FINER FRACTAL GEOMETRY

OF RADIAL JULIA SETS OF MEROMORPHIC FUNCTIONS

AND

CONFORMAL GRAPH DIRECTED MARKOV SYSTEMS;

REAL ANALYTICITY OF HAUSDORFF DIMENSION

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ABSTRACT. We prove several results establishing real analticity of Hausdorff dimensions of limit sets of analytic families of conformal graph directed Markov systems. With this tool and with iterated functions systems resulting from the existence nice sets in the sense of Rivera-Letelier, we prove that the the canonical Hausdorff measure restricted to the radial Julia set of a tame meromorphic function is σ-finite and that the Hausdorff dimension of the radial Julia sets for fairly general families of meromorphic functions is real–analytic.

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1. Introduction

Complex dynamics is a field originated in the works of Pierre Fatou and Gaston Julia. Of course, the problem of linearization for a fixed point was studied before (Böttcher, Koenings and others) and definitely it was an inspiration for the idea of creating this separate branch of mathematics, but numerous and extensive works of Fatou and Julia were the place where complex dynamics was born and maturated. The field became wildly known and popular when about three decades ago first computer images of Mandelbrot set and Julia sets appeared. Complex dynamics got an interest of many researches who started to investigate a variety of interesting and exiting topics in this field. One of them is the geometry of Julia sets and one of the ways to describe and analyze the complex nature of this object is its Hausdorff dimension. In this paper we study the behavior of this dimension under analytic perturbations.

Probably the first result indicating how the Hausdorff dimension of Julia sets changes under analytic perturbations is the result of Ruelle in [14]. He studied the family $z \mapsto z^2 + c$ and showed that the Hausdorff dimension of the Julia set is a real-analytic function for a complex parameter c sufficiently close to zero. The main technique Ruelle used was thermodynamic formalism. We refer the reader to the books of Zinsmister [19] and Przytycki & Urbański [11] for a modern exposition of thermodynamic formalism and contemporary approach to the problem of real analyticity of Hausdorff dimension.

The problem of real analyticity of the Hausdorff dimension was further studied for many families of rational and meromorphic functions (see e.g. [18], [17], [16], [8], [1] and [9]). In the present paper we continue this line of investigation. Our two main theorems are the following Theorem A and Theorem B. In these theorems we establish real–analyticity of Hausdorff dimension of radial Julia sets under weakest, up to our knowledge, conditions.

Theorem A. *Let* $f : \mathbb{C} \to \hat{\mathbb{C}}$ *be a nicely strongly regular tame meromorphic function. Let* $\Lambda \subset \mathbb{C}^d$ *be an open set and let* $\{f_\lambda\}_{\lambda \in \Lambda}$ *be an analytic family of meromorphic functions such that*

(1) $f_{\lambda_0} = f$ *for some* $\lambda_0 \in \Lambda$,

(2) *there exists an holomorphic motion* $H : \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \mathbb{C}$ *such that each* H_λ *is a topological conjugacy between* f_{λ_0} *and* f_λ *on* $\mathcal{J}_{\lambda_0}$.

Then the map

$$\lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda)$$

is real–analytic on some neighborhood of λ_0.

Theorem B. *Suppose that* $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ *is a dynamically regular meromorphic function of divergence type which belongs to class* $\mathcal{S}$. *If* $\Lambda \subseteq \mathbb{C}$ *is an open set,* $\{f_\lambda\}_{\lambda\in\Lambda}$ *is an analytic family (in the sense of Section 5) mermorphic functions and* $f_{\lambda_0} = f$ *for some* $\lambda_0 \in \Lambda$, *then the function* $\Lambda \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$ *is real–analytic in some open neighborhood of* λ_0 *contained in* Λ.

One of our two main techniques employed in the proofs of these theorems is the, recently emerging, concept of nice sets. These sets were introduced and extensively studied by Przytycki and Rivera-Letelier ([12], [10]) in the context of Collet-Eckmann rational mappings.

Nice sets in transcendental meromorphic dynamics were used to show that there is no absolutely continuous invariant probability for Misurewicz exponential maps (see [3]). A general construction of nice sets for transcendental functions can be found in [2]. In our present paper we use them to construct appropriate conformal iterated function systems and then to apply the developed machinery of graph directed Markov systems from [6] and [7]. While doing this, as an actually auxiliary step, we obtain new, up to our knowledge, results about real analyticity of the Hausdorff dimension of limits sets of (infinite) conformal graph directed Markov systems. The following Theorem C and Theorem D in particular extend those from [16] and [1]. The number $b(S_\lambda)$ refers here to the Bowen's parameter of the system S_λ.

Theorem C. *If* $\{S_\lambda\}_{\lambda\in\Lambda}$ *is a weakly regularly analytic family of finitely primitive conformal graph directed Markov systems, then the function* $\Lambda \ni \lambda \mapsto b(S_\lambda) \in \mathbb{R}$ *is real–analytic on some neighborhood of every strongly regular parameter* $\lambda_0 \in \Lambda$. *In addition, if the Bowen's parameter is equal to the Hausdorff dimension of the limit set (due to Theorem 4.1 guaranteed for example by the Open Set Condition), we thus automatically get real analyticity of Hausdorff dimension.*

Theorem D. *If* $\Lambda \subseteq \mathbb{C}^d$ *is an open set and* $\{S_\lambda\}_{\lambda\in\Lambda}$ *is an analytic family of finitely primitive conformal graph directed Markov systems such that* S_{λ_0} *is strongly regular for some* $\lambda_0 \in \Lambda$ *and there exists a holomorphic motion* $H\colon \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z))$$

for all $\lambda \in \Lambda$ *and all* $z \in \mathcal{J}_{\lambda_0}$*, then the Bowen's parameter function* $\Lambda \ni \lambda \to b(S_\lambda)$ *is real–analytic on some sufficiently small neighborhood of* λ_0*. In addition, if the Bowen's parameter is equal to the Hausdorff dimension of the limit set (due to Theorem 4.1 guaranteed for example by the Open Set Condition), we thus automatically get real analyticity of Hausdorff dimension.*

Note that although we assume in the latter theorem seemingly more, namely the existence of an appropriate holomorphic motion, however, on the other hand, we merely assume here analyticity of the family of graph directed Markov systems, which is much weaker than weakly regular analyticity required in Theorem C. Staying in the realm of abstract Conformal Graph Directed Markov Systems we are able to provide a very mild sufficient condition, called periodical separation, which entails the existence of a suitable holomorphic motion. Theorem D gets then very weak hypotheses indeed. This is however not quite the end of the story about directed Markov systems. The point is that those systems constructed in the proof of Theorem A are not known to satisfy the Open Set Condition. To remedy this we invoke the theory of conformal Walters expanding maps developed in in [4].

Having Conformal Iterated Function Systems produced with the help of nice sets, we were also able to show (see Theorem 3.4), as a straightforward consequence of the theory of Conformal Graph Directed Markov Systems, that the canonical Hausdorff measure restricted to the radial Julia set of a tame meromorphic function is σ-finite.

As the last remark we would like to say that all our considerations about tame meromorphic functions also do apply to rational functions, and also for them are new. We do not assume that our meromorphic functions are transcendental.

2. Nice Sets and Corresponding Conformal Iterated Function Systems

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The Fatou set of f consists of all points $z \in \mathbb{C}$ that admit an open neighborhood U_z such that all the forward iterates f^n, $n \geq 0$, of f are well-defined on U_z and the family of maps $\{f^n|_{U_z} : U_z \to \mathbb{C}\}_{n=0}^\infty$ is normal. The Julia set $\mathcal{J}(f)$ is then defined as the complement of the Fatou set of f in $\mathbb{C}$. By $\operatorname{sing}(f^{-1})$ we denote the set of singularities of f^{-1}. We define the *postsingular set* of $f : \mathbb{C} \to \hat{\mathbb{C}}$ as

$$\mathcal{P}(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\operatorname{sing} f^{-1})}.$$

Given $z \in \mathbb{C}$ we say that a complex number w is in $\omega(z)$ if all the forward iterates $f^n(z)$, $n \geq 0$, are well-defined and w is a cluster point of the sequence $\{f^n(z)\}_{n=0}^{\infty}$. The set $\omega(z)$ is then refereed to as the ω-limit set of z. Note that $\omega(z) = \emptyset$ if and only if either z is eventually mapped to infinity or $\lim_{n\to\infty} f^n(z) = \infty$. The primary object of our study in this paper, the *radial Julia set* $\mathcal{J}_r(f)$ of f is defined as

$$\mathcal{J}_r(f) := \{z \in \mathcal{J}(f) : \omega(z) \setminus \mathcal{P}(f) \neq \emptyset\}.$$

Given a set $F \subset \hat{\mathbb{C}}$ and $n \geq 0$, by $\mathrm{Comp}(f^{-n}(F))$ we denote the collection of all connected components of $f^{-n}(F)$. A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called *tame* if its postsingular set does not contain its Julia set. Unless otherwise stated all meromorphic functions considered in the sequel will be tame. As noted in the introduction, J. Rivera-Letelier introduced in [12] the concept of nice sets in the realm of the dynamics of rational maps of the Riemann sphere. In [2] N. Dobbs proved their existence for tame meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$. We quote now his theorem.

Theorem 2.1. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function. Fix $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$, $L > 1$ and $K > 1$. Then there exists $\kappa > 1$ such that for all $r > 0$ sufficiently small, there exists an open connected set $U = U(z, r) \subset \mathbb{C} \setminus \mathcal{P}(f)$ such that*

(a) *If $V \in \mathrm{Comp}(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then $V \subset U$.*
(b) *If $V \in \mathrm{Comp}(f^{-n}(U))$ and $V \cap U \neq \emptyset$, then, for all $w, w' \in V$,*

$$|(f^n)'(w)| > L \text{ and } \frac{|(f^n)'(w)|}{|(f^n)'(w')|} < K.$$

(c) $\overline{B(z, r)} \subset U \subset B(z, \kappa r) \subset \mathbb{C} \setminus \mathcal{P}(f)$.

