DIOPHANTINE APPROXIMATION FOR CONFORMAL MEASURES OF ONE-DIMENSIONAL ITERATED FUNCTION SYSTEMS

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Abstract. Recall that a Borel measure μ on $I\!R$ is is called extremal if μ-almost every number in $I\!R$ is not very well approximable. In this paper, we investigate extremality (and implying it the exponentially fast decay property (efd)) of conformal measures, induced by regular infinite conformal iterated function systems. We then give particular attention to the class of such systems generated by the continued fractions algorithm with restricted entries. It is proved that if the index set of entries has bounded gaps, then the corresponding conformal measure satisfies the (efd) property and is extremal. Also a class of examples of index sets with unbounded gaps is provided for which the corresponding conformal measure also satisfies the (efd) property and is extremal.

1. Introduction

A point $x \in I\!R$ is called very well approximable if there exist $\delta > 0$ and infinitely many integers $p, q \in \boldsymbol{Z}$, $q \geq 1$, such that

$$|qx - p| \leq q^{-(1+\delta)}.$$

It is a classical result that the set of all very well approximable numbers has the Lebesgue measure zero but the Hausdorff dimension equal to 1. Thus the natural question arises about other measures. To be more precise, a Borel measure μ on $I\!R$ is is called extremal if μ-almost every number in $I\!R$ is not very well approximable. Barak Weiss has provided in [15] a nice sufficient condition for a Borel probability measure on $I\!R$ to be extremal. We will use this result heavily. We would like to add that Weiss's result has its multidimensional counterparts (see [3], [2], [9] and [13] for more examples).

In this paper, we investigate extremality (and implying it the exponentially fast decay property (efd)) of conformal measures, induced by regular infinite conformal iterated function systems. We recall that the iteration of infinitely many conformal maps naturally arises in several contexts. For example, one is naturally led to consider infinite systems when one is dealing with parabolic systems, i.e. finite systems of conformal maps which have some cusps (see [8] and the reference therein). Another natural class of examples, investigated in detail in our article, arise from the sets of continued fractions with restricted entries.

In the present paper our ultimate goal is to explore extremality of conformal measures naturally associated to regular continued fraction iterated function systems with restricted entries. Our starting point is the above mentioned result of Barak Weiss (see Theorem 3.2, comp. [15]) saying that the exponentially fast decay property (efd) introduced in [14] by W. Veech, is in the one-dimensional case (the support of the reference measure is contained in $I\!R$) sufficient for extremality. In our paper the extremality of conformal measures we are dealing with is always proven via the (efd) property. In [4] the authors derive some additional diophantine consequences of the exponentially fast decay property.

Our paper is organized as follows. In Section 2 (Preliminaries) we collect some basic properties of conformal infinite iterated function systems. In Section 3 (General Sufficient Conditions) we establish the (efd) property of several classes of measures and, as the main result of the section, we prove Theorem 3.7, a sufficient condition for the (efd) property to hold, heavily used in further sections. In Section 4 we consider the maps $\phi_n : [0,1] \to [0,1]$, for $n \in I\!N$ given by the formula

$$\phi_n(x) = \frac{1}{x+n},$$

and an arbitrary subset I of positive integers $I\!N$. We investigate the corresponding iterated function system $S_I = \{\phi_i\}_{i\in I}$ called a continued fraction iterated function system. Its limit set J_I consists of all those $x \in (0,1)$ that each partial denominator x_i, $i \ge 1$ in the continued fraction expansion

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots}}}}$$

is in I. In Theorem 5.1 an effective sufficient condition for the conformal measure of the system S_I to satisfy the (efd) property (and as a consequence to be extremal) is provided. This condition is expressed in terms of the arithmetic properties of the infinite set I of positive integers. This continues our theme from [6], [11], and [7] that many geometric measure theoretic properties of these systems are reflected in the arithmetic properties of the index set I. The last section, Section 5 is occupied by the proof of the following two fact. First, that each index set I with bounded gaps gives rise to the conformal measure (up to a multiplicative constant the packing measure on J_I) which satisfies the (efd) property, and in consequence, is extremal. Secondly, that there exist infinite regular systems with unbounded gaps whose conformal measures satisfy the (efd) and are extremal.

