)$, $i = 1, ..., n(x_1)$ (in fact there could be more than one point from E inside $W^s_\delta(x_i)$, but we choose one).

We will show that $y_i \neq y_j$ for $i \neq j$. Suppose this is not true, then $y_i = y_j$ for some $i \neq j$, $\Rightarrow x_i, x_j \in W^s_\delta(y_i)$. But $f^n x_i = f^n x_j \Rightarrow \exists m$ such that $f^m(x_i) \neq f^m(x_j)$, $f^m(x_i), f^m(x_j) \in W^s_\delta(f^m(y_i))$, and $f^{m+1}(x_i) = f^{m+1}(x_j)$.

However $C_f \cap \Lambda = \emptyset$; hence $\exists \varepsilon_0 > 0$ such that $f\colon\ B(x', \varepsilon_0) \to \mathbb{P}^2$ injective, $(\forall) x' \in \Lambda$, and $\delta \ll \varepsilon_0$. We can choose $\delta << \varepsilon_0$, so one cannot have $f(f^m(x_i)) = f(f^m(x_j)), f^m(x_i) \neq f^m(x_j)$ and $d(f^m(x_i), f^m(x_j)) < 2\delta$. In conclusion $y_i \neq y_j$, $(\forall) i \neq j$.

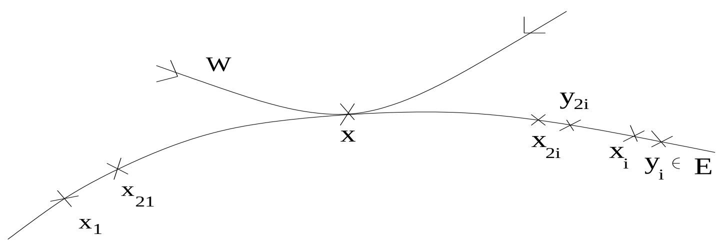
We assumed that $\|Df^n|_{E^s_{x_{i_1}}}\| = \max\{\|Df^n|_{E^s_{x_i}}\|, i = 1, ..., n(x_1)\}$ which implies that $f^n(W^s_\delta(x_i)) \subset f^n(W^s_{C\cdot\delta}(x_{i_1}))$, $(\forall) i = 1, ..., n(x_1)$ with C a universal constant which does not depend on x_1, x_i or n.

We have $\bigcup_{i=1}^{n}(x_1) f^n(W^s_\delta(y_i)) \subset f^n(W^s_{2C\delta}(x_1))$, hence we choose only the point $x_1 = y_1$ out of the set $\{y_1, .., y_{n(x_1)}\}$.

If for all points $z \in E$, we have $f^n(z) \in f^n(W^s_{2C\delta}(x_{i_1}))$, then $W \subset f^n(W^s_{(2C+1)\delta}(x_{i_1}))$ for n large enough and the procedure stops.

So, suppose there exists a point $x_{2,1} \in E$ such that $f^n(x_{2,1}) \in W \backslash f^n(W^s_{2\delta C}(x_{i_1}))$.

$f^n(x_{2,1})$ has $n(x_{2,1})$ preimages $x_{2,1}, x_{2,2}, ..., x_{2,n(x_{2,1})}$ in $f^{-n}(f^n(x_{2,1})) \cap \Lambda$, where $n(x_{2,1}) \geq d'^n$.



Now for each $x_{2,i}$, a preimage of $f^n(x_{2,1})$, $\exists y_{2,i} \in E$ such that $x_{2,i} \in W^s_\delta(y_{2,i})$, $(\forall) i$ (equivalently $y_{2,i} \in W^s_\delta(x_{2,i})$, $(\forall) i$). By the same argument as before, $y_{2,i} \neq y_{2,j}$, $i \neq j$.

Is it possible to have $y_i = y_{2,j}$ for some i, j?

If $y_i = y_{2,j} \Rightarrow x_i, x_{2,j} \in W^s_\delta(y_i)$.

$$x_{2,j} \in W^s_{2\delta}(x_i) \Rightarrow f^n(x_{2,j}) \in f^n(W^s_{2\delta}(x_i)).$$

But $f^n(x_{2,j}) = f^n(x_{2,1}) \notin f^n(W^s_{2C\delta}(x_1))$ and $f^n(W^s_{2\delta}(x_i)) \subset f^n(W^s_{2C\delta}(x_1))$. So $y_i \neq y_{2,j}, \forall i, j$.