Let $\mathcal{U}$ be the collection of all nice sets of $f : \mathbb{C} \to \hat{\mathbb{C}}$, i.e. all the sets U satisfying the above proposition with some $z \in \mathcal{J}(f) \setminus \mathcal{P}(f)$ and some $r > 0$. Note that if $U = U(z, r) \in \mathcal{U}$ and $V \in \mathrm{Comp}(f^{-n}(U))$ satisfies the requirements (a), (b) and (c) from Proposition 2.1 then there exists a unique holomorphic inverse branch $f_V^{-n} : B(z, \kappa r) \to \mathbb{C}$ such that $f_V^{-n}(U) = V$. The collection S_U of all such inverse branches forms obviously an iterated function system in the sense of [6] and [7]. In particular, it clearly satisfies the Open Set Condition. We denote its limit set by $\mathcal{J}_U$.

3. Hausdorff Dimension; Hausdorff, Conformal and Invariant Measures

Recall that $\mathcal{U}$ is the collection of all nice sets of a tame meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$. Since, by Theorem 2.1, $\mathcal{U}$ forms a basis of topology for $\mathcal{J}(f) \setminus \mathcal{P}(f)$ and since this metric space is separable, it follows from Lindelöf's Theorem that $\mathcal{U}$ contains a countable cover of $\mathcal{J}(f) \setminus \mathcal{P}(f)$. We start with the following.

Lemma 3.1. *If $\mathcal{W}$ is a subcover of $\mathcal{U}$, than*

$$\mathcal{J}_r(f) = \bigcup_{U \in \mathcal{W}} \bigcup_{k=0}^{\infty} f^{-k}(\mathcal{J}_U).$$

Proof. Since $\mathcal{J}_U \subset \mathcal{J}_r$ for all U,

$$\bigcup_{U \in \mathcal{W}} \bigcup_{k=0}^{\infty} f^{-k}(\mathcal{J}_U) \subset \mathcal{J}_r.$$

On the other hand, if $x \in \mathcal{J}_r$, then there exists $y \in \omega(x) \setminus \mathcal{P}(f)$ and therefore $U \in \mathcal{W}$ with $y \in U$ such that the set $\{n \geq 0 : f^n(x) \in U\}$ is infinite. So, $x \in f^{-k}(\mathcal{J}_u)$ for some $k \geq 0$. This finishes the proof. □

Now, we aim to prove that for a tame meromorphic the Hausdorff dimension of limit sets of all nice sets is the same and is equal to the Hausdorff dimension of the radial Julia set. To do this we need the following proposition, concerning general Conformal Iterated Function Systems, which is also interesting on its own. Here we use the notation from [7].

Proposition 3.2. *Let $S = \{\varphi_e\}_{e \in E}$ be a Conformal Iterated Function System. For every $\tau \in E^*$, let*

$$\mathcal{J}_\tau^\infty = \pi(\{\omega \in E^\infty \colon \omega \text{ contains infinitely many copies of the block } \tau\}).$$

Then $\mathrm{HD}(\mathcal{J}_\tau^\infty) = \mathrm{HD}(\mathcal{J}_s)$.

Proof. Let $F \subseteq E$ be an arbitrary finite subset of E (containing all letters of τ). Let $\tilde{m}_F$ and $\tilde{\mu}_F$ be respectively the corresponding symbolic geometric and invariant h_s-conformal measures. Let

$$F_\tau^\infty = \{\omega \in F^\infty \colon \omega \text{ contains infinitely many copies of the block } \tau\}$$

since $\mathrm{supp}(\tilde{\mu_F}) = F^\infty$, it follows from Birkhoff Ergodic Theorem that $\tilde{\mu}_F(F_\tau^\infty) = 1$. Since the measures $\tilde{m_F}$ and $\tilde{\mu}_F$ are equivalent, we conclude that $m_f(F_\tau^\infty) = 1$. Thus

$$m_F(\pi(F_\tau^\infty) = m_F \circ \pi^{-1}(\pi(F_\tau^\infty) \geq m_F(F_\tau^\infty) = 1.$$

Since F is finite, the measure m_F coincides on $\mathcal{J}_F$ up to a multiplicative constant with the Hausdorff measure $\mathcal{H}_{h_F}$. So, $\mathcal{H}_{h_F}(\pi(F_\tau^\infty)) > 0$, whence $\mathrm{HD}(\pi(F_\tau^\infty)) = h_F$. Thus,

$$\mathrm{HD}(\mathcal{J}_\tau^\infty) \geq \sup \mathrm{HD}(\pi(F_\tau^\infty)) = \sup h_F = h_E.$$

where both suprema are taken over all F being a finite subsets of E containing all elements of the finite word τ. This finishes the proof. $\square$

Coming back to meromorphic functions, we prove the following.

Lemma 3.3. *If U and W are two arbitrary nice sets of a tame meromorphic function $f\colon \mathbb{C} \to \bar{\mathbb{C}}$, then $\mathrm{HD}(\mathcal{J}_W) = \mathrm{HD}(\mathcal{J}_U)$.*

Proof. Let $S_U = \{\phi_i^U : i \in I_U\}$ be the iterated function system induced by the nice set U. Since $U \cap \mathcal{J}(f) \neq \emptyset$, there exists $q \geq 0$ so large that

$$f^q(U) \cap \mathcal{J}(f) \supseteq \mathcal{J}(f) \cap W.$$

Since

$$\lim_{n\to\infty} \sup\{\mathrm{diam}(\varphi_\omega^U(U)\colon |\omega| = n\} = 0$$

(in fact the rate of convergence is exponential), and since W is an open set, there thus exists $\tau \in E_U^*$ such that

$$f^q(\varphi_\tau^U(U \cap \mathcal{J}(f)) \subseteq \mathcal{J}(f) \cap W.$$

Hence

$$f^q(\mathcal{J}_{U,\tau}^\infty) \subseteq \mathcal{J}_W. \tag{3.1}$$

Therefore, applying Proposition 3.2, we get that

$$\mathrm{HD}(\mathcal{J}_U) = \mathrm{HD}(\mathcal{J}_{U,\tau}^\infty) = \mathrm{HD}(f^q(\mathcal{J}_{U,\tau}^\infty)) \leq \mathrm{HD}(\mathcal{J}_W).$$

Exchanging the roles of the U and W we also get that $\mathrm{HD}(\mathcal{J}_W) \leq \mathrm{HD}(\mathcal{J}_U)$ and the proof is complete. $\square$

Now, as the main result of this section, we show that the number

$$h := \mathrm{HD}(\mathcal{J}_r(f))$$

is equal to the common value of Hausdorff dimensions of the limits sets of all the iterated function systems induced by all the nice sets and that the the corresponding Hausdorff measure on the radial limit set $\mathcal{J}(f)$ is σ-finite.

Theorem 3.4. *Let $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ be a tame meromorphic function. Then the following is true.*

(a) $h = \mathrm{HD}(\mathcal{J}_r(f))) = \mathrm{HD}(\mathcal{J}_U)$ *for every nice set U.*

(b) *The h-dimensional Hausdorff measure $\mathcal{H}_h$ restricted to each nice limit set $\mathcal{J}_U, U \in \mathcal{U}$, is finite.*

(c) *The h-dimensional Hausdorff measure $\mathcal{H}_h$ restricted to $\mathcal{J}_r(f)$ is σ-finite.*

Proof. Fixing $U \in \mathcal{U}$, and choosing a countable subcover of $\mathcal{U}$ containing U, Part (a) follows immediately from Lemma 3.3 and Lemma 3.1. Part (b) follows from Theorem 4.5.1 and 4.5.11 from [7] and Part (c) is an immediate consequence of part (b) and Lemma 3.1 applied with an arbitrary countable subcover $\mathcal{W}$ of $\mathcal{U}$. We are done. $\square$

We call a tame meromorphic function $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ *nicely strongly regular* if there exist at least one nice set $U \in \mathcal{U}$ giving rise to a strongly regular iterated function system (IFS) S_U. In Section 6 we shall provide same sufficient conditions for a meromorphic function to be nicely (strongly) regular. Nice strong regularity will turn out to be a much harder issue than mere regularity.

Let us now recall another fundamental concept. A Borel σ-finite measure m_h on $\mathcal{J}(f)$ is called h-conformal for $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ if

$$m_h(f(A)) = \int_A |f'|^h dm_h \tag{3.2}$$

for every Borel set $A \in \mathcal{J}(S)$ (or $\subseteq \mathbb{C}$) such that $f|_A$ is one–to–one. The existence of this kind measures is of enormous help in the investigation of geometric properties of Julia sets. Therefore, we now prove their existence and present some of the properties of these measures.

Theorem 3.5. *If $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ is a tame nicely regular meromorphic function, then the following holds.*

(a) *Each nicely set $W \in \mathcal{U}$ gives rise to a regular IFS.*

(b) *There exist a σ-finite h-conformal measure m_h for $f\colon \mathbb{C} \to \bar{\mathbb{C}}$. In addition, $m_h(\mathcal{J}(f)\backslash\mathcal{J}_r(f)) = 0$, and for every nice set $U \in \mathcal{U}$, $m_h(\mathcal{J}_U) > 0$ and the measure $m_h|_{\mathcal{J}_U}$ is h-conformal for the IFS S_U.*

(c) *There exist a Borel σ-finite f-invariant measure μ_h on $\mathcal{J}(f)$ such that $\mu_h(\mathcal{J}(f)\backslash\mathcal{J}_r(f)) = 0$, $0 < \mu_h(\mathcal{J}_U) < +\infty$, for every nice set $U \in \mathcal{U}$, and $\mu_h|_{\mathcal{J}_U} = \mu_U$ is equivalent to the h-conformal probability measure m_U on $\mathcal{J}_U$.*

(d) *The Radon-Nikodyn derivative $\frac{d\mu_h}{dm_h}$ has a real-analytic extension to an open neighborhood of the Julia set $\mathcal{J}(f)$.*

