ERGODIC OPTIMIZATION FOR NON-COMPACT DYNAMICAL SYSTEMS

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Abstract. We consider maximizing orbits and maximizing measures for continuous maps $T : X \to X$ and functions $f : X \to \mathbb{R}$, where X need not be compact. We give sufficient conditions for the function f to have a normal form, which allows a characterisation of f-maximizing measures in terms of their support. If $T : X \to X$ is a countable state subshift of finite type and f has a Gibbs measure, then f does have a normal form, and hence a maximizing measure. The family of Gibbs measures $(\mu_{tf})_{t\geq 1}$ for the functions tf is shown to be tight. Any weak*-accumulation point as $t \to \infty$ is f-maximizing, and has maximal entropy among all f-maximizing measures.

Contents

1. Introduction

Let $T : X \to X$ be a continuous map of a topological space X. Given a continuous function $f : X \to \mathbb{R}$, an orbit $\{x, T(x), T^2(x), \ldots\}$ is called *f-maximizing* if the time average $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ is larger than along any other orbit. If $\mathcal{M}$ denotes the set of T-invariant probability measures then the measure $\mu \in \mathcal{M}$ is called *f-maximizing* if

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$\int f\, d\mu = \sup_{m\in\mathcal{M}} \int f\, dm$. We can define f-minimizing orbits and measures in a similar way.

By *ergodic optimization* we mean the circle of problems relating to the search for maximizing (or minimizing) orbits and measures, and the determination of the maximum ergodic average. The resolution of these problems for specific systems has applications to other disciplines. The subject evolved during the 1990s, and for much of this time researchers worked independently, unaware of others' contributions. In 1990 Coelho [C] studied zero temperature limits of equilibrium states, and their connection with certain maximizing measures. He showed that for Hölder functions f on subshifts of finite type, an f-maximizing measure cannot be fully supported unless f is cohomologous to a constant. In an unpublished manuscript from around 1993, Conze & Guivarc'h [CG] improved Coelho's result by proving that such an f has a *normal form*: there exists a continuous function φ such that $\tilde{f} := f + \varphi - \varphi \circ T \leq \sup_{m\in\mathcal{M}} \int f\, dm$. The f-maximizing measures are thereby identified as precisely those invariant probability measures whose support lies in the set of global maxima of the normal form $\tilde{f}$.

Conze & Guivarc'h also gave a preliminary analysis of a specific non-trivial model for ergodic optimization: $T(x) = 2x \pmod 1$ and $f_\theta(x) = \cos 2\pi(x-\theta)$. A little later this parametrized family of functions was studied independently by Hunt & Ott [HO] and by Jenkinson [J1, J2]. Jenkinson proposed a complete characterisation of the maximizing measures for the family f_θ, in terms of Sturmian measures. This conjecture was proved by Bousch [B1], who had himself independently arrived at the same characterisation. Bousch also re-discovered the normal form theorem of Conze & Guivarc'h, exploiting it in a key way for his proof. An alternative proof of the normal form theorem was given by Contreras, Lopes & Thieullen [CLT], using techniques inspired by Mañé [M1, M2], who had established a normal form theorem in the context of Lagrangian systems. A subsequent strengthening of Mañé's result by Fathi [F1] is closer in spirit to the approach of Bousch, constructing the normal form from the fixed point of a certain nonlinear operator.

In the present article we consider ergodic optimization for dynamical systems $T : X \to X$ where X is not necessarily compact. In this general context a maximizing measure need not even exist, so it is convenient (cf. §2) to distinguish the successively weaker notions of maximizing measure, maximizing orbit, and limsup maximizing orbit. In §3 an appropriately strengthened notion of normal form is introduced, which in particular guarantees the existence of a maximizing measure. In §§4–6 we proceed to derive sufficient conditions for a function to have such a strong normal form. In the very general context of §4 the sufficient condition is a rather abstract notion of *essential compactness*. Our main motivation for studying dynamical systems on non-compact spaces is the case of countable state subshifts of finite type, and the smooth systems, such as Gauss's continued fraction map, that they model. From §5 onwards we assume $T : X \to X$ is a countable state subshift of finite type, and f is a function of summable variation. Theorem 3 gives an easily checkable sufficient condition for such a function to have a strong normal form. In the case of the full shift on $\mathbb{N}$ this condition is that for some $I \in \mathbb{N}$,

$$\sum_{j=2}^{\infty} \mathrm{var}_j(f) < 2 \inf f|_{[I]} - \sup f - \sup f|_{[i]}$$

for all sufficiently large i (cf. Corollary 2); here $[k]$ denotes the set of sequences whose first entry is k, and $\mathrm{var}_j(f) = \sup\{f(x) - f(y) :$ the first j entries of x and y agree$\}$. In particular, if $\sup f|_{[i]} \to -\infty$ as $i \to \infty$ then f has a strong normal form. It follows (Theorem 4, Corollary 3) that if f has an invariant Gibbs measure then it also has a strong normal form, and hence a maximizing measure; this is because the existence of a Gibbs measure implies that $\sup f|_{[i]} \to -\infty$ as $i \to \infty$.

A different sufficient condition for the existence of a strong normal form is given in Theorem 5, involving a more subtle analysis of the behaviour of f "at infinity".

In the final §7 we consider accumulation points of the family μ_{tf} of Gibbs measures for tf, where $t \geq 1$. We prove (Theorem 6) that this family is tight, hence precompact in the weak* topology. Any weak* accumulation point as $t \to \infty$ is shown to be f-maximizing, and to have maximum entropy among all f-maximizing measures.

2. Maximizing orbits and maximizing measures

Let X be a topological space, not necessarily compact. For a continuous transformation $T : X \to X$, let $\mathcal{M}$ denote the set of T-invariant Borel probability measures on X. In general $\mathcal{M}$ might be empty, though if X is a compact metrizable space then $\mathcal{M} \neq \emptyset$, by the Krylov-Bogolioubov Theorem ([Wa2], Cor. 6.9.1).

Definition 1. (Three types of maximum ergodic average)

By convention we define the supremum of the empty set to be $-\infty$.

If X is a topological space and $f : X \to I\!R$ is continuous then for $x \in X$ we define $\gamma(f,x) \in [-\infty,\infty]$ by

$$\gamma(f,x) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)\,.$$

Then define the *maximum limsup time average* $\gamma(f)$ by

$$\gamma(f) = \sup_{x\in X} \gamma(f,x)\,.$$

A point $x \in X$ will be called (T,f)*-regular* (or simply *regular*) if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ exists (we allow divergence to either $-\infty$ or $+\infty$), and the set of regular points is denoted by $\mathrm{Reg}(X,T,f)$

Define the *maximum time average* $\beta(f)$ by

$$\beta(f) = \sup_{x\in \mathrm{Reg}(X,T,f)} \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)\,.$$

The function f being continuous is Borel measurable, so for any Borel measure μ the integral $\int f\,d\mu \in [-\infty,\infty]$ is defined provided $\int f^+\,d\mu$ and $\int f^-\,d\mu$ are not both infinite, where $f^\pm := \max(\pm f, 0)$. Let $\mathcal{M}_f := \{\mu \in \mathcal{M} : \int f\,d\mu \text{ is defined}\}$, and define the *maximum space average* $\alpha(f)$ to be

$$\alpha(f) = \sup_{m\in\mathcal{M}_f} \int f\,dm\,.$$

Remark 1.
(a) If f is bounded either above or below then $\mathcal{M}_f = \mathcal{M}$.
(b) If X is compact then $\mathcal{M}_f = \mathcal{M}$ for all continuous functions f.
(c) If $\mathcal{M}_f$ is empty then $\alpha(f) = -\infty$. In particular, if $\mathcal{M}$ is empty then $\alpha(f) = -\infty$ for all continuous functions f.
(d) If $\mathcal{M}$ is non-empty, and $f \in L^1(\mu)$ for some $\mu \in \mathcal{M}$, then $\alpha(f) > -\infty$.
(e) If f is bounded above then $\alpha(f) < \infty$. From §5 onwards we assume f to be bounded above.

Definition 2. (Maximizing orbits and measures)
Any $x \in X$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \gamma(f)$$

is called *limsup maximizing* for the function f. Its T-orbit $\mathcal{O}(x) = \{x, Tx, T^2x, \ldots\}$ is also called *limsup maximizing* for f.

If $x \in \mathrm{Reg}(X,T,f)$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \gamma(f)\,,$$

we say that x, and its orbit $\mathcal{O}(x)$, are *f-maximizing.*

If there exists $\mu \in \mathcal{M}_f$ such that

$$\int f\, d\mu = \gamma(f)\,,$$

then μ is called an *f-maximizing measure*, or simply a *maximizing measure.* Let $\mathcal{M}_{\max}(f)$ denote the set of f-maximizing measures.

For most of our results we shall assume that X is a Polish space, i.e. it is separable and can be metrized by means of a complete metric.[1] Under this hypothesis the three maximum ergodic averages can be related as follows, thereby explaining the privileged role of $\gamma(f)$ (rather than $\alpha(f)$ or $\beta(f)$) in Definition 2.

Proposition 1. *Let X be a Polish space. Suppose that $T : X \to X$ and $f : X \to I\!R$ are both continuous. Then*
(i) (General case)

$$\alpha(f) \le \beta(f) \le \gamma(f)\,.$$

(ii) (Compact case)
If furthermore X is compact then $\alpha(f) = \beta(f) = \gamma(f) \ne \pm\infty$, and $\mathcal{M}_{\max}(f)$ is non-empty.

Proof. (i) Suppose that $\alpha(f) > \beta(f)$. Then there exists a measure $\mu \in \mathcal{M}_f$ for which $\int f\, d\mu > \beta(f)$. In particular $\int f\, d\mu > -\infty$. Since X is a Polish space then the triple $(X, \mathcal{B}, \mu)$ is a Lebesgue space, where $\mathcal{B}$ is the completion of the Borel σ-algebra by μ ([Ro], p. 174). So $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure-preserving automorphism of a Lebesgue space, and consequently admits an ergodic decomposition ([Ro] pp. 178, 194, [Wa2] p. 34): there is a Borel probability measure P_μ on the set $\mathcal{E} \subset \mathcal{M}$ of T-ergodic measures, such that if $g \in L^1(\mu)$ then $g \in L^1(m)$ for P_μ almost every $m \in \mathcal{E}$, and

$$\int g\, d\mu = \int_{m\in\mathcal{E}} \int g\, dm\, dP_\mu(m)\,.$$

If $f \in L^1(\mu)$ this gives $\int f\, d\mu = \int_{m\in\mathcal{E}} \int f\, dm\, dP_\mu(m)$, where $f \in L^1(m)$ for P_μ-a.e. $m \in \mathcal{E}$. So there exists an ergodic measure μ' such

[1] This is preferable to assuming that X is a complete separable metric space, since a Polish space need not have a particularly natural or simple complete metric; eg. $X = [0,1)$ is Polish but the usual metric $d(x,y) = |x-y|$ is not complete.

that $\int f\,d\mu' \geq \int f\,d\mu > \beta(f)$ and $f \in L^1(\mu')$ (so in particular $\mu' \in \mathcal{M}_f$). By the ergodic theorem we know that μ'-almost every x satisfies $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \int f\,d\mu'$. In particular there is at least one $x \in \mathrm{Reg}(X,T,f)$ for which $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \int f\,d\mu' > \beta(f)$, contradicting the definition of $\beta(f)$. So in the case where $f \in L^1(\mu)$ we in fact have $\alpha(f) \leq \beta(f)$.