Therefore $\{y_1, \dots, y_{n(x_1)}\} \cap \{y_{2,1}, \dots, y_{2,n(x_{2,1})}\} = \emptyset$. Now, assuming for instance that y_{2,i_2} satisfies

$$\|Df^n|_{E^s_{y_{2,i_2}}}\| = \max\{\|Df^n|_{E^s_{y_{2,i}}}\|, i = 1, ..., n(x_{2,1})\},$$

we can cover all $f^n(W^s_\delta(y_{2,i}))$ with $f^n(W^s_{2C\delta}(y_{2,i_2}))$. So, from the set $\{y_{2,1}, \dots, y_{2,n(x_{2,1})}\} \subset E$ we can keep only y_{2,i_2}, and throw away the other points. Also, from $\{y_1, \dots, y_{n(x_1)}\}$ we kept only y_{i_1}, as long as $\|Df^n|_{E^s_{y_{i_1}}}\| = \max\{\|Df^n|_{E^s_{y_i}}\|, i = 1, ..., n(x_1)\}$. This procedure is repeated until we exhaust the entire finite set E. But, since E is (n, δ)-spanning $\Rightarrow$

$$\Rightarrow W \subset \bigcup_{y \in E} f^n W^s_\delta(y) \subset \bigcup_{z_k \in \widetilde{E}} f^n(W^s_{(2C+1)\delta}(z_k)),$$

where $\widetilde{E}$ is the subset obtained by the above procedure: $y_{i_1} \in \widetilde{E}$, $y_{2,i_2} \in \widetilde{E}$, etc. (points with maximum $\|Df|_{E^s}\|$ among preimages).

Now $\sum\limits_{z_k \in \widetilde{E} \subset E} [\text{diam } f^n(W^s_{(2C+1)\delta}(z_k))]^t \le A \cdot \sum\limits_{z_k \in \widetilde{E}} \|Df^n|_{E^s_{z_k}}\|^t \le A \cdot \frac{\alpha^t}{d'^n} \cdot \sum\limits_{y \in E} \|Df^n|_{E^s_y}\|^t$.

From the Variational Principle, $P(t\phi^s) \le h_{\text{top}}(f|_\Lambda) + t \log \gamma$, $\gamma := \sup\limits_{x \in \Lambda} \|Df|_{E^s_x}\|$.

Assume that $t > \frac{\log d' - h_{\text{top}}(f|_\Lambda)}{\log \alpha + \log \gamma}$ and let $t > t' > \frac{\log d' - h_{\text{top}}(f|_\Lambda)}{\log \alpha + \log \gamma}$. Then $P(t\phi^s) < P(t' \cdot \phi^s) \le h_{\text{top}}(f|_\Lambda) + t' \cdot \log \gamma$, so $\sum\limits_{y \in E} \|Df^n|_{E^s_y}\|^t < e^{n(h_{\text{top}} f|_\Lambda) + t' \log \gamma)}$, for all n large enough .

This implies that $\sum\limits_{z_k \in \widetilde{E}} [\text{diam } f^n(W^s_{(2C+1)\delta}(z_k))]^t \le e^{n(h_{\text{top}}(f|_\Lambda) + t' \log \gamma + t \log \alpha - \log d')}$.

But $t > \frac{\log d' - h_{\text{top}}(f|_\Lambda)}{\log \alpha + \log \gamma}$ and we can choose t' as close to t as we want. Hence, using also here that $\alpha\gamma \le 1$, we can make $h_{\text{top}}(f|_\Lambda) + t' \log \gamma + t \log \alpha - \log d' < 0 \Rightarrow \sum\limits_{z_k \in \widetilde{E}} [\text{diam}(f^n(W^s_{(2C+1)\delta}(z_k))]^t \xrightarrow[n \to \infty]{} 0 \Rightarrow$ $HD(W) \le t$.

But t has been chosen arbitrarily larger than $\frac{\log d' - h_{top}(f|_\Lambda)}{\log \alpha + \log \gamma}$, therefore we obtain the desired inequality.

□

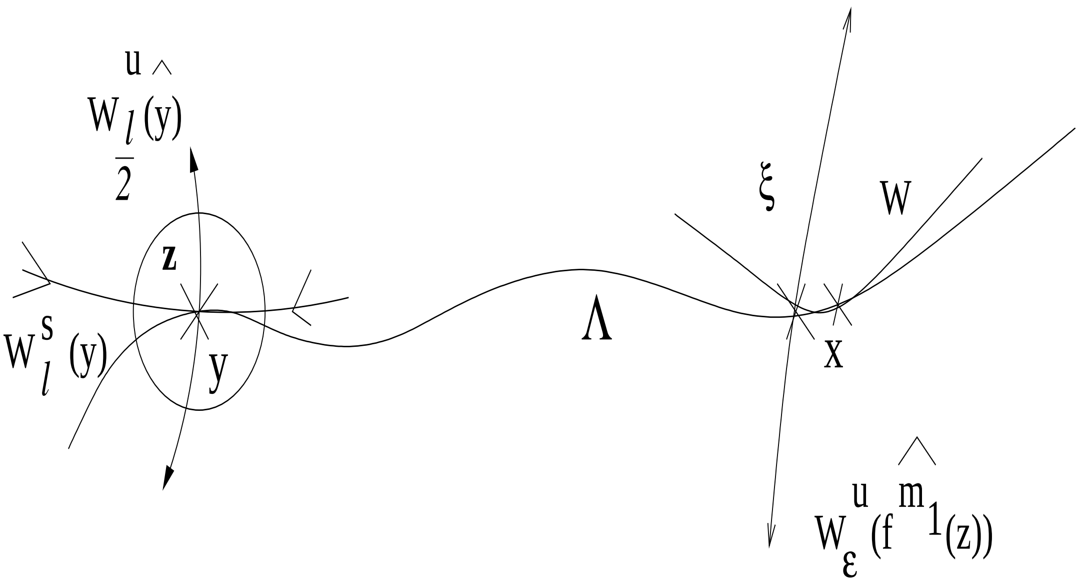
3 Lower Estimate for $HD(W^s_\varepsilon(x) \cap \Lambda)$