Proof. Let $U \in \mathcal{U}$ be a nice set giving rise to a regular IFS $S_U = \{\varphi_e^U : e \in E\}$. Denote $\|e\|$ the number $n \in \mathbb{N}$ such that $f^n \circ \varphi_e^U = \mathrm{id}_U$. For every $n \geq 1$ let I_n parametrize all holomorphic branches $\{f_i^{-n}\}_{i \in I_n}$ of f^{-n} that are defined on U, let $I = \bigcup_{n=1}^{\infty} I_n$ and for every $i \in I$ let $n(i)$ be a unique integer $k \geq 0$ such that $i \in I_k$. Let I_* be the subset of I consisting all elements i such that $f^k(f_i^{-n(i)}(U)) \cap U = \emptyset$ for all $0 \le k \le n(i)-1$. Notice that the family $\{f_i^{-n(i)}(U)\}_{i \in I_*}$ consists of mutually disjoint sets and define the measure m_h on $U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(U)$ by the following formula. If $i \in I_*$, $A \subseteq f_i^{-n(i)}(U)$ is an arbitrary Borel set, then

$$m_h(A) = \int_{f^{n(i)}(A)} |(f_i^{-n(i)})'(z)|^h dm_U(z). \tag{3.3}$$

Otherwise, if $A \subseteq U$ is a Borel set, then

$$m_h(A) = m_U(A). \tag{3.4}$$

It immediately follows (3.3) that (3.2) holds for all Borel set $A \subseteq f_i^{-n(i)}(U)$ where $i \in I_*$, since $n(i) \ge 1$. Now, for any $z \in \mathcal{J}_U$ let $N(z) \ge 1$ be the first return time to U, *i.e.* $N(z) \ge 1$ is the least integer such that $f^{N(z)}(z) \in U$. Note that $N(z) < +\infty$ and $f^{N(z)}(z) \in \mathcal{J}_U$. For every Borel set $A \subseteq \mathcal{J}_U$ and every $n \ge 1$ let

$$A_n = \{z \in A \colon N(z) = n\},$$

then $\{A_n\}_{n=1}^\infty$ is a partition of A into measurable sets. Notice that

$$A_n = \bigcup_{||e||=n} \varphi_e^U(f^n(A_n)).$$

Assume that $f|_A$ is 1-1. Then also $f|_{A_n}$ is also 1-1, and by (3.3), (3.4) and conformality of the measure m_U for the IFS S_U,

$$\begin{aligned}
m_h(f(A)) &= \sum_{n=1}^\infty m_h(f(A_n)) = \sum_{n=1}^\infty \sum_{e:||e||=n} m_h(f \circ \varphi_e^U(f^n(A_n))) \\
&= \sum_{n=1}^\infty \sum_{e:||e||=n} \int_{f^n(A_n)} |(f \circ \varphi_e^U)'|^h dm_U \\
&= \sum_{n=1}^\infty \sum_{e:||e||=n} \int_{f^n(A_n)} |f' \circ \varphi_e^U|^h |(\varphi_e^U)'|^h dm_U \\
&= \sum_{n=1}^\infty \sum_{e:||e||=n} \int_{\varphi_e^U(f^n(A_n))} |f'|^h dm_U \\
&= \sum_{n=1}^\infty \sum_{||e||=n} \int_{\varphi_e^U(f^n(A_n))} |f'|^h dm_h \\
&= \sum_{n=1}^\infty \int_{\bigcup\{\varphi_e^U(f^n(A_n)):||e||=n\}} |f'|^h dm_h = \sum_{n=1}^\infty \int_{A_n} |f'|^h dm_h
\end{aligned}$$

$$= \int_{\bigcup_{n=1}^\infty A_n} |f'|^h dm_h = \int_A |f'|^h dm_h.$$

Thus, (3.2) holds for all Borel sets

$$A \subseteq U \cup \bigcup_{i \in I_*} f^{-n(i)}(U)$$

such that $f|_A$ is 1-1. Observe that then all sets $f(A \cap U)$ and $f(A \cap f^{-n(i)}(U))$, $i \in I_*$ are mutually disjoint and $m_h(A \cap U) = m_U(A \cap \mathcal{J}_U)$ as well as $m_h(f(A \cap U)) = m_h(f(A \cap \mathcal{J}_U))$. Since also

$$m_h(\mathbb{C} \backslash (\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U))) = 0 \tag{3.5}$$

and since

$$f\big(\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U)\big) = f^{-1}\big(\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U)\big) = \mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U),$$

we conclude that m_h is a Borel σ-finite h-conformal measure for $f\colon \mathbb{C} \to \bar{\mathbb{C}}$ such that $m_h|_{\mathcal{J}_U} = m_U$. Since

$$\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U) \subseteq \mathcal{J}_r \tag{3.6}$$

we further conclude from (3.5) that $m_h(\mathcal{J}(f) \setminus \mathcal{J}_r(f) = \emptyset$.

If now W is an arbitrary nice set, then it follow from (3.1), conformality property (3.2), and the fact that $m_h(\mathcal{J}_{U,\tau}^\infty) = m_U(\mathcal{J}_{U,\tau}^\infty) = 1$, that $m_h(\mathcal{J}_\omega) > 0$. Items (a) and (b) of our theorem are then proved.

It is known that one can spread out the measure μ_U and get a unique ergodic and conservative f-invariant measure μ_h on $\mathcal{J}_U \cup \bigcup_{i \in I_*} f_i^{-n(i)}(\mathcal{J}_U)$ such that $\mu_h|_{\mathcal{J}_U} = \mu_U$. Hence, by (3.6), $\mu_h(\mathcal{J}(f) \backslash \mathcal{J}_r(f)) = 0$ and the item (c) is proved.

Let $z \in \mathcal{J}_r(f)$ and let U_z be a nice set containing z. Then $\frac{d\mu_h}{dm_h} = k\frac{d\mu_h}{dm_h}$ on U_z with some constant $k > 0$. Since $\frac{d\mu_{U_z}}{dm_{U_z}}$ is a real-analytic function on U_z (see Theorem 6.1.3 from [7]), item (d) is also proved. We are done. □

4. Real Analyticity of Hausdorff Dimension for Conformal Graph Directed Markov Systems

The results of this section are far going strengthenings of existing theorems about real analyticity (see [16] and references therein) or even continuity (see [13]) of the Hausdorff dimension of limit sets of Conformal Graph Directed Markov Systems. Although we try to explain all notions used in this section, or at least to give references as to where these are defined, as a general rule we follow the terminology of [7] and the reader can always consult this book in case of doubt. We would also like to emphasize that our results in this section will primarily concern Conformal Graph Directed Markov Systems (conformal GDMSs) without assuming the Open Set Condition to hold and will be primarily formulated as real analyticity of Bowen's parameter defined below. Real analyticity of the Hausdorff dimension of limit sets will be then obtained as an immediate corollary in the case when the Open set Condition holds. Indeed, the following theorem was proved in [7] as Theorem 4.2.13.

Theorem 4.1. *If S is a finitely primitive conformal GDMS, then $b(S) = \mathrm{HD}(\mathcal{J}_S)$, where*

$$b(S) := \inf\{t : P(t) < 0\}$$

is called the Bowen's parameter of the system S.

However in Section 5 devoted to proving real analyticity of the Hausdorff dimension of radial Julia sets of nicely strongly regular tame meromorphic functions, we will construct conformal Graph Directed Markov Systems which will not be known to satisfy the Open Set Condition. Real analyticity of Bowen's parameter will come from the present section whereas its equality to Hausdorff dimension will come from the theory of conformal Walters expanding maps laid down in [4].

Let $\Lambda \subseteq \mathbb{C}^d$ be a complex manifold. Let $\Gamma = (E, V, t, i, A)$ be a finitely primitive multigraph with edges E, vertices V, initial and terminal function t and i, and a incidence matrix $A\colon E \times E \to \{0,1\}$ (see [7]). For every vertex $v \in V$ let bounded open sets $W_v, W_r' \subseteq \mathbb{C}$ be given satisfying that $\overline{W_r} \subset W_r'$. Further more, for every $\lambda \in \Lambda$, let

$$S_\lambda = \{\varphi_e^\lambda\colon W_{t(e)} \to W_{i(e)}\}$$

be a conformal GDMS generated over the multigraph Γ with the properties that W_r is connected, $\overline{W_r} \subset W_r'$, $\varphi_e : W_{t(e)}' \to W_{i(e)}'$ and $\varphi_e : W_{t(e)} \to W_{i(e)}$. Although for our applications to meromorphic dynamics considered in this paper all the sets X_r^λ will be independent of λ, here we do not assume that the corresponding compact seed sets $X_v^\lambda \subset W_v$ are independent of λ.

Fix $\lambda_o \in \Lambda$ and for every $\omega \in E_A^\infty$, let $\Psi_\omega\colon \Lambda \to \mathbb{C}$ be given by the following formula

$$\Psi_\omega(\lambda) = \frac{(\varphi_{\omega_1}^\lambda)'(\pi_\lambda(\sigma_\omega))}{(\varphi_{\omega_1}^{\lambda_0})'(\pi_{\lambda_0}(\sigma_\omega))}$$

where $\pi_\lambda \colon E_A^\infty \to \mathcal{J}_{S_\lambda}$ is the canonical projection induced by the GDMS S_λ. The family $\{S_\lambda\}_{\lambda\in\Lambda}$ is called analytic if

(raa) For any $e \in E$ and every $z \in W_{t(e)}$, the function $\Lambda \ni \lambda \mapsto \varphi_e^\lambda(z) \in \mathbb{C}$, $\lambda \in \Lambda$ is holomorphic.

The analytic family $\{S_\lambda\}_{\lambda\in\Lambda}$ is called loosely regularly analytic if the following conditions are satisfied.

(rab) The GDMS S_{λ_0} is strongly regular (we simply say the parameter λ_0 is strongly regular).

(rac) There exists a function $\kappa \colon E \to (0, +\infty)$ such that

$$\sup\{\|(\varphi_e^\lambda)'\| \exp(\kappa(e)) \colon e \in E, \lambda \in \Lambda\} < +\infty.$$

(rad) The family of real-valued function $\Lambda \ni \lambda \mapsto \kappa(\omega_1)^{-1} \log|\Psi_\omega(\lambda)|$, $\omega \in E_A^\infty$, $\lambda \in \Lambda$, is bounded.

There are also two differences of the above as related to Section 4 of [16]. The first one, inessential, is that we do not require in here the sets X_V^λ to be independent of λ, and the second one, more important, is that condition (rad) involves $\log|\Psi_\omega(\lambda)|$ rather than $\log\Psi_\omega(\lambda)$. A family of such maps was called in [16] *weakly regularly analytic.* Then Theorem 4.2 from [16] can be reformulated as follows.

