THE DOUBLING PROPERTY OF CONFORMAL MEASURES OF INFINITE ITERATED FUNCTION SYSTEMS

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ABSTRACT. We provide sufficient conditions for the conformal measures induced by regular conformal infinite iterated function systems to satisfy the doubling property. We apply these conditions to iterated function systems derived from the continued fraction algorithm -continued fractions with restricted entries. For these systems our conditions are expressed in terms of the asymptotic density properties of the allowed entries. As examples, we give some relatively large classes of sets of continued fractions with restricted entries for which the corresponding conformal measures have the doubling property. Similarly, we give some other classes for which the conformal measure does not have the doubling property.

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Recall that a Borel probability measure m defined on a compact metric space (Y, ρ) satisfies the doubling property provided that there exists a constant $C > 0$ and $\tau > 1$ such that

$$m(B(x, \tau r)) \le Cm(B(x, r)), \tag{0.1}$$

for m almost all $x \in Y$ (or equivalently for all x in the topological support of m) and all $r \le 1$. A trivial observation is that instead of 1 we can equivalently require r to be less than or equal to any fixed positive number. Since the property (0.1) holds for some $\tau > 1$ if and only if it holds when $\tau = 2$, it is appropriate to call this the doubling property although it has been given several names. The doubling property and some closely related variants play an important role in geometric measure theory: Vitali and differentiation type theorems (see, for examples [Fa], [Fe], [M1]). For some additional consequences of the doubling property see for example [M2] (John-Nirenberg inequality), [GS] and [JW]. Thus, it is important to know whether a measure has this property or not. Some prime examples of such measures arise in conformal dynamics. If one has either a finite conformal iterated function system satisfying the open set condition or a hyperbolic rational function of the Riemann sphere $\overline{\mathbb{C}}$, then there is a natural probability measure m called the conformal measure defined on J, the limit set of this system or the Julia set respectively. The measure m satisfies the doubling condition. In fact, letting h denote the Hausdorff dimension of J, it is well-known there is a constant $C \ge 1$ such that for all $x \in J$,

$$C^{-1} \le \frac{m(B(x, r))}{r^h} \le C,$$

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for all sufficiently small r. See, for example, [MU1], Lemma 3.14 for the iterated function system case. Going beyond the hyperbolic case, the h-dimensional conformal measure m for a parabolic rational function or a finite parabolic conformal iterated function system has been shown to satisfy the doubling property respectively in [SU1] and [SU2].

In this paper, we investigate the conformal measure, if there is one, induced by an infinite conformal iterated function system. We recall that the iteration of infinitely many conformal maps naturally arises in several contexts. For example, one is naturally led to consider infinite systems when one is dealing with a finite system of conformal maps which have some cusps or singularities, the parabolic systems mentioned above, see e.g. [MU3], [MU4], [SU2]. Another natural class of examples arise from the investigation of sets of continued fractions with restricted entries. We determine here some easily verifiable conditions under which the conformal measure satisfies (or does not satisfy) the doubling condition. This continues our study of the geometric properties of these systems [MU1], [MU2], [MMU], [JJM] and [Ur]. As a special case, we give particular attention to the systems generated by continued fractions with restricted entries. Our sufficient conditions for general systems then have interpretations in terms of the arithmetic properties of the allowed entries. The case where the each entry must be a prime remains unsolved.

1. Preliminaries

Let us describe first the setting of conformal (infinite) iterated function systems introduced in [MU1]. Let I be a countable index set or alphabet with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric space X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \le s\rho(x, y)$, for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We define the limit set, J, of this system as the image of the coding space under a coding map as follows. Let I^n denote the space of words of length n, I^∞ the space of infinite sequences of symbols in I, $I^* = \bigcup_{n\ge1} I^n$ and for $\omega \in I^n$, $n \ge 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \dots \omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \ge 1$, converge to zero and since they form a decreasing family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = J_S = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the Open Set Condition (OSC) if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. (We do not exclude the possibility that $\overline{\phi_i(U)} \cap \overline{\phi_j(U)} \neq \emptyset$.) A system satisfying (OSC) is said to satisfy the Strong Open Set Condition (SOSC) if $J_S \cap U \neq \emptyset$ and it is said to satisfy the Super Strong Open Set Condition (SSOSC)if $\overline{J_S} \subset U$.

An iterated function system S satisfying the Open Set Condition is said to be conformal if $X \subset I\!R^d$ for some $d \geq 1$ and the following conditions are satisfied.

- **(1a):** $U = \mathrm{Int}_{I\!R^d}(X)$.
- **(1b):** There exists an open connected set V such that $X \subset V \subset I\!R^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V. (Note that for $d = 1$ this just means that all the maps ϕ_i, $i \in I$, are C^1 monotone diffeomorphisms, for $d \geq 2$ the words C^1 conformal mean holomorphic or antiholomorphic, and for $d > 2$ the maps ϕ_i, $i \in I$ are Möbius transformations. The proof of the last statement can be found in [BP] for example, where it is called Liouville's theorem.)
- **(1c):** There exist $\gamma, l > 0$ such that for every $x \in X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.
- **(1d):** Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets C of V

$$\mathrm{diam}(\phi_\omega(C)) \leq ||\phi'_\omega||\mathrm{diam}(C) \tag{1.1}$$

and

$$\mathrm{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||, \tag{1.2}$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \geq 1$ is a constant depending only on V. Moreover,

$$\mathrm{diam}(\phi_\omega(X)) \geq D^{-1}||\phi'_\omega|| \tag{1.3}$$

and

$$\phi_\omega(B(x, r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r) \tag{1.4}$$

for every $x \in X$, every $0 < r \leq \mathrm{dist}(X, \partial V)$, and every word $\omega \in I^*$.