We are now ready to prove:

Theorem 2. *If Λ is a basic set of unstable index 1 for the Axiom A map $f \colon \mathbb{P}^2 \to \mathbb{P}^2$, and $C_f \cap \Lambda = \emptyset$, then $HD(W^s_\varepsilon(x) \cap \Lambda) \ge \left| \log \inf\limits_{y \in \Lambda} |Df|_{E^s_y}| \right|^{-1} \cdot h^-(f|_\Lambda)$, for all $x \in \Lambda$*

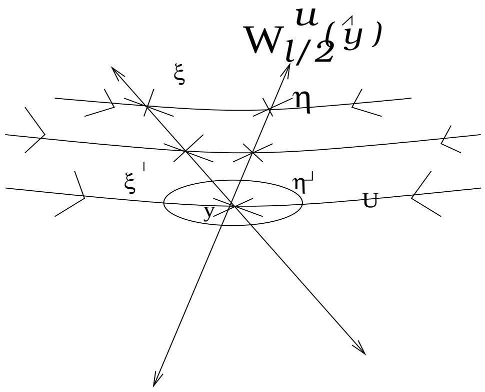
.

Proof. Let $W := W^s_\varepsilon(x) \cap \Lambda$ for a point x in Λ, and consider $y \in \Lambda$. Let also $\mathcal{A}$ be an open cover of Λ with $mesh(\mathcal{A}) =: \ell$. We will prove that there exists an integer m such that $f^{-m}(W) \cap \Lambda$ intersects any unstable local manifold $W^u_{\ell/2}(\hat{y})$, $(\forall)\hat{y} \in \hat{\Lambda}$. Denote by $B(y, \ell/4)$ the $\ell/4$-ball around y.



Since Λ is a basic set, hence $f|_\Lambda$ is transitive, it follows that $\exists\ m_1$ integer and $z \in B(y, \ell/4) \cap \Lambda$ such that $f^{m_1}(z) \in B\left(x, \frac{\varepsilon}{2}\right)$. Now take the unstable manifold $W^u_\varepsilon(\hat{f}^{m_1}(z))$, where $\widehat{f^{m_1}(z)}$ is any prehistory of $f^{m_1}(z)$ such that $(f^{m_1}(z))_{-m_1} = z$. From the local product structure, $W^u_\varepsilon(\widehat{f^{m_1}(z)}) \cap W = \xi$. The point ξ has a prehistory $\hat{\xi}$ with $d(\xi_{-m_1}, z) < \ell/4$, $\xi_{-m_1} \in \Lambda$, if m_1 is big enough. So, $W^s_{M\cdot\ell}(\xi_{-m_1})$ intersects $W^u_{\ell/2}(\hat{y})$ transversally, for any prehistory $\hat{y}$ of y, where M is some constant which does not depend on y, giving the maximum inclination of the unstable spaces w.r.t. stable ones.

Also, $f^{m_1}(\xi_{-m_1}) \in W$ and $f^{m_1}(W^s_{M\cdot\ell}(\xi_{-m_1}) \cap \Lambda) \subset W$, so $W^s_{M\cdot\ell}(\xi_{-m_1}) \cap \Lambda \subset f^{-m_1}(W) \cap \Lambda$. We cover now the set $f^{-m}(W) \cap \Lambda$ with open sets, $\mathcal{U} = \{U\}$, such that for some $\varepsilon' > 0$, $\sum_{U \in \mathcal{U}} (\text{diam } U)^{\delta+\varepsilon'} < \frac{1}{2}$, with $\delta := HD(W^s_\varepsilon(x) \cap \Lambda)$. For each $U \in \mathcal{U}$, let $U^* := \left(\bigcup_{y \in U} W^u_{\ell/2}(\hat{y})\right) \cap \Lambda$, obtaining in this way a cover of Λ.



Let us take a point $\xi \in U^*$; then $W^s_{\ell'}(\xi) \cap W^u_{\ell/2}(\hat{y}) = \{\eta\}, \ell' := M \cdot \ell$.

Assume that in the prehistory $\hat{y}$, we have $\hat{y}_{-k} = y^i_k$, where $\{y^i_k\}$ is a way to index the preimages of y under f^k.