Theorem 4.2. *If $\{S_\lambda\}_{\lambda\in\Lambda}$ is a weakly regularly analytic family of finitely primitive conformal GDMS, then the function $\Lambda \ni \lambda \mapsto b(S_\lambda) \in \mathbb{R}$ is real–analytic on some neighborhood of every strongly regular parameter $\lambda_0 \in \Lambda$.*

Note also apart differences in hypotheses there is also a difference in the assertion as to the original theorem. The point is that since we did not assume the Open Set Condition to hold, we changed the Hausdorff dimension to the Bowen's parameter; the Open Set Condition was used exclusively in Section 4 of [16] to know that Bowen's parameter is equal to the Hausdorff dimension of the limit set. Our strengthened version of Theorem 4.2 is the following.

Theorem 4.3. *If $\{S_\lambda\}_{\lambda\in\Lambda}$ is a losely regularly analytic family of finitely primitive conformal GDMS, then the function $\Lambda \ni \lambda \mapsto b(S_\lambda) \in \mathbb{R}$ is real–analytic on some neighborhood of every strongly regular parameter $\lambda_0 \in \Lambda$.*

Proof. Fix a strongly regular parameter $\lambda_0 \in \Lambda$. We shall show that on same sufficiently small open neighborhood of λ_0, the family of functions

$$\{\lambda \mapsto (\kappa(\omega_1))^{-1} \log\Psi_\omega(\lambda)\}_{\omega\in E_A^\infty},$$

is uniformly bounded. Then the theorem follows from Theorem 4.2. So assume without loss of generality that Λ is simply connected. First, for every $\omega \in E_A^\infty$, choose an analytic branch of logarithm $\log_\omega(\Psi_\omega)\colon \Lambda \to \mathbb{C}$ such that

$$\log_\omega \Psi_\omega(\lambda_0) = 0. \tag{4.1}$$

Then set

$$\Psi_\omega^{1/\kappa(\omega_1)}(\lambda) := \exp\left(\frac{1}{\kappa(\omega_1)}\log_\omega \Psi_\omega(\lambda)\right). \tag{4.2}$$

Let $B > 0$ be the bound coming from (rad). We then have that, for all $\omega \in E_A^\infty$ and all $\lambda \in \Lambda$,

$$\begin{aligned} \left|\Psi_\omega^{1/\kappa(\omega_1)}(\lambda)\right| &= \exp\left(\mathrm{Re}\left(\frac{1}{\kappa(\omega_1)}\log_\omega \Psi_\omega(\lambda)\right)\right) \\ &= \exp\left(\frac{1}{\kappa(\omega_1)}\mathrm{Re}\left(\log_\omega \Psi_\omega(\lambda)\right)\right) \\ &= \exp\left(\frac{1}{\kappa(\omega_1)}\log|\Psi_\omega(\lambda)|\right) \le e^B \end{aligned} \tag{4.3}$$

Put

$$g_\omega := \Psi_\omega^{1/\kappa(\omega_1)}.$$

Fix an arbitrary $r > 0$ so small that $B(\lambda_0, 2r) \subseteq \Lambda$ and let $\lambda \in B(\lambda_0, r)$. Set

$$\Gamma_r = \{\gamma \in \Lambda\colon |\gamma_j - \lambda_j| = r \text{ for all } j = 1, \dots, d\}.$$

In virtue of Cauchy's formula, and of (4.3), we have

$$\begin{aligned} \left|\frac{dg_\omega}{d\lambda}(\lambda)\right| &= \left|\frac{1}{(2\pi i)^d}\int_{\Gamma_r}\frac{g_\omega(\gamma)}{(\gamma_1-\lambda_1)^2\dots(\gamma_d-\lambda_d)^2}d\gamma_1\dots d\gamma_d\right| \\ &\le \frac{1}{(2\pi)^d}\int_{\Gamma_r}\frac{|g_\omega(\gamma)|}{|\gamma_1-\lambda_1|^2\dots|\gamma_d-\lambda_d|^2}|d\gamma_1|\dots|d\gamma_d| \\ &= \frac{1}{(2\pi r^2)^d}\int_{\Gamma_r}|g_\omega(\gamma)||d\gamma_1|\dots|d\gamma_d| \\ &\le \frac{e^B r^d}{(2\pi r^2)^d} = \frac{e^B}{(2\pi r)^d}. \end{aligned}$$

Since $g_\omega(\lambda_0) = 1$, we therefore get, for all $\lambda \in B(\lambda_0, r)$, that

$$|g_\omega(\lambda) - 1| = |g_\omega(\lambda) - g_\omega(\lambda_0)| \le \frac{e^B}{(2\pi r)^d}|\lambda - \lambda_0|.$$

Fix now $\delta \in (0, r)$ so small that $e^B(2\pi r)^{-d}\delta < \frac{1}{4}$. Let $\log_0\colon B(1, \frac{1}{2}) \to \mathbb{C}$ be an analytic branch of logarithm such that $\log_0(1) = 0$. Then $\log_0 \circ g_\omega\colon B(\lambda_0, \delta) \to \mathbb{C}$ is an analytic branch of logarithm of g_ω, and, by

(4.2). It follows from (4.1) and the fact that $\log_0 \circ g_\omega(\lambda_0) = \log_0(1) = 0$, that

$$\frac{1}{\kappa(\omega_1)} \log_\omega \Psi_\omega(\lambda) = \log_0 \circ g_\omega(\lambda)$$

for all $\omega \in E_A^\infty$ and all $\lambda \in B(\lambda_0, \delta)$. Then

$$\left| \frac{1}{\kappa(\omega_1)} \log_\omega \Psi_\omega(\lambda) \right| \le \sup\{|\log \circ g_0(z)|\colon z \in B(1, 1/4)\} < +\infty.$$

We are done. □

Remark 4.4. With the hypotheses of Theorem 4.3, if know that the Bowen's parameter is equal to the Hausdorff dimension of the limit set (due to Theorem 4.1 guaranteed for example by the Open Set Condition), we automatically have a corresponding real analyticity result for Hausdorff dimension.

We now provide a useful sufficient condition for an analytic family of GDMS to be loosely regularly analytic.

Definition 4.5. An analytic family $\{S_\lambda\}_{\lambda \in \Lambda}$ of finitely primitive conformal GDMS is called *uniformly Hölderly conjugate* if

(a) there exist $\lambda_0 \in \Lambda$ (called the center of Λ) such that S_{λ_0} is strongly regular,

(b) there are two constant $c > 0$ and $\alpha \in (0, 1)$ and for every $\lambda \in \Lambda$ there exist a homeomorphism $H_\lambda\colon \mathcal{J}_{\lambda_0} \to \mathcal{J}_\lambda$ such that

$$C^{-1}|z - \omega|^{\frac{1}{\alpha}} \le |H_\lambda(z) - H_\lambda(\omega)| \le C|z - \omega|^\alpha \text{ for all } z, w \in \mathcal{J}_{\lambda_0}$$

and

$$\varphi_e^\lambda \circ H_\lambda = H_\lambda \circ \varphi_e^{\lambda_0} \text{ for all } e \in E.$$

Proposition 4.6. *Each uniformly Hölderly conjugate analytic family of finitely primitive conformal GDMS is loosely regularly analytic and the associated Bowen's parameter function is real–analytic in some sufficiently small neighborhood of its center.*

Proof. In order to show the first part we need to prove that there exists a function $\kappa\colon E \to (o, +\infty)$ such that (rac) and (rad) holds. Indeed, it follows from condition (b) of Definition 4.5 that

$$C^{-1} \operatorname{diam}^{\frac{1}{\alpha}}(\varphi_e^{\lambda_0}(\mathcal{J}_{\lambda_0})) \le \operatorname{diam}(\varphi_e^\lambda(\mathcal{J}_\lambda)) \le C \operatorname{diam}^\alpha(\varphi_e^{\lambda_0}(\mathcal{J}_{\lambda_0}))$$

for all $e \in E$ and all $\lambda \in \Lambda$. Therefore, by uniform distortion, there exists $\hat{C} > 0$ such that

$$\hat{C}^{-1}|(\varphi_e^{\lambda_0})'(z)|^{\frac{1}{\alpha}} \le |(\varphi_e^\lambda)'(\omega)| \le \hat{C}|(\varphi_e^{\lambda_0})'(z)|^\alpha \tag{4.4}$$

for all $e \in E$, all $z \in \mathcal{J}_{\lambda_0}$, all $\lambda \in \Lambda$ and all $\omega \in \mathcal{J}_\lambda$. It follows

$$\begin{aligned}-\log\hat{C}+(1/\alpha-1)\log|(\varphi_e^{\lambda_0})'(z)| &\le \log|(\varphi_e^{\lambda})'(\omega)|-\log|(\varphi_e^{\lambda_0})'(z)|\\ &\le \log\hat{C}+(\alpha-1)\log|(\varphi_e^{\lambda_0})'(z)|,\end{aligned}$$

and then for the distortion constant $K>1$ we have

$$\begin{aligned}&-\log\hat{C}+(1-1/\alpha)\log K+(1/\alpha-1)\log\|(\varphi_e^{\lambda_0})'\|\\ &\qquad\le \log|(\varphi_e^{\lambda})'(\omega)|-\log|(\varphi_e^{\lambda_0})'(z)|\\ &\qquad\qquad\le \log\hat{C}+(\alpha-1)\log K+(\alpha-1)\log\|(\varphi_e^{\lambda_0})'\|.\end{aligned}$$

Hence

$$\begin{aligned}-M &\le \frac{(1-\frac{1}{\alpha})}{\alpha}+\frac{\log\hat{C}+(1-\frac{1}{\alpha})\log K}{-\alpha\log||(\varphi_e^{\lambda_0})'||}\\ &\le \frac{1}{-\alpha\log||(\varphi_e^{\lambda_0})'||}\log\frac{|(\varphi_e^{\lambda})'(\omega)|}{|(\varphi_e^{\lambda_0})'(z)|}\\ &\le \frac{(1-\alpha)}{\alpha}+\frac{(1-\alpha)\log K+\log\hat{C}}{-\alpha\log||(\varphi_e^{\lambda_0})'||}\le M\end{aligned}$$

for some $M>0$ large enough since, for all $e\in E$, $||(\varphi_e^{\lambda_0})'||\le S<1$ and $\lim_{e\to\infty}||(\varphi_e^{\lambda_0})'||=0$. Since in addition, by (4.4),

$$\begin{aligned}||(\varphi_e^{\lambda})'|| \le \hat{C}||(\varphi_e^{\lambda_0})'||^{\alpha} &= \hat{C}\exp\left(\alpha\log||(\varphi_e^{\lambda_0})'||\right)\\ &= \hat{C}\left(\exp(-(-\alpha\log||(\varphi_e^{\lambda_0})'||))\right)\end{aligned}$$

we conclude that conditions (rac) and (rad) are satisfied if we set

$$\kappa(e)=-\alpha\log||(\varphi_e^{\lambda_0})'||.$$

We are thus done with the first assertion of our proposition. The second are follows now immediately from the Theorem 4.3 ☐

Remark 4.7. With the hypotheses of Proposition 4.6, if know that the Bowen's parameter is equal to the Hausdorff dimension of the limit set (due to Theorem 4.1 guaranteed for example by the Open Set Condition), we automatically have a corresponding real analyticity result for Hausdorff dimension.