Next suppose $f \notin L^1(\mu)$. The assumption that $\alpha(f) > \beta(f)$ then implies that $\int f\,d\mu = +\infty$ and $\beta(f) < \infty$. The function $f_k(x) := \min(k, f(x))$ is in $L^1(\mu)$, and as above there exists an ergodic measure $\mu_k \in \mathcal{M}_{f_k}$ with $\int f_k\,d\mu_k \geq \int f_k\,d\mu$. By the ergodic theorem we can find a μ_k-generic point $x_k \in X$ such that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(T^i x_k) = \int g\,d\mu_k$ for all $g \in L^1(\mu_k)$.

We claim that each $x_k \in \mathrm{Reg}(X,T,f)$. First note that each $\mu_k \in \mathcal{M}_f$, with $\int f\,d\mu_k > -\infty$, because $\mu_k \in \mathcal{M}_{f_k}$ and $f^- = f_k^-$. If $f \in L^1(\mu_k)$ then $x_k \in \mathrm{Reg}(X,T,f)$ by the ergodic theorem. If $\int f\,d\mu_k = \infty$ we must show that $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x_k) \to \infty$ as $n \to \infty$. In fact let x be an m-generic point for any ergodic measure m, and suppose $\int f\,dm = \infty$. For any $R > 0$ we shall find $N \in \mathbb{N}$ such that $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) > R$. The monotone convergence theorem implies that $\int f_k\,dm \to \int f\,dm = \infty$ as $k \to \infty$, so for any $R > 0$ there exists $K = K(R) \in \mathbb{N}$ such that $\int f_K\,dm > 2R$. Now $f_K \in L^1(m)$, so the ergodic theorem guarantees an $N = N(K(R),R) \in \mathbb{N}$ such that $|\frac{1}{n}\sum_{i=0}^{n-1} f_K(T^i x) - \int f_K\,dm| < R$ for all $n \geq N$. Hence $\frac{1}{n}\sum_{i=0}^{n-1} f_K(T^i x) > R$ for all $n \geq N$. But $f \geq f_K$, so $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) > R$ for all $n \geq N$, as required. So indeed each $x_k \in \mathrm{Reg}(X,T,f)$.

Since $f \geq f_k$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x_k) \geq \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f_k(T^i x_k) = \int f_k\,d\mu_k \geq \int f_k\,d\mu\,.$$

Therefore

$$\begin{aligned}\beta(f) &= \sup_{x\in\mathrm{Reg}(X,T,f)} \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)\\ &\geq \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x_k) \geq \lim_{k\to\infty}\int f_k\,d\mu = \infty\,,\end{aligned}$$

contradicting the fact that $\beta(f) < \infty$. This contradiction means that in the case where $f \notin L^1(\mu)$ we again have $\alpha(f) \leq \beta(f)$.

It is immediate from their definitions that $\beta(f) \leq \gamma(f)$, so part (i) is proved.

(ii) To prove $\alpha(f)=\beta(f)=\gamma(f)$ it suffices, by (i), to show that $\alpha(f)\geq\gamma(f)$. This is the content of Lemmas 2.3 and 2.4 in [YH], but for completeness we give the argument here. The compactness of X means that $\mathcal{M}$ is compact for the weak* topology ([Wa2], Thm. 6.10). Consequently for every $x\in X$ there exists $\mu_x\in\mathcal{M}_f=\mathcal{M}$ such that $\gamma(f,x)=\int f\,d\mu_x$. This is because there is an increasing sequence (n_i) such that $\gamma(f,x)=\lim_i\frac{1}{n_i}\sum_{j=0}^{n_i-1}f(T^jx)=\lim_i\int f\,d\mu_i$, where $\mu_i=\frac{1}{n_i}\sum_{j=0}^{n_i-1}\delta_{T^jx}$, but (μ_i) converges weak* subsequentially to some $\mu_x\in\mathcal{M}$, so $\lim_i\int f\,d\mu_i=\int f\,d\mu_x$. Now $\gamma(f)=\sup_{x\in X}\gamma(f,x)=\sup_{x\in X}\int f\,d\mu_x=\lim_k\int f\,d\mu_{x_k}$ for some sequence $x_k\in X$, so if $\mu\in\mathcal{M}$ is any weak* accumulation point of (μ_{x_k}) then $\gamma(f)=\int f\,d\mu\leq\alpha(f)$, as required.

The common maximum ergodic average is finite because f is bounded, and $\mathcal{M}_{\max}(f)\neq\emptyset$ because the map $\mu\mapsto\int f\,d\mu$ is continuous for the weak* topology. □

Remark 2.
If f has a maximizing measure then it also has a maximizing orbit, and hence a limsup maximizing orbit, by an argument similar to the proof of Proposition 1 (i). The existence of a maximizing measure is equivalent to the existence of a maximizing orbit $\mathcal{O}(x)$ such that $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^i(x)}$ has a weak* accumulation point as $n\to\infty$, where δ_z denotes the Dirac measure at the point z. If f has a *compactly supported* maximizing measure μ then $\gamma(f)\neq\pm\infty$, since $\int f\,d\mu\neq\pm\infty$ by continuity. If a limsup maximizing orbit has compact closure then there exists a (compactly supported) maximizing measure, and hence a maximizing orbit.

If X is compact then Proposition 1 (ii) means the study of (limsup) maximizing orbits reduces to, and is more elegantly formulated as, the study of maximizing measures. If X is non-compact then this is not the case; the following examples illustrate that $\alpha(f)$, $\beta(f)$, $\gamma(f)$ need not coincide, nor be finite, and (limsup) maximizing orbits or maximizing measures need not exist.

Example 1. (f has no limsup maximizing orbit)
If T is the identity map on $[0,1)$ and f is strictly increasing, say, then there are no limsup maximizing orbits; $\gamma(f)=\infty$ if and only if f is unbounded.

Example 2. ($\alpha(f)<\beta(f)$; f has a maximizing orbit, $\mathcal{M}_{\max}(f)=\emptyset$)
Let $T:\mathbb{R}\to\mathbb{R}$ be the translation $T(x)=x+1$, and $f:\mathbb{R}\to\mathbb{R}$ any function which only takes negative values and has $\lim_{x\to\infty}f(x)=0$. Clearly $\beta(f)=\gamma(f)=0$, while $\alpha(f)=-\infty$ because $\mathcal{M}$ is empty.

If T is modified so as to create some recurrence then $\mathcal{M}$ will be non-empty yet $\alpha(f) < \beta(f)$ may still occur. If $T(x) = x/2$ on $[-2, 0]$, $T(x) = 2x$ on $[0, 1]$, and $T(x) = x + 1$ elsewhere then $\mathcal{M}$ contains only the Dirac measure at 0, but if f is as above then $\alpha(f) = \int f \, d\delta_0 = f(0) < 0 = \beta(f) = \gamma(f)$.

Example 3. ($\beta(f) < \gamma(f)$; f has a limsup maximizing orbit, but no maximizing orbit)

Equip $X = \mathbb{Z}_2 \times \mathbb{N}$ with the discrete topology, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Let $T((i, n)) = (i+1, n+1)$ if $n = 2^k$ and $T((i, n)) = (i, n+1)$ otherwise. If $f((i, n)) = i$ for each $i \in \mathbb{Z}_2$ then $\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \frac{1}{3}$ and $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \frac{2}{3}$ for all $x \in X$. Therefore $\gamma(f) = 2/3$, and every orbit is limsup maximizing. However $\mathrm{Reg}(X, T, f)$ is empty, so there are no maximizing orbits.

Example 4. ($-\infty < \alpha(f) < \beta(f) < \gamma(f) < \infty$)

Example 3 can easily be modified so as to obtain $-\infty < \alpha(f) < \beta(f) < \gamma(f) < \infty$. Let X be the set of non-negative integers, with $T(0) = 0$ and $T(n) = n + 2$ otherwise. Then $f : X \to \mathbb{R}$ can be chosen to take only the values 0 or 1, with $f(0) = 0$, $\liminf_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(2)) = 0$, $\limsup_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(2)) = 1$, and the limit $\lim \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(1)) = 1/2$ existing. So $\alpha(f) = 0$, $\beta(f) = 1/2$, and $\gamma(f) = 1$.

Example 5. ($\alpha(f) = \beta(f) = \gamma(f) = -\infty$)

If $T(x) = x + 1$ on $\mathbb{R}$ and $\lim_{x\to\infty} f(x) = -\infty$ then every orbit is maximizing, with $\alpha(f) = \beta(f) = \gamma(f) = -\infty$.

3. Normal forms

Let $C(X)$ denote the space of continuous real-valued functions on the Polish space X. The topology of uniform convergence on compact subsets makes this a complete topological vector space, though if X is non-compact then $C(X)$ is not metrisable. Its closed subspace $CB(X)$, consisting of those continuous functions which are bounded, is however metrisable: the metric $d(\varphi, \psi) = \sup_{x\in X} |\varphi(x) - \psi(x)|$ generates the subspace topology. Let $CB^{\wedge}(X) \subset C(X)$ denote the (in general non-metrisable) subspace consisting of those continuous functions which are bounded above.

Definition 3. (Dynamical cohomology and normal forms)

If $\varphi \in CB(X)$ then a function of the form $\varphi - \varphi \circ T$ is called a *(continuous) coboundary*. Two functions f, g which differ by a coboundary are *cohomologous*, and we write $f \sim g$. This is an equivalence relation on $C(X)$; the corresponding equivalence classes are called *cohomology classes*.

A function $\tilde{f} \sim f$ is a *weak normal form* for f if $\tilde{f} \leq \gamma(f)$. It is a *strong normal form* for f if in addition its set of maxima $\tilde{f}^{-1}(\gamma(f))$ contains the support of some T-invariant probability measure. (Recall that the *support* of a measure μ, denoted $\mathrm{supp}(\mu)$, is by definition the smallest closed subset $Y \subset X$ with $\mu(Y) = 1$).

Ergodic averages are well defined on cohomology classes: $\int f\,d\mu$ (for $\mu \in \mathcal{M}_f$) and $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ (if it exists) and $\overline{\lim}_n \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ do not depend on the cohomology class representative f. Consequently $\alpha(f)$, $\beta(f)$ and $\gamma(f)$ are well-defined on cohomology classes, and (limsup) maximizing orbits and maximizing measures are the same for all functions in a given cohomology class. If the cohomology class has a strong normal form, the following result implies that a maximizing measure is completely characterized by its *support*.