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2. Preliminaries

Let us describe first the setting of conformal (infinite) iterated function systems introduced in [5]. Let I be a countable index set or alphabet with at least two elements and let $S = \{\phi_i : X \to X\}_{i\in I}$ be a collection of injective contractions from a compact metric space X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \le s\rho(x,y)$, for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We define the limit set J of this system as the image of the coding space under a coding map as follows. Let I^n denote the space of words of length n, I^∞ the space of infinite sequences of symbols in I, $I^* = \bigcup_{n\ge1} I^n$ and for $\omega \in I^n$, $n \ge 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2\ldots\omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = J_S = \pi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i \in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.

An iterated function system $S = \{\phi_i : X \to X\}_{i \in I}$ is said to satisfy the Open Set Condition (OSC) if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. (We do not exclude the possibility that $\overline{\phi_i(U)} \cap \overline{\phi_j(U)} \neq \emptyset$.) A system satisfying (OSC) is said to satisfy the Strong Open Set Condition (SOSC) if $J_S \cap U \neq \emptyset$ and it is said to satisfy the Super Strong Open Set Condition (SSOSC) if $\overline{J_S} \subset U$.

An iterated function system S satisfying the Open Set Condition is said to be conformal if $X \subset I\!R^d$ for some $d \geq 1$ and the following conditions are satisfied.

(1a) $U = \mathrm{Int}_{I\!R^d}(X)$.

(1b) There exists an open connected set V such that $X \subset V \subset I\!R^d$ and such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V. (Note that for $d = 1$ this just means that all the maps ϕ_i, $i \in I$, are C^1 monotone diffeomorphisms, for $d \geq 2$ the words C^1 conformal mean holomorphic or antiholomorphic, and for $d > 2$ the maps ϕ_i, $i \in I$ are Möbius transformations. The proof of the last statement can be found in [1] for example, where it is called Liouville's theorem.)

(1c) There exist $\gamma, l > 0$ such that for every $x \in X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.

(1d) Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

In fact we will need the following stronger condition than (1d).

(1e) There exists a function $K : [0, \infty) \to I\!R$ such that $\lim_{r \searrow 0} K(r) = 1$ and

$$|\phi'_\omega(y)| \leq K(||x - y||)|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$.

This condition is always satisfied if $d \geq 2$ (see [8], comp. [12]) and if the system S is finite with $\phi_i \in C^{1+\varepsilon}$ for all $i \in I$.

Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets C of V

$$\mathrm{diam}(\phi_\omega(C)) \leq ||\phi'_\omega||\mathrm{diam}(C) \tag{2.1}$$

and

$$\mathrm{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||, \tag{2.2}$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \geq 1$ is a constant depending only on V. In addition,

$$\mathrm{diam}(\phi_\omega(J)) \geq D^{-1}||\phi'_\omega|| \tag{2.3}$$

$$\phi_\omega(B(x,r)) \subset B(\phi_\omega(x), ||\phi'_\omega||r) \tag{2.4}$$

and

$$\phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r) \tag{2.5}$$

for every $x \in X$, every $0 < r \leq \mathrm{dist}(X, \partial V)$, and every word $\omega \in I^*$.

As was demonstrated in [5], infinite conformal iterated function systems, unlike finite systems, may not possess a conformal measure. There are even continued fraction systems which do not possess a conformal measure, see [6], Example 6.5. Thus, the infinite systems naturally break into two main classes, irregular and regular. This dichotomy can be determined from either the existence of a zero of the topological pressure function or, equivalently, the existence of a conformal measure. The topological pressure function, $\mathrm{P}(t)$, $t \geq 0$, is defined as follows. For every integer $n \geq 1$, define

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi'_\omega||^t.$$

and set

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(t).$$

For a conformal system S, we sometimes write $\psi_S = \psi_1 = \psi$. The finiteness parameter, θ_S, of the system S is defined by $\inf\{t : \psi(t) < \infty\} = \theta_S$. In [5], it was shown that the topological pressure function $\mathrm{P}(t)$ is non-increasing on $[0, \infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$ and $\mathrm{P}(d) \leq 0$. Of course, $\mathrm{P}(0) = \infty$ if and only if I is infinite. In [5] (see Theorem 3.15) we have proved the following characterization of the Hausdorff dimension of the limit set J_S, which will be denoted by $h = h_S$.