As was demonstrated in [MU1], infinite conformal iterated function systems, unlike finite systems, may not possess a conformal measure. There are even continued fraction systems which do not possess a conformal measure, see [MU2], Example 6.5. Thus, the infinite systems naturally break into two main classes, irregular and regular. This dichotomy can be determined from either the existence of a zero of the topological pressure function or, equivalently, the existence of a conformal measure. The topological pressure function, $\mathrm{P}(t)$, $t \geq 0$, is defined as follows. For every integer $n \geq 1$, define

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi_\omega'||^t.$$

and set

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(t).$$

For a conformal system S, we sometimes write $\psi_S = \psi_1 = \psi$. The finiteness parameter, θ_S, of the system S is defined by $\inf\{t : \psi(t) < \infty\} = \theta_S$. In [MU1], it was shown that the topological pressure function $\mathrm{P}(t)$ is non-increasing on $[0, \infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$ and $\mathrm{P}(d) \leq 0$. Of course, $\mathrm{P}(0) = \infty$ if and only if I is infinite. In [MU1] (see Theorem 3.15) we have proved the following characterization of the Hausdorff dimension of the limit set J_S, which will be denoted by $h = h_S$.

Theorem 1.1.

$$h_S = \sup\{h_{(J_F)} : F \subset I \textit{ is finite }\} = \inf\{t : \mathrm{P}(t) \leq 0\}.$$

If $\mathrm{P}(t) = 0$, *then* $t = h_S$.

We called the system S regular provided that there is some t such that $\mathrm{P}(t) = 0$. It follows from [MU1] that t is unique. Also, the system is regular if and only if there is a t-conformal measure. A Borel probability measure m is said to be t-conformal provided $m(J_S) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \neq j$.

There are some natural subclasses within the regular systems. A system $S = \{\phi_i\}_{i\in I}$ is said to be strongly regular if $0 < \mathrm{P}(t) < \infty$ for some $t \geq 0$. As an immediate application of Theorem 1.1 we get the following.

Theorem 1.2. *A conformal system S is strongly regular if and only if $h > \theta$.*

Also, in [MU1] we called a system $S = \{\phi_i\}_{i\in I}$ hereditarily regular or cofinitely regular provided every nonempty subsystem $S' = \{\phi_i\}_{i\in I'}$, where I' is a cofinite subset of I, is regular. A finite system is cofinitely regular and for an infinite system, we showed in [MU1] that whether a system is cofinitely regular can be also determined from the pressure function:

Theorem 1.3. *An infinite system S is cofinitely or hereditarily regular if and only if* $\mathrm{P}(\theta) = \infty \Leftrightarrow \psi(\theta) = \infty \Leftrightarrow \{t : \mathrm{P}(t) < \infty\} = (\theta, \infty) \Leftrightarrow \{t : \psi(t) < \infty\} = (\theta, \infty)$.

We end this section with a short description of the contents of this paper. In Section 2 we provide some general sufficient conditions for conformal measures to satisfy the doubling property. In Section 3 we consider the maps $\phi_n : [0,1] \to [0,1]$, for $n \in I\!N$ given by the formula

$$\phi_n(x) = \frac{1}{x+n},$$

and an arbitrary subset I of positive integers $I\!N$. We investigate the corresponding iterated function system $S_I = \{\phi_i\}_{i\in I}$ called there a continued fraction iterated function system. Its limit set J_I consists of all those $x \in (0,1)$ that each partial denominator x_i, $i \geq 1$ in the continued fraction expansion

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots}}}}$$

is in I. In Theorem 3.4 an effective sufficient condition for the conformal measure of the system S_I to satisfy the doubling property is provided. This condition is in terms of the arithmetic properties of the infinite set I of positive integers. This continues our theme from [MU2] that many geometric measure theoretic properties of these system are reflected in the arithmetic properties of the index set I. In Section 4 we describe some general classes of cofinitely regular continued fraction systems whose conformal measures are proven to either satisfy or fail to satisfy the doubling property.

2. General results

Let us recall that a Borel measure ν defined on a metric space (Y, ρ) satisfies the doubling property provided that there exists a constant $C > 0$ such that

$$m(B(x, 2r)) \leq Cm(B(x, r))$$

for all $x \in Y$ and all $r \geq 0$.

Theorem 2.1. *Suppose that $S = \{\phi_i\}_{i\in I}$ is a regular conformal iterated function system satisfying the (SSOSC) and that m is the corresponding conformal measure. Suppose also that there exist constant $\gamma \geq 1$ and $C > 0$ with the following two properties.*

- **(a):** $\forall i \in I \, \forall x \in \phi_i(J) \quad m\Big(B(x, 2K\gamma \mathrm{diam}(\phi_i(X)))\Big) \leq C||\phi_i'||^h$.
- **(b):** $\forall i \in I \ \forall x \in \phi_i(J) \ \forall r \geq \gamma \mathrm{diam}(\phi_i(X)) \qquad m(B(x, 2r)) \leq Cm(B(x, r))$.

Then the conformal measure m satisfies the doubling property.

Proof. The (SSOSC) says that

$$\zeta = \mathrm{dist}(\overline{J}, I\!R^d \setminus X) > 0.$$

Put $\alpha = (4\zeta^{-1}K^2D^3)^{-1}$. First, we shall show that

$$\exists C_1 > 0 \; \forall i \in I \; \forall x \in \phi_i(J) \; \forall r \geq \alpha \mathrm{diam}(\phi_i(X)) \quad m(B(x, 2Kr)) \leq C_1 m(B(x, r)). \tag{2.1}$$

Indeed, if $r \geq \gamma \mathrm{diam}(\phi_i(X))$, then with $C_1 = C^{\frac{\log K}{\log 2}+1}$ we are done by assumption (b). So, we may assume that

$$\alpha \mathrm{diam}(\phi_i(X)) \leq r \leq \gamma \mathrm{diam}(\phi_i(X)). \tag{2.2}$$