Proposition 2. *Suppose $T : X \to X$ is a continuous map on the Polish space X, and the continuous function $f : X \to I\!R$ has a strong normal form $\tilde{f}$. Then*

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset\,.$$

Proof. Since the set of maxima $\tilde{f}^{-1}(\gamma(f))$ is non-empty, and $\tilde{f}$ is continuous, then $\gamma(f) \neq \pm\infty$. So $\tilde{f}$ is bounded, hence $\mathcal{M}_{\tilde{f}} = \mathcal{M}$ (cf. Remark 1 (a)). But $f \sim \tilde{f}$, so $\int f\,dm = \int \tilde{f}\,dm$ for all $m \in \mathcal{M}_{\tilde{f}} = \mathcal{M}$. Hence $\mathcal{M}_f = \mathcal{M}$ as well.

Now $\tilde{f} \leq \gamma(f)$, so $\int f\,dm = \int \tilde{f}\,dm \leq \gamma(f)$ for all $m \in \mathcal{M}$. If $\mu \in \mathcal{M}$ satisfies $\mathrm{supp}(\mu) \subset \tilde{f}^{-1}(\gamma(f))$ then $\int f\,d\mu = \int \tilde{f}\,d\mu = \gamma(f)$, so μ is f-maximizing. But $\tilde{f}$ is a strong normal form, so there exists at least one such μ, hence there exists at least one f-maximizing measure.

If $m \in \mathcal{M}$ is such that $\mathrm{supp}(m) \subsetneq \tilde{f}^{-1}(\gamma(f))$, then in fact $\int f\,dm = \int \tilde{f}\,dm < \gamma(f)$, because $\tilde{f} \leq \gamma(f)$, so f is not f-maximizing. □

Proposition 2 clearly demonstrates the usefulness of finding strong normal forms, as they essentially resolve the ergodic optimization problem: maximizing measures are identified as those invariant probability measures whose support lies inside the set of maxima of the normal form. In particular the existence of a strong normal form guarantees the existence of a maximizing measure. Not every continuous function has a strong normal form; indeed the absence of strong normal forms is in a sense typical (cf. Remarque 7 in [B2], §3 in [BJ]). However in many interesting cases f does have a strong normal form: Bousch [B1], and independently Contreras, Lopes & Thieullen [CLT], proved this for Lipschitz functions f and expanding maps T on the circle. Jenkinson [J4] established the analogous fact for functions of summable variation defined on finite alphabet subshifts of finite type, and Bousch [B2] further extended the result to functions f satisfying Walters' condition (cf. [Wa1]) and T either weakly expanding or with weak local product structure. An unpublished manuscript of Conze & Guivarc'h [CG] seems to be the earliest treatment of normal forms for ergodic optimization, containing in particular a version of our Proposition 2. In all of the above articles the space X is assumed to be compact, whereas in this paper we seek sufficient conditions for f to have a strong normal form in the case where X is *not necessarily compact*.

For compact X, every weak normal form $\tilde{f}$ is actually a strong normal form: the existence of an f-maximizing measure μ (cf. Proposition 1 (ii)) forces $\mathrm{supp}(\mu) \subset \tilde{f}^{-1}(\gamma(f))$, since if not then $\int f \, d\mu = \int \tilde{f} \, d\mu$ is *strictly* less than $\gamma(f)$. In the non-compact case, a weak normal form need not be a strong normal form (see Example 6), so that weak normal forms are less useful for our purposes.

One approach to proving the existence of a strong normal form for f is to search for fixed points of a certain nonlinear operator M_f, which we now define. This operator was introduced by Bousch [B1] to study maximizing measures in the case where X is compact. In the context of Lagrangian flows an analogous construction is the Lax-Oleinik semigroup of operators, as studied in a recent series of papers by Fathi [F1, F2, F3, F4]. Fixed points of M_f can be used to construct a different kind of normal form for f, the *Bousch-Fathi normal form*.

Definition 4. (Bousch-Lax-Oleinik operator; Bousch-Fathi normal form)

Let $T : X \to X$ be a continuous surjection on the topological space X, and suppose that $f : X \to \mathbb{R}$ is continuous. If $\varphi \in C(X)$ then for each $x \in X$, define $M_f\varphi(x) \in (-\infty, \infty]$ by

$$M_f\varphi(x) := \sup_{y \in T^{-1}x} (f + \varphi)(y). \tag{1}$$

If f is bounded above then this defines a nonlinear operator $M_f : CB^\wedge(X) \to CB^\wedge(X)$, the *Bousch-Lax-Oleinik operator*.

Iterates of M_f can be expressed as

$$M_f^n\varphi(x) = \sup_{y \in T^{-n}(x)} \left(S_n f(y) + \varphi(y)\right) , \tag{2}$$

where

$$S_n f(y) := \sum_{i=0}^{n-1} f(T^i y) .$$

Any $(\varphi, c) \in CB^\wedge(X) \times I\!R$ satisfying $M_f\varphi = \varphi + c$ is called a *fixed point pair* for M_f; if $\varphi \in CB(X)$ then the constant c is uniquely determined (cf. Lemma 1), in which case we call φ an *essential fixed point* of M_f. If M_f has an essential fixed point $\varphi \in CB(X)$ then f is necessarily bounded above. The function $\tilde{f} := f + \varphi - \varphi \circ T$, where $\varphi \in CB(X)$ is an essential fixed point for M_f, is called a *Bousch-Fathi normal form* for f.

Given a continuous function $f : X \to I\!R$, define

$$c(f) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sup_{x\in X} S_n f(x) \in [-\infty, \infty] .$$

If $c(f) \in [-\infty, \infty)$ then $\sup_{x\in X} S_n f(x)$ is finite for all sufficiently large n, and is a subadditive sequence of reals, so in fact the limit $\lim_n \frac{1}{n} \sup_{x\in X} S_n f(x) \in [-\infty, \infty)$ exists and equals $\inf_n \frac{1}{n} \sup_{x\in X} S_n f(x)$. Clearly $c(f) \geq \gamma(f) = \sup_{x\in X} \overline{\lim}_n \frac{1}{n} S_n f(x)$, but for non-compact X it is possible that $c(f) > \gamma(f)$; this arises if each point has a preimage where f is large, but is itself homoclinic to a part of X where f is small (cf. Example 7).

Lemma 1. *Let $T : X \to X$ be a continuous surjection on the topological space X. Suppose the continuous function $f : X \to I\!R$ is such that M_f has a fixed point pair $(\varphi, c) \in CB(X) \times I\!R$.*

Then $c = c(f) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} S_n f(x)$.

Proof. The equation $\varphi + c = M_f\varphi$ is equivalent to $M_{f-c}\varphi = \varphi$, which implies that

$$\varphi(x) = M^n_{f-c}\varphi(x) = -nc + \sup_{y\in T^{-n}(x)} (S_n f(y) + \varphi(y))$$

for all $n \in I\!N$, $x \in X$. Now φ is bounded, and writing $a = \inf \varphi$, $b = \sup \varphi$ we have

$$\frac{a-b}{n} + c \ \leq \ \frac{1}{n} \sup_{y\in T^{-n}(x)} S_n f(y) \ \leq \ c + \frac{b-a}{n}$$

for all $n \in I\!N$, $x \in X$. Therefore for all $n \in I\!N$,

$$\frac{a-b}{n} + c \ \leq \ \frac{1}{n} \sup_{x\in X} \sup_{y\in T^{-n}(x)} S_n f(y) \ \leq \ c + \frac{b-a}{n} .$$

But T is surjective, so this is equivalent to

$$\frac{a-b}{n} + c \ \leq \ \frac{1}{n} \sup_{y\in X} S_n f(y) \ \leq \ c + \frac{b-a}{n} \quad \forall n \in I\!N .$$

Letting $n \to \infty$ gives the result. $\square$

If X is compact then every Bousch-Fathi normal form is a strong normal form; the strategy of Bousch [B1, B2], and in a different context Fathi [F1, F2, F3, F4], for establishing existence of strong normal forms was to show that M_f has fixed point pairs $(\varphi, c(f))$, where necessarily $c(f) = \alpha(f) = \beta(f) = \gamma(f)$. For non-compact X, a Bousch-Fathi normal form need not even be a weak normal form (cf. Example 7). In Theorem 2 we prove that for suitable functions f over countable state subshifts of finite type, a Bousch-Fathi normal form does always exist. To promote this to a *strong* normal form requires extra hypotheses on f, however, and these are explored in §§5, 6. In the following §4 we introduce the intermediate concept of *essential compactness* as a sufficient condition for the existence of a strong normal form.

4. Essentially compact functions

The importance of a strong normal form for a function was demonstrated by Proposition 2: existence of a strong normal form implies existence of a maximizing measure, and gives a characterisation of maximizing measures in terms of their support. In this section we seek sufficient conditions for a function to have a strong normal form, in the general context of Polish spaces X. In §§5-6 this question is pursued in the special case of countable state subshifts of finite type.

Definition 5. (Essentially compact functions)
Let $T : X \to X$ be a continuous map on the topological space X. A continuous function $f : X \to \mathbb{R}$ is *essentially compact* if there is an essential fixed point $\varphi \in CB(X)$ for M_f, and a subset $Y \subset X$ such that
(a) $\widetilde{Y} := \cap_{n=0}^{\infty} T^{-n} Y$ is non-empty and compact,
(b) $T^{-1}(x) \cap Y$ is non-empty and finite for every $x \in X$, and the map $x \mapsto \mathrm{card}(T^{-1}(x) \cap Y)$ is locally constant,
(c) for each $x \in X$,

$$\varphi(x) + c(f) = \max_{y \in T^{-1}(x) \cap Y} (f + \varphi)(y)\,. \tag{3}$$

Remark 3. It is often straightforward to find a subset Y satisfying conditions (a) and (b). For example if X is a countable state subshift of finite type then Y can be chosen to be an appropriate finite union of length-one cylinders (see §5). For suitable subshifts of finite type the class of essentially compact functions is a large one; for example it contains all functions to which the thermodynamic formalism applies (cf. Theorem 4, Corollary 3).

Clearly every essentially compact function has a Bousch-Fathi normal form; the following result, a non-compact generalization of Lemme B in [B1], tells us more.

Theorem 1. *Let $T : X \to X$ be a continuous surjection on a Polish space X, where $T^{-1}(x)$ is at most countable for every $x \in X$.*

If the continuous function $f : X \to I\!R$ is essentially compact then it has a strong normal form $\tilde{f}$, and hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset .$$

Proof. It suffices to show that f has a strong normal form; the characterization of $\mathcal{M}_{\max}(f)$ then follows by Proposition 2. If φ is the essential fixed point in Definition 5 then $\tilde{f} := f+\varphi-\varphi\circ T$ is a Bousch-Fathi normal form for f, by definition. Moreover, $c(f) \geq \gamma(f)$, so in particular $\gamma(f) < \infty$. To see that $\tilde{f}$ is a strong normal form for f we shall show that $\tilde{f}^{-1}(\gamma(f))$ contains a compact T-invariant set, from which it follows that $\tilde{f}^{-1}(\gamma(f))$ contains the support of some T-invariant measure, by the Krylov-Bogolioubov Theorem ([Wa2], Cor. 6.9.1).