Theorem 2.1.

$$h_S = \sup\{h_F : F \subset I \textit{ is finite }\} = \inf\{t : \mathrm{P}(t) \leq 0\}.$$

If $\mathrm{P}(t) = 0$*, then* $t = h_S$*.*

We called the system S regular provided that there is some t such that $\mathrm{P}(t) = 0$. It follows from [5] that t is unique. Also, the system is regular if and only if there is a t-conformal measure. A Borel probability measure m is said to be t-conformal provided $m(J_S) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \neq j$.

There are some natural subclasses within the regular systems. A system $S = \{\phi_i\}_{i \in I}$ is said to be strongly regular if $0 < \mathrm{P}(t) < \infty$ for some $t \geq 0$. As an immediate application of Theorem 2.1 we get the following.

Theorem 2.2. *A conformal system S is strongly regular if and only if $h > \theta$.*

Also, in [5] we called a system $S = \{\phi_i\}_{i \in I}$ hereditarily regular or cofinitely regular provided every nonempty subsystem $S' = \{\phi_i\}_{i \in I'}$, where I' is a cofinite subset of I, is regular. A finite system is cofinitely regular and for an infinite system, we showed in [5] that whether a system is cofinitely regular can be also determined from the pressure function:

Theorem 2.3. *An infinite system S is cofinitely or hereditarily regular if and only if* $\mathrm{P}(\theta) = \infty \Leftrightarrow \psi(\theta) = \infty \Leftrightarrow \{t : \mathrm{P}(t) < \infty\} = (\theta, \infty) \Leftrightarrow \{t : \psi(t) < \infty\} = (\theta, \infty)$.

We would like to mention that cofinitely regular systems are important since they often appear as natural examples and checking cofinite regularity (if it holds) is usually much easier than checking regularity of systems which are not cofinitely regular.

3. General Sufficient Conditions

We start with the following simple but useful result.

Proposition 3.1. *Let (X, ρ) be a metric space and let μ be a Borel probability measure on X. Then the following two conditions are equivalent.*

(a)

$$\exists(\alpha \in (0,1))\, \exists(\beta > 1)\, \exists(\xi > 0)\, \forall(x \in X)\, \forall(r \leq \xi)$$
$$\mu(B(x,r)) \leq \alpha\mu(B(x, \beta r)).$$

(b) *There exists a Borel set $Y \subset X$ with $\mu(Y) = 1$ such that the condition (a) is satisfied with X replaced by Y.*

Proof. The implication (a)⇒(b) is obvious. In order to prove the opposite implication fix $x \in X$ and $r \le \xi/2$. If $B(x,r) \cap Y = \emptyset$, then $\mu(B(x,r)) = 0$, and we are done. Otherwise, fix $y \in B(x,r) \cap Y$. Then

$$\mu(B(x,r)) \le \mu(B(y,2r)) \le \alpha\mu(B(y,\beta 2r)) \le \alpha\mu(B(x,2\beta r + r)) = \alpha\mu(B(x,(2\beta+1)r)),$$

and we are done. ■

Any Borel probability measure satisfying condition (a) or, equivalently, condition (b) of Proposition 3.1, is said to satisfy the exponentially fast decay (efd) property. The significance of this property, though interesting itself, results from the following fact, essentially proven in [15] and crucial for our approach.

Theorem 3.2. *(B. Weiss) Every Borel probability measure on $I\!R$ satisfying the (efd) property is extremal.*

A Borel probability measure μ is said to be geometric if there exist $\xi > 0$, $C > 0$, and $h > 0$ such that

$$C^{-1}r^h \le \mu(B(x,r)) \le Cr^h$$

for all $x \in supp(\mu)$ and all $r \le \xi$. Our next simple result is the following.