Now, using (a), we obtain

$$m(B(x, 2Kr)) \leq m\big(B(x, 2K\gamma \mathrm{diam}(\phi_i(X)))\big) \leq C||\phi_i'||^h. \tag{2.3}$$

On the other hand, using (1.3), we get

$$B(x, r) \supset B(x, \alpha \mathrm{diam}(\phi_i(X))) \supset B(x, \alpha D^{-1}||\phi_i'||) \supset \phi_i(B(\phi_i^{-1}(x), \alpha D^{-1})).$$

Since m is positive on non-empty open sets of $\overline{J}$, $M = \inf\{m(B(z, \alpha D^{-1})) : z \in \overline{J}\} > 0$. Noting $\phi_i^{-1}(x) \in J$ and $B(\phi_i^{-1}(x), \alpha D^{-1}) \subset X$, by conformality of m, we obtain

$$m(B(x, r)) \geq K^{-h}||\phi_i'||^h m(B(\phi_i^{-1}(x), \alpha D^{-1})) \geq MK^{-h}||\phi_i'||^h.$$

Combining this and (2.3), we obtain $m(B(x, 2Kr)) \leq M^{-1}CK^h m(B(x, r))$ and formula (2.1) is proved. We now turn to the doubling property. Fix $x \in J$ and $0 < r < (4KD)^{-1}\zeta \mathrm{diam}(X)$. Write $x = \pi(\omega)$, $\omega \in I^\infty$ and let $n \geq 0$ be the least integer such that $\phi_{\omega|_n}(X) \subset B(x, 2KD\zeta^{-1}r)$. In view of the requirement imposed on r, we have $n \geq 1$ and by the definition of n, $\mathrm{diam}(\phi_{\omega|_{n-1}}(X)) \geq 2KD\zeta^{-1}r$. Using (1.1), $2Kr||\phi_{\omega|_{n-1}}'||^{-1} \leq \zeta$. Therefore, $B(\pi(\sigma^{n-1}(\omega)), 2rK||\phi_{\omega|_{n-1}}'||^{-1})$ X and consequently,

$$B(x, 2r) \subset \phi_{\omega|_{n-1}}\Big(B(\pi(\sigma^{n-1}(\omega)), 2Kr||\phi_{\omega|_{n-1}}'||^{-1})\Big)$$

and

$$m(B(x, 2r)) \leq ||\phi_{\omega|_{n-1}}'||^h m\Big(B(\pi(\sigma^{n-1}(\omega)), 2Kr||\phi_{\omega|_{n-1}}'||^{-1})\Big). \tag{2.4}$$

On the other hand,

$$m(B(x, r)) \geq K^{-h}||\phi_{\omega|_{n-1}}'||^h m\Big(B(\pi(\sigma^{n-1}(\omega)), r||\phi_{\omega|_{n-1}}'||^{-1})\Big). \tag{2.5}$$

In addition, $\pi(\sigma^{n-1}(\omega)) \in \phi_{\omega_n}(X)$ and since $\phi_{\omega|_n}(X) \subset B(x, 2KD\zeta^{-1}r)$, we get

$$\begin{aligned} r||\phi_{\omega|_{n-1}}'||^{-1} &\geq rK^{-1}||\phi_{\omega|_n}'||^{-1} \cdot ||\phi_{\omega_n}'|| \geq K^{-1}D^{-2}r\mathrm{diam}^{-1}(\phi_{\omega|_n}(X))\mathrm{diam}(\phi_{\omega_n}(X)) \\ &\geq K^{-1}D^{-2}r(4KD\zeta^{-1}r)^{-1}\mathrm{diam}(\phi_{\omega_n}(X)) \\ &= (4\zeta^{-1}K^2D^3)^{-1}\mathrm{diam}(\phi_{\omega_n}(X)) = \alpha \mathrm{diam}(\phi_{\omega_n}(X)). \end{aligned}$$

Thus, (2.1) may be applied and combining this with (2.4) and (2.5), we get

$$m(B(x,2r)) \le C_1K^h m(B(x,r))$$

and the proof is complete. ■

Corollary 2.2. *Suppose that $S=\{\phi_i\}_{i\in I}$ is a regular conformal iterated function system satisfying the (SSOSC) and that m is the corresponding conformal measure. Suppose that the following two conditions are satisfied.*

(a): $\forall \kappa \le 1\, \exists C_\kappa > 0\, \forall i \in I\, \exists x_i \in \phi_i(X) \quad m\Big(B(x_i, \kappa \mathrm{diam}(\phi_i(X)))\Big) \le C_\kappa ||\phi_i'||^h.$

(b): $\exists \gamma \ge 1\, \exists C > 0\, \forall i \in I\, \exists y_i \in \phi_i(X)\, \forall r \ge \gamma \mathrm{diam}(\phi_i(X)) \quad m(B(y_i,2r)) \le Cm(B(y_i,r)).$

Then the conformal measure m satisfies the doubling property.

Proof. Let us first show that the condition (a) can be strengthened by replacing the quantifier $\exists x_i \in \phi_i(X)$ by the quantifier $\forall x \in \phi_i(X)$. Indeed, if $x \in \phi_i(X)$, then $||x-x_i|| \le \mathrm{diam}(\phi_i(X))$ and therefore

$$m\Big(B(x,\kappa \mathrm{diam}(\phi_i(X)))\Big) \le m\Big(B(x_i,(\kappa+1)\mathrm{diam}(\phi_i(X)))\Big) \le C_{\kappa+1}||\phi_i'||^h.$$