Now we may prove the following main general result of this section.

Theorem 4.8. *If* $\Lambda\subseteq\mathbb{C}^d$ *is an open set and* $\{S_\lambda\}_{\lambda\in\Lambda}$ *is an analytic family of finitely primitive GDMS such that* S_{λ_0} *is strongly regular for some* $\lambda_0\in\Lambda$ *and there exists a holomorphic motion* $H\colon\Lambda\times\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ *such that*

$$\varphi_e^{\lambda}(H(\lambda,z))=H(\lambda,\varphi_e^{\lambda_0}(z))$$

for all $\lambda \in \Lambda$ *and all* $z \in \mathcal{J}_{\lambda_0}$*, then the Bowen's parameter function* $\Lambda \ni \lambda \to b(S_\lambda)$ *is real–analytic on some sufficiently small neighborhood of* λ_0*.*

Proof. Fix a radius $r > 0$ such that $B(\lambda_0, r) \subseteq \Lambda$. Then by the λ-lemma (see [5], [15]) there exist $K \geq 1$ such that for every $\lambda \in B(\lambda_0, r)$, the map $\hat{\mathbb{C}} \ni z \to H(\lambda, z) \in \hat{\mathbb{C}}$ are uniformly Hölder continuous (with the same exponent λ and constant c). Hence, the analytic family $\{S_\lambda\}_{\lambda \in B(\lambda_0,r)}$ is uniformly Hölderly conjugate. Proposition 4.6 finishes then the proof. □

Remark 4.9. With the hypotheses of Theorem 4.8, if know that the Bowen's parameter is equal to the Hausdorff dimension of the limit set (due to Theorem 4.1 guaranteed for example by the Open Set Condition), we automatically have a corresponding real analyticity result for Hausdorff dimension.

For our applications to meromorphic functions, we will need Theorem 4.8 in the form as stated above, explicitly involving holomorphic motion. Seeing however only graph directed Markov systems and to make this theorem easily applicable, we can already now provide very mild general sufficient conditions for an analytic family of conformal GDMSs to admit a holomorphic motion.

Definition 4.10. Let S be a finitely primitive GDMS and let

$$E_p^* = \{\omega \in E^* : \omega_1 = \omega_{|\omega|} \text{ and } \omega \neq \tau^k \text{ for any } \tau \in E \text{ and } k \geq 2\}.$$

For every $\omega \in E_p^*$ let $x_\omega \in \overline{W_{t(\omega)}}$ be the only fixed point of the map $\varphi_\omega : \overline{W}_{t(\omega)} \to \overline{W}_{t(\omega)}$. We say that the system S is *periodically separated*, if $x_\omega \neq x_\tau$ wherever $\omega, \tau \in E_p^*$ and the words are *incomparable* (that is none of them is an extension of the other).

Let us now record two obvious sufficient conditions for a GDMS to be periodically separated. The following proposition will not be used later.

Proposition 4.11. *If either*

(a) *for every* $\omega \in E_p^*$, $x_\omega \in W_{t(\omega)}$ *and* $\varphi_a(W_{t(a)}) \cap \varphi_b(W_{t(b)}) = \emptyset$ *whenever* $a, b \in E$ *with* $a \neq b$

or

(b) $\varphi_a(\overline{W_{t(a)}}) \cap \varphi_b(\overline{W_{t(b)}}) = \emptyset$ *whenever* $a, b \in E$ *with* $a \neq b$,

then S *is a periodically separated.*

Lemma 4.12. *If $\Lambda \subseteq \mathbb{C}^d$ is an open simply connected set and $\{S_\lambda\}_{\lambda\in\Lambda}$ is an analytic family of finitely primitive periodically separated conformal GDMS, then for every $\lambda_0 \in \Lambda$ there exist a holomorphic motion $H\colon \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \hat{\mathbb{C}}$ such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z)) \text{ for all } \lambda \in \Lambda \text{ and all } z \in \overline{\mathcal{J}_{\lambda_0}}.$$

In addition

$$H(\{\lambda\} \times \overline{\mathcal{J}_{\lambda_0}}) = \overline{\mathcal{J}_\lambda} \text{ and } H(\{\lambda\} \times \mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda \text{ for all } \lambda \in \Lambda.$$

Proof. Fix $\omega \in E_p^*$. Since the map

$$\Lambda \times W_{t(\omega)} \ni (\lambda, z) \mapsto \varphi_\omega^\lambda(z) \in \mathbb{C}$$

is holomorphic and since $(\varphi_\omega^\lambda)'(\xi) \neq 1$ for all $\xi \in W_{t(\omega)}$, it follows from the implicit function theorem that for $r_{\lambda_0,\omega} > 0$ small enough there exist a unique holomorphic function

$$B(\lambda_0, r_{\lambda_0,\omega}) \ni \lambda \mapsto x_{\lambda_0,\omega}^\lambda \in \ W_{t(\omega)}$$

such that $\varphi_\omega^\lambda(x_{\lambda_0,\omega}^\lambda) = x_{\lambda_0,\omega}^\lambda$ and $x_{\lambda_0,\omega}^\lambda$ is (of course) the unique fixed point of the map $\varphi_\omega^\lambda\colon W_{t(\omega)} \to W_{t(\omega)}$. By this uniqueness, all the maps $x_{\lambda_0,\omega}^{(\cdot)}$ glue together to a unique holomorphic function $\Lambda \ni \lambda \mapsto x_\omega^\lambda \in W_{t(\omega)}$ such that

$$\varphi_\omega^\lambda(x_\omega^\lambda) = x_\omega^\lambda. \tag{4.5}$$

Let

$$Y_\lambda := \{x_\omega^\lambda \colon \omega \in E_p^*\}.$$

Since all the systems $\{S_\lambda\}_{\lambda\in\Lambda}$ are periodically separate, there exist a bijection $P_\lambda\colon Y_\lambda \to E_p^*$ sending each point $x \in Y_\lambda$ to the unique $\omega \in E_p^*$ such that $x_\omega^\lambda = P_\lambda(x)$ and for every fixed $\lambda \in \Lambda$, the map $Y_{\lambda_0} \ni z \mapsto W_{P_{\lambda_0}(z)}^\lambda \in Y_\lambda$ is bijective. Thus the map $H\colon \Lambda \times Y_{\lambda_0} \to \mathbb{C}$ given by the formula

$$H(\lambda, z) = x_{P_{\lambda_0}(z)}^\lambda \in \mathbb{C}$$

is a holomorphic motion, and by the λ-lemma it uniquely extends to a holomorphic motion $H\colon \Lambda \times \overline{Y_{\lambda_0}} \to \mathbb{C}$. But, by finite primitivity, $\overline{Y_\lambda} = \overline{\mathcal{J}_\lambda}$. By (4.5) on the other hand only by continuity of the map H, we have that

$$\varphi_\omega^\lambda(H(\lambda, z)) = H(\lambda, \varphi_\omega^{\lambda_0}(z)) \tag{4.6}$$

for all $\lambda \in \Lambda$ and all $z \in \overline{\mathcal{J}_{\lambda_0}}$. Also, for all $\lambda \in \Lambda$, we have

$$H(\{\lambda\} \times \overline{\mathcal{J}_{\lambda_0}}) = \overline{H(\{\lambda\} \times Y_{\lambda_0})} = \overline{Y_\lambda} = \overline{\mathcal{J}_\lambda}.$$

Finally, it follows from (4.6) that $\pi_\lambda(\omega) = H(\lambda, \pi_{\lambda_0}(\omega))$, and therefore

$$\mathcal{J}_\lambda = \pi_\lambda(E_A^\infty) = H(\{\lambda\} \times \pi_{\lambda_0}(E_A^\infty)) = H(\{\lambda\} \times \mathcal{J}_{\lambda_0}).$$

We are done. □

Theorem 4.13. *If $\Lambda \subseteq \mathbb{C}$ is an open simply connected set whose complement $\mathbb{C}\backslash\Lambda$ contains at least two points and $\{S_\lambda\}_{\lambda\in\Lambda}$ is an analytic family of finitely primitive periodically separated conformal GDMS, then for every $\lambda_0 \in \Lambda$ there exists a holomorphic motion $H\colon \Lambda\times\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z)) \text{ for all } \lambda \in \Lambda \text{ and all } z \in \overline{\mathcal{J}_{\lambda_0}}.$$

In addition,

$$H(\{\lambda\} \times \mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda \text{ for all } \lambda \in \Lambda,$$

and if the system S_{λ_0} is strongly regular, then the Bowen's parameter function $\lambda \mapsto b(S_\lambda)$ is real–analytic on some sufficiently small neighborhood of λ_0.

Proof. In virtue of Lemma 4.12 there exist a holomorphic motion on $\lambda\times\overline{\mathcal{J}_{\lambda_0}}$ satisfying the required properties. By Slodkowski's Theorem [15] it can be extended to a holomorphic motion on $\Lambda \times \hat{\mathbb{C}}$ with uniformly bounded dilatation. The last assertion of the theorem follows now immediately from Proposition 4.8. □

Remark 4.14. With the hypotheses of Theorem 4.13, if know that the Bowen's parameter is equal to the Hausdorff dimension of the limit set (due to Theorem 4.1 guaranteed for example by the Open Set Condition), we automatically have a corresponding real analyticity result for Hausdorff dimension.