Proposition 3.3. *Every geometric measure satisfies the (efd) property.*

Proof. Fix $\beta > 1$ to be specified later and consider an arbitrary point $x \in supp(\mu)$ and $r \le \xi/\beta$. Then

$$\mu(B(x,r)) \le Cr^h = C^2\beta^{-h}(C^{-1}\big(\beta r)^h\big) \le C^2\beta^{-h}\mu(B(x,\beta r)),$$

and it suffices to take $\beta > 1$ so large that $C^2\beta^{-h} < 1$. We are therefore done since the condition (b) of Proposition 3.1 is satisfied with $Y = supp(\mu)$. ■

As an immediate consequence of this proposition, Lemma 3.14 from [5] and Theorem 3.2, we get the following.

Theorem 3.4. *If $S = \{\phi_i : X \to X\}_{i\in I}$ is a finite conformal iterated function system and $h := \mathrm{HD}(J_S)$, then all the measures: Hausdorff $\mathrm{H}^s|_{J_S}$, packing $\mathrm{P}^s|_{J_S}$ and the h-conformal measure m on J_S satisfy the (efd) property. If in addition, $X \subset I\!R$, then all these measures are extremal.*

As an immediate consequence of Proposition 3.3 and a well known Theorem from [10], we get the following.

Theorem 3.5. *If $f : X \to X$ is a mixing expanding repeller and $h := \mathrm{HD}(X)$, then the h-dimensional Hausdorff measure on X and h-dimensional packing measure on X satisfy the (efd) property. If in addition, $X \subset I\!\!R$, then all these measures are extremal.*

We shall prove now in the context of iterated function systems a sufficient condition for a conformal measure to satisfy the (efd) property. This condition will be the starting point of all our more specific results establishing the (efd) property of conformal measures for some conformal iterated function systems.

Theorem 3.6. *Suppose that the system $S = \{\phi_i\}_{i\in I}$ is regular, and denote by m the corresponding h-conformal measure. Suppose also that there exists an integer $q \geq 0$ and $\kappa \in (0, \min\{1, \mathrm{dist}(X, \partial V)\})$ such that the following three conditions are satisfied.*

(a) *There exist three real constants: $\gamma \geq 1$, $\beta > 1$, $\alpha \in (0,1)$, and a finite set $F \subset I$ such that for all $i \in I \setminus F$, all $x \in \phi_i(J)$ and all $r \in [\gamma||\phi_i'||, \kappa]$*
$$m(B(x,r)) \leq \alpha\mu(B(x,\beta r)).$$
(b) *For all $\omega \in I^*$ with $|\omega| \leq q$, all $x \in J$, and all $r \in (0, \kappa)$*
$$m(\phi_\omega(B(x,r))) \leq \alpha^{-\frac{1}{3}}|\phi_\omega'(x)|^h m(B(x,r)).$$
(c) *For all $\omega \in I^q$ and all $x \in J$,*
$$\phi_\omega(B(x,\kappa)) \subset X.$$

Then the h-conformal measure m satisfies the (efd) property.

Proof. Let
$$G = F \cup \{i \in I : \gamma||\phi_i'|| > \kappa]\}.$$
Since $\lim_{i\in I}||\phi_i'|| = 0$, the set G is finite. Take now an arbitrary $\rho > 0$, $i \in I \setminus G$, $x \in \phi_i(J)$, and $r \in (\rho||\phi_i'||, \kappa)$. If $r \geq \gamma||\phi_i'||$, then it follows from (a) that $m(B(x,r)) \leq \alpha\mu(B(x,\beta r))$. If $r < \gamma||\phi_i'||$, then $\rho < \gamma$, and it follows from (a) that
$$\begin{aligned} m(B(x,r)) &\leq m(B(x,\gamma||\phi_i'||)) \leq \alpha m(B(x,\beta\gamma||\phi_i'||)) \leq \alpha m\left(B\left(x, \beta\frac{\gamma}{\rho}\rho||\phi_i'||\right)\right) \\ &\leq \alpha m\left(B\left(x, \beta\frac{\gamma}{\rho}r\right)\right). \end{aligned}$$
In any case
$$m(B(x,r)) \leq \alpha m(B(x, \beta\max\{1, \gamma/\rho\}r)). \tag{3.1}$$