Let us show the same about condition (b). Put $\beta = \max\{2, 6\gamma^{-1}\}$. Then

$$\frac{\frac{3}{2}+(\beta\gamma)^{-1}}{2}\beta \ge 1,$$

and using (b) we find for every $x \in \phi_i(X)$ and for every $r \ge \beta\gamma \mathrm{diam}(\phi_i(X)))$ that

$$\begin{aligned}
m\left(B\left(x,\frac{3}{2}r\right)\right) &\le m\left(B\left(y_i,\frac{3}{2}r+\frac{r}{\beta\gamma}\right)\right) = m(B(y_i,(3/2+(\beta\gamma)^{-1})r)) \\
&= m\left(B\left(y_i, 2\left(\frac{3/2+(\beta\gamma)^{-1}}{2}\right)r\right)\right) \\
&\le Cm\left(B\left(y_i,\frac{3/2+(\beta\gamma)^{-1}}{2}r\right)\right) \\
&\le Cm\left(B\left(x,\frac{3/2+(\beta\gamma)^{-1}}{2}r+\mathrm{diam}(\phi_i(X))\right)\right) \\
&\le Cm\left(B\left(x,\frac{1}{2}(3/2+(\beta\gamma)^{-1})+(\beta\gamma)^{-1})r\right)\right) \\
&= Cm\left(B\left(x,\left(\frac{3}{4}+\frac{3}{2}(\beta\gamma)^{-1}\right)r\right)\right) \\
&\le Cm\left(B\left(x,\left(\frac{3}{4}+\frac{3}{2}\cdot\frac{1}{6}\right)r\right)\right) = m(B(x,r)).
\end{aligned}$$

Thus, $m(B(x,2r)) \le C^2m(B(x,r))$. We finish the proof by directly applying Theorem 2.1. ■

3. Continued Fraction Iterated Function Systems

For every $n \geq 1$ we consider the map $\phi_n : [0,1] \to [0,1]$ given by the formula

$$\phi_n(x) = \frac{1}{x+n}.$$

If $I \subset I\!N$ contains at least two elements, we call $\{\phi_i\}_{i\in I}$ a c.f. (continued fraction) iterated function system. Many geometrical properties of these systems have been thoroughly investigated in [MU2] and [Ur]. In this section, we deal with the doubling property of conformal measures of c.f. systems. There are three arithmetic properties of the index set I under which we can guarantee that the conformal measure exists and is doubling. One is simply that $1 \notin I$. The other two involve the asymptotic density properties of I. In the next section we will give some examples of index sets which satisfy these conditions. We start here with the simplest.

Lemma 3.1. *If $1 \notin I$ and $S = \{\phi_i\}_{i\in I}$ is a c.f. iterated function system, then S satisfies the Super Strong Open Set Condition.*

Proof. Since

$$\phi_n\left(\left[-\frac{1}{4}, \frac{2}{3}\right]\right) = \left[\frac{1}{n+\frac{2}{3}}, \frac{1}{n-\frac{1}{4}}\right],$$

we have $\phi_n([-1/4, 2/3]) \cap \phi_{n-1}([-1/4, 2/3]) = \emptyset$. Since obviously

$$\phi_n([-1/4, 2/3]) \cap \phi_k([-1/4, 2/3]) = \emptyset$$

if $|n-k| \geq 2$, the open set condition is satisfied with the set $X = [-1/4, 2/3] \subset V = (-1/2, 1)$. Since in addition $\overline{J_S} \subset [0, 1/2]$, the (SSOSC) is satisfied. ■

Next, we show that regular c.f. systems automatically satisfy one part of our general sufficient conditions for doubling.

Lemma 3.2. *If $S = \{\phi_i\}_{i\in I}$ is a regular c.f. iterated function system and m is the corresponding conformal measure, then condition (a) of Corollary 2.2 is satisfied.*

Proof. Let $X = [0,1]$. Fix $\kappa \geq 1$. We are to prove that there exists $C > 0$ such that for all $n \in I$ and for all $x \in \phi_n(J)$,

$$m(B(x, \kappa \mathrm{diam}(\phi_n(X)))) \leq C n^{-2h}. \tag{3.1}$$

Clearly, it suffices to prove this inequality for all n large enough. In fact, if $\phi_k(X) \cap B(x, \kappa \mathrm{diam}(\phi_n(X))) \neq \emptyset$, then $\left|\frac{1}{k} - \frac{1}{n}\right| \leq C_1 n^{-2}$ and $k \leq C_1 n$ for some constant $C_1 > 0$ depending on κ. Hence, $|k-n| \leq \frac{C_1 k}{n} \leq C_1^2$. Therefore, using the conformality of m, we have

$$m(B(x, \kappa \mathrm{diam}(\phi_n(X)))) \leq C_2 \frac{C_1^2}{(n - C_1^2)^{2h}} \leq C_3 n^{-2h} = C_3 ||\phi_n'||^h.$$

for some positive constants C_2 and C_3 and all n large enough. ■

Next, let us define one of the asymptotic density properties we need. Given $0 \leq \alpha \leq 1$ and $g > 1$ we call an infinite subset of $I\!N$, (α, g)-*evenly distributed* provided there exist constants $A \geq 1$ and $q \geq 1$ such that

$$\#(I \cap [n, gn]) \leq An^{\alpha} \tag{3.2}$$

for all $n \geq q$, and in addition if $n \in I$, then

$$\min\{\#(I \cap [n, gn]), \#(I \cap [g^{-1}n, n])\} \geq A^{-1}n^{\alpha}. \tag{3.3}$$

Our first result about (α, g)-evenly distributed sets is the following.