5. REAL ANALYTICITY OF HAUSDORFF DIMENSION FOR MEROMORPHIC FUNCTIONS

Recall that a meromorphic function $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ belongs to Speiser class $\mathcal{S}$ if the set $\text{sing}(f^{-1})$ of all singularities of f^{-1} is finite. Let $\Lambda \subseteq \mathbb{C}^d$ be an open set. We say that a family $\{f_\lambda\}_{\lambda\in\Lambda}$ of meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$ is analytic if

(a) The function $f_{\lambda_0} : \mathbb{C} \to \hat{\mathbb{C}}$ belongs to Speiser class $\mathcal{S}$.
(b) The function $\Lambda \ni \lambda \mapsto \text{sing}(f_\lambda^{-1})$ is continuous.
(c) Each point of $\text{sing}(f_{\lambda_0}^{-1}) \setminus \mathcal{J}(f_{\lambda_0})$ belongs to the attraction basin of some attracting periodic orbit of f_{λ_0}.
(d) The function $\Lambda \ni \lambda \mapsto f_\lambda(z) \in \hat{\mathbb{C}}$ is meromorphic for all $z \in \mathbb{C}$.

The main result of this section is the following.

Theorem 5.1. *Let* $f : \mathbb{C} \to \hat{\mathbb{C}}$ *be a nicely strongly regular tame meromorphic function. Let* $\Lambda \subset \mathbb{C}^d$ *be an open set and let* $\{f_\lambda\}_{\lambda \in \Lambda}$ *be an analytic family of meromorphic functions such that*

1. $f_{\lambda_0} = f$ *for some* $\lambda_0 \in \Lambda$,
2. *there exists an holomorphic motion* $H : \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \mathbb{C}$ *such that each* H_λ *is a topological conjugacy between* f_{λ_0} *and* f_λ *on* $\mathcal{J}_{\lambda_0}$.

Then the map

$$\lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda)$$

is real–analytic on some neighborhood of λ_0.

Proof. The idea of the proof of this theorem is to associate, by means of nice sets, with the analytic family $f_\lambda : \mathbb{C} \to \hat{\mathbb{C}}$ of meromorphic maps a loosely regularly analytic family of conformal Iterated Function Systems (the simplest subclass of conformal Graph Directed Markov Systems) that have the same Hausdorff dimensions of their limit sets as the Hausdorff dimensions of the radial Julia sets of the corresponding maps f_λ. Then to use the real analyticity results of Section 4.

Since the function f_{λ_0} is tame, it has at least one nice set U. Let $S_{\lambda_0} = \{\varphi_e^{\lambda_0}\}_{e \in E}$ be the associated to U iterated function system. We can require that

$$B(\xi, R) \subseteq U \subseteq B(\xi, 2R) \tag{5.1}$$

with same non-periodic point $\xi \in \mathcal{J}_{\lambda_0}$. Because of analyticity of our family $f_\lambda : \mathbb{C} \to \hat{\mathbb{C}}$, $\lambda \in \Lambda$, (this takes care of singular points of f_λ^{-1} lying in the Fatou set of f_λ) and because of topological conjugacy guaranteed by (2) (this takes care of singular points of f_λ^{-1} lying in the Julia set of f_λ), we may further require that

$$B(\xi, 12R) \cap \bigcup_{n=0}^{\infty} f_\lambda^n \left(\mathrm{sing}(f_\lambda^{-1})\right) = \emptyset$$

and

$$|(f_\lambda^k)'(z)| \geq 6K \text{ whenever } z \in B(\xi, 6R) \text{ and } f_\lambda^k(z) = \xi \tag{5.2}$$

for all $\lambda \in \Gamma_{\lambda_0}^2$, where $\Gamma_{\lambda_0}^2 \subseteq \Lambda$ is a sufficiently small open neighborhood of $\lambda_0 \in \Lambda$. The number $K \geq 1$ is here the Koebe's constant corresponding to the scale 1/2. Now for every $\lambda \in \Gamma_{\lambda_0}^2$ form an iterated function system S_λ acting on $\overline{B}(\xi, 6R)$ as follows. If $e \in E$, let φ_e^λ be the unique holomorphic inverse branch of $f^{\|e\|}$ defined on $B(\xi, 6R)$ and sending ξ to $H_\lambda(\varphi_e^{\lambda_0}(\xi))$. We shall prove the following

Claim 1. *For any* $\lambda \in \Gamma_{\lambda_0}^2$ *sufficiently close to* λ_0, $S_\lambda = \{\varphi_e^\lambda\}_{e \in E}$ *is a strongly regular conformal iterated function system on* $B(\xi, 6R)$.

Proof. Conformality of the maps φ_e^λ, $e \in E$, is immediate from their definitions. The distortion properties follows immediately from Koebe's Distortion Theorems and the fact that all maps φ_e^λ have unique univalent holomorphic extensions to $B(\xi, 12R)$. In order to complete the proof that S_λ is a conformal IFS it thus suffices to show that

$$\varphi_e^\lambda(B(\xi, 5R)) \subseteq B(\xi, 5R).$$

Indeed, since for any $e \in E$, $\varphi_e^{\lambda_0}(U) \subseteq U$, we get that $\varphi_e^{\lambda_0}(\mathcal{J}(S_{\lambda_0})) \cap U \neq \emptyset$, and therefore (see also (5.1)) $\varphi_e^\lambda(H_\lambda(\mathcal{J}(S_{\lambda_0})) \cap B(\xi, 3R) \neq \emptyset$ for all λ sufficiently close to λ_0 (independently of e), say $\lambda \in B(\lambda_0, \delta_1) \subseteq \Gamma^2_{\lambda_0}$. Since $\|(\varphi_e^\lambda)'\| \leq \frac{1}{6}$ (see (5.2)), using the triangle inequality, we conclude that

$$\begin{aligned}\varphi_e^\lambda\left(B(\xi, 6R)\right) &\subseteq B\left(\xi, 3R + 12R\|(\varphi_e^\lambda)'\|\right)\\ &\subseteq B(\xi, 3R + 2R)\\ &= B(\xi, 5R).\end{aligned}$$

We are left to show that all the systems S_λ are strongly regular for all λ sufficiently close to λ_0. Indeed, since the system S_λ and S_{λ_0} are quasi-conformally conjugated on $\overline{\mathcal{J}(S_{\lambda_0})}$ and $\overline{\mathcal{J}(S_\lambda)}$ respectively, we get that

$$C^{-1}||(\varphi_\omega^{\lambda_0})'||^{\frac{1}{\alpha_\lambda}} \leq ||(\varphi_\omega^\lambda)'|| \leq C||(\varphi_\omega^{\lambda_0})'||^{\alpha_\lambda} \tag{5.3}$$

for all $\omega \in E^*$ and some C and numbers $\alpha_\lambda \in (0, 1)$ such that

$$\lim_{\lambda \to \lambda_0} \alpha_\lambda = 1. \tag{5.4}$$

For every $\lambda \in B(\lambda_0, \delta_1)$ put

$$b_\lambda = b(S_\lambda).$$

Since the function $f_{\lambda_0} : \mathbb{C} \to \hat{\mathbb{C}}$ is nicely strongly regular, the system S_{λ_0} is strongly regular, and so, there exist $t_0 < b_{\lambda_0}$ and $0 < \kappa < 1$ such that

$$0 < P_{\lambda_0}(t) < +\infty \tag{5.5}$$

for all $t \in (t_0 - \kappa(b_{\lambda_0} - t_0), t_0 + \kappa(b_{\lambda_0} - t_0))$. In view of (5.4) there exists $\delta_2 \in (0, \delta_1)$ such that

$$\alpha_\delta t_0, \alpha_\delta^{-1} t_0 \in (t_0 - \kappa(b_{\lambda_0} - t_0), t_0 + \kappa(b_{\lambda_0} - t_0))$$

for all $\lambda \in B(\lambda_0, \delta_2)$. Formulas (5.3) and (5.5) imply then that

$$P_\lambda(t) \leq P_{\lambda_0}(\alpha_\lambda t_0) < +\infty$$

and

$$P_\lambda(t) \geq P_{\lambda_0}(\alpha_\lambda^{-1} t) > 0$$

for all $\lambda \in B(\lambda_0, \delta_2)$. this is that all system S_λ, $\lambda \in B(\lambda_0, \delta_2)$ are strongly regular and the proof of Claim 1 is complete. □

Claim 2. $\mathrm{HD}(\mathcal{J}_\lambda) = b_\lambda$ *for all* $\lambda \in B(\lambda_0, \delta_2)$.

Proof. By the very definition of the nice sets all the sets $\varphi_e^{\lambda_0}(\overline{U})$, $e \in E$, are mutually disjoint, and therefore, so are the sets $\{\varphi_e^{\lambda_0}(\overline{\mathcal{J}(S_{\lambda_0})}\}_{e\in E}$. Finally, became of topological conjugacy, the sets $\{\varphi_e^{\lambda}(\overline{\mathcal{J}(S_\lambda)}\}_{e\in E}$ are mutually disjoint for every fixed $\lambda \in B(\lambda_0, \delta_2)$. This means that the global map $F_\lambda \colon \bigcup_{e\in E} \varphi_e^\lambda(\overline{\mathcal{J}(S_\lambda)}) \to \overline{\mathcal{J}(S_\lambda)}$ is well-defined as given by the formula

$$F_\lambda(\varphi_e^\lambda(z)) = z \text{ where } z \in \overline{\mathcal{J}_\lambda}.$$

It is plain to see that the maps F_λ are Walter expanding conformal maps in the sense of [4]. Therefore, Theorem 2.7 in [4] yields $\mathrm{HD}(\mathcal{J}(S_\lambda)) = b_\lambda$ for all $\lambda \in B(\lambda_0, \delta_2)$. The proof of Claim 2 is complete. □

Claim 3. *There exits* $\delta_3 \in (0, \delta_2)$ *such that, for every* $\lambda \in B(\lambda_0, \delta_3) \subset \Lambda$, *we have that*

$$\mathrm{HD}(\mathcal{J}(S_\lambda)) = \mathrm{HD}(\mathcal{J}_r(f_\lambda)).$$

Proof. Clearly $\mathcal{J}(S_\lambda) \subseteq \mathcal{J}_r(f_\lambda)$, so

$$\mathrm{HD}(\mathcal{J}(S_\lambda)) \le \mathrm{HD}\left(\mathcal{J}_r(f_\lambda)\right). \tag{5.6}$$

