PRESSURE AND RECURRENCE

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1. Introduction

Let $T : X \to X$ be an ergodic measure preserving transformation with respect to a probability measure μ on X. Let α be a countable measurable generating partition of X with finite entropy and for every $x \in X$ let $\alpha^n(x)$ denote the only element of the partition $\alpha^n = \vee_{j=0}^{n-1} T^{-j}(\alpha)$ containing x. Put

$$\tau_n(x) = \min\{j \geq 1 : T^j(x) \in \alpha^n(x)\}. \tag{1.1}$$

Ornstein an Weiss proved in [OW] that for μ-a.e. $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \log \tau_n(x) = \mathrm{h}_\mu, \tag{1.2}$$

where h_μ is the measure-theoretical entropy of the mapping $T : X \to X$ with respect to the measure μ.

Let us assume now additionally that $T : X \to X$ is a subshift of finite type, $\phi : X \to X$ is a Hölder continuous potential, $\mu = \mu_\phi$ is a unique equilibrium state (Gibbs measure) of T and ϕ, and α is the partition of X into initial cylinders of length 1. Let

$$S_n\phi = \sum_{j=0}^{n-1} \phi \circ T^j. \tag{1.3}$$

We shall use the following important property of Gibbs measures for Hölder continuous potentials (see, e.g.[Bo]): there exists a constant C such that for every $x \in X$

$$C^{-1} \exp(S_n\phi(x) - n\mathrm{P}(\phi)) \leq \mu(\alpha^n(x)) \leq C \exp(S_n\phi(x) - n\mathrm{P}(\phi)) \tag{1.4}$$

where $\mathrm{P}(\phi)$ is the topological pressure of T and ϕ.

$$\mathrm{P}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{A\in\alpha^n} \sup\Big(S_n\phi|_A\Big)\right).$$

Another important property is the following

$$\mathrm{P}(\phi) = \mathrm{h}_\mu + \int_X \phi d\mu \tag{1.5}$$

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The following question arises. Given x, consider the sum

$$\sum_{j=0}^{\tau_n(x)} \mu(\alpha^n(T^i(x))) \tag{1.6}$$

(the sum of measures of cylinders of the partition α^n along the trajectory of x until the time τ_n when this trajectory reaches the initial cylinder $\alpha^n(x)$). The question is whether the limit

$$\frac{1}{n}\log \sum_{j=0}^{\tau_n(x)} \mu(\alpha^n(T^i(x))) \tag{1.7}$$

exists and what its value is equal to. We shall give an answer for μ being a Gibbs measure for a Hölder continuous potential.

There are at least two naive ways to answer the above question. First, according to (1.2), the time τ_n is approximately $\exp(n\mathrm{h}_\mu)$, while the measure of the typical cylinder is close to $\exp(-n\mathrm{h}_\mu)$, so it seems that the limit should be equal to 0.

On the other hand, an atom of the partition α^n should be visited by the trajectory of x with a frequency close to $\mu(\alpha^n)$, thus the sum (1.6) should be rather close to

$$\sum_{A\in\alpha^n} \mu(A)\tau_n(x)\mu(A). \tag{1.8}$$

This suggests that the limit (1.7) should be equal to

$$\lim_{n\to\infty}\left(\frac{1}{n}\log(\tau_n(x)) + \frac{1}{n}\log\sum_{A\in\alpha^n}\mu(A)^2\right) = \lim_{n\to\infty}\left(\frac{1}{n}\log(\tau_n(x)) + \frac{1}{n}\log\int_X \mu(\alpha^n(x))d\mu(x)\right) \tag{1.9}$$

provided that this limit exists. It is then easily seen that the limit (1.9) really exists if μ is a Gibbs measure for a Hölder continuous potential ϕ for a subshift of finite type and equals $\mathrm{h}_\mu + \mathrm{P}(2\phi) - 2\mathrm{P}(\phi)$.

Indeed, the first summand in (1.9) tends to h_μ by (1.2), while the sum $\sum_{A\in\alpha^n}\mu(A)^2$ can be estimated, using the property of Gibbs measures

$$\sum_{A\in\alpha^n}\mu(A)^2 \asymp \sum_{A\in\alpha^n}\sup\Big(\exp(2S_n\phi|_A - 2n\mathrm{P}(\phi)\Big)$$

and we conclude that the limit in (1.7) should be rather equal to $\mathrm{h}_\mu + \mathrm{P}(2\phi) - 2\mathrm{P}(\phi)$, which is greater than 0 provided ϕ is not homologous to a constant (see Remark 1.2 below).