Now let $x \in X$ be arbitrary. The surjectivity of T means we can replace x by $T(x)$ in (3) to obtain

$$\begin{aligned} \varphi(Tx) + c(f) &= \max_{y \in Y \cap T^{-1}(T(x))} (f+\varphi)(y) \\ &= \varphi(x) + f(x) + r_\varphi(x) \end{aligned} \tag{4}$$

where

$$r_\varphi(x) := \max_{y \in Y \cap T^{-1}(T(x))} (f+\varphi)(y) - (f+\varphi)(x).$$

Note that r_φ is both continuous and non-negative. We shall see later that in fact $r_\varphi^{-1}(0) = \tilde{f}^{-1}(\gamma(f))$.

Let S denote the restriction of the map T to the nonempty compact invariant set $\tilde{Y}$. We first claim that

$$Z := \bigcap_{n=0}^{\infty} S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$$

is a non-empty compact T-invariant set contained in the zero set $r_\varphi^{-1}(0)$. The fact that $Z \subset r_\varphi^{-1}(0)$ is clear from the definition, as is T-invariance, because $S = T|_{\widetilde{Y}}$, so we concentrate on showing that Z is non-empty and compact. We can write

$$Z = \bigcap_{N=0}^{\infty} Z_N, \quad \text{where } Z_N := \bigcap_{n=0}^{N-1} S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y}) .$$

Now $Z_1 \supset Z_2 \supset \ldots$, so that if each Z_N is non-empty and compact then the same will be true of Z. To prove the compactness of Z_N, note that $r_\varphi^{-1}(0)$ is closed, since r_φ is continuous, and $\widetilde{Y}$ is compact. So $r_\varphi^{-1}(0) \cap \widetilde{Y}$ is compact, and in particular closed. Hence each $S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$ is closed, since S^n is continuous. But $S = T|_{\widetilde{Y}}$, so each $S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$ is a subset of the compact set $\widetilde{Y}$, and therefore itself compact. Consequently the intersection $Z_N = \cap_{n=0}^{N-1} S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$ is also compact.

Now we show that each Z_N is non-empty. Let z be any point in $\widetilde{Y} = \cap_{n=0}^{\infty} T^{-n} Y$. Then $T^N(z) \in \widetilde{Y}$ as well. By part (b) of Definition 5, the set $T^{-1}(T^N(z)) \cap Y$ is non-empty and finite. So there exists $x_{N-1} \in T^{-1}(T^N(z)) \cap Y$ such that

$$(f+\varphi)(x_{N-1}) = \max_{y \in T^{-1}(T^N(z)) \cap Y} (f+\varphi)(y).$$

But $T^N(z) = T(x_{N-1})$, so x_{N-1} satisfies

$$(f+\varphi)(x_{N-1}) = \max_{y \in T^{-1}(T(x_{N-1})) \cap Y} (f+\varphi)(y).$$

In other words,

$$r_\varphi(x_{N-1}) = 0\,.$$

Now in general if $A \subset \widetilde{Y}$ then $T^{-1}A \cap Y \subset \widetilde{Y}$. So since $T^N(z) \in \widetilde{Y}$, we have $x_{N-1} \in T^{-1}(T^N(z)) \cap Y \subset \widetilde{Y}$. That is,

$$x_{N-1} \in r_\varphi^{-1}(0) \cap \widetilde{Y}\,.$$

Now $T^{-1}(x_{N-1})$ is finite and non-empty, by part (b) of Definition 5, so there exists $x_{N-2} \in T^{-1}(x_{N-1}) \cap Y \subset \widetilde{Y}$ such that

$$(f+\varphi)(x_{N-2}) = \max_{y \in T^{-1}(x_{N-1}) \cap Y} (f+\varphi)(y).$$

But $x_{N-1} = T(x_{N-2})$, so x_{N-2} satisfies

$$(f+\varphi)(x_{N-2}) = \max_{y \in T^{-1}(T(x_{N-2})) \cap Y} (f+\varphi)(y).$$

That is,

$$r_\varphi(x_{N-2}) = 0\,.$$

So we have that

$$x_{N-2} \in r_\varphi^{-1}(0) \cap \widetilde{Y} \quad , \quad \text{and} \quad T(x_{N-2}) = x_{N-1}\,.$$

Continuing in this way we find a finite sequence $x_0, x_1, \ldots, x_{N-1}$ of points with the property that

$$x_n \in r_\varphi^{-1}(0) \cap \widetilde{Y} \quad , \quad \text{and} \quad x_n = T^n(x_0) = S^n(x_0)$$

for all $0 \leq n \leq N-1$. Therefore $T^n(x_0) \in r_\varphi^{-1}(0) \cap \widetilde{Y}$ for all $0 \leq n \leq N-1$. So $Z_N = \cap_{n=0}^{N-1} S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$ contains the point x_0, and in particular is non-empty, as required. It follows that $Z := \cap_{n=0}^{\infty} S^{-n}(r_\varphi^{-1}(0) \cap \widetilde{Y})$ is a non-empty compact T-invariant subset of $r_\varphi^{-1}(0)$, by the argument given above. But any non-empty compact T-invariant set supports a T-invariant probability measure. That is, there exists a measure $m \in \mathcal{M}$ with

$$\operatorname{supp}(m) \subset Z = r_\varphi^{-1}(0) \cap \widetilde{Y}. \tag{5}$$

If we integrate equation (4) with respect to any invariant measure $\mu \in \mathcal{M}_f$ we obtain

$$\int f\, d\mu = c(f) - \int r_\varphi\, d\mu \leq c(f)\,,$$

whereas setting $\mu = m$ gives

$$\int f\, dm = c(f)\,.$$

It follows that $c(f) = \alpha(f)$. But $c(f) \geq \gamma(f)$, and $\gamma(f) \geq \alpha(f)$ by Proposition 1 (i), so in fact $c(f) = \gamma(f)$. Therefore m is an f-maximizing measure. Moreover, equation (4) now reads

$$\gamma(f) - r_\varphi = \tilde{f}\,,$$

so that

$$r_\varphi^{-1}(0) = \tilde{f}^{-1}(\gamma(f))\,.$$

Combining this with equation (5) shows that $\operatorname{supp}(m) \subset \tilde{f}^{-1}(\gamma(f))$, so that $\tilde{f}$ is indeed a strong normal form for f. □

5. Countable state subshifts of finite type

Definition 6. The space $\Sigma = \mathbb{N}^{\mathbb{N}}$, equipped with the product topology, is the *full shift* on the alphabet $\mathbb{N}$. Given an *adjacency matrix* $A : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$, the associated *subshift of finite type* Σ_A is the subspace of Σ defined by

$$\Sigma_A = \{\omega \in \Sigma : A(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \geq 1\}.$$

All subshifts of finite type are Polish spaces, and we shall always use the complete metric $\delta(x, y) = 2^{-\min\{n : x_n \neq y_n\}}$.

A finite word $w \in \mathbb{N}^n$ is *A-admissible* if $A(w_i, w_{i+1}) = 1$ for all $1 \leq i \leq n-1$. Given $w \in \mathbb{N}^n$ and $\omega \in \Sigma_A$, the *concatenation* $w\omega$ is the sequence defined by

$$(w\omega)_i = \begin{cases} w_i & \text{if } 1 \leq i \leq n \\ \omega_{i-n} & \text{if } i \geq n+1 \end{cases} .$$

The *shift map* $T : \Sigma \to \Sigma$ is defined by $(T\omega)_n = \omega_{n+1}$. If Σ_A is a subshift of finite type then $T(\Sigma_A) \subset \Sigma_A$, and we again write $T : \Sigma_A \to \Sigma_A$ to denote the corresponding restriction of the shift map. In general $T : \Sigma_A \to \Sigma_A$ is not surjective, but we shall only consider those Σ_A satisfying primitivity conditions (see Definition 7), which imply surjectivity.

For any $n \in I\!N$ we define $\Pi_{A,n} : \Sigma_A \to I\!N^n$ by $\Pi_{A,n}(\omega) = (\omega_1, \ldots, \omega_n)$ (projection onto the first n coordinates), and $\pi_{A,n} : \Sigma_A \to I\!N$ by $\pi_{A,n}(x) = x_n$ (projection onto the n-th coordinate).

If $w \in I\!N^n$ then the corresponding *cylinder set* in Σ_A is defined by

$$[w] = [w]_A = \Pi_{A,n}^{-1}(w) = \{x \in \Sigma_A : \Pi_{A,n}(x) = w\} .$$

Define

$$I\!N_A = \{i \in I\!N : [i]_A \neq \emptyset\} ,$$

the set of those symbols which actually appear as an entry in some element of Σ_A. If $I\!N_A$ is finite then Σ_A is compact, and called a *finite state subshift of finite type*. If $I\!N_A$ is infinite then Σ_A is non-compact, and called a *countable state subshift of finite type*.

Henceforth our Polish space X will be a subshift of finite type Σ_A, and $T : \Sigma_A \to \Sigma_A$ will be the shift map. To discuss ergodic optimization and normal forms we require extra assumptions on the matrix A; these will be appropriate infinite alphabet analogues of aperiodicity, ensuring the shift map enjoys some *mixing* properties.

Definition 7. An adjacency matrix A, and the corresponding subshift of finite type Σ_A, are called *primitive* if there is an $n \in I\!N$ such that for all $x \in \Sigma_A$ and all $i \in I\!N_A$ there exists $w \in I\!N^n \cap [i]_A$ with $wx \in \Sigma_A$. The smallest such n is called the *primitive constant* for A, denoted n_A.

An adjacency matrix A, and the corresponding subshift of finite type Σ_A, are called *finitely primitive* if there exists $N \in I\!N$, and a finite subalphabet $\mathbb{M} \subset I\!N$, such that for all $x \in \Sigma_A$ and all $i \in I\!N_A$ there exists $w \in \mathbb{M}^N \cap [i]_A$ with $wx \in \Sigma_A$. Neither N nor $\mathbb{M}$ are unique. Any such pair $(N, \mathbb{M})$ is a *finitely primitive pair* for A; N is a *finitely primitive constant* and $\mathbb{M}$ a *finitely primitive alphabet*. The smallest finitely primitive constant N is of course well-defined, but for our purposes (cf. eg. Theorems 2 and 3) it is useful to consider all possible finitely primitive pairs.

Σ_A has the *finite universal preimage property* if there is a finite sub-alphabet $\mathbb{L} \subset I\!N$ such that every $x \in \Sigma_A$ has a preimage $ix \in \Sigma_A$ for some $i \in \mathbb{L}$. Any such $\mathbb{L}$ is a *finite universal preimage alphabet* for Σ_A.

Remark 4. Finite primitivity implies both primitivity and the finite universal preimage property; any finitely primitive alphabet $\mathbb{M}$ is a finite universal preimage alphabet. Each of the conditions in Definition 7 implies that the shift map $T : \Sigma_A \to \Sigma_A$ is surjective.