In order to prove the opposite inequality take $\lambda \in B(\lambda_0, \delta_2)$ and consider a nice set $U_\lambda \subseteq B(\xi, R)$ for the tame meromorphic map f_λ. If ψ_e^λ is a member of the iterated functions system S'_λ induced by the nice set U_λ, then $\psi_e^\lambda(U_\lambda) \subseteq U_\lambda$. So, if λ is taken sufficiently close to λ_0 (independently of e), say $\lambda \in B(\lambda_0, \delta_3)$,with $0 < \delta_3 \le \delta_2$, then

$$H_\lambda^{-1} \circ \psi_e^\lambda\left(\mathcal{J}_r(f_\lambda) \cap U_\lambda\right) \subseteq B(\xi, 2R).$$

Thus the map $\psi_e^{\lambda_0} \colon U \to \mathbb{C}$, the unique holomorphic inverse branch of $f_{\lambda_0}^{||e||}$, determined by the condition that $\psi_e^{\lambda_0}(\xi) = H_\lambda^{-1} \circ \psi_e^\lambda(\xi)$ is a member of S_{λ_0}. But then $\psi_e^\lambda = \varphi_e^\lambda$, where $\varphi_e^\lambda \in S_\lambda$. Consequently, the limit set $\mathcal{J}(S'_\lambda)$ of S'_λ, $\lambda \in B(\lambda_0, \delta_3)$, is contained in $\mathcal{J}(S_\lambda)$. Hence $\mathrm{HD}(\mathcal{J}(S'_\lambda)) \le \mathrm{HD}(\mathcal{J}(S_\lambda))$, and, in virtue of Theorem 3.4, $\mathrm{HD}(\mathcal{J}_r(f_\lambda)) \le \mathrm{HD}(\mathcal{J}(S_\lambda))$. Along with (5.6), this finishes the proof of Claim 3. □

Conclusion of the proof of Theorem 7.1. This is now straightforward. Since the family $(f_\lambda)_{\lambda\in\Lambda}$ is analytic, so is the family $\{S_\lambda\}_{\lambda\in B(\lambda_0,\delta_3)}$. By the very definition of the systems S_λ, the map $H|_{B(\lambda_0,\delta_3)\times\mathcal{J}(S_{\lambda_0})}$ forms a holomorphic motion such that $H_\lambda(\mathcal{J}(S_{\lambda_0})) = \mathcal{J}(S_\lambda)$ and $\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z))$ for all $\lambda \in B(\lambda_0, \delta)$ and all $z \in \mathcal{J}(S_{\lambda_0})$. By Slodkowski's

Theorem ([15]) this holomorphic motion extends to a holomorphic motion of the entire extended complex plane $\hat{\mathbb{C}}$. Thus Proposition 4.8 with the help of the Claims 1, 2 and 3, completes the proof of our theorem. □

Notice that, unlike [9] we did not have to assume our family $\{f_\lambda\}_{\lambda\in\Lambda}$ to be anything like of uniformly balanced growth, or of bounded deformation.

6. Nice Strong Regularity of Dynamically Regular Meromorphic Functions

In this section we deal with *dynamically regular functions* as defined in [9]. Our goal is to show that they are nicely strongly regular and then, in the next section, to prove real analyticity of the Hausdorff dimension of radial Julia set of analytic families of dynamically regular meromorphic functions. We refer the reader to [9] for definition and specific facts about *dynamically regular functions.* In what follows we use the notation of that article.

Let $f\colon \mathbb{C} \to \overline{C}$ be a dynamically regular meromorphic function. Let $|d\tau(z)|$ be the Riemannian metric defined in section 5.1 of [9]. Remember that metric $|d\tau|$ is conformally equivalent to the standard Euclidean metric $|dz|$. It was proved in [9] that the limit

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbf{1}(z)$$

exist for any $z \in \mathcal{J}(f)$, where $\mathcal{L}_t\colon C_b(\mathcal{J}(f)) \to C_b(\mathcal{J}(f))$ is the bounded linear operator defined by the formula

$$\mathcal{L}_t g(z) = \sum_{w\in f^{-1}(z)} g(w)|f'(w)|_\tau^{-t}$$

and referred to as *the Perron-Frobenius operator associated to the parameter* t and the number $P(t)$ is called *the topological pressure at* t. In [9] it is proved that there is a certain number c, that if $t > c$, then

$$\|\mathcal{L}_t \mathbf{1}\|_\infty < \infty \text{ and therefore } P(t) < \infty.$$

For every open set $U \subseteq \mathbb{C}$, we define

$$K(U) = \bigcap_{n=0}^{\infty} f^{-n}(U^c \cap \mathcal{J}(f)) = \{z \in \mathcal{J}(f)\colon f^n(z) \notin U \text{ for all } n \geq 0\}$$

and

$$K_r(U)\colon = \bigcap_{n=0}^{\infty} f^{-n}(U^c \cap \mathcal{J}_r(f)) = \{z \in \mathcal{J}_r(f)\colon f^n(z) \notin U \text{ for all } n \geq 0\}.$$

Of course $K(U)$ is a closed subset of $\mathcal{J}(f)$ and $K_r(U)$ is a closed subset of $\mathcal{J}_r(f)$. Both are $K(U)$ and $K_r(U)$ are forward invariant in the sense that

$$f(K(U)) \subseteq K(U) \text{ and } f(K_r(U)) \subseteq K_r(U).$$

Put

$$U_n^c = \bigcap_{j=0}^{n} f^{-j}(U^c)$$

and define

$$\overline{P}_U^c(t) = \sup\{\overline{P}_U^c(z,t)\colon z \in \mathcal{J}(f) \cap U^c\},$$

where

$$\overline{P}_U^c(z,t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in U_n^c \cap f^{-n}(z)} |(f^n)'(\omega)|_\tau^{-t} = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbf{1}_{U_n^c}(z).$$

We shall prove the following.

Lemma 6.1. *If U is an open subset of $\mathcal{J}(f)$ and $\overline{P}_U^c(t) < 0$, then* $\mathrm{HD}(K_r(U)) \leq t$.

Proof. For every $k \geq 0$ set

$$K_r^k(U) = \{z \in K_r(U)\colon \limsup_{n\to\infty} |f^n(z)| < k\}.$$

From topological hyperbolicity of f (here we have dynamical regularity) there exist $\delta > 0$ such that each open ball $B(z, 2\delta)$, $z \in \mathcal{J}(f)$, is disjoint from the forward orbit of the singular set of f^{-1}. Cover the ball $\mathcal{J} \cap \overline{B}(0,k)$ with finitely many balls $\{B(x_j,\delta)\}_{x\in E}$ where the set $E \subset \mathcal{J}(f) \cap U^c$. Fix arbitrary $\eta > 0$. By hyperbolicity of f, the definition of $\overline{P}_U^c(t)$, there exist $l = l(\eta) \geq 0$ such that, for all $n \geq l$, all $x \in E$ and all $w \in f^{-n}(x)$,

$$|(f^n)'(w)|_\tau \geq 2K\eta^{-1}\delta$$

and

$$\sum_{y \in f^{-n}(x)} |(f^n)'(y)|_\tau^{-t} \leq \exp\big(P(t)/2\big).$$

But the family

$$\Big\{f_w^{-n}\big(B(x,\delta)\big)\colon n \geq l,\, x \in E,\, w \in f^{-n}(x)\Big\}$$

covers $K_r^k(U)$ and

$$\operatorname{diam}\big(f_w^{-n}\left(B(x,\delta)\right)\big) \leq K2\delta |(f^n)'(w)|_\tau^{-1} \leq \eta$$

for all n, x and w as above. Also

$$\sum_{n=l}^{\infty}\sum_{x\in E}\sum_{w\in f^{-n}(x)}\operatorname{diam}_\tau^t\left(f_w^{-n}(B(x,\delta))\right)\le\sum_{n=l}^{\infty}\sum_{x\in E}\sum_{w\in f^{-n}(x)}(2K\delta)^t|(f^n)'(w)_\tau|^{-t}$$
$$\le(2K\delta)^t\sum_{n=l}^{\infty}\sum_{e\in E}\exp\left(\frac{1}{2}P(t)n\right)=(2K\delta)^t\#E\frac{\exp(P(t)l(\eta)/2)}{1-\exp\left(P(t)/2\right)}$$

since $\lim_{\eta\to 0}l(\eta)=+\infty$, we thus get that the Hausdorff measure $\mathcal{H}_t\left(K_r^k(U)\right)=0$. Since

$$K_r(U)=\bigcup_{k=0}^{\infty}K_r^k(U),$$

we conclude that $\mathcal{H}_t\left(K_r(U)\right)=0$. Whence $\mathrm{HD}\left(K_r(U)\right)\le t$. We are done. ☐

We now need the following standard auxiliary fact.

Lemma 6.2. *Let $F\subseteq\mathbb{C}$ be a closed set, let $R\in(0,\infty)$ and let K be a closed subset of $B(0,R)\backslash F$. there then exist a smooth C^∞ function $g\colon\mathbb{C}\to[0,+\infty)$ with the following tow properties*

(a) $\mathbf{1}_F\le g\le\mathbf{1}$
(b) $g|_K\equiv 0$

The main technical result of this section is the following.