Of course, both "proofs" are wrong. They use some limit estimates for a given time τ_n and a growing number of cylinders α^n. In particular, in the second "proof" we see that the time τ_n is certainly too short to visit all cylinders α^n even once. However, this wrong proof leads us to a correct formula. More precisely, we have the following.

Theorem 1.1. *If $T : X \to X$ is a subshift of finite type, $\phi : X \to X$ is a Hölder continuous potential, $\mu = \mu_\phi$ is a unique equilibrium state of T and ϕ and α is the partition of X into initial cylinders of length 1, then for μ-a.e. $x \in X$*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \exp\Big(S_n\phi \circ T^j(x)\Big) = \mathrm{h}_\mu + \mathrm{P}(2\phi) - \mathrm{P}(\phi). \tag{1.10}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \mu(\alpha^n(T^j(x))) = \mathrm{h}_\mu + \mathrm{P}(2\phi) - 2\mathrm{P}(\phi). \tag{1.11}$$

Notice that for $\phi = 0$ the above theorem follows immediately from the result of Ornstein and Weiss. Thus it can be understood as its generalization.

Remark 1.2. *The value of the second limit in Theorem 1.1:* $\mathrm{h}_\mu + \mathrm{P}(2\phi) - 2\mathrm{P}(\phi)$ *is non-negative. Moreover,* $\mathrm{h}_\mu + \mathrm{P}(2\phi) - 2\mathrm{P}(\phi) = 0$ *iff ϕ is homologous to a constant, i.e. there exists a Hölder –continuous function $g : X \to \mathbf{R}$ and a constant $c \in \mathbf{R}$ such that $\phi = g \circ T - g + c$.*
Indeed, it is well-known that (under the assumptions of Theorem 1.1) the function $t \mapsto P(t\phi)$ is convex and smooth and $\frac{d}{dt}P(t\phi) = \int_X \phi d\mu_{t\phi}$. Moreover, this function is strictly convex unless ϕ is homologous to a constant. Thus, using (1.5),we can write

$$\mathrm{P}(2\phi) = \mathrm{P}(\phi) + \int_1^2 \frac{d}{dt}\mathrm{P}(t\phi) \geq \mathrm{P}(\phi) + \int_X \phi d\mu_\phi = \mathrm{P}(\phi) + \mathrm{P}(\phi) - \mathrm{h}_\mu$$

and the inequality is strict if ϕ is not homologous to a constant. This gives the required inequality.

Roughly speaking, Theorem 1.1 means that the wrong "proof" above gives the correct answer because large cylinders α_n (i.e. cylinders of big measure) are visited by the trajectory of x up to time $\tau_n(x)$ with a frequency close to the limit one (given by the Birkhoff Ergodic Theorem). It turns out that (typically) the time $\tau_n(x)$ is sufficiently long for the integral $\int_X \mu(\alpha^n(x))d\mu(x)$ to be well approximated by the time average $\frac{1}{\tau_n(x)} \sum_{j=0}^{\tau_n(x)} \mu(\alpha^n(T^i(x)))$. The main tool in the (real) proof of Theorem 1.1 is provided by a detailed analysis of large deviations of the sums $S_n(\phi)$.

After writing this note we found out that the questions of this spirit have been considered before for sequences of independent identically distributed random variables. It seems that this research was originated by "A new law of large numbers" (see [ER]), where the average $\frac{1}{k}U_n$, $U_n = \max_{0\leq i\leq n-k}(X_i + \cdots + X_{i+k})$, $k = [c\log n]$ was considered. See also [DDL]. A result analogous to Theorem 1.1 in the context of independent equally distributed random variables has appeared in [To].

2. Proofs

Let $T : X \to X$ be an ergodic measure preserving transformation with respect to a probability measure μ on X. Let ϕ be a bounded measurable function defined on X.