Proposition 3.3. *Let $I \subset I\!N$ be an (α, g)-evenly distributed set. Then the system S_I is cofinitely regular and $\theta_S = \alpha/2$.*

Proof. It follows from (3.2) that for every $t \geq 0$,

$$\psi(t) \asymp \sum_{k=0}^{\infty} \#(I \cap [g^k, g^{k+1}])(g^k)^{-2t} \preceq \sum_{k=0}^{\infty} A(g^k)^{\alpha}(g^k)^{-2t} = A\sum_{k=0}^{\infty}(g^{\alpha-2t})^k. \tag{3.4}$$

If $\alpha = 0$, then it immediately follows from this formula that $\theta_S = 0$ and since I is infinite, $\psi(0) = \infty$. In particular, S_I is cofinitely regular. (In fact, the system is absolutely regular-every subsystem is regular, see [MU2].) So, we may assume that $\alpha > 0$. Then it follows from (3.3) that for all n large enough $[n, gn] \cap I \neq \emptyset$ and consequently $\#(I \cap [n, g^2n]) \geq A^{-1}n^{\alpha}$. Thus,

$$\begin{aligned} \psi(t) &\asymp \sum_{k=0}^{\infty} \#(I \cap [g^{2k}, g^{2k+2}])(g^{2k})^{-2t} \succeq \sum_{k=0}^{\infty} A^{-1}(g^{2k})^{\alpha}(g^{2k})^{-2t} \\ &= A^{-1}\sum_{k=0}^{\infty}(g^{2(\alpha-2t)})^k. \end{aligned} \tag{3.5}$$

This inequality along with (3.4) implies that S_I is cofinitely regular and $\theta_S = \alpha/2$. ■

Let us now define the second density property we need. We say that an infinite subset I of $I\!N$ is *slowly growing* provided there exists a constant $C > 0$ such that

$$\forall n \in I\, \forall 0 \leq k \leq n \quad \#(I \cap [n, n+3k]) \leq C\#(I \cap [n, n+k]) \tag{3.6}$$

and

$$\forall n \in I\, \forall 0 \leq k \leq n/4 \quad \#(I \cap [n-3k, n]) \leq C\#(I \cap [n-k, n]) \tag{3.7}$$

The main result of this section is the following.

Theorem 3.4. *Let $I \subset I\!N \setminus \{1\}$ be an infinite $(\alpha, 4/3)$-evenly distributed subset and let I be slowly growing. Then the corresponding conformal measure m satisfies the doubling property.*

Proof. In view of Lemma 3.2, it suffices to check condition (b) of Corollary 2.2. To this end, fix $n \in I$ and set $y_n = 1/n \in \phi_n(X)$. (Recall we can take $X = [-1/4, 2/3]$.) For each $r > 0$, put

$$I_+(r) = I \cap \left\{k \geq n : \frac{1}{n} - \frac{1}{k} < r\right\} = I \cap \left[n, \frac{n}{1-rn}\right]$$

and

$$I_-(r) = I \cap \left\{k \leq n : \frac{1}{k} - \frac{1}{n} < r\right\} = I \cap \left[\frac{n}{1+rn}, n\right].$$

First, note the symmetric difference of $J \cap B(x,r)$ and $\bigcup_{k \in I_+(r) \cup I_-(r)} \phi_k(J)$ is contained in the union of two sets of the form $\phi_j(J)$, for some two $j \in [\min\{I_+(r) \cup I_-(r)\} - 1, \max\{I_+(r) \cup I_-(r)\} + 1]$. From this we easily deduce that

$$m(B(1/n, r)) \asymp \sum_{k \in I_+(r) \cup I_-(r)} m(\phi_k(J)) = \sum_{k \in I_+(r)} k^{-2h} + \sum_{k \in I_+(r)} k^{-2h}.$$

Our goal is to compare the two sums on the hand side for the values r and $2r$. To do this we use both of the arithmetic properties of I.

Let us deal first with the set $I_+(r)$. Consider two cases.

Case 1+: $r \leq \frac{1}{4n}$.
Then $I_+(2r) = I \cap [n, \frac{n}{1-2rn}] = I \cap [n, n + (\frac{n}{1-2rn} - n)]$. But, since $r \leq 1/4$, $3\left(\frac{n}{1-rn} - n\right) \geq \left(\frac{n}{1-2rn} - n\right)$. Therefore, using (3.6), we obtain that

$$\#I_+(2r) = \#\left(I \cap \left[n, \frac{n}{1-2rn}\right]\right) \leq C\#\left(I \cap \left[n, \frac{n}{1-rn}\right]\right) = C\#(I_+(r)).$$

Using in addition the fact that $\frac{n}{1-2rn} \leq 2n$, we conclude that

$$\sum_{k \in I_+(2r)} k^{-2h} \asymp n^{-2h}\#(I_+(2r)) \leq Cn^{-2h}\#(I_+(r)) \asymp \sum_{k \in I_+(r)} k^{-2h}. \tag{3.8}$$

Case 2+: $r > \frac{1}{4n}$.
Then

$$\frac{n}{1-rn} \geq \frac{n}{1-\frac{1}{4}} = \frac{4}{3}n$$

and since I is $(\alpha, 4/3)$-evenly distributed, we have

$$\sum_{k \in I_+(r)} k^{-2h} \geq \sum_{k \in I \cap [n, \frac{4}{3}n]} k^{-2h} \asymp n^{\alpha - 2h}. \tag{3.9}$$

Since by Proposition 3.3, $2h > 2\theta = \alpha$, we find

$$\begin{aligned}\sum_{k \in I_+(2r)} k^{-2h} &\leq \sum_{k \in I \cap [n, \infty)} k^{-2h} \asymp \sum_{j=0}^{\infty} \left(\left(\frac{4}{3}\right)^j n\right)^{\alpha} \left(\left(\frac{4}{3}\right)^j n\right)^{-2h} \\ &\asymp n^{\alpha-2h} \sum_{j=0}^{\infty} \left(\left(\frac{4}{3}\right)^{\alpha-2h}\right)^j = \frac{1}{1 - \left(\frac{3}{4}\right)^{2h-\alpha}} n^{\alpha-2h}.\end{aligned}$$

Combining this and (3.9), we obtain

$$\sum_{k\in I_+(2r)} k^{-2h} \preceq \sum_{k\in I_+(r)} k^{-2h}. \tag{3.10}$$

Let us deal in turn with the set $I_-(r)$.

Case 1-: $r \le \frac{3}{2}\cdot\frac{1}{n}$.