Primitivity is equivalent to *topological mixing* of $T : \Sigma_A \to \Sigma_A$ (i.e. for all non-empty open sets $Y, Z \subset \Sigma_A$ there exists $M \in I\!N$ such that $T^{-m}Y \cap Z \neq \emptyset$ for all $m \geq M$). If $I\!N_A = I\!N$ then A is primitive if and only if there exists $n \in I\!N$ such that every entry of the matrix A^n is strictly positive.

Sarig [Sa2] defines the big images and preimages property as the existence of a finite $\mathbb{M} \subset I\!N$ such that for all $j \in I\!N$ there exists $i, k \in \mathbb{M}$ with $A(i,j) = 1 = A(j,k)$. If A is primitive and $I\!N_A = I\!N$ then this condition is equivalent to finite primitivity.

Similarly we can define the big preimages property to be the existence of a finite $\mathbb{L} \subset I\!N$ such that for all $j \in I\!N$ there exists $i \in \mathbb{L}$ with $A(i,j) = 1$. The finite universal preimage property is weaker than the big preimages property, though if $I\!N_A = I\!N$ then the two are equivalent.

So far we have considered arbitrary continuous functions f. Henceforth, however, we require further control on the modulus of continuity.

Definition 8. The n^{th} *variation* of $f : \Sigma_A \to I\!R$ is defined by

$$\mathrm{var}_n(f) = \sup_{\Pi_{A,n}(x) = \Pi_{A,n}(y)} \{f(x) - f(y)\}.$$

We say f has *summable variations* if

$$V(f) := \sum_{n=1}^{\infty} \mathrm{var}_n(f) < \infty.$$

Note that $\mathrm{var}_0(f) = \sup_{x,y \in \Sigma_A} \{f(x) - f(y)\}$ is not included in this sum, so summable variations does not imply boundedness.

The following theorem guarantees the existence of a Bousch-Fathi normal form for functions of summable variation which are bounded (above) on a (finitely) primitive subshift of finite type. Its proof is inspired by the analogous result of Bousch [B2], Théorème 1, where an essential fixed point for M_f is obtained by approximating M_f by contraction mappings.

Theorem 2. *Let Σ_A be a subshift of finite type. Suppose that $f : \Sigma_A \to I\!R$ has summable variations and is bounded above. If*

either (i) Σ_A is primitive and f is also bounded below,
or (ii) Σ_A is finitely primitive,
then the Bousch-Lax-Oleinik operator M_f has an essential fixed point $\varphi \in CB(\Sigma_A)$, and f has a Bousch-Fathi normal form $\tilde{f} = f + \varphi - \varphi \circ T$.

If $\varphi \in CB(\Sigma_A)$ is any such essential fixed point then

$$var_i(\varphi) \le \sum_{j=i+1}^{\infty} var_j(f) \quad \text{for } i \ge 1. \tag{6}$$

In case (i), φ also satisfies

$$var_0(\varphi) \le n_A var_0(f) + \sum_{j=2}^{\infty} var_j(f) \tag{7}$$

where $n_A \in I\!N$ is the primitive constant for A.

In case (ii), φ also satisfies

$$var_0(\varphi) \le N(\sup f - \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})}) + \sum_{j=2}^{\infty} var_j(f) \tag{8}$$

for every finitely primitive pair $(N, \mathbb{M})$ for A.

Proof. We first introduce some common notation so as to treat cases (i) and (ii) simultaneously. If Σ_A is finitely primitive then let $(N, \mathbb{M})$ be any finitely primitive pair. If Σ_A is primitive then set $\mathbb{M} = I\!N$, and let N denote the primitive constant n_A.

In both cases (i) and (ii), the restriction $f|_{\pi_{A,1}^{-1}(\mathbb{M})}$ is bounded below. In case (i) this is because f itself is bounded below, while in case (ii) it is because $\mathbb{M}$ is finite and $\mathrm{var}_1(f) < \infty$.

Define $M_{f,\lambda} : CB^\wedge(\Sigma_A) \to CB^\wedge(\Sigma_A)$ by

$$M_{f,\lambda}\varphi(x) = \sup_{y \in T^{-1}(x)} (f + \lambda\varphi)(y).$$

The iterates of $M_{f,\lambda}$ can be expressed as

$$M_{f,\lambda}^n \varphi(x) = \sup\{S_{\lambda,n} f(w_x x) + \lambda^n \varphi(w_x x) : w_x \in I\!N^n, w_x x \in \Sigma_A\},$$

where $S_{\lambda,n} f(z) := \sum_{j=0}^{n-1} \lambda^{n-1-j} f(T^j z)$.

In fact we claim that $M_{f,\lambda}(CB(\Sigma_A)) \subset CB(\Sigma_A)$. In case (i) this is clear, since f itself is bounded. To see that it holds in case (ii), note that if $\varphi \in CB(\Sigma_A)$ then $M_{f,\lambda}\varphi$ is bounded above, because f is, so we only need check that $M_{f,\lambda}\varphi$ is bounded below. But the finitely primitive alphabet $\mathbb{M}$ is a finite universal preimage alphabet, so $M_{f,\lambda}\varphi \ge \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})} + \lambda \inf \varphi|_{\pi_{A,1}^{-1}(\mathbb{M})} > -\infty$.

The nonlinear operator $M_{f,\lambda} : CB(\Sigma_A) \to CB(\Sigma_A)$ is λ-Lipschitz with respect to the complete metric $d(\varphi,\psi) = \sup_{x\in\Sigma_A}(\varphi-\psi)(x)$, so for $0 \le \lambda < 1$ it has a unique fixed point $\varphi_\lambda \in CB(\Sigma_A)$.

Now suppose that $\varphi \in CB(\Sigma_A)$. For any $\varepsilon > 0$ and any $x, y \in \Sigma_A$, we can find words $w_{x,\varepsilon}, w_{y,\varepsilon} \in \mathbb{N}^N$ with $w_{x,\varepsilon}x, w_{y,\varepsilon}y \in \Sigma_A$ such that

$$0 \le M_{f,\lambda}^N\varphi(x) - \big(S_{\lambda,N}f(w_{x,\varepsilon}x) + \lambda^N\varphi(w_{x,\varepsilon}x)\big) < \varepsilon \tag{9}$$

and

$$0 \le M_{f,\lambda}^N\varphi(y) - \big(S_{\lambda,N}f(w_{y,\varepsilon}y) + \lambda^N\varphi(w_{y,\varepsilon}y)\big) < \varepsilon\,. \tag{10}$$

Therefore

$$\begin{aligned} &M_{f,\lambda}^N\varphi(x) - M_{f,\lambda}^N\varphi(y) \\ &< \varepsilon + S_{\lambda,N}f(w_{x,\varepsilon}x) + \lambda^N\varphi(w_{x,\varepsilon}x) - \big[S_{\lambda,N}f(w_{y,\varepsilon}y) + \lambda^N\varphi(w_{y,\varepsilon}y)\big]\,. \end{aligned} \tag{11}$$

Since Σ_A is (finitely) primitive, we can find $w'_{y,\varepsilon} \in \mathbb{M}^N$ such that $w'_{y,\varepsilon}y \in \Sigma_A$ and $\pi_{A,1}(w'_{y,\varepsilon}y) = \pi_{A,1}(w_{x,\varepsilon}x)$. By (10) we have

$$S_{\lambda,N}f(w'_{y,\varepsilon}y)+\lambda^N\varphi(w'_{y,\varepsilon}y) \le M_{f,\lambda}^N\varphi(y) < S_{\lambda,N}f(w_{y,\varepsilon}y)+\lambda^N\varphi(w_{y,\varepsilon}y)+\varepsilon\,,$$

so substituting into (11) gives

$$\begin{aligned} &M_{f,\lambda}^N\varphi(x) - M_{f,\lambda}^N\varphi(y) \\ &< 2\varepsilon + S_{\lambda,N}f(w_{x,\varepsilon}x) - S_{\lambda,N}f(w'_{y,\varepsilon}y) + \lambda^N\left[\varphi(w_{x,\varepsilon}x) - \varphi(w'_{y,\varepsilon}y)\right]\,. \end{aligned} \tag{12}$$

Now f is bounded above, so $S_{\lambda,N}f(w_{x,\varepsilon}x) \le N\sup f$. As previously observed, the restriction of f to $\pi_{A,1}^{-1}(\mathbb{M})$ is bounded below, so $S_{\lambda,N}f(w'_{y,\varepsilon}y) \ge \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})}\sum_{r=0}^{N-1}\lambda^r$. Since $\pi_{A,1}(w'_{y,\varepsilon}y) = \pi_{A,1}(w_{x,\varepsilon}x)$, we have

$$\lambda^N\left(\varphi(w_{x,\varepsilon}x) - \varphi(w'_{y,\varepsilon}y)\right) < \max\{0, \varphi(w_{x,\varepsilon}x) - \varphi(w'_{y,\varepsilon}y)\} \le \mathrm{var}_1(\varphi)\,.$$

Combining these observations and substituting into (12) gives

$$M_{f,\lambda}^N\varphi(x) - M_{f,\lambda}^N\varphi(y) < 2\varepsilon + N\sup f - \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})}\sum_{r=0}^{N-1}\lambda^r + \mathrm{var}_1(\varphi)\,.$$

But $\varepsilon > 0$ was arbitrary, as were $x, y \in \Sigma_A$, so

$$\mathrm{var}_0(M_{f,\lambda}^N\varphi) \le N\sup f - \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})}\sum_{r=0}^{N-1}\lambda^r + \mathrm{var}_1(\varphi) \tag{13}$$

for any $\varphi \in CB(\Sigma_A)$. In particular, the fixed point φ_λ of $M_{f,\lambda}$ satisfies

$$\mathrm{var}_0(\varphi_\lambda) \leq N \sup f - \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})} \sum_{r=0}^{N-1} \lambda^r + \mathrm{var}_1(\varphi_\lambda)\,. \tag{14}$$

Now suppose that $j \geq 1$. We shall consider the j^{th} variation of $M_{f,\lambda}\varphi$. Suppose $x, y \in \Sigma_A$ satisfy $\Pi_{A,j}(x) = \Pi_{A,j}(y)$. For any $\varepsilon > 0$ we can find $i_x, i_y \in I\!N$ with $i_x x, i_y y \in \Sigma_A$ such that

$$0 \leq M_{f,\lambda}\varphi(x) - (f + \lambda\varphi)(i_x x) < \varepsilon \tag{15}$$

and

$$0 \leq M_{f,\lambda}\varphi(y) - (f + \lambda\varphi)(i_y y) < \varepsilon\,. \tag{16}$$

Then

$$M_{f,\lambda}\varphi(x) - M_{f,\lambda}\varphi(y) < \varepsilon + (f + \lambda\varphi)(i_x x) - (f + \lambda\varphi)(i_y y)\,. \tag{17}$$