We introduce the following notation.

$$c_{\phi,\mu}(t) = \limsup_{n\to\infty} \frac{1}{n} \log \int \exp(S_n t\phi) d\mu.$$

where $S_n(\phi)$ is defined in 1.3. In statistical mechanics, $c_{\phi,\mu}(t)$ is usually called the free energy function, provided that lim sup can be replaced by lim. The notion of free energy is closely related to the notion of topological pressure. We have the following simple

Lemma 2.1. *If $T : X \to X$ is a subshift of finite type, $\phi : X \to X$ is a Hölder continuous potential and μ_ϕ is a unique equilibrium state of T and ϕ, then $c_{\phi,\mu_\phi}(t) = \mathrm{P}((t+1)\phi) - \mathrm{P}(\phi)$, where $\mathrm{P}((t+1)\phi)$ is the topological pressure of the function $(t+1)\phi$. In particular, in this case,* lim sup *can be replaced by* lim *in the definition of c_{ϕ,μ_ϕ}.*

Proof. As before, denote by α the partition into cylinders of length 1. It is well-known that for every $x \in X$

$$\mu_\phi(\alpha^n(x)) \asymp \exp\Big(S_n\phi(x) - \mathrm{P}(\phi)n\Big),$$

(see (1.4)) where the comparability relation $A \asymp B$ means that the quotients A/B and B/A are uniformly bounded from above (and so also from below). Let α be the partition of X into cylinders of length 1 and for every cylinder $A \in \alpha^n$ choose one point $x_A \in A$. We then get

$$\begin{aligned}
\log \int \exp(S_n t\phi) d\mu &\asymp_+ \log \sum_{A\in\alpha^n} \mu_\phi(A) \exp\Big(S_n t\phi(x_A)\Big) \\
&\asymp_+ \log \sum_{A\in\alpha^n} \exp\Big(S_n\phi(x_A) - \mathrm{P}(\phi)n\Big) \exp\Big(S_n t\phi(x_A)\Big) \\
&= \log \sum_{A\in\alpha^n} \exp\Big(S_n((t+1)\phi)(x_A) - \mathrm{P}(\phi)n\Big),
\end{aligned}$$

where the comparability relation $A \asymp_+ B$ means that the differences $A - B$ and $B - A$ are uniformly bounded from above (and so also from below). It now immediately follows from the definition of $c_{\phi,\mu_\phi}(t)$ and from the definition of topological pressure that

$$c_{\phi,\mu_\phi}(t) = \mathrm{P}((t+1)\phi) - \mathrm{P}(\phi).$$

■

Remark 2.2. *We use both free energy and pressure even though these notions are very closely related to each other in our case. The free energy is usually used in the statement of Large Deviation Theorem which will be our main tool in the proof of Lemma 2.4.*

In order to prove Theorem 1.1, we shall estimate the sum $\sum_{j=0}^{\tau_n(x)} \exp\Big(S_n\phi\circ T^j(x)\Big)$ from above (Proposition 2.3) and from below (Lemma 2.4). Notice that the estimate from above works under much weaker assumptions than the estimate from below.

Proposition 2.3. *If $T : X \to X$ is an ergodic measure preserving transformation with respect to a probability measure μ, the partition α and the time $\tau_n(x)$ are defined as in (1.1). Let ϕ be a bounded measurable function. Then for μ-a.e. $x \in X$*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \exp\Big(S_n\phi \circ T^j(x)\Big) \le \mathrm{h}_\mu + c_{\phi,\mu}(1),$$

Proof. Put

$$g_n(x) = \exp\Big(S_n\phi(x)\Big).$$

Fix $\epsilon > 0$ and for every $n \ge 1$ consider the set

$$B_n(\epsilon) = \left\{ x \in X : \sum_{j=0}^{\exp\left((\mathrm{h}_\mu+\frac{\epsilon}{3})n\right)} g_n \circ T^j(x) > \exp\Big((\mathrm{h}_\mu + c_{\phi,\mu}(1) + \epsilon)n\Big) \right\}.$$

Applying Tchebyschev's inequality we obtain

$$\begin{aligned}
\mu(B_n(\epsilon)) &\le \exp\Big(-(\mathrm{h}_\mu + c_{\phi,\mu}(1) + \epsilon)n\Big) \int \sum_{j=0}^{\exp\left((\mathrm{h}_\mu+\frac{\epsilon}{3})n\right)} g_n \circ T^j d\mu \\
&\le \exp\Big(-(\mathrm{h}_\mu + c_{\phi,\mu}(1) + \epsilon)n\Big) \exp\Big((\mathrm{h}_\mu + \frac{\epsilon}{3})n\Big) \int g_n d\mu \\
&= \exp\Big(-(c_{\phi,\mu}(1) + \frac{2}{3}\epsilon)n\Big) \int g_n d\mu.
\end{aligned}$$