Now η has a preimage η^i_{-k} by f^k such that y^i_{-k} and η^i_{-k} are very close to each other, where $f^k(y^i_{-k}) = y$. From the definition of $n^-_{\mathcal{A}}(U^*), \xi$ has a prehistory $(\xi^i_{-1}, \dots, \xi^i_{-k})$ such that ξ^i_{-k} and η^i_{-k} are in the same set A^i_{-k} from $\mathcal{A}$, for all $0 < k < n_{\mathcal{A}}(U^*)$. Since mesh $(\mathcal{A}) \le \ell$ for example, we get

$$\begin{aligned}\xi^i_{-k} \in W^s_\ell(\eta^i_{-k}) \Rightarrow d(\xi,\eta) &= d(f^k\xi^i_{-k}, f^k\eta^i_{-k}) \\ &\le d(\xi^i_{-k}, \eta^i_{-k}) \cdot \|Df^k|_{E^s_\omega}\| \text{ for some } \omega \in W^s_\ell(\eta^i_{-k}).\end{aligned}$$

Similarly, if ξ, ξ' are arbitrary points in U^*, and if they have prehistories such that ξ^i_{-k}, ξ'^i_{-k} are in the same set A^i_{-k} of $\mathcal{A}$, then

$$d(\xi^i_{-k}, \xi'^i_{-k}) \le d(\xi^i_{-k}, \eta^i_{-k}) + d(\eta^i_{-k}, \eta'^i_{-k}) + d(\xi'^i_{-k}, \eta'^i_{-k}).$$

Since $\eta^i_{-k}, \eta'^i_{-k}$ follow the unstable manifold $W^u_{\ell/2}(\hat{y})$, one gets $d(\eta^i_{-k}, \eta'^i_{-k}) < \frac{\ell}{2}$ and for $d(\xi^i_{-k}, \eta^i_{-k})$ and $d(\xi'^i_{-k}, \eta'^i_{-k})$ we use the estimate (3).

Hence,

$$d(\xi^i_{-k}, \xi'^i_{-k}) \le \frac{\ell}{2} + d(\xi,\eta) \cdot \|Df^k|_{E^s_\omega}\|^{-1} + d(\xi',\eta') \cdot \|Df^k|_{E^s_{\omega'}}\|^{-1},$$

and diam $W^s_\ell(\eta) \cap U^* \le M \cdot$ diam U, $k < \eta_{\mathcal{A}}(U^*)$. Again by using the maximality of $n = n^-_{\mathcal{A}}(U^*)$, we will obtain for some points ξ, ξ' and some of their prehistories that

$$\ell < d(\xi^i_{-n}, \xi'^i_{-n}) \le \frac{\ell}{2} + 2M \cdot \text{diam } U \cdot \sup_\omega \|Df^n|_{E^s_\omega}\|^{-1}.$$

So, $\frac{\ell}{2} \le 2M \cdot \text{diam } U \cdot \left(\inf_{y \in W^s_\ell(\Lambda)} \|Df|_{E^s_y}\| \right)^{-n^-_{\mathcal{A}}(U^*)}$. Denote $\inf_{y \in W^s_\ell(\Lambda)} \|Df\|_{E^s_y}\|$ by λ_s; since $C_f \cap \Lambda = \emptyset$ it follows $\lambda_s > 0$ and then,

$$\sum_{U^*} \left(\lambda_s^{n^-_{\mathcal{A}}(U^*)} \cdot \frac{\ell}{4M} \right)^{\delta+\varepsilon'} \le \sum_{U \in \mathcal{U}} (\text{diam } U)^{\delta+\varepsilon'} < \frac{1}{2}.$$

Therefore, $h^-_{\mathcal{A}} \le |\log \lambda_s| \cdot (\delta + \varepsilon')$.

And now, since $h^- = \overline{\lim}_{\text{mesh}(\mathcal{A}) \to 0} h^-_{\mathcal{A}}$, and ε' was chosen arbitrarily,

$$\begin{aligned}h^-(f|_\Lambda) &\le |\log \lambda_s| \cdot \delta \\ \Rightarrow HD(W^s_\varepsilon(x) \cap \Lambda) &\ge h^-(f|_\Lambda) \cdot |\log \lambda_s|^{-1}\end{aligned}$$

□

We conclude this section with the remark that $h^-(f|_\Lambda) \ne h^-(\hat{f}|_{\hat{\Lambda}})$ and that the value of h^- is not stable under perturbations.

Remark: Indeed, let us take Λ, a basic set for an Axiom A endomorphism f without cycles among the basic sets of its saddle part S_1. Then, if f_ε is sufficiently close to f, f_ε will have a basic set Λ_ε close to Λ in the Hausdorff metric. And we know from the Stability Theorem that there exists a homeomorphism $\hat{g}\colon\ \hat{\Lambda}\to\hat{\Lambda}_\varepsilon$ which commutes with a surjection $g\colon\ \hat{\Lambda}\to\Lambda_\varepsilon$

$$\begin{tikzcd} \hat{\Lambda} \arrow[r, "\hat{g}"] \arrow[d, "p"'] \arrow[dr] & \hat{\Lambda}_\varepsilon \arrow[d, "p"] \\ \Lambda & \Lambda_\varepsilon \end{tikzcd} \qquad g \qquad p \text{ is the canonical projection.}$$