Since $j \geq 1$ then in particular $\Pi_{A,1}(x) = \Pi_{A,1}(y)$, and because $i_x x \in \Sigma_A$ then also $i_x y \in \Sigma_A$, since Σ_A is a subshift of finite type. From (16),

$$(f + \lambda\varphi)(i_x y) \leq M_{f,\lambda}\varphi(y) < (f + \lambda\varphi)(i_y y) + \varepsilon\,,$$

so (17) gives

$$\begin{aligned} M_{f,\lambda}\varphi(x) - M_{f,\lambda}\varphi(y) &< 2\varepsilon + (f + \lambda\varphi)(i_x x) - (f + \lambda\varphi)(i_x y) \\ &\leq 2\varepsilon + \mathrm{var}_{j+1}(f + \lambda\varphi) \\ &\leq 2\varepsilon + \mathrm{var}_{j+1}(f) + \mathrm{var}_{j+1}(\varphi)\,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we in fact have

$$\mathrm{var}_j(M_{f,\lambda}\varphi) \leq \mathrm{var}_{j+1}(f) + \mathrm{var}_{j+1}(\varphi)\,,$$

so in particular

$$\mathrm{var}_j(\varphi_\lambda) \leq \mathrm{var}_{j+1}(f) + \mathrm{var}_{j+1}(\varphi_\lambda)\,. \tag{18}$$

Iteration of (18) yields

$$\mathrm{var}_i(\varphi_\lambda) \leq \sum_{j=i+1}^{\infty} \mathrm{var}_j(f) \quad \text{for all } i \geq 1\,. \tag{19}$$

In particular $\mathrm{var}_1(\varphi_\lambda) \leq \sum_{j=2}^{\infty} \mathrm{var}_j(f)$, so substituting into (14) gives

$$\mathrm{var}_0(\varphi_\lambda) \leq N \sup f - \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})} \sum_{r=0}^{N-1} \lambda^r + \sum_{j=2}^{\infty} \mathrm{var}_j(f)\,. \tag{20}$$

In particular, the 0^{th} variation of each φ_λ, $0 \leq \lambda < 1$, is bounded independently of λ, so defining $\varphi_\lambda^* := \varphi_\lambda - \inf \varphi_\lambda$ we see that $(\varphi_\lambda^*)_{\lambda\in[0,1)}$ is a bounded subset of $C(\Sigma_A)$. By (19) we also know that $(\varphi_\lambda^*)_{\lambda\in[0,1)}$

is an equicontinuous family, so by Ascoli-Arzela it is pre-compact in $C(\Sigma_A)$.

Let φ now denote any accumulation point of this family as $\lambda \nearrow 1$. Now $M_{f,\lambda}\varphi_\lambda^* = \varphi_\lambda^* + (1-\lambda)\inf \varphi_\lambda$ for all $0 \le \lambda < 1$, so

$$M_f\varphi = \varphi + c$$

for some $c \in \mathbb{R}$, and (φ, c) is the required fixed point pair. Since $\varphi \in CB(X)$ then $c = c(f) = \lim_n \frac{1}{n}\sup_{x\in X} S_n f(x)$, by Lemma 1. Letting $\lambda \nearrow 1$ in (19) shows that (6) holds, while letting $\lambda \nearrow 1$ in (20) gives

$$\mathrm{var}_0(\varphi) \le N(\sup f - \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})}) + \sum_{j=2}^{\infty} \mathrm{var}_j(f)\,,$$

so that the appropriate one of (7) and (8) holds. □

Remark 5. If Σ_A is finitely primitive and $f \in CB(\Sigma_A)$ then both of inequalities (7) and (8) hold. Which one of these gives the sharper estimate on the oscillation of essential fixed points φ will depend on the particular space Σ_A and function f.

The Bousch-Fathi normal form guaranteed by Theorem 2 is in general not a strong normal form; it is straightforward to exhibit bounded functions of summable variation without maximizing measures, as in the following example where the Bousch-Fathi normal form is found explicitly.

Example 6. (Bousch-Fathi normal form but no strong normal form)

Let $T : \Sigma \to \Sigma$ be the full shift on the alphabet $\mathbb{N}$, and let $f : \Sigma \to \mathbb{R}$ be constant on length-2 cylinder sets, with $f[m,n] = \frac{-1}{n(n+1)}$ if $m = n+1$ and $f[m,n] = -1$ otherwise. First we claim that $\alpha(f) = \beta(f) = \gamma(f) = 0$, and that there exist f-maximizing orbits but no f-maximizing measures. Certainly $\gamma(f) \le 0$, since $f < 0$. If ν_n denotes the unique invariant measure supported on the periodic orbit generated by $\omega^{(n)} := \overline{(n, n-1, \ldots, 1)}$, then $\sup_{n\ge 0} \int f\, d\nu_n = 0$, so that $\alpha(f) = 0$. Therefore $\alpha(f) = \beta(f) = \gamma(f) = 0$, by Proposition 1 (i). If $\omega = (1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots)$, formed by concatenating, in order of increasing n, all words of the form $(n, n-1, \ldots, 1)$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i\omega) = 0$, so that ω has an f-maximizing orbit. It is not hard to see that f does not have any f-maximizing measures. Therefore, by Proposition 2, f does not have a strong normal form.

The function f is already in weak normal form: $f \le \gamma(f)$. By Theorem 2 we know that f has a Bousch-Fathi normal form. To see this explicitly, let $\varphi \in CB(\Sigma)$ be constant on length-1 cylinder sets, defined by $\varphi([n]) = -1/n$ for all $n \in \mathbb{N}$. A short calculation reveals that $f + \varphi - \varphi \circ T = 0 = \gamma(f)$ on cylinder sets of the form $[n+1, n]$, whereas $(f + \varphi - \varphi \circ T)([m, n]) = -1 - \frac{1}{m} + \frac{1}{n} < 0 = \gamma(f)$ if $m \neq n+1$. So $(\varphi, 0)$ is a fixed point pair for M_f, and $\tilde{f} = f + \varphi - \varphi \circ T$ is a Bousch-Fathi normal form for f.

In fact a Bousch-Fathi normal form need not even be a weak normal form, as the following example shows.

Example 7. ($c(f) > \gamma(f)$; Bousch-Fathi normal form but no weak normal form)

It is notationally convenient to take $\mathbb{Z}$ rather than $\mathbb{N}$ as the alphabet for the following subshift of finite $\Sigma_A \subset \mathbb{Z}^{\mathbb{N}}$. Define the adjacency matrix A by $A(i, i+1) = 1\ \forall i \in \mathbb{Z}$, $A(0, i) = 1\ \forall i \geq 1$, and $A(i, j) = 0$ otherwise. Define f to be constant on cylinder sets of length one, with $f[i]_A = 0$ for $i \leq 0$ and $f[i]_A = -1$ otherwise. Clearly $\lim_n \frac{1}{n} S_n f(x) = -1$ for all $x \in \Sigma_A$, so $\gamma(f) = -1$. But $c(f) = 0$; indeed f is already in Bousch-Fathi normal form, since $(\varphi, c(f)) = (0, 0)$ is a fixed point pair for M_f. However f does not have a weak normal form: if $\tilde{f} \sim f$ with $\tilde{f} \leq \gamma(f) = 0$ then $c(\tilde{f}) \leq 0$, whereas in fact $c(\tilde{f}) = c(f) = -1$ since it is easily seen that $c(\cdot)$ is well-defined on cohomology classes.

The key idea in the following result is that if the oscillation of f is *large* compared to the oscillation of essential fixed points φ of M_f, then a Bousch-Fathi normal form can be promoted to a strong normal form.

Proposition 3. *Let Σ_A be any subshift of finite type. Suppose f is continuous and M_f has an essential fixed point $\varphi \in CB(\Sigma_A)$ with*

$$var_0(\varphi) < \inf f|_V - \sup f|_{[i]_A} \quad \text{for all sufficiently large } i \in \mathbb{N}, \quad (21)$$

where $V \subset \Sigma_A$ satisfies $\cap_{n=0}^{\infty} T^{-n} V \neq \emptyset$ and $T^{-1}(x) \cap V \neq \emptyset$ for all $x \in \Sigma_A$.
Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset .$$

Proof. It suffices to prove the essential compactness of f; since Σ_A is a Polish space and each $T^{-1}(x)$ is at most countable the other conclusions follow from Theorem 1 and Proposition 2. If $I \in \mathbb{N}$ is such that (21) holds for all $i > I$, then in particular

$$\inf f|_V - \sup f|_{[i]_A} > 0 \quad \text{for all } i > I.$$

Therefore $\pi_{A,1}(V)$ is a bounded subset of $\mathbb{N}$, with least upper bound $J \leq I$. Let us define $Y := \cup_{i \leq I} [i]_A$. Clearly $V \subset Y$. Then $\cap_{n=0}^{\infty} T^{-n} Y$ is compact, and contains the non-empty set $\cap_{n=0}^{\infty} T^{-n} V$, so is itself non-empty. Moreover, $T^{-1}(x) \cap Y$ is finite for each $x \in \Sigma_A$, and contains

$T^{-1}(x) \cap V$ so is non-empty, and its cardinality is locally constant in x. So Y satisfies conditions (a) and (b) of Definition 5. It remains to check property (c), that

$$\varphi(x) + c = \max_{y \in T^{-1}(x) \cap Y} (f + \varphi)(y) \quad \text{for all } x \in \Sigma_A .$$

Let $x \in \Sigma_A$ be arbitrary, and suppose that $ix \in \Sigma_A$ for some $i > I$. Now x has at least one pre-image $jx \in V \subset Y$, and by (21) we know that

$$\begin{aligned} f(jx) - f(ix) &> \mathrm{var}_0(\varphi) \\ &\geq \varphi(ix) - \varphi(jx) . \end{aligned}$$

That is,

$$(f + \varphi)(jx) - (f + \varphi)(ix) > 0 .$$

So the supremum $\sup_{y \in T^{-1}(x)} (f + \varphi)(y) = M_f \varphi(x)$ cannot be attained by any $y \in \cup_{i>I} [i]_A = \Sigma_A \setminus Y$. Therefore the supremum must be attained by one of the finitely many preimages of x lying in Y. That is,

$$M_f \varphi(x) = \sup_{y \in T^{-1}(x)} (f + \varphi)(y) = \max_{y \in T^{-1}(x) \cap Y} (f + \varphi)(y)$$

for all $x \in \Sigma_A$, as required. □

Corollary 1. *Suppose Σ_A has a finite universal preimage alphabet $\mathbb{L}$. Suppose f is continuous, and M_f has an essential fixed point $\varphi \in CB(\Sigma_A)$ with*

$$var_0(\varphi) < \inf f|_{\cup_{k \in \mathbb{L}} [k]_A} - \sup f|_{[i]_A} \quad \text{for all sufficiently large } i \in I\!N . \tag{22}$$

Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset .$$

Proof. $V = \cup_{k \in \mathbb{L}} [k]_A$ satisfies the hypotheses of Proposition 3. □

6. Normal form theorems

Our first normal form theorem is the following.