But it follows from the definition of $c_{\phi,\mu}(1)$ that for all n large enough, say $n \ge n_\epsilon$, $\int g_n d\mu \le \exp\Big((c_{\phi,\mu}(1) + \frac{\epsilon}{3})n\Big)$. Consequently

$$\mu(B_n(\epsilon)) \le \exp\Big(-\frac{\epsilon}{3}n\Big)$$

for all $n \ge n_\epsilon$. Thus the series $\sum_{n=1}^{\infty} \mu(B_n(\epsilon))$ converges and it follows from the Borel-Cantelli lemma that there exists a measurable set A'_ϵ such that $\mu(A'_\epsilon) = 1$ and each point form A'_ϵ belongs to finitely many sets $B_n(\epsilon)$ only. In particular

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\exp\left((\mathrm{h}_\mu+\frac{\epsilon}{3})n\right)} g_n \circ T^j(x) \le \mathrm{h}_\mu + c_{\phi,\mu}(1) + \epsilon$$

for all $x \in A'_e$. Since by (1.2), $\lim_{n\to\infty} \frac{1}{n} \log \tau_n(x) = \mathrm{h}_\mu$ for μ-a.e. $x \in X$, we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \leq \mathrm{h}_\mu + c_{\phi,\mu}(1) + \epsilon$$

for all points x in some measurable set A_ϵ with $\mu(A_\epsilon) = 1$. Putting $A = \bigcap_{k\geq 1} A_{1/k}$, we therefore have $\mu(A) = 1$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \leq \mathrm{h}_\mu + c_{\phi,\mu}(1)$$

for all $x \in A$. We are done. ■

Our main technical result is the following.

Lemma 2.4. *If $T : X \to X$ is a subshift of finite type, $\phi : X \to X$ is a Hölder continuous potential and $\mu = \mu_\phi$ is a unique equilibrium state of T and ϕ, then for μ-a.e. $x \in X$*

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \exp\Big(S_n\phi \circ T^j(x)\Big) \geq \mathrm{h}_\mu + c_{\phi,\mu}(1). \tag{2.1}$$

Proof. Replacing ϕ by $\phi - \mathrm{P}(\phi)$ if necessary, we may assume without loss of generality that $\mathrm{P}(\phi) = 0$. Now, we can also assume that $\phi < 0$ in X. Indeed, since $\mathrm{P}(\phi) = 0$, it follows that there exists $k \in \mathbb{N}$ such that for every x we have $S_k(\phi) < 0$. So, we can replace ϕ by $\phi' = \frac{S_k\phi}{k}$. The Gibbs states μ_ϕ and $\mu_{\phi'}$ are the same. Since $\exp\Big(S_n\phi \circ T^j(x)\Big)$ differs from $\exp\Big(S_n\phi' \circ T^j(x)\Big)$ by a bounded factor, the left-hand side of the inequality 2.1 does not change when ϕ is replaced by ϕ'. By the same reason, the right-hand side does not change either.

From now on we assume that $\mathrm{P}(\phi) = 0$ and $\phi < 0$ in X.

Let us assume that ϕ is homologous to a constant. In this case the pressure function $t \mapsto P(t\phi)$ is affine and (see Remark 1.2) $\mathrm{h}_\mu + P(2\phi) - 2P(\phi) = 0$. Moreover, in this case $\mu = \mu_\phi$ is simply the measure of maximal entropy and $\mu(\alpha^n) \asymp \exp(-n\mathrm{h}_\mu)$. Thus, in this case the statement of Theorem 1.1 follows directly from the result of [OW].
So from now on, we also assume that ϕ is not homologous to a constant.

Put $\psi = \phi + \mathrm{h}_\mu$. We then have

$$\begin{aligned}\sum_{i=0}^{\tau_n(x)} \exp S_n\phi(T^i(x)) &= \exp(-n\mathrm{h}_\mu) \sum_{i=0}^{\tau_n(x)} \exp(S_n\phi(T^i x) + n\mathrm{h}_\mu) \\ &= \exp(-n\mathrm{h}_\mu) \sum_{i=0}^{\tau_n(x)} \exp(S_n\psi(T^i(x))).\end{aligned} \tag{2.2}$$

Fix $\delta > 0$. Then, using (2.2), we get

$$\sum_{i=0}^{\tau_n} \exp S_n\phi(T^i(x)) \geq \exp(-n\mathrm{h}_\mu)\exp(n\delta)\cdot \#\{i \in (0,\tau_n) : S_n\psi(x) > n\delta\}.$$