It follows that $n-(n/(1+rn)) \le n/4$. So, using (3.7), we get

$C\#I_-(r) = C\#I\cap\Big[n-(n-\frac{n}{1+rn}),n\Big] \ge \#I\cap\Big[n-3\Big(n-\frac{n}{1+rn}\Big),n\Big]$. Since $n-3\Big(n-\frac{n}{1+rn}\Big) \le \frac{n}{1+2rn}$, we have $\#(I_-(2r)) \le C\#(I_-(r))$. Using

$$\frac{n}{1+2rn} \ge \frac{n}{1+3} = \frac{n}{4} \tag{3.11}$$

we deduce that

$$\sum_{k\in I_-(2r)} k^{-2h} \asymp n^{-2h}\#(I_-(2r)) \le Cn^{-2h}\#(I_-(r)) \asymp \sum_{k\in I_-(r)} k^{-2h}. \tag{3.12}$$

Case 2-: $\frac{3}{2}\cdot\frac{1}{n} < r \le 1/2$.

Then

$$\frac{n}{1+rn} \le \frac{n}{1+\frac{3}{2}} = \frac{2}{5}n \le \frac{3}{4}n$$

and since I is $(\alpha,4/3)$-evenly distributed, we get that

$$\begin{aligned}\sum_{k\in I_-(r)} k^{-2h} &\ge \sum_{k\in I\cap\left[\frac{n}{1+rn},n\right]} k^{-2h} \asymp \sum_{j=0}^{\log_{4/3}(1+rn)} \left(\left(\frac{3}{4}\right)^j n\right)^{\alpha}\left(\left(\frac{3}{4}\right)^j n\right)^{-2h}\\ &= n^{\alpha-2h}\sum_{j=0}^{\log_{4/3}(1+rn)}\left(\left(\frac{4}{3}\right)^{2h-\alpha}\right)^j \asymp n^{\alpha-2h}\left(\frac{4}{3}\right)^{(2h-\alpha)\log_{4/3}(1+rn)}\\ &= n^{\alpha-2h}(1+rn)^{2h-\alpha} \ge n^{\alpha-2h}(rn)^{2h-\alpha} = r^{2h-\alpha}.\end{aligned} \tag{3.13}$$

Also,

$$\begin{aligned}\sum_{k\in I_-(2r)} k^{-2h} &\le \sum_{k\in I\cap\left[\frac{n}{1+2rn},n\right]} k^{-2h} \asymp \sum_{j=0}^{\log_{4/3}(1+2rn)} \left(\left(\frac{3}{4}\right)^j n\right)^{\alpha}\left(\left(\frac{3}{4}\right)^j n\right)^{-2h}\\ &= n^{\alpha-2h}\sum_{j=0}^{\log_{4/3}(1+2rn)}\left(\left(\frac{4}{3}\right)^{2h-\alpha}\right)^j \asymp n^{\alpha-2h}\left(\frac{4}{3}\right)^{(2h-\alpha)\log_{4/3}(1+2rn)}\\ &= n^{\alpha-2h}(1+2rn)^{2h-\alpha} \le n^{\alpha-2h}\left(\frac{2}{3}rn+2rn\right) = \left(\frac{8}{3}\right)^{2h-\alpha} r^{2h-\alpha}.\end{aligned}$$

Combining this and (3.13), we conclude that

$$\sum_{k\in I_-(2r)} k^{-2h} \preceq \sum_{k\in I_-(r)} k^{-2h}. \tag{3.14}$$

Recalling that

$$m(B(1/n,r)) \asymp \sum_{k\in I_+(r)\cup I_-(r)} m(\phi_k(J)) = \sum_{k\in I_+(r)} k^{-2h} + \sum_{k\in I_+(r)} k^{-2h},$$

the combination of formulae (3.8), (3.10), (3.12), and (3.14) shows that if $r \le 1/2$, then $m(B(1/n,2r)) \preceq m(B(1/n,r))$. Since for $r \ge 1/2$ such an estimate is obvious, we complete the proof by applying Corollary 2.2. ■

4. Examples of Restricted Entry Continued Fraction Systems

In the first part of this section we provide a list of cofinitely regular systems derived from the continued fraction algorithm whose corresponding conformal measures satisfy the doubling property. Afterwards, in the last part of this section, we describe a class of such systems for which the doubling property fails.

Our first theorem concerns index sets I which are growing geometrically.

Theorem 4.1. *Let $a \ge 2$ be an integer and b a nonnegative integer. Let $I = \{a^n + b\}_{n\ge 1}$, then the system $S = \{\phi_n\}_{n\in I}$ is cofinitely regular and the corresponding conformal measure (which with the methods of* [MU2] *can be shown to be the normalized h-dimensional Hausdorff measure restricted to the limit set J_I) has the doubling property.*

Proof. We shall prove first that the set I is $(0,4/3)$-evenly distributed. And indeed, since $a^{n+1}+b \ge a^{n+1} > \left(\frac{4}{3}\right)(a^n+b)$ for all n large enough, and since $a^{n-1}+b < \left(\frac{3}{4}\right)(a^n+b)$ for all n large enough, we conclude that for all n large enough

$$I \cap \left[\frac{3}{4}(a^n+b), \frac{4}{3}(a^n+b)\right] = \{a^n+b\}$$

and therefore the set I is $(0,4/3)$-evenly distributed. In particular, in view of Proposition 3.3, it is hereditarily regular. Fix $n \ge 1$. If $a^j + b \in [a^n+b, 4(a^n+b)]$, then $j \ge n$ and $a^j \le a^{n+3}$ assuming n to be large enough. Consequently, $j \in [n, n+3]$ and

$$\#(I \cap [a^n+b, 4(a^n+b)]) \le 4. \tag{4.1}$$

If $a^j + b \in [\frac{1}{4}(a^n+b), a^n+b]$, then $j \le n$ and $a^j \ge \frac{1}{4}a^n - \frac{3}{4}b \ge a^{n-3}$. Hence $j \in [n-3, n]$ and

$$\#\left(I \cap \left[\frac{1}{4}(a^n+b), a^n+b\right]\right) \le 4. \tag{4.2}$$

In view of (4.1) and (4.2), the formulae (3.6) and (3.7) are respectively satisfied, and we conclude the proof by applying Theorem 3.4. ■

Our next examples consist of families which are growing less rapidly.