Put $\eta = \min\{||\phi_i'|| : i \in G\}$. Since the measure m is positive on non-empty open subsets of J, we have $P := \inf\{m(B(x, \gamma\beta\eta)) : x \in J\} > 0$. Since the measure m has no atoms, there exists $\zeta \in (0, \kappa)$ so small that

$$m(B(x,\zeta)) \leq \alpha P \tag{3.2}$$

for all $x \in J$. In view of condition (1e) we may assume $\zeta \in (0, \kappa)$ to be so small that

$$\alpha^{\frac{1}{6h}} \leq \frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \leq \alpha^{-\frac{1}{6h}} \tag{3.3}$$

for all $\omega \in I^*$, all $x \in J$, and all $y \in B(x, \zeta)$. Similarly as above, $Q := \inf\{m(B(x,\zeta)) : x \in J\} > 0$ and there exists $\theta \in (0, \zeta/K)$ so small that

$$m(B(x, K\theta)) \leq \alpha Q \tag{3.4}$$

for all $x \in J$. Fix now $x \in J$ and $r \in (0, \theta)$. Write $x = \pi(\omega)$, where $\omega \in I^\infty$, and let $n \geq 0$ be the least integer such that $||\phi_{\omega|_n}'|| \leq \theta^{-1} r$. Then $n \geq 1$ and $||\phi_{\omega|_{n-1}}'|| > \theta^{-1} r$. Consequently

$$Kr||\phi_{\omega|_{n-1}}'||^{-1} \geq r||\phi_{\omega|_n}'||^{-1}||\phi_{\omega_n}'|| \geq \theta||\phi_{\omega_n}'|| \tag{3.5}$$

and

$$Kr||\phi_{\omega|_{n-1}}'||^{-1} < K\theta < \zeta < \kappa < \mathrm{dist}(X, \partial V). \tag{3.6}$$

This inequality along with formula (2.5) imply that

$$B(x,r) \subset \phi_{\omega|_{n-1}}\Big(B\Big(\pi(\sigma^{n-1}(\omega)), Kr||\phi_{\omega|_{n-1}}'||^{-1}\Big)\Big),$$

and if $n - 1 \leq q$, then it follows from condition (b) that

$$m(B(x,r)) \leq \alpha^{-\frac{1}{3}}|\phi_{\omega|_{n-1}}'(\pi(\sigma^{n-1}(\omega)))|^h m\Big(B\Big(\pi(\sigma^{n-1}(\omega)), Kr||\phi_{\omega|_{n-1}}'||^{-1}\Big)\Big). \tag{3.7}$$

If $n - 1 > q$, then write $\omega|_{n-1} = \tau\eta$, where $|\eta| = q$. It then follows from (3.6), (c), (3.3), and (b) that

$$\begin{aligned} m(B(x,r)) &\leq m\Big(\phi_\tau\Big(\phi_\eta\Big(B\Big(\pi(\sigma^{n-1}(\omega)), Kr||\phi_{\omega|_{n-1}}'||^{-1}\Big)\Big)\Big)\Big) \\ &\leq \alpha^{-\frac{1}{6}}|\phi_\tau'\Big(\phi_\eta(\pi(\sigma^{n-1}(\omega)))\Big)|^h m\Big(\phi_\eta\Big(B\Big(\pi(\sigma^{n-1}(\omega)), Kr||\phi_{\omega|_{n-1}}'||^{-1}\Big)\Big)\Big) \\ &\leq \alpha^{-\frac{1}{6}}|\phi_\tau'\Big(\phi_\eta(\pi(\sigma^{n-1}(\omega)))\Big)|^h \alpha^{-\frac{1}{3}}|\phi_\eta'(\pi(\sigma^{n-1}(\omega)))|^h m\Big(B\Big(\pi(\sigma^{n-1}(\omega)), Kr||\phi_{\omega|_{n-1}}'||^{-1}\Big)\Big) \qquad (3.8) \\ &= \alpha^{-\frac{1}{2}}|\phi_{\omega|_{n-1}}'(\pi(\sigma^{n-1}(\omega)))|^h m\Big(B\Big(\pi(\sigma^{n-1}(\omega)), Kr||\phi_{\omega|_{n-1}}'||^{-1}\Big). \end{aligned}$$