Since $\mathrm{P}(\phi) = 0$, we have $\int \psi d\mu = 0$. Thus the Large Deviation Theorem (see [El], Th. II.6.1) gives

$$\lim_{n\to\infty} \frac{1}{n}\log\mu\left(\left\{x : \frac{S_n\psi(x)}{n} > \delta\right\}\right) = -\hat{I}(\delta), \tag{2.3}$$

where $\hat{I}(\delta)$ is the Legendre–Fenchel transform of the free energy function

$$c_{\psi,\mu}(t) = \lim_{n\to\infty}\frac{1}{n}\log\int \exp S_n(t\psi)d\mu.$$

Notice that $\hat{I}(\delta) = I(-\mathrm{h}_\mu + \delta)$ where I is the Legendre–Fenchel transform of the free energy function

$$c_{\phi,\mu}(t) = \lim_{n\to\infty}\frac{1}{n}\log\int \exp S_n(t\phi)d\mu.$$

For every $n \geq 1$ put

$$B_\delta(n) = \left\{x : \text{there exists } y \in \alpha^n(x) \text{ such that } \frac{S_n\psi(y)}{n} > \delta\right\}$$

Since ψ is a Hölder continuous function, there exists a constant C independent of n such that if $y \in \alpha^n(x)$ then $|S_n\psi(x) - S_n\psi(y)| < C$. Fix $\epsilon > 0$. Since the transform $\hat{I}$ is continuous, it follows from (2.3) that for all n large enough

$$\mu(B_\delta(n)) \geq \mu\left(\left\{x : \frac{S_n\psi(x)}{n} > \delta - \frac{C}{n}\right\}\right) > \exp(-n(\hat{I}(\delta)+\epsilon)). \tag{2.4}$$

The idea of the computation below is the following. For an integer $M = M(n)$ we shall estimate from below the number of points in the trajectory of x under T^n: $x, T^n(x), \ldots, T^{Mn}(x)$ which fall into the set $B_\delta(n)$. As a tool, we use the Tchebyschev's inequality together with weak dependence of random variables $\chi_{B_\delta(n)} \circ T^{nj}$. More precisely, we conclude that the frequency of "times" $k \in \{1,\ldots,M\}$ such that $T^{nk}(x) \in B_\delta(n)$ is close to the measure of $B_\delta(n)$. This estimate works for all x outside some set A_n, where $\mu(A_n)$ is close to 0 (see (2.7),(2.8),(2.10)). The "time" M under consideration depends on n and is related to the typical return time τ_n (see (2.8)). If x is chosen so that for every $n > n_0 = n_0(x)$ the point $x \notin A_n$, we will get the estimate of

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{i=0}^{\tau_n(x)} \exp S_n\phi(T^i(x))$$

from below in terms of the value of Legendre-Fenchel transform of the free energy function evaluated at δ (see (2.13)). Finally, we examine the range of possible δ's. This will lead us (using the Legendre-Fenchel transform again) to the inequality (2.1).

So, let $Y_n = \chi_{B_\delta(n)}$, where $\chi_{B_\delta(n)}$ is the characteristic function of the set $B_\delta(n)$. Notice that the function Y_n is constant on each cylinder of n-th generation. We have $E(Y_n) = \mu(B_\delta(n))$ and $D^2(Y_n) = \mu(B_\delta)(n)(1 - \mu(B_\delta(n)))$. For every integer $M \geq 1$ set

$$L_{M,n} = Y_n + Y_n \circ T^n, \dots, Y_n \circ T^{Mn}.$$

Our aim now is to estimate the variance $D^2(L_{M,n})$ from above. Let $\tilde{Y}_n = Y_n - E(Y_n)$. Then

$$\begin{aligned} D^2(L_{M,n}) &= E((\tilde{Y}_n + \tilde{Y}_n \circ T^n + \dots + \tilde{Y}_n \circ T^{Mn})^2) \\ &= (M+1)E(\tilde{Y}_n^2) + 2(ME(\tilde{Y}_n(\tilde{Y}_n \circ T^n)) + (M-1)E(\tilde{Y}_n(\tilde{Y}_n \circ T^{2n})) + \dots \\ &\quad \dots + E(\tilde{Y}_n(\tilde{Y}_n \circ T^{Mn}))). \end{aligned} \tag{2.5}$$