We know that $h(f|_\Lambda)=h(\hat{f}|_{\hat{\Lambda}})=h(\hat{f}_\varepsilon|_{\hat{\Lambda}_\varepsilon})=h(f_\varepsilon|_{\Lambda_\varepsilon})$, so the value of the entropy remains constant w.r.t. small perturbations. However $h^-(f|_\Lambda)\neq h^-(\hat{f}|_{\hat{\Lambda}})$ in general. For example when $f(z,w)=(z^2+c,w^2),(|c|$ small, $\Lambda=\{z_0\}\times J_{(w^2)})$ we have $h_i(f|_\Lambda)=0$ as we showed in the Remark from §1, which implies $h^-(f|_\Lambda)=0$; but at the same time Proposition 2 implies that $h^-(\hat{f}|_{\hat{\Lambda}})=h_i(\hat{f}|_{\hat{\Lambda}})=h(\hat{f}|_{\hat{\Lambda}})$ since $\hat{f}\colon\ \hat{\Lambda}\to\hat{\Lambda}$ is a homeomorphism. Combining this with $h(\hat{f}|_{\hat{\Lambda}})=h(f|_\Lambda)=\log 2$ shows $h^-(f|_\Lambda)\neq h^-(\hat{f}|_{\hat{\Lambda}})$.

If we consider the example discussed in Section 4 of a perturbation f_ε such that $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism, we see that $h^-(f_\varepsilon|_{\Lambda_\varepsilon})=h(f_\varepsilon|_{\Lambda_\varepsilon})=\log 2$. However $h^-(f)=0$ since Λ is a quasicircle, hence the value of h^- is not stable under perturbation.

4 Examples and applications

First we study a large class of maps obtained as perturbations of (z^2+c,w^2) $(0\neq|c|$ small) and identify those elements of the class which are injective on their respective basic sets.

Theorem 3. *Given the map $f_\varepsilon(z,w)=(z^2+a\varepsilon z+b\varepsilon w+c+d\varepsilon zw+e\varepsilon w^2,w^2)$, there exist small positive constants $c(a,b,d,e)$ and $\varepsilon(a,b,c,d,e)$ such that, for $0\neq|c|<c(a,b,d,e)$ and $0<\varepsilon<\varepsilon(a,b,c,d,e)$ we have that f_ε is injective on its basic set Λ_ε close to $p_0(c)\times S^1$ (where $p_0(c)$ is the attracting fixed point for z^2+c), if and only if $b\neq 0$ or $d\neq 0$.*

Proof. Assume that $f_\varepsilon(z,w)=f_\varepsilon(z',w')$ for two points $(z,w),(z',w')\in\Lambda_\varepsilon$

$$\Rightarrow z^2+a\varepsilon z+b\varepsilon w+c+d\varepsilon zw+e\varepsilon w^2=z'^2+a\varepsilon z'+b\varepsilon w'+c+d\varepsilon z'w'+e\varepsilon w'^2$$
$$\Rightarrow (z^2-z'^2)+\varepsilon[a(z-z')+b(w-w')+d(zw-z'w')+e(w^2-w'^2)]=0.$$

Assume $w\neq w'$. Then, $w'=-w$ since $w^2=w'^2$

$$\Rightarrow (z-z')(z+z'+\varepsilon a)+\varepsilon[2bw+dw(z+z')]=0.$$

$$\Rightarrow (z-z')(z+z'+\varepsilon a)=-\varepsilon[2bw+dw(z+z')]. \tag{1}$$

Let $p_0(c)$ be the fixed attracting point for $z\to z^2+c$, $0\neq|c|$ small. And consider $\alpha:=\sup_{(z,w)\in\Lambda_\varepsilon}|z-p_0(c)|$. Let (z_0,w_0) a point in Λ_ε, where the supremum in Λ_ε is attained. Hence

$\alpha = |z_0 - p_0(c)|$, and $p_0^2(c) + c = p_0(c)$. Now, we can find a point $(z, w) \in \Lambda_\varepsilon$ such that

$$\begin{aligned} f_\varepsilon(z,w) = (z_0, w_0) \Rightarrow z_0 &= z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 \\ \Rightarrow z_0 - p_0(c) &= z^2 - p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \\ &= (z - p_0(c))^2 + 2zp_0(c) - 2p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \Rightarrow \\ \alpha &\le \alpha^2 + 2|p_0(c)|\alpha + K\varepsilon, \end{aligned}$$

with K, a positive constant depending on the parameters of the map. Now $\alpha^2 + \alpha(2|p_0(c)| - 1) + K\varepsilon \geq 0$, and since $\alpha \ll 1$ (since Λ_ε is very close to $\{p_0(c)\} \times S^1$), we obtain (for some constant K'):

$$0 \le \alpha \le \frac{2K\varepsilon}{1 - 2|p_0(c)| + \sqrt{(1 - 2|p_0(c)|)^2 - 4K\varepsilon}} \le K' \cdot \varepsilon.$$

From (2) one gets $2\alpha(2|p_0(c)| + \varepsilon|a|) > |2bw + dw(z + z')|$ for $\varepsilon < \varepsilon(a, b, c, d, e)$, since z, z' are both ε-close to $p_0(c)$.