Theorem 4.2. *If $p, q \geq 1, b \geq 0$ are integers, $I = \{n^p + b\}_{n \geq q}$ and $1 \notin I$, then the system $S = \{\phi_n\}_{n \in I}$ is cofinitely regular and the corresponding conformal measure possesses the doubling property.*

Proof. We shall prove first that the set I is $(1/p, 4/3)$-evenly distributed. And indeed, fix $n \geq q$. Then $k^p + b \in [n^p + b, \frac{4}{3}(n^p + b)]$ which equivalently means that

$$k \geq n \text{ and } k^p \leq \frac{4}{3}n^p + \frac{1}{3}b. \tag{4.3}$$

In particular, if $k^p + b \in [n^p + b, \frac{4}{3}(n^p + b)]$ and n is large enough, then $k^p \leq 2^p n^p$ and, in consequence, $k \leq 2n \leq 2(n^p + b)^{1/p}$. Thus, we have proved that

$$\#\left(I \cap \left[n^p + b, \frac{4}{3}(n^p + b)\right]\right) \leq 2(n^p + b)^{1/p} \tag{4.4}$$

On the other hand, if $n \leq k \leq \left(\frac{5}{4}\right)^{1/p} n$, then $k^p \leq \frac{5}{4}n^p \leq \frac{4}{3}n^p + \frac{1}{3}b$ for all n large enough, and consequently

$$\#\left(I \cap \left[n^p + b, \frac{4}{3}(n^p + b)\right]\right) \geq \left(\left(\frac{5}{4}\right)^{1/p} - 1\right) n \geq \frac{1}{2}\left(\left(\frac{5}{4}\right)^{1/p} - 1\right)(n^p + b)^{1/p} \tag{4.5}$$

for all n large enough. Keep now $n \geq q$ fixed. Then $k^p + b \in [\frac{3}{4}(n^p + b), n^p + b]$ if and only if

$$k \leq n \text{ and } k^p \geq \frac{3}{4}n^p - \frac{1}{4}b. \tag{4.6}$$

In particular, if $k^p + b \in [\frac{3}{4}(n^p + b), n^p + b]$ and n is large enough, then $k^p \geq \left(\frac{1}{2}\right)^p n^p$ and, in consequence, $k \geq \frac{1}{2}n \geq \frac{1}{4}(n^p + b)^{1/p}$ for all n large enough. Thus

$$\#\left(I \cap \left[\frac{3}{4}(n^p + b), n^p + b\right]\right) \leq \frac{3}{4}(n^p + b). \tag{4.7}$$

On the other hand, if $(3/4)^{1/p} n \leq k \leq n$, then $k^p \geq \frac{3}{4}n^p \geq \frac{3}{4}n^p - \frac{1}{4}b$, and therefore

$$\#\left(I \cap \left[\frac{3}{4}(n^p + b), n^p + b\right]\right) \geq \left(1 - \left(\frac{3}{4}\right)^{\frac{1}{p}}\right) n \geq \frac{1}{2}\left(1 - \left(\frac{3}{4}\right)^{\frac{1}{p}}\right)(n^p + b)^{\frac{1}{p}}$$

for all n large enough. Combining this inequality along with (4.7), (4.5) and (4.4), we conclude that I is $(1/p, 4/3)$-evenly distributed.
Let us now show that the set I is slowly growing. Towards this end fix $n \geq q$ and $0 \leq k \leq n^p + b$. Let $l \geq n$ be the largest integer such that $l^p + b \leq n^p + b + k$. That is,$\#(I \cap [n, n+k]) = l - n + 1$. Then $(l+1)^p + b > n^p + b + k$ and therefore $l + 1 > (n^p + k)^{1/p}$. Hence

$$\#(I \cap [n^p + b, n^p + b + k]) > (n^p + k)^{1/p} - n.$$

On the other hand, if $s > (n^p + 3k)^{1/p}$, then $s^p + b > n^p + b + 3k$ and consequently

$$\#(I \cap [n^p + b, n^p + b + 3k]) \leq (n^p + 3k)^{1/p} - n.$$

Using the Mean Value Theorem we get

$$(n^p+k)^{1/p}-n=(n^p+k)^{1/p}-(n^p)^{1/p}=\frac{1}{p}x^{\frac{1}{p}-1}k$$

for some $x\in[n^p,n^p+k]\subset[2n^p+b]\subset[n^p,3n^p]$, where the last inclusion holds for all n large enough. Hence,

$$\#(I\cap[n^p+b,n^p+b+k])\geq\frac{3^{\frac{1-p}{p}}}{p}n^{1-p}k. \tag{4.8}$$

Similarly

$$(n^p+3k)^{1/p}-n=(n^p+k)^{1/p}-(n^p)^{1/p}=\frac{1}{p}3ky^{\frac{1}{p}-1}$$

for some $y\in[n^p,n^p+k]$ and therefore

$$\#(I\cap[n^p+b,n^p+b+3k])\leq\frac{3}{p}n^{1-p}k.$$

Combining this inequality and (4.8), we get

$$\#(I\cap[n^p+b,n^p+b+3k])\leq 3^{\frac{2p-1}{p}}\#(I\cap[n^p+b,n^p+b+k]). \tag{4.9}$$