For every $l \geq 0$ let

$$\gamma(l) = \sup\left\{ \frac{|\mu(C_n \cap T^{-(l+n)}(C'_n) - \mu(C_n)\mu(C'_n)|}{\mu(C_n)\mu(C'_n)} : n \geq 1,\, C_n, C'_n \in \alpha^n \right\}.$$

Since the random variable $\tilde{Y}_n$ is constant on each cylinder of length n, we get that

$$\begin{aligned} E(\tilde{Y}_n(\tilde{Y}_n \circ T^{jn})) &= \sum_{C_n}\sum_{C'_n} \mu(C_n \cap T^{-jn}(C'_n))\tilde{Y}_n|_{C_n}\tilde{Y}_n|_{C'_n} \\ &= \sum_{C_n}\sum_{C'_n} \mu(C_n)\mu(C'_n)\tilde{Y}_n|_{C_n}\tilde{Y}_n|_{C'_n}\frac{\mu(C_n \cap T^{-jn}(C'_n))}{\mu(C_n)\mu(C'_n)} \\ &= \sum_{C_n}\sum_{C'_n} \mu(C_n)\mu(C'_n)\tilde{Y}_n|_{C_n}\tilde{Y}_n|_{C'_n} \\ &+ \sum_{C_n}\sum_{C'_n} \mu(C_n)\mu(C'_n)\tilde{Y}_n|_{C_n}\tilde{Y}_n|_{C'_n}\frac{\mu(C_n \cap T^{-nj}(C'_n)) - \mu(C_n)\mu(C'_n)}{\mu(C_n)\mu(C'_n)} \end{aligned}$$

The first summand is equal to $(E(\tilde{Y}_n))^2 = 0$. The second summand can be estimated from above by $\gamma((j-1)n))(E(|\tilde{Y}_n|))^2 \leq \gamma((j-1)n)E(\tilde{Y}_n^2)$. The sequence $\{\gamma(k)\}_{k=0}^\infty$ converges to 0 exponentially fast, this is a well-known property of Gibbs measures, see e.g. [Bo]. Using (2.5) we obtain that

$$D^2(L_{M,n}) \leq (M+1)E(\tilde{Y}_n^2)(1 + 2\sum_{j=1}^\infty \gamma((j-1)n)) \leq C_1 M E(\tilde{Y}_n^2), \tag{2.6}$$

where C_1 is some universal constant independent of M and n. By Tchebyschev's inequality we get

$$\mu\left(\left\{x : \left|\frac{L_{M,n}(x)}{M} - \mu(B_\delta(n))\right| > \eta\right\}\right) < \frac{D^2\left(\frac{L_{M,n}}{M}\right)}{\eta^2}. \tag{2.7}$$

Put

$$M = M(n) = \left[\frac{\exp n(\mathrm{h}_\mu - \epsilon)}{n}\right] \text{ and } \eta = \frac{1}{2}\mu(B_\delta(n)) \tag{2.8}$$

and set

$$A_n = \left\{ x : \left| \frac{L_{M(n),n}(x)}{M(n)} - \mu(B_\delta(n)) \right| > \frac{1}{2}\mu(B_\delta(n)) \right\}. \tag{2.9}$$

Since $E(\tilde{Y}_n^2) = D^2(Y_n) = \mu(B_\delta(n))(1 - \mu(B_\delta(n)))$, using (2.4 and (2.6), for all n large enough, we get that

$$\begin{aligned} \mu\Big(A_n\Big) &\leq \frac{D^2\left(\frac{L_{M(n),n}}{M(n)}\right)}{(\frac{1}{2}\mu(B_\delta(n)))^2} \leq \frac{C_2 M(n)\mu(B_\delta(n))(1-\mu(B_\delta(n)))}{M(n)^2(\mu(B_\delta(n)))^2} \\ &\leq C_3 \frac{n}{\exp(n(\mathrm{h}_\mu - \epsilon))\exp(-n(\hat{I}(\delta)+\epsilon))} = C_3 n \exp\Big(n(\hat{I}(\delta) - \mathrm{h}_\mu + 2\epsilon)\Big), \end{aligned} \tag{2.10}$$