But for $(z, w) \in \Lambda_\varepsilon$, one has $|w| = 1$, and $|z + z'|$ is ε-close to $2|p_0(c)|$. Therefore, assuming first that $b \neq 0$, if $|c|$ is small enough in comparison to $|b|$, then $|p_0(c)|$ becomes so small that $|2bw + dw(z + z')| > |b| > 0$. Hence $2K'\varepsilon(2|p_0(c)| + \varepsilon|a|) \geq |b| > 0$, which is a contradiction if $|c|$ is small enough (since $|c|$ small will imply that $|p_0(c)|$ is also small, and we can always reduce ε accordingly).

Hence, we proved that $w' = w$. Then, from $f_\varepsilon(z, w) = f_\varepsilon(z', w')$ it follows that $z^2 - z'^2 = -\varepsilon[a(z - z') + dw(z - z')]$. If $z \neq z'$, we would then get $z + z' = -\varepsilon(a + dw)$.

But z, z' are both ε -close to $p_0(c) \neq 0$, so if we choose $\varepsilon < \varepsilon(a, b, c, d, e)$ appropiately, then $|z + z'| > |p_0(c)| > 0$. On the other hand, if $|a + dw| \neq 0$, we can take ε small enough such that $\varepsilon|a + dw| < |p_0(c)|$, which gives a contradiction. If $|a + dw| = 0$, then we get a contradiction again since $p_0(c) \neq 0$.

In conclusion $z = z', w = w'$ and in consequence $f_\varepsilon|_{\Lambda_\varepsilon} \colon\ \Lambda_\varepsilon \to \Lambda_\varepsilon$ is an injective map. The case when $b = 0, d \neq 0$ is treated similarly.

We now only have to show that if $b = 0, d = 0$, then f_ε is non-injective. Actually we prove that each point in Λ_ε has at least two preimages in Λ_ε.

Let us fix a point $(z_0, w_0) \in \Lambda_\varepsilon$ and show that $(z_0, -w_0) \in \Lambda_\varepsilon$ too. Consider a neighbourhood of (z_0, w_0), $U = D(z_0, \delta) \times D(w_0, \delta)$. Then from the fact that (z_0, w_0) is a nonwandering point, we obtain that there exists an increasing sequence of integers n_k such that $f_\varepsilon^{n_k}(U) \cap U \neq \emptyset$. If $p_2(z, w) = w$ is the projection on the second factor, then for n_k large enough, $p_2 \circ f_\varepsilon^{n_k}(U) = S^1$. So if we take $V := D(z_0, \delta) \times D(-w_0, \delta)$ as a neighbourhood of $(z_0, -w_0)$, then since w does not appear in the first term of f_ε, we get $f_\varepsilon^{n_k}(V) \cap V \neq \emptyset$, and as $\delta > 0$ was arbitrary, it follows that $(z_0, -w_0) \in \Lambda_\varepsilon$.

Since obviously $f_\varepsilon(z_0, w_0) = f_\varepsilon(z_0, -w_0)$, f_ε is at least 2-to-1 on Λ_ε.

In conclusion, the only instances when f_ε is injective are when $b \neq 0$ or $d \neq 0$. $\square$

Corollary 1. *In the setting of Theorem 3, if $f_\varepsilon(z, w)$ is injective on Λ_ε and $|c|$ is small enough, then $h^-(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$ and $HD(W^s_{\varepsilon'}(x) \cap \Lambda_\varepsilon) > \frac{\log 2}{\log|\frac{1+\sqrt{1-4c}}{2c}|}$.*

Moreover, in this case Λ_ε is not a Jordan curve.

Proof. If $f_\varepsilon|_{\Lambda_\varepsilon}$ is injective, then it is a homeomorphism of Λ_ε and hence $h^-(f_\varepsilon|_{\Lambda_\varepsilon}) = h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = h(f_\varepsilon|_{\Lambda_\varepsilon})$. But

$$h(f_\varepsilon|_{\Lambda_\varepsilon}) = h(\hat{f}_\varepsilon) = h(\hat{f}) = h(f) = \log 2$$

In the above, $\hat{f}$ represents the lifting of f to a homeomorphism of $\hat{\Lambda}$ and we can use the conjugacy between $\hat{f}$ and $\hat{f}_\varepsilon$ ([14]); also the last equality is due to the fact that $f|_\Lambda$ is just the map $w \to w^2$ whose entropy is $\log 2$.

The estimate for the stable dimension in our claim is proved by using Theorem 2 and the fact that $\left|\inf_{y\in\Lambda}|Df|_{E^s_y}|\right| > |p_0(c)|$, where $p_0(c) = \frac{2c}{1+\sqrt{1-4c}}$ is the unique fixed attracting point of the map $z^2 + c$, for $|c|$ small.

The last consequence to show is that the basic set Λ_ε is not a Jordan curve.

Indeed, if Λ_ε were a Jordan curve, the Nitecki-Przytycki Theorem in Section 1 would imply that $h_i(f_\varepsilon|_{\Lambda_\varepsilon})$ is zero, but as we saw above, $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = \log 2$.

□

In the case of diffeomorphisms the stable dimension depends continuously on the map, even real analytically in the case of Henon maps ([16]). However this property is not present anymore for holomorphic endomorphisms.