In order to check condition (3.7), fix $0\leq k\leq n/4$. Let $u\leq n$ be the least integer such that $n^p+b-k\leq u^p+b$. Hence $\#(I\cap[n^p+b-k,n^p+b])=n-u+1$. Then $(u-1)^p+b<n^p+b-k$ and therefore $u-1<(n^p-k)^{1/p}$. Hence, using the Mean Value Theorem we get

$$\#(I\cap[n^p+b-k,n^p+b])\geq n-(n^p-k)^{1/p}=(n^p)^{1/p}-(n^p-k)^{1/p}=\frac{1}{p}\xi^{\frac{1-p}{p}}k$$

for some $\xi\in[n^p-k,n^p]$. Thus

$$\#(I\cap[n^p+b-k,n^p+b])\geq\frac{1}{p}n^{1-p}k. \tag{4.10}$$

Similarly, if $v<(n^p-3k)^{1/p}$, then $v^p+b<n^p+b-3k$ and using the Mean Value Theorem we get

$$\#(I\cap[n^p+b-3k,n^p+b])\leq n-(n^p-3k)^{1/p}=\frac{1}{p}\eta^{\frac{1-p}{p}}3k,$$

where $\eta\in[n^p-3k,n^p]\subset[n^p-3\frac{1}{4}(n^p+b),n^p]=[\frac{1}{4}n^p-\frac{3}{4}b,n^p]\subset[\frac{1}{8}n^p,n^p]$ for all n large enough. Hence, using (4.10), we obtain

$$\#(I\cap[n^p+b-3k,n^p+b])\leq\frac{3}{p}8^{\frac{p-1}{p}}n^{1-p}k\leq\frac{3}{p}8^{\frac{p-1}{p}}\#(I\cap[n^p+b-k,n^p+b]).$$

This and (4.9) demonstrate that the set I satisfies the slowly growing property (3.6) and (3.7). Invoking now Theorem 3.4 finishes the proof. ■

Remark 4.3. *Obviously the methods employed in Theorem 4.1 and Theorem 4.2 can be applied to prove the doubling property for the sets of the form* $\{ca^n+b\}_{n\geq 1}$ *and* $\{cn^p+b\}_{n\geq 1}$.

Recall a set $I = \{a_n\}_{n\geq 1}$ has *bounded gaps* if $\sup_{n\geq 1}\{a_{n+1} - a_n\} < \infty$. It is obvious that I is slowly growing and that it is $(1, 4/3)$-evenly distributed. It therefore follows from Theorem 3.4 that

Theorem 4.4. *If $I \subset I\!N \setminus \{1\}$ has bounded gaps, then the corresponding conformal measure satisfies the doubling property.*

Corollary 4.5. *If $I \subset I\!N\setminus\{1\}$ is an arithmetic progression, then the corresponding conformal measure satisfies the doubling property.*

We remark that in case I is an arithmetic progression, it was shown in [MU2] that m is, up to a constant the packing measure, restricted to the set J_I and the Hausdorff measure of J_I is 0.
We shall now describe a class of examples which do not satisfy the doubling property. An increasing sequence $\{a_n\}_{n\geq 1}$ is said to belong to the class PNDP if the following conditions are satisfied

- **(a):** $1 \notin \{a_n\}_{n\geq 1}$.
- **(b):** $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$.
- **(c):** If $\theta = \inf\{t \geq 0 : \xi(t) := \sum_{n\geq 1} a_n^{1-2t} < \infty\}$, then $\xi(\theta) = \infty$.

As an example of a PNDP sequence take for instance the sequence $\{2^{n^2}\}_{n\geq 1}$.

Our examples are obtained by fattening up a PNDP sequence somewhat. We call a subset $I \subset I\!N\setminus\{1\}$ NDP if there exists a PNDP sequence $\{a_n\}_{n\geq 1}$ such that the symmetric difference of I and the union $\bigcup_{n\geq 1}[a_n, 2a_n]$ is finite. We shall prove the following.

Theorem 4.6. *If $I \subset I\!N$ is an NDP subset of $I\!N$ and $h > 1/2$, then the system S_I is cofinitely regular and the corresponding conformal measure does not satisfy the doubling property.*

Proof. The fact that the system S_I is cofinitely regular follows immediately from the item (c) since $\xi(t) \asymp \psi_S(t)$. For every $n \geq 1$ large enough, we have

$$m(B(0, 1/a_n)) \succeq \sum_{j=a_n}^{2a_n} j^{-2h} \geq 4^{-h} a_n^{1-2h}$$

and

$$m\left(B\left(0, \frac{1}{2a_n}\right)\right) \asymp \sum_{I\cap[a_{n+1},\infty)} j^{-2h} \leq \sum_{j\geq a_{n+1}} j^{-2h} \asymp a_{n+1}^{1-2h}.$$

Since it follows from item (b) that

$$\lim_{n\to\infty} \frac{a_n^{1-2h}}{a_{n+1}^{1-2h}} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^{2h-1} = \infty,$$

we conclude that the measure m fails to satisfy the doubling property for the point $0 \in \overline{J}$. ■

Remark 4.7. *Notice that if I is an NDP set and if we add to I finitely many elements all different from 1, then the resulting set I_1 remains in the class NDP, and if we add a sufficiently long initial segment of $\mathbb{N} \setminus \{0\}$ so that $h_{I_1} > 1/2$ (see Theorem 1.1 and recall that the Hausdorff dimension of the set of all continued fractions with all partial denominators greater than 1 had Hausdorff dimension greater than 1/2), then the conformal measure of the system S_{I_1} fails to satisfy the doubling property.*

Question 4.8. There are many interesting index sets I for which we were not able to determine whether the doubling property holds. In particular, if the index set I is the set of prime numbers, then, as is shown in [MU2] using results about the asymptotic behavior of the size two sided gaps in the primes, the system is hereditarily regular, but the Hausdorff measure of the corresponding set of continued fractions in dimension h_I is 0 and the packing measure is infinite. *Does the conformal measure generated from the index set consisting of all prime numbers have the doubling property?*

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