Theorem 3. *Suppose $f : \Sigma_A \to I\!R$ has summable variations, and either (i) f is bounded, and Σ_A is primitive and has a finite universal preimage alphabet $\mathbb{L} \subset I\!N$ such that*

$$n_A var_0(f) + \sum_{j=2}^{\infty} var_j(f) < \inf f|_{\cup_{k \in \mathbb{L}} [k]_A} - \sup f|_{[i]_A} \tag{23}$$

for all sufficiently large $i \in I\!N$, where $n_A \in I\!N$ is the primitive constant for A, *or (ii) f is bounded above, Σ_A is finitely primitive, and*

$$N(\sup f - \inf f|_{\cup_{k\in\mathbb{M}}[k]_A}) + \sum_{j=2}^{\infty} var_j(f) < \inf f|_{\cup_{k\in\mathbb{M}}[k]_A} - \sup f|_{[i]_A} \quad (24)$$

for all sufficiently large $i \in I\!N$, where $(N, \mathbb{M})$ is some finitely primitive pair for A.

Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset\,.$$

Proof. Under either hypothesis the Bousch-Lax-Oleinik operator M_f has a bounded essential fixed point φ, by Theorem 2.

In case (i) we know that $\text{var}_0(\varphi) \le n_A \text{var}_0(f) + \sum_{j=2}^{\infty} \text{var}_j(f)$, by (7), and the conclusion follows from Corollary 1. In case (ii), by (8) we have $\text{var}_0(\varphi) \le N(\sup f - \inf f|_{\cup_{k\in\mathbb{M}}[k]_A}) + \sum_{j=2}^{\infty} \text{var}_j(f)$, and the conclusion again follows from Corollary 1, since finite primitivity implies the finite universal preimage property. □

In the case where the subshift of finite type is the full shift, Theorem 3 can be reformulated as:

Corollary 2. *Let $\Sigma = I\!N^{I\!N}$ be the full shift. Suppose $f : \Sigma \to I\!R$ is bounded above, has summable variations, and that there exists $I \in I\!N$ such that*

$$\sum_{j=2}^{\infty} var_j(f) < 2\inf f|_{[I]} - \sup f - \sup f|_{[i]}$$

for all i sufficiently large.

Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset\,.$$

Proof. By Theorem 3 it suffices to note that Σ is finitely primitive and that $(N, \mathbb{M}) = (1, \{I\})$ is a finitely primitive pair for any $I \in I\!N$. □

Apart from regularity (summable variations) and boundedness, the key condition in Theorem 3 is that the value of f on some finite union of cylinders is sufficiently larger than its values "at infinity". An extreme example of this is when $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$. This occurs in particular when the *summability condition*

$$\sum_{i\in I\!N} \exp(\sup f|_{[i]_A}) < \infty \quad (25)$$

holds. This condition plays a key role in the development of the thermodynamic formalism for countable state subshifts of finite type, allowing us to define the *Ruelle operator* $\mathcal{L}_f\varphi(x) = \sum_{Ty=x} e^{f(y)}\varphi(y)$. If Σ_A is finitely primitive and f has summable variations then (25) is equivalent (cf. [MU2], Prop. 2.7) to the finiteness of the topological pressure

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n y = y} \exp\left(\sup_{\omega\in[\Pi_{A,n}(y)]} \sum_{i=0}^{n-1} f(T^i\omega) \right) ,$$

and is a necessary condition for the existence of an invariant Gibbs measure for f. Any f satisfying the summability condition (25) is necessarily bounded above, and if $\mathbb{N}_A$ is infinite then f is unbounded below.

Theorem 4. *Let Σ_A be finitely primitive. Suppose f has summable variations and satisfies the summability condition (25).*
Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{ m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f)) \right\} \neq \emptyset \,.$$

Proof. By Theorem 2 we know that the Bousch-Lax-Oleinik operator M_f has an essential fixed point $\varphi \in CB(\Sigma_A)$. The summability condition (25) and the finiteness of $\mathrm{var}_1(f)$ together imply that $\sup f|_{[i]_A} \to -\infty$ as $i \to \infty$, with the convention that $\sup f|_{[i]_A} = -\infty$ whenever $[i]_A$ is empty. So the righthand side of (22) equals $+\infty$, whereas the lefthand side is finite since φ is bounded, therefore the result follows from Corollary 1. $\square$

In particular we deduce

Corollary 3. *Let Σ_A be finitely primitive. If f has summable variations and an invariant Gibbs measure then it also has a maximizing measure.*

We now proceed to derive a more refined normal form theorem. Suppose $f : \Sigma_A \to \mathbb{R}$ is bounded above. We call $\sup_{y\in T^{-1}(x)} f(y)$ the *preimage supremum*, and $\overline{\lim}_{i\to\infty} f(ix)$ the *preimage limit supremum* of the function f at the point x. By convention $f(ix) = -\infty$ if $x \notin T([i]_A)$. If $f_i : \Sigma_A \to \mathbb{R} \cup \{-\infty\}$ is defined as $f_i(x) = f(ix)$ then the function $\bar{f}(x) = \overline{\lim}_{i\to\infty} f_i(x)$ is well-defined pointwise. Let X_f denote those $x \in \Sigma_A$ for which $\bar{f}(x) > -\infty$.

We see below (Theorem 5) that if preimage suprema are uniformly sufficiently larger than preimage limit suprema then, provided f satisfies additional uniformity conditions (cf. Definition 9) "at infinity", it again has a strong normal form.

Definition 9. The function $f : \Sigma_A \to I\!R$ *uniformly attains preimage suprema* if there exists $J \in I\!N$ such that for all $x \in \Sigma_A$ the preimage supremum $\sup_{y\in T^{-1}(x)} f(y)$ is attained at a point $y_x \in T^{-1}(x) \cap (\cup_{j=1}^J [j]_A)$.

f *uniformly almost attains preimage suprema* if for all $\varepsilon > 0$ there exists $J \in I\!N$ such that for all $x \in \Sigma_A$ there is some $y_x \in T^{-1}(x) \cap \cup_{j=1}^J [j]_A$ such that $f(y_x) > \sup_{y\in T^{-1}(x)} f(y) - \varepsilon$.

f *uniformly attains preimage limit suprema* if $\overline{\lim}_{j\to\infty} f_j = \bar{f}$ uniformly; more precisely, for all $\varepsilon > 0$ and $C \in I\!R$ there exists $I \in I\!N$ such that if $i > I$ then

$$f(ix) < \overline{\lim_{j\to\infty}} f(jx) + \varepsilon \quad \text{for all } x \in X_f \cap T([j]_A)\,,$$

and

$$f(ix) < -C \quad \text{for all } x \in \Sigma_A \setminus X_f\,.$$

The condition of uniformly almost attaining preimage suprema is clearly weaker than uniformly attaining preimage suprema, and is sufficient for the following normal form theorem; however in practice the condition of uniformly attaining preimage suprema is often easier to verify.

Theorem 5. *Suppose $f : \Sigma_A \to I\!R$ has summable variations, uniformly attains preimage limit suprema, uniformly almost attains preimage suprema, and*
either (i) f is bounded and Σ_A is primitive and has a finite universal preimage alphabet $\mathbb{L} \subset I\!N$ such that

$$n_A var_0(f) + \sum_{j=2}^{\infty} var_j(f) < \inf_{x\in\Sigma_A}\left[\sup_{y\in T^{-1}(x)} f(y) - \overline{\lim_{j\to\infty}} f(jx)\right] \tag{26}$$

where $n_A \in I\!N$ is the primitive constant for A,
or (ii) f is bounded above, Σ_A is finitely primitive, and

$$N(\sup f - \inf f|_{\pi_{A,1}^{-1}(\mathbb{M})}) + \sum_{j=2}^{\infty} var_j(f) < \inf_{x\in\Sigma_A}\left[\sup_{y\in T^{-1}(x)} f(y) - \overline{\lim_{j\to\infty}} f(jx)\right], \tag{27}$$

where $(N, \mathbb{M})$ is some finitely primitive pair for A.
Then f is essentially compact, hence has a strong normal form $\tilde{f}$, hence

$$\mathcal{M}_{max}(f) = \left\{m \in \mathcal{M} : supp(m) \subset \tilde{f}^{-1}(\gamma(f))\right\} \neq \emptyset\,.$$

Proof. It suffices to prove essential compactness; the other conclusions follow from Theorem 1 and Proposition 2. In either case (i) or (ii), by

Theorem 2 we deduce there exists an essential fixed point φ for the Bousch-Lax-Oleinik operator M_f such that

$$\mathrm{var}_0(\varphi) < \inf_{x\in\Sigma_A}\left[\sup_{y\in T^{-1}(x)} f(y) - \overline{\lim_{j\to\infty}} f(jx)\right]. \tag{28}$$

Define $\varepsilon = \inf_{x\in\Sigma_A}\left[\sup_{y\in T^{-1}(x)} f(y) - \overline{\lim}_{j\to\infty} f(jx)\right] - \mathrm{var}_0(\varphi)$.

If $\mathbb{L}$ is a finite universal preimage alphabet for Σ_A (in particular if $\mathbb{L}$ is a finitely primitive alphabet) then for all $x \in \Sigma_A$,

$$\sup_{y\in T^{-1}(x)} f(y) \geq \inf f|_{\cup_{j\in\mathbb{L}}[j]_A} > -\infty\,,$$

since f is bounded above and $\mathrm{var}_1(f) < \infty$. Define

$$C = \frac{\varepsilon}{2} - \inf f|_{\cup_{j\in\mathbb{L}}[j]_A} + \mathrm{var}_0(\varphi)\,. \tag{29}$$

Since f uniformly attains preimage limit suprema there exists $I \in \mathbb{N}$ such that if $i > I$ then

$$f(ix) < \overline{\lim_{j\to\infty}} f(jx) + \frac{\varepsilon}{2} \quad \text{for all } x \in X_f \cap T([i]_A) \tag{30}$$

and

$$f(ix) < -C \quad \text{for all } x \in \Sigma_A \setminus X_f\,. \tag{31}$$

Since f uniformly almost attains preimage suprema, we can find $J \in \mathbb{N}$ such that for all $x \in \Sigma_A$ there exists $i_x \in \{1,\dots,J\}$ such that $i_x x \in \Sigma_A$ and

$$f(i_x x) > \sup_{y\in T^{-1}(x)} f(y) - \varepsilon/2\,. \tag{32}$$

If $V := \cup_{j\leq J}[j]_A$ then $\cap_{n=0}^{\infty} T^{-n}(V)$ is compact and non-empty, and $T^{-1}(x)\cap V$ is non-empty for each $x \in \Sigma_A$. Let $K = \max(I,J)$. Then for all $x \in X_f$, and for all $i > K$ such that $ix \in \Sigma_A$, the inequalities (28), (30), (32) yield

$$\begin{aligned} f(i_x x) - f(ix) &> \sup_{y\in T^{-1}(x)} f(y) - \frac{\varepsilon}{2} - \left(\overline{\lim_{j\to\infty}} f(jx) + \frac{\varepsilon}{2}\right) \\ &> \mathrm{var}_0(\varphi) \\ &\geq \varphi(ix) - \varphi(i_x x)\,, \end{aligned}$$

while for $x \in \Sigma_A \setminus X_f$ and $i > K$, (29), (31), (32) give

$$\begin{aligned} f(i_x x) - f(ix) &> \sup_{y \in T^{-1}(x)} f(y) - \frac{\varepsilon}{2} + C \\ &\geq \inf f|_{\cup_{j \in \mathbb{L}}[j]_A} - \frac{\varepsilon}{2} + C \\ &\geq \mathrm{var}_0(\varphi) \\ &\geq \varphi(ix) - \varphi(i_x x) . \end{aligned}$$

That is,

$$(f+\varphi)(i_x x) > (f+\varphi)(ix) \quad \forall x \in \Sigma_A, \forall i > K \text{ s.t. } ix \in \Sigma_A . \qquad (33)$$

Let $Y := \cup_{j \leq K}[j]_A \supset V$. Clearly $T^{-1}(x) \cap Y$ is finite for all $x \in \Sigma_A$, and is non-empty because it contains $T^{-1}(x) \cap V$. The invariant set $\cap_{n=0}^{\infty} T^{-n} Y$ is certainly compact, and is non-empty because it contains $\cap_{n=0}^{\infty} T^{-n} V \neq \emptyset$.