Indeed, from Corollary 5, if f_ε is a perturbation of $f(z, w) = (z^2 + c, w^2)$, such that f_ε injective on its basic set Λ_ε, then $HD(W^s_\delta(x_\varepsilon) \cap \Lambda_\varepsilon)$ is bounded below by a constant which does not depend on ε, no matter how small ε is.

On the other hand $HD(W^s_\delta(x) \cap \Lambda) = 0$, if Λ is the basic set of f. Therefore we get a contradiction, since in any neighbourhood of f we were able to find a map having stable dimension bounded below by a constant not depending on ε. This will be formulated in the following:

Corollary 2. *If f is a holomorphic map on $\mathbb{P}^2$ which has a basic set Λ, f_ε is an ε- perturbation of f and Λ_ε is the basic set of f_ε close to Λ, then $HD(W^s_\delta(x_\varepsilon)\cap\Lambda_\varepsilon)$ does not converge to $HD(W^s_\delta(x)\cap\Lambda)$, when $\varepsilon \to 0$ and $x_\varepsilon \in \Lambda_\varepsilon, x \in \Lambda, x_\varepsilon \to x$. Hence the dependence of the stable dimension on the map is not continuous.*

Now we ask what can be said about the number of preimages that a point in the basic set Λ can have in Λ, in order to shed more light on the upper estimate from Theorem 1.

Lemma 1. *Let f be a holomorphic map on $\mathbb{P}^2$, and Λ one of its basic sets of infinite cardinality, such that $C_f \cap \Lambda = \emptyset$, where C_f is the critical set of f. Then the set $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ is open with respect to the induced topology on Λ.*

Proof. If $C_f \cap \Lambda = \emptyset$, then there exists $\delta > 0$ such that ,if $f(z_1) = f(z_2), z_1, z_2 \in \Lambda, z_1 \neq z_2$, then $d(z_1, z_2) > \delta$. First let us notice that Λ cannot have isolated points. This follows from the fact that $f|_\Lambda$ is transitive and since Λ is assumed to be infinite. Now assume that the statement of our lemma is not true. Then there would exist $x \in \Lambda$ such that $f^{-1}(x) \cap \Lambda = \{z\}$ and, since we saw that Λ does not have isolated points, there would exist a sequence $y_n \in \Lambda$, $y_n \to x$, with $\{z_n^1, z_n^2\} \subset f^{-1}(y_n) \cap \Lambda, z_n^1 \neq z_n^2, \forall n$. From the discussion above, it follows that $d(z_n^1, z_n^2) > \delta$.

But since Λ is compact, there exists a subsequence of $(z_n^1)_n$, denoted also by $(z_n^1)_n$ such that $z_n^1 \to \xi^1 \in \Lambda$.

By taking yet another subsequence, one can assume also that $z_n^2 \to \xi^2 \in \Lambda$.

Since $f(z_n^1) = y_n$ and $y_n \to x$, we have $f(\xi^1) = x$ and similarly $f(\xi^2) = x$. But $d(z_n^1, z_n^2) > \delta$, hence $d(\xi^1, \xi^2) \geq \delta$. Therefore we get a contradiction with the fact that the set $f^{-1}(x) \cap \Lambda$ has only one element. So the set of points $x \in \Lambda$ having only one preimage in Λ, is open in the induced topology. $\square$

Lemma 2. *In the same setting as in Lemma 1, the set* $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ *is closed.*

Proof. First we will show that $f|_\Lambda : \Lambda \to \Lambda$ is open.

Indeed, $f|_\Lambda$ can be lifted to a homeomorphism $\hat{f}$ of $\hat{\Lambda}$ on the inverse limit set, such that $p \circ \hat{f} = f \circ p$, where $p : \hat{\Lambda} \to \Lambda$ is the canonical projection $p(x_n)_{n \leq 0} = x_0$.

Let U an open set in the topology induced on Λ from the usual topology on $\mathbb{P}^2$.

Then, since p is continuous, $p^{-1}(U)$ is open in $\hat{\Lambda}$. But $\hat{f}$ is a homeomorphism, hence $\hat{f}(p^{-1}(U))$ is open in $\hat{\Lambda}$.

Now let us prove that p is open. For this we need to show that for any $\delta > 0$, there is some $\delta' > 0$ such that for any $\hat{x} \in \hat{\Lambda}$, $p(B(\hat{x}, \delta)) \supset B(x, \delta')$.

Indeed, if M denotes the diameter of $\mathbb{P}^2$, then take an integer N large enough such that $\frac{M}{2^N} < \frac{\delta}{2}$ and $\delta' > 0$ such that for any $x, y \in \mathbb{P}^2$ with $d(x, y) < \delta'$, and any prehistory of x, $\hat{x}$, there is a prehistory $\hat{y}$ of y with

$$d(x, y) + \frac{d(x_{-1}, y_{-1})}{2} + ... + \frac{d(x_{-n}, y_{-n})}{2^N} < \frac{\delta}{2}$$

Therefore $d(\hat{x}, \hat{y}) < \delta$ and $p(B(\hat{x}, \delta)) \supset B(x, \delta')$. This shows that p is open.