where C_2 and C_3 are some universal constants. We therefore conclude that if $\hat{I}(\delta) < \mathrm{h}_\mu$ and $\epsilon > 0$ is small enough, then the series $\sum \mu(A_n)$ converges. Hence, by Borel-Cantelli Lemma, for μ-a.e x there exists $n_0 = n_0(x)$ such that for all $n \geq n_0(x)$, $x \notin A_n$. In view of (1.2) we may assume without loss of generality that $\tau_n(x) > \exp(n(\mathrm{h}_\mu - \epsilon))$ for all $n \geq n_0(x)$. Thus, for all $n \geq n_0$ we get that

$$\begin{aligned} \#\{i \in \{0,..\tau_n(x)\} : S_n\psi(T^i(x)) > n\delta\} &\geq \#\{i \in \{0,\dots,n^{-1}\exp(n(\mathrm{h}_\mu - \epsilon))\} : T^i(x) \in B_\delta(n)\} \\ &\geq \frac{1}{2}\mu(B_\delta(n))\exp(n(\mathrm{h}_\mu - \epsilon)). \end{aligned}$$

Finally, using (2.2) and (2.4) we obtain

$$\sum_{i=0}^{\tau_n(x)} \exp S_n\phi(T^i(x)) \geq \frac{1}{2}\exp(-n\mathrm{h}_\mu)\exp(n\delta)\exp(-n(\hat{I}(\delta)+\epsilon))\exp(n(\mathrm{h}_\mu - \epsilon)) \tag{2.11}$$

Therefore,

$$\liminf_{n\to\infty} \frac{1}{n}\log \sum_{i=0}^{\tau_n(x)} \exp S_n\phi(T^i(x)) \geq -\mathrm{h}_\mu + \delta - \hat{I}(\delta) - 2\epsilon + \mathrm{h}_\mu \tag{2.12}$$

Letting $\epsilon \searrow 0$, we get

$$\liminf_{n\to\infty} \frac{1}{n}\log \sum_{i=0}^{\tau_n(x)} \exp S_n\phi(T^i(x)) \geq \delta - \hat{I}(\delta) \tag{2.13}$$

The reasoning above works for every $\delta > 0$ such that the series $\sum \mu(A_n)$ is convergent. As we have noticed, the sufficient condition for this is that $\hat{I}(\delta) < \mathrm{h}_\mu$. So, in particular, one can take an arbitrary δ such that $\hat{I}(\delta) < \delta < \mathrm{h}_\mu$. Notice that the domain of I (the Legendre-Fenchel transform of $c_{\phi,\mu}(t)$) is contained in $(-\infty, 0)$, (roughly speaking, z is in the domain of I if there exists t such that $c'_{\phi,\mu}(t) = z$). Consequently, the domain of $\hat{I}$ is contained in $(-\infty, \mathrm{h}_\mu)$. Therefore, the estimate (2.13) is fulfilled for an arbitrary δ in the domain of $\hat{I}(\delta)$ for which $\hat{I}(\delta) < \delta$. We shall argue now that such δs exist. Indeed, $\hat{I}$ is differentiable ([El] Th. VI.5.6) and it attains its minimum $\hat{I}(0) = 0$ (so also $\hat{I}'(0) = 0$). Therefore, we can write the following estimate

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{i=0}^{\tau_n(x)} \exp S_n\phi(T^i(x)) \geq \sup_\delta\{\delta - \hat{I}(\delta)\}.$$

where the supremum is evaluated over all δ in the domain of the function $\hat{I}$. But, again by definition of Legendre transform, this supremum is precisely the value of the Legendre-Fenchel transform of $\hat{I}$ at the point 1. The Legendre-Fenchel transform of I is again $c_{\phi,\mu}(t)$ (see e.g. [El],Th. VI.5.3) and the Legendre-Fenchel transform of $\hat{I}$ evaluated at the point t is equal to

$$c_{\phi,\mu}(t) + \mathrm{h}_\mu \cdot t.$$

So, its value at 1 equals $c_{\phi,\mu}(1) + \mathrm{h}_\mu$. This shows that

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{i=0}^{\tau_n(x)} \exp S_n\phi(T^i(x)) \geq \mathrm{h}_\mu + c_{\phi,\mu}(1).$$

We are done. ■

We now get the main result of this paper, Theorem 1.1, as an immediate consequence of Proposition 2.3, Lemma 2.4 and Lemma 2.1.

The following proposition along with (1.2) shows that Theorem 1.1 can be used to calculate topological pressure provided that given are generic points of equilibrium states of Hölder continuous potentials $2^{-j}\phi$, $j \geq 0$.