It now follows from (3.7), (3.8), (3.6) and (3.4) that

$$\begin{aligned} m(B(x,r)) &\leq \alpha^{-\frac{1}{2}}|\phi_{\omega|_{n-1}}'(\pi(\sigma^{n-1}(\omega)))|^h \alpha m\Big(B\Big(\pi(\sigma^{n-1}(\omega)), \zeta\Big)\Big) \\ &= \alpha^{\frac{1}{2}}|\phi_{\omega|_{n-1}}'(\pi(\sigma^{n-1}(\omega)))|^h \alpha m\Big(B\Big(\pi(\sigma^{n-1}(\omega)), \zeta\Big)\Big). \end{aligned}$$

Consider now the case when $\gamma\theta^{-1}\beta Kr||\phi'_{\omega|_{n-1}}||^{-1} \geq \zeta$. Then, in view of (2.4) and (3.3), we get that

$$m(B(x,\gamma\theta^{-1}\beta Kr)) \geq \alpha^{\frac{1}{6}}|\phi'_{\omega|_{n-1}}(\pi(\sigma^{n-1}(\omega)))|^h m\Big(B\Big(\pi(\sigma^{n-1}(\omega)),\zeta\Big)\Big).$$

Combining these last two inequalities, we obtain

$$m(B(x,r)) \leq \alpha^{\frac{1}{3}} m\Big(B\Big(x,\gamma\theta^{-1}\beta Kr\Big)\Big). \tag{3.9}$$

So, suppose that

$$\gamma\theta^{-1}\beta Kr||\phi'_{\omega|_{n-1}}||^{-1} < \zeta.$$

Then in view of (3.3),

$$m\Big(B\Big(x,\gamma\theta^{-1}\beta Kr\Big)\Big) \geq \alpha^{\frac{1}{6}}|\phi'_{\omega|_{n-1}}(\pi(\sigma^{n-1}(\omega)))|^h m\Big(B\Big(\pi(\sigma^{n-1}(\omega)),\gamma\theta^{-1}\beta Kr||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big). \tag{3.10}$$

Assuming now that $\omega_n \notin G$ and taking into account that $\pi(\sigma^{n-1}(\omega)) \in \phi_{\omega_n}(J)$, it follows from (3.1) applied with $\rho = \theta$, along with (3.5) and (3.6), that

$$m\Big(B\Big(\pi(\sigma^{n-1}(\omega)),Kr||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) \leq \alpha m\Big(B\Big(\pi(\sigma^{n-1}(\omega)),\gamma\theta^{-1}\beta Kr||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big).$$

Combining this along with (3.7), (3.8) and (3.10), we get

$$m(B(x,r)) \leq \alpha^{\frac{1}{3}} m\Big(B\Big(x,\gamma\theta^{-1}\beta Kr\Big)\Big). \tag{3.11}$$

Suppose finally that $\omega_n \in G$. It then follows from (3.5) that $\gamma\theta^{-1}\beta Kr||\phi'_{\omega|_{n-1}}||^{-1} \geq \gamma\beta\eta$, and consequently that

$$m\Big(B\Big(\pi(\sigma^{n-1}(\omega)),\gamma\theta^{-1}\beta Kr||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) \geq P.$$

Therefore, in view of (3.6) and (3.2), we get

$$m\Big(B\Big(\pi(\sigma^{n-1}(\omega)),Kr||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big) \leq \alpha m\Big(B\Big(\pi(\sigma^{n-1}(\omega)),\gamma\theta^{-1}\beta Kr||\phi'_{\omega|_{n-1}}||^{-1}\Big)\Big).$$

Now, exactly the same argument leads to (3.11). This formula and (3.9) complete the proof as $\gamma\theta^{-1}\beta K > 1$. ■

A rather straightforward argument leads to the following slightly stronger form of Theorem 3.6, which is however much more convenient for further applications.