Coming back to the set $\hat{f}(p^{-1}(U))$ which we proved to be open, it follows immediately that $f(U) = p(\hat{f}(p^{-1}(U)))$ is open also in Λ since p is an open map.

In conclusion $f|_\Lambda : \Lambda \to \Lambda$ is open.

Now assume that the set of points with only one preimage in Λ is not closed.

Then, since Λ does not have isolated points, there would exist $x \in \Lambda$ having two preimages y_1, y_2 in Λ and a sequence $x_n \in \Lambda, x_n \to x$ with $f^{-1}(x_n) \cap \Lambda = \{z_n\}$.

By taking eventually a subsequence, one can suppose that $d(y_1, z_n) > \delta, \forall n$, for some positive constant δ.

Let now U an open ball around y_1 of radius $\frac{\delta}{2}$. Since $f|_\Lambda$ is open and Λ has no isolated points, it follows that $x_n \in f(U)$ for n large enough. So, there exists $\xi_n \in U$ such that $f(\xi_n) = x_n$. Also, since we assumed $d(y_1, z_n) > \delta$, it is clear that $z_n \neq \xi_n$.

So, this implies that $\{\xi_n, z_n\} \subset f^{-1}(x_n) \cap \Lambda$, which is a contradiction with our assumption.

Therefore the set $\{x \in \Lambda, Card\{f^{-1}(x) \cap \Lambda\} = 1\}$ is closed in Λ.

$\square$

Lemmas 1, 2 imply the following:

Corollary 3. *In the same setting as in Lemma 1, and assuming also that the basic set Λ is connected, one has that, if there exists a point $x \in \Lambda$ with only one preimage in Λ, then every point in Λ has exactly one preimage in Λ.*

Similarly one can show a more general result:

Corollary 4. *In the same setting as in Lemma 1, and assuming also that Λ is connected, the number of preimages that a point from Λ has in Λ is constant.*

As can be seen from the proofs of Lemmas 1, 2, this statement is true in a more general setting: f does not have to be necessarily holomorphic for example.

A natural question is how we can check which basic sets are connected.

In fact it turns out that it is enough to perturb a map with a basic set which we know to be connected in order to obtain other such examples.

Lemma 3. *In the same setting as in Lemma 1, if Λ is connected and g is sufficiently close to f, with its corresponding basic set Λ_g, then Λ_g is connected as well.*

Proof. We know from a result of Przytycki ([14]) that there exists a conjugacy at the level of inverse limits, between $\hat{\Lambda}$ and $\widehat{\Lambda_g}$ if g is close enough to f.

Also an easy result from topology shows that Λ is connected if and only if $\hat{\Lambda}$ is connected. Hence if we assumed Λ to be connected, it follows also that $\hat{\Lambda}$, $\widehat{\Lambda_g}$, and Λ_g are all connected.

□

Corollary 3 and Lemma 3, together with Theorems 1 and 2 give now a lot of information now about the stable dimension of perturbations of the map $(z^2 + c, w^2)$.

Corollary 5. *For a perturbation g of the map $(z^2 + c, w^2)$, where $0 \neq |c|$ small, and the basic set Λ_g close to $p_0(c) \times S^1$, we have that $HD(W^s_\varepsilon(x) \cap \Lambda_g)$ is either equal to zero, and in this case $g|_{\Lambda_g}$ not injective, or $\frac{\log 2}{|\log \inf_{y \in \Lambda_g} |Df|_{E^s_y}||} \leq HD(W^s_\varepsilon(x) \cap \Lambda_g) \leq \frac{log 2}{|\log \frac{(\sup_{\xi \in \Lambda_g} |Df|_{E^s_\xi}|)^2}{\inf_{z \in \Lambda_g} |Df|_{E^s_z}|}|}$, and in this case $g|_{\Lambda_g}$ is injective.*

Proof. Let us first notice that the condition $\alpha\gamma < 1$ from Theorem 1 is satisfied automatically for g close enough to the original map.

Now, since the basic set $\Lambda = \{p_o(c)\} \times S^1$ is connected, Lemma 3 implies that Λ_g will be connected too if g is a small perturbation of the map $(z^2 + c, w^2)$, hence, from Corollary 4, the number of preimages that a point can have in Λ_g is constant and will be denoted by d'.

To find this constant we notice that, since $h(g|_{\Lambda_g}) = \log 2$ and $h(g|_{\Lambda_g}) \geq \log d'$ (from §1), we get $1 \leq d' \leq 2$. So, either every point of Λ_g has one preimage in Λ_g, or every point from Λ_g has exactly two different preimages in this set.

In the first case g is a homeomorphism on Λ_g and then Theorems 1 and 2 give the required inequalities for $HD(W^s_\varepsilon(x) \cap \Lambda_g)$; in the second case, $g|_{\Lambda_g}$ is a 2-1 map, therefore Theorem 1 gives $HD(W^s_\varepsilon(x) \cap \Lambda_g) = 0$.

$\square$

Obviously the same reasoning can be applied to other perturbations of maps f on $\mathbb{P}^2$ for which $C_f \cap \Lambda = \emptyset$, (see for example [5] for more examples, like product maps, solenoids, etc.)

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