Proposition 2.5. *If $T : X \to X$ is a continuous map of a compact metric space X and if $\phi : X \to X$ is a continuous potential, then*

$$\mathrm{P}(\phi) = \mathrm{h}_{\mathrm{top}}(T) + \sum_{j=0}^{\infty}(\mathrm{P}(2^{-j}\phi) - \mathrm{P}(2^{-(j+1)}\phi)).$$

Moreover

$$\left|\mathrm{P}(\phi) - \Big(\mathrm{h}_{\mathrm{top}}(T) + \sum_{j=0}^{n}(\mathrm{P}(2^{-j}\phi) - \mathrm{P}(2^{-(j+1)}\phi))\Big)\right| \leq 2^{-(n+1)}||\phi||_\infty.$$

Proof. Since for every $n \geq 0$

$$\mathrm{P}(\phi) = \sum_{j=0}^{n}(\mathrm{P}(2^{-j}\phi) - \mathrm{P}(2^{-(j+1)}\phi)) + \mathrm{P}(2^{-(n+1)}\phi),$$

we obtain

$$\left|\mathrm{P}(\phi) - \left(\mathrm{h_{top}}(T) + \sum_{j=0}^{n}(\mathrm{P}(2^{-j}\phi) - \mathrm{P}(2^{-(j+1)}\phi))\right)\right| = |\mathrm{P}(2^{-(n+1)}\phi) - \mathrm{h_{top}}(T)|$$
$$= |\mathrm{P}(2^{-(n+1)}\phi) - \mathrm{P}(0)|$$
$$\leq 2^{-(n+1)}||\phi||_\infty.$$

We are done. ■

Finally, let us discuss what can be proved in a more general context. In general, without the strong assumption of Theorem 1.1 we are only able to prove the following straightforward lower bound:

Proposition 2.6. *Suppose that $T : X \to X$ is a transformation preserving an ergodic probability measure μ. Let α be a countable measurable generating partition of X with finite entropy and let τ_n be defined as in (1.1). Let $\phi : X \to \mathbf{R}$ be a bounded measurable function. Then for μ-almost every $x \in X$ we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \exp(S_n\phi \circ T^j(x)) \geq \mathrm{h}_\mu + \int_X \phi d\mu$$

Proof. Fix $M > 0$ such that $|\phi| < M$. Using Jensen's inequality, we have

$$\frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \exp(S_n\phi \circ T^j(x)) = \frac{1}{n}\log(\tau_n(x) + 1) + \frac{1}{n} \log \frac{1}{\tau_n(x) + 1} \sum_{j=0}^{\tau_n(x)} \exp S_n\phi(T^j(x))$$
$$\geq \frac{1}{n}\log(\tau_n(x) + 1) + \frac{1}{n}\frac{1}{\tau_n(x) + 1} \sum_{j=0}^{\tau_n(x)} S_n\phi(T^j(x))$$

The first summand tends to h_μ as n tends to ∞. The second one can be written as

$$\frac{1}{n}\frac{1}{\tau_n(x) + 1}\left(\sum_{j=n}^{\tau_n(x)-n} n\phi(T^j(x)) + \sum_{j=0}^{n-1}(j + 1)\phi(T^j(x)) + \sum_{j=\tau_n(x)-n+1}^{\tau_n(x)} (\tau_n - j + 1)\phi(T^j(x))\right)$$

It is easy to see that

$$\frac{1}{n}\frac{1}{\tau_n(x) + 1}\left(\sum_{j=n}^{\tau_n(x)-n} n\phi(T^j(x))\right) = \frac{1}{\tau_n(x) + 1}\left(\sum_{j=n}^{\tau_n(x)-n} \phi(T^j(x))\right)$$

tends to $\int_X \phi d\mu$, while the remaining part of the sum can be estimated by $\frac{1}{\tau_n(x)+1} \cdot 2nM$. This tends to 0 a.e. since $\tau_n(x)$ grows exponentially fast for almost every x. ■

Let us recall the general estimate from above (see Proposition 2.3):

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{\tau_n(x)} \exp\Big(S_n\phi \circ T^j(x)\Big) \le \mathrm{h}_\mu + c_{\phi,\mu}(1),$$

Of course, $c_{\phi,\mu}(1) \ge \int \phi d\mu$, but usually the inequality is sharp and, in general we do not get any precise formula analogous to (1.11).

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