Theorem 3.7. *Suppose that the system $S = \{\phi_i\}_{i\in I}$ is regular, and denote by m the corresponding h-conformal measure. Suppose also that there exists an integer $q \geq 0$ and $\kappa \in (0,\min\{1,\operatorname{dist}(X,\partial V)\})$ such that the following three conditions are satisfied.*

(a) *There exist three real constants: $\gamma \geq 1$, $\beta > 1$, $\alpha \in (0,1)$, and a finite set $F \subset I$ such that for all $i \in I \setminus F$ there exists $y_i \in \phi_i(X)$ such that for all $r \in [\gamma||\phi_i'||,\kappa]$*

$$m(B(y_i,r)) \leq \alpha\mu(B(y_i,\beta r)).$$

(b) *For all* $\omega \in I^*$ *with* $|\omega| \le q$*, all* $x \in J$*, and all* $r \in (0,\kappa)$

$$m(\phi_\omega(B(x,r))) \le \alpha^{-\frac{1}{3}}|\phi_\omega'(x)|^h m(B(x,r)).$$

(c) *For all* $\omega \in I^q$ *and all* $x \in J$*,*

$$\phi_\omega(B(x,\kappa)) \subset X.$$

Then the h*-conformal measure* m *satisfies the (efd) property.*

Proof. In view of Theorem 3.6 it suffices to check condition (a) of this theorem with appropriate $\kappa > 0$ (smaller) and $\beta > 1$ (bigger). All other constants will remain unchanged. Indeed, for every $i \in I$, every $r \ge \gamma||\phi_i'||$ and every $x \in \phi_i(X)$, we have

$$B(x,r) \subset B\Big(y_i, r+\mathrm{diam}(\phi_i(X))\Big) \subset B\Big(y_i, r+D||\phi_i'||\Big) \subset B\left(y_i, r+\frac{D}{\gamma}r\right) = B\left(y_i, \left(1+\frac{D}{\gamma}\right)r\right).$$

Putting $\beta' = \beta\left(1+\frac{D}{\gamma}r\right) + \frac{D}{\gamma} > \beta > 1$, we get

$$\begin{aligned} B(x,\beta' r) &\supset B\Big(y_i, \beta' r - \mathrm{diam}(\phi_i(X))\Big) \supset B\Big(y_i, \beta' r - D||\phi_i'||\Big) \supset B\left(y_i, \beta' r - \frac{D}{\gamma}r\right) \\ &= B\left(y_i, \left(\beta' - \frac{D}{\gamma}\right)r\right) = B\left(y_i, \beta\left(1+\frac{D}{\gamma}\right)r\right). \end{aligned}$$

Assuming now in addition that $i \in I \setminus F$ and that $r \in \left(0, \left(1+\frac{D}{\gamma}\right)\kappa\right)$, we therefore obtain from condition (a) that

$$m(B(x,r)) \le m\left(B\left(y_i, \left(1+\frac{D}{\gamma}\right)r\right)\right) \le \alpha m\left(B\left(y_i, \beta\left(1+\frac{D}{\gamma}\right)r\right)\right) \le \alpha m(B(x,\beta' r)).$$

We are done. ■

Thinking about applications of Theorem 3.7 let us prove the following.

Proposition 3.8. *If* $S = \{\phi_i : X \to X\}_{i\in I}$ *is a regular conformal iterated function system satisfying the super strong open set condition, then the conditions (b) and (c) of Theorem 3.7 are satisfied.*

Proof. Condition (c) is satisfied by taking $q = 0$ and any positive $\kappa < \mathrm{dist}(J, \partial X)$. Condition (b) is then obviously satisfied. ■

4. Real Continued Fractions; General Results

In this section we examine the (efd) property of conformal measures of continued fractions iterated function systems with restricted entries. For every $n \geq 1$ we consider the map $\phi_n : [0,1] \to [0,1]$ given by the formula

$$\phi_n(x) = \frac{1}{x+n}$$

If $I \subset I\!N$ contains at least two points, we call $S_I = \{\phi_n\}_{n \in I}$ a continued fraction iterated function system. It is straightforward to see that the limit set of this system consists of all irrational whose continued fraction expansion has all entries in I. Many geometrical properties of such limit sets have been thoroughly investigated in [6], [11] and [7]. We start our considerations by recalling the following result proven in [7] as Lemma 3.2.

Proposition 4.1. *If $1 \notin I$ and $S = \{\phi_i\}_{i \in I}$ is a regular continued fraction iterated function system, then S satisfies the super strong open set condition, and consequently, due to Proposition 3.8 the conditions (b) and (c) of Theorem 3.7 are satisfied.*

In order to get rid of the rather restrictive, fairly artificial and irritating assumption that $1 \notin I$, we shall prove the following.

Lemma 4.2. *Assume that $S = \{\phi_i\}_{i \in I}$ is a regular continued fraction iterated function system and let m the corresponding h-conformal measure. If*

$$\limsup_{n \to \infty} \frac{m\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right)}{m\left(\left[0, \frac{1}{n+1}\right]\right)} < \infty,$$

then there exist $\alpha \in (0,1)$ and $\kappa > 0$ such that the conditions (b) and (c) of Theorem 3.7 are satisfied with $q = 2$.

Proof. Fix $\kappa \in (0, 1/3)$. For every $x \in [1/2, 1]$ and every $n \geq 1$, we have

$$\phi_n(B(x,\kappa)) = \left(\frac{1}{n+x+\kappa}, \frac{1}{n+x-\kappa}\right) \subset (0,1).$$

If $x \in [0, 1/2]$, then

$$\phi_n(B(x,\kappa)) = \left(\frac{1}{n+x+\kappa}, \frac{1}{n+x-\kappa}\right) \subset (0,\infty)$$

and therefore, for every $k \geq 1$

$$\phi_{kn}(B(x,\kappa)) \subset \phi_k((0,\infty)) = (0, 1/k) \subset (0,1).$$

(kn is here the concatenation of the letters k and n, and not their product). Hence the condition (c) of Theorem 3.7 is satisfied with $q = 2$. In order to verify condition (b) put

$$B(z,r,R) = \{x \in I\!R : z - r \leq x \leq z + R\}.$$

Fix now $x \in (1/2, 1] \cap J$ and fix $r \in (0, \kappa]$. Without loss of generality we may assume that $x + r > 1$. Fix $R \in [r, 4r]$. Then for every $n \in I$

$$\phi_n(B(x, r, R)) = \phi_n((x - r, 1]) \cup \phi_n([1, x + R)). \tag{4.1}$$

Now

$$\begin{aligned} m\Big(\phi_n((x-r,1])\Big) &\le \left(\frac{n+x}{n+x-r}\right)^{2h}\left(\frac{1}{n+x}\right)^{2h} m((x-r,1]) \\ &= \left(\frac{1}{1-\frac{r}{n+x}}\right)^{2h} |\phi_n'(x)|^h m((x-r,1]) \le 2|\phi_n'(x)|^h m((x-r,1]), \\ &= 2|\phi_n'(x)|^h m(B(x,r,R)) \end{aligned} \tag{4.2}$$

where the last inequality sign was written assuming that $\kappa > 0$ (and consequently $r < \kappa$) is small enough. Also

$$\phi_n([1, x+R)) = \left(\frac{1}{n+x+R}, \frac{1}{n+1}\right] = \phi_{n+1}\Big([0, x+R-1)\Big).$$

Hence (note that if $n + 1 \notin I$, then $m\Big(\phi_n([1, x + R))\Big) = m\Big(\phi_{n+1}\Big([0, x + R - 1)\Big)\Big) = 0$)

$$\begin{aligned} m\Big(\phi_n([1,x+R))\Big) &\le \left(\frac{1}{n+1}\right)^{2h} m\Big([0,x+R-1)\Big) = \left(\frac{n+x}{n+1}\right)^{2h}\left(\frac{1}{n+x}\right)^{2h} m\Big([0,x+R-1)\Big) \\ &\le (n+x)^{-2h} m\Big([0,x+R-1)\Big) = |\phi_n'(x)|^h m\Big([0,x+R-1)\Big). \end{aligned} \tag