Proposition 6.3. *If $f\colon\mathbb{C}\to\hat{\mathbb{C}}$ is a dynamically regular function, $t>\frac{\varrho}{\alpha_1+\tau}$ and U is an arbitrary open subset of $\mathbb{C}$ intersecting the Julia set $\mathcal{J}(f)$, then $\bar{P}_U^c(t)<P(t)$.*

Proof. Since $\lim_{n\to\infty}m_t(U_n^c)=0$, there exist $q\ge 1$ so large that

$$m_t(U_q^c)\le\frac{1}{5}||\varrho_t||_\infty^{-1},\text{ where }\varrho_t=\frac{d\mu_t}{dm_t}.$$

Let $R>0$ be so large that $m_t\left(B^c(0,R)\right)<\frac{1}{8}m_t(U_q^c)$. Let K be a compact subset of $B(0,R)\backslash U_q^c$ such that

$$m_t\left(B(0,R)\backslash\left(U_q^c\cup K\right)\right)<\frac{1}{8}m_t(U_q^c)$$

Finally, let g be the function associated to R,K and $F=U_q^c$ according to Lemma 6.2. Then

$$\begin{aligned}\int_{\mathcal{J}(f)}gdm_t &\le \int_{B^c(0,R)}gdm_t+\int_{U_q^c\cap B(0,R)}gdm_t+\int_K gdm_t+\int_{B(0,R)\backslash(K\cup U_q^c)}gdm_t\\ &= \int_{B^c(0,R)}gdm_t+\int_{U_q^c\cap B(0,R)}gdm_t+\int_{B(0,R)\backslash(K\cup U_q^c)}gdm_t\end{aligned}$$

$$\begin{aligned} &\le \frac{1}{8}m_t(U_q^c) + m_t(U_q^c) + \frac{1}{8}m_t(U_q^c) \\ &= \frac{5}{4}m_t(U_q^c) \\ &\le \frac{1}{4}||\varrho_t||_\infty^{-1}. \end{aligned}$$

Now, since the function g is bounded and Hölder continuous, it follows from Theorem 6.5 in [9] that there exist $s \ge q$ such that

$$\|\mathcal{L}_t^s g - \varrho_t \int g dm_t\|_\infty \le \frac{1}{4}.$$

Consequently

$$\|\mathcal{L}_t^s \mathbf{1}_{U_s^c}\|_\infty \le \|\mathcal{L}_t^s \mathbf{1}_{U_q^c}\|_\infty \le ||\mathcal{L}_t^s g||_\infty \le ||\varrho_t||_\infty \int g dm_t + \frac{1}{4} \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Hence, for any $n \ge 0$ and any $z \in \mathcal{J}(f)$,

$$\begin{aligned} &e^{-P(t)(n+1)s}\mathcal{L}_t^{(n+1)s}\mathbf{1}_{U_{(n+1)s}^c}(z) = \exp(-P(t)(n+1)s)\mathcal{L}_t^{ns}\big(\mathcal{L}_t^s\mathbf{1}_{U_{(n+1)s}^c}(z)\big) \\ &= \exp(-P(t)(n+1)s) \sum_{w\in f^{-ns}(z)} |(f^{ns})'(w)|_\tau^{-t}\mathcal{L}_t^s\mathbf{1}_{U_{(n+1)s}^c}(w) \\ &= \exp(-P(t)(n+1)s) \sum_{w\in f^{-ns}(z)} \Big(|(f^{ns})'(w)|_\tau^{-t}\cdot \sum_{x\in f^{-s}(w)\cap U_{(n+1)s}^c} |(f^s)'(x)|_\tau^{-t}\Big) \\ &= \exp(-P(t)(n+1)s) \sum_{w\in f^{-ns}(z)\cap U_{ns}^c} \Big(|(f^{ns})'(w)|_\tau^{-t} \sum_{x\in f^{-s}(w)\cap U_s^c} |(f^s)'(x)|_\tau^{-t}\Big) \\ &= e^{-P(t)ns} \sum_{w\in f^{-ns}(z)} |(f^{ns})'(w)|_\tau^{-t}\mathbf{1}_{U_{ns}^c}(w)\Big(e^{-P(t)s} \sum_{x\in f^{-s}(w)} |(f^s)'(x)|_\tau^{-t}\mathbf{1}_{U_s^c}(x)\Big) \\ &\le e^{-P(t)ns} \sum_{w\in f^{-ns}(z)} \Big(|(f^{ns})'(w)|_\tau^{-t}\mathbf{1}_{U_{ns}^c}(w)\|e^{-P(t)s}\mathcal{L}_t^s\mathbf{1}_{U_s^c}\|_\infty\Big) \\ &\qquad \le ||e^{-P(t)ns}\mathcal{L}_t^{ns}\mathbf{1}_{U_n^s}||_\infty||e^{-P(t)s}\mathcal{L}_t\mathbf{1}_{U_s^c}||_\infty \le \frac{1}{2}||e^{-P(t)ns}\mathcal{L}_t^{ns}\mathbf{1}_{U_n^c}||_\infty. \end{aligned}$$

Therefore

$$|| \exp(-P(t)(n+1)s)\mathcal{L}_t^{(n+1)s}\mathbf{1}_{U_{(n+1)s}^c}||_\infty \le \frac{1}{2}||e^{-P(t)ns}\mathcal{L}_t^{ns}\mathbf{1}_{U_n^s}||_\infty.$$

So, by indication,

$$||e^{-P(t)ns}\mathcal{L}_t^{ns}\mathbf{1}_{U_{ns}^c}||_\infty \le 2^{-n} \tag{6.1}$$

for any $n \ge 0$. Now, for any integer $k \ge 0$ write $k = ns + r$, $0 \le r \le s - 1$. Formula (6.1) implies then that

$$||\mathcal{L}_t^k\mathbf{1}_{U_k^c}||_\infty \ \le \ ||\mathcal{L}_t^k\mathbf{1}_{U_{ns}^c}||_\infty \le ||\mathcal{L}_t^r||_\infty||\mathcal{L}_t^{ns}\mathbf{1}_{U_{ns}^c}||_\infty$$

$$\begin{aligned}
&\le Q_s 2^{-n} e^{P(t)ns}\\
&\le Q_s 2^{-\frac{k-r}{s}} e^{P(t)r} e^{-P(t)k}\\
&= Q_s 2^{\frac{r}{s}} e^{-P(t)r} 2^{-\frac{k}{s}} e^{P(t)k}\\
&\le M_s 2^{-\frac{k}{s}} e^{-P(t)k},
\end{aligned}$$

where $M_s = 2^{\frac{s-1}{s}} Q_s \max\{e^{-P(t)r} \colon 0 \le r \le s-1\}$. Thus

$$\overline{P}_U^c \le \overline{\lim}_{n\to\infty} \frac{1}{k} \log ||\mathcal{L}_t^k \mathbf{1}_{U_k^c}||_\infty \le P(t) - \frac{1}{s}\log 2 < P(t).$$

We are done. □

Corollary 6.4. *If $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular function and U is an arbitrary open subset of $\mathbb{C}$ intersecting the Julia set $\mathcal{J}(t)$, then $\mathrm{HD}(K_r(U)) < \mathrm{HD}(\mathcal{J}_r)$.*

Proof. We know that the topological pressure is finite for all $t > \frac{\varrho}{\alpha_1+\tau}$. We also know (see theorem 8.3 in [9]) that $P(\mathrm{HD}(\mathcal{J}_r)) = 0$. Since in addition the function $t \to \underline{P}_U^c(t) \le P(t)$ is continuous (as convex) throughout $\left(\frac{\varrho}{\alpha_1+\tau}, +\infty\right)$, we therefore conclude from Proposition 6.3 that there exist $t \in \left(\frac{\varrho}{\alpha_1+\tau}, \mathrm{HD}(\mathcal{J}_r)\right)$ such that $\underline{P}_U^c(t) < 0$. Lemma 6.1 then yields that $\mathrm{HD}(K_r(U)) \le t < \mathrm{HD}(\mathcal{J}_r)$. □

Corollary 6.5. *If $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular meromorphic function of divergence type then each nice set $U \in \mathcal{U}$ gives rise to a strongly regular IFS. In particular, the function $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is nicely strongly regular.*

Proof. Let $S_U = \{\varphi_e\}_{e\in E}$ be the conformal IFS generated by the nice set U. Fix one $b \in E$ and let $S_{U,b} = \{\varphi_e\}_{e\in E\setminus\{b\}}$. Then $\mathrm{HD}(\mathcal{J}_{S_{U,b}}) \le \mathrm{HD}(K_r(\varphi_b(U))) < \mathrm{HD}(\mathcal{J})) = \mathrm{HD}(\mathcal{J}_{S_U})$, where the inequality sing " $<$ " follows from Corollary 6.4 and the equality sign " $=$ " comes from Theorem 3.4. The system S_U is thus strongly regular because of Theorem 4.3.10 from [7]. □

7. REAL ANALYTICITY OF HAUSDORFF DIMENSION FOR DYNAMICALLY REGULAR MEROMORPHIC FUNCTIONS

Taking fruits of the previous section, in the present short section we provide concrete examples of analytic families that satisfy the hypotheses of Theorem 5.1. In consequence the Hausdorff dimension of their radial set varies in a real–analytic fashion.

Theorem 7.1. *Suppose that $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular meromorphic function of divergence type which belongs to class $\mathcal{S}$. If $\Lambda \subseteq \mathbb{C}$*

is an open set, $\{f_\lambda\}_{\lambda\in\Lambda}$ *is an analytic family (in the sense of Section 5) mermorphic functions and* $f_{\lambda_0} = f$ *for some* $\lambda_0 \in \Lambda$*, then the function* $\Lambda \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$ *is real–analytic in some open neighborhood of* λ_0 *contained in* Λ*.*

Proof. Since our family is analytic, for every $a_{\lambda_0} \in \operatorname{sing}(f_{\lambda_0}^{-1})$ there exists a meromorphic function $\lambda \mapsto a_\lambda \in \operatorname{sing}(f_\lambda^{-1})$ defined on some sufficiently small neighborhood of λ_0. Furthermore, the analyticity of the family $\{f_\lambda\}_{\lambda\in\Lambda}$ applied again entails the functions $\{\lambda \mapsto f_\lambda^n(a_\lambda)\}_{n=0}^\infty$ to form a normal family on some sufficiently small neighborhood of λ_0 for every point a_{λ_0} in $\operatorname{sing}(f_{\lambda_0}^{-1})$. Therefore, see Lemma 9.3 in [9], there exist a holomorphic motion $H\colon \Gamma_{\lambda_0} \times \mathcal{J}_{\lambda_0} \to \hat{\mathbb{C}}$ over same neighborhood $\Gamma_{\lambda_0} \subseteq \Lambda$ of λ_0 such that $H_\lambda(\mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda$ and $H_\lambda \circ f_{\lambda_0} = f_{\lambda_0} \circ H_\lambda$ on $\mathcal{J}_{\lambda_0}$ for all $\lambda \in \Gamma_{\lambda_0}$. Since also the meromorphic function $f_{\lambda_0} : \mathbb{C} \to \hat{\mathbb{C}}$ as dynamically regular is tame, and since, by Corollary 6.5, it is nicely strongly regular, invoking Theorem 5.1 completes the proof. □

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