By (33), each supremum $\sup_{y \in T^{-1}(x)}(f+\varphi)(y) = M_f\varphi(x)$ must be attained by one of the finitely many preimages of x lying in Y. In particular each supremum is a maximum, and

$$M_f\varphi(x) = \sup_{y \in T^{-1}(x)} (f+\varphi)(y) = \max_{y \in T^{-1}(x) \cap Y} (f+\varphi)(y)$$

for all $x \in \Sigma_A$. Therefore f is essentially compact, as required. □

7. Zero temperature Gibbs measures are maximizing

Several authors, notably [CG], [CLT], [J1, J3, J4], have noted that, under suitable hypotheses, any weak* accumulation point as $t \to \infty$ of the family μ_{tf} of Gibbs (or equilibrium) measures is an f-maximizing measure. Moreover, this accumulation point has maximal entropy among all maximizing measures. The thermodynamic interpretation of the parameter t is as an inverse temperature, so that letting $t \to \infty$ corresponds to letting the temperature tend to zero. Here we establish a version of this result in the context of countable state subshifts of finite type; the principal novelty is a tightness argument which guarantees the pre-compactness of the family μ_{tf}.

Definition 10. An adjacency matrix A, and the corresponding subshift of finite type Σ_A, are *finitely irreducible* if there exists $N \in I\!N$, and a finite sub-alphabet $\mathbb{M} \subset I\!N$, such that for all $x \in \Sigma_A$ and all $i \in I\!N_A$ there exists $w \in [i]_A \cap (\cup_{n=1}^{N} \mathbb{M}^n)$ with $wx \in \Sigma_A$.

Clearly finite primitivity implies finite irreducibility. However primitivity neither implies nor is implied by finite irreducibility.

The following facts are classical results in the thermodynamic formalism of Ruelle and Bowen (see [Ru], [Bo]) in the case of finite state subshifts of finite type. For countable state subshifts of finite type they are due to Mauldin & Urbański [MU2].[2]

Lemma 2. *Let Σ_A be finitely irreducible. Suppose $g : \Sigma_A \to I\!R$ has summable variations and satisfies the summability condition (25). Then*

$$P(g) = \sup\{h(m) + \int g\,dm : m \in \mathcal{M}_g, \int g\,dm > -\infty\}\,, \tag{34}$$

where $h(m)$ denotes the metric entropy of m.

There exists a unique $\mu_g \in \mathcal{M}_g$, called the Gibbs measure *for g, such that*

$$C \le \frac{\mu_g([w]_A)}{\exp\big(\inf S_{|w|}g|_{[w]_A} - |w|P(g)\big)} \le \exp\left(4V(g)\right) \tag{35}$$

for all A-admissible finite words w, where $C > 0$ is a constant independent of w, and $|w|$ denotes the length of w.

If furthermore Σ_A is finitely primitive, and

$$\sum_{i\in I\!N} \inf(-g|_{[i]_A}) \exp(\inf g|_{[i]_A}) < \infty\,, \tag{36}$$

then $\int g\,d\mu_g > -\infty$ and μ_g is the unique equilibrium state *for g; i.e. it is the unique measure in $\mathcal{M}$ which realises the supremum in (34):*

$$P(g) = h(\mu_g) + \int g\,d\mu_g\,. \tag{37}$$

Proof. The variational principle (34) is Theorem 2.6 of [MU2], the Gibbs property (35) follows from [MU2], Theorems 2.11 and 2.17, while the fact that μ_g is the unique equilibrium state follows from Lemma 2.13 and Theorem 2.14 of [MU2]. $\square$

If f has summable variations and satisfies the summability condition (25), then for each $t \ge 1$ the same is true of the function tf, so that tf has a unique invariant Gibbs measure μ_{tf}. We recall that a family $\{\nu_t\}_t \in F$ of Borel probability measures on a metric space X is called *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\nu_t(K) > 1 - \varepsilon$ for all $t \in F$. Prohorov's theorem (cf. [Bi], p. 37) asserts that any such family is pre-compact in the weak* topology.

[2]The results in [MU2] are stated under the slightly stronger hypothesis that f is locally Hölder, though in fact summable variations is sufficient.

Theorem 6. *Let Σ_A be finitely primitive. Suppose f has summable variations, satisfies the summability condition (25), and satisfies (36). Then the family of Gibbs measures $\{\mu_{tf}\}_{t\geq 1}$ is tight, hence has a weak* accumulation point as $t \to \infty$. Any such accumulation point μ is an f-maximizing measure, and moreover satisfies $h(\mu) = \sup_{m\in\mathcal{M}_{max}(f)} h(m)$.*

Proof. Suppose μ is a weak* accumulation point of $\{\mu_{tf}\}_{t\geq 1}$. If $p(t) = P(tf)$ for $t \geq 1$ then $p'(t) = \int f\, d\mu_{tf}$ (cf. [MU2], Prop. 2.46). But (34) implies that p is convex, so that $t \mapsto p'(t) = \int f\, d\mu_{tf}$ is non-decreasing, and bounded above by $\sup f$. It follows that the limit $\lim_{t\to\infty} p'(t) = \lim_{t\to\infty} \int f\, d\mu_{tf} = \int f\, d\mu$ exists.

If μ is not f-maximizing then there exists $\nu \in \mathcal{M}_f$ with $\int f\, d\nu > \int f\, d\mu$. So the gradient of the line $l_\nu : t \mapsto h(\nu) + t\int f\, d\nu$ is strictly larger than the gradient at any point on the curve $t \mapsto p(t)$, $t \geq 1$, therefore $l_\nu(t) > p(t)$ for all sufficiently large t. That is, $h(\nu)+\int tf\, d\nu > P(tf)$ for all sufficiently large t, contradicting (34). Therefore μ is f-maximizing.

The proof that μ maximizes entropy among f-maximizing measures follows the same argument as in Lemma 14 of [J3].

It remains to show that $\{\mu_{tf}\}_{t\geq 1}$ is tight, from which it follows that accumulation points μ exist. Given $\varepsilon > 0$, we claim there is an increasing sequence (n_k) in $I\!N$ such that the compact set $K = \{x \in \Sigma_A : 1 \leq x_k \leq n_k\ \forall k \in I\!N\}$ satisfies $\mu_{tf}(K) > 1 - \varepsilon$ for all $t \geq 1$. Now

$$\begin{aligned} \mu_{tf}(K) &= \mu_{tf}\left(\Sigma_A \setminus \cup_{k=1}^{\infty}\{x \in \Sigma_A : x_k > n_k\}\right) \\ &\geq 1 - \sum_{k=1}^{\infty} \mu_{tf}\left(\{x \in \Sigma_A : x_k > n_k\}\right) \\ &= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_{tf}(\pi_{A,k}^{-1}(i)), \end{aligned}$$

so to ensure $\mu_{tf}(K) > 1 - \varepsilon$ it suffices to choose (n_k) such that

$$\sum_{i=n_k+1}^{\infty} \mu_{tf}(\pi_{A,k}^{-1}(i)) < \frac{\varepsilon}{2^k} \quad \forall k \in I\!N\ \forall t \geq 1. \tag{38}$$

The shift invariance of μ_{tf} and the righthand inequality in (35) (with $g = tf$, $|w| = |i| = 1$) give

$$\begin{aligned} \mu_{tf}(\pi_{A,k}^{-1}(i)) &= \mu_{tf}([i]_A) \\ &\le e^{4V(tf)} \exp\left(\inf\{tf|_{[i]_A}\} - P(tf)\right) \qquad (39) \\ &= e^{4tV(f)} \exp\left(\inf\{tf|_{[i]_A}\} - P(tf)\right) . \end{aligned}$$

If $m \in \mathcal{M}_f$ is such that $I := \int f\, dm$ is finite then (34) gives $P\left(t(f-I)\right) \ge \int t(f-I)\, dm + h(m) \ge 0$. But $P\left(t(f-I)\right) = P(tf) - tI$, so by (39),

$$\begin{aligned} \mu_{tf}(\pi_{A,k}^{-1}(i)) &\le e^{4tV(f)} \exp\left(\inf\{t(f-I)|_{[i]_A}\}\right) e^{-P(t(f-I))} \\ &\le e^{4tV(f)} \exp\left(\inf\{t(f-I)|_{[i]_A}\}\right) \qquad (40) \\ &= \exp\left(t\left(4V(f) - I + \inf\{f|_{[i]_A}\}\right)\right) . \end{aligned}$$

The summability condition (25) implies that $\inf f|_{[i]_A} \to -\infty$ as $i \to \infty$, with our usual convention that $f|_{[i]_A} = -\infty$ if $[i]_A = \emptyset$, so there exists $J \in \mathbb{N}$ such that if $i \ge J$ then $4V(f) - I + \inf f|_{[i]_A} < 0$. So if $t \ge 1$ and $i \ge J$ then $t(4V(f) - I + \inf f|_{[i]_A}) < 4V(f) - I + \inf f|_{[i]_A} < 0$, and from (40) we obtain

$$\mu_{tf}(\pi_{A,k}^{-1}(i)) \le e^{4V(f)-I} e^{\inf f|_{[i]_A}} . \qquad (41)$$

The summability condition (25) means there exists $n_k \ge J$ such that

$$\sum_{i=n_k+1}^{\infty} e^{\inf f|_{[i]}} < \frac{\varepsilon}{2^k} e^{I-4V(f)} ,$$

and combined with (41) this proves (38), as required. □

Remark 6. Even in the case of Σ_A compact it is unknown whether the family μ_{tf} always *converges* as $t \to \infty$. The best result in this direction is due to Brémont [Br], who proved that if f depends on only finitely many coordinates then there is convergence. This followed earlier partial results in [C], [J3], [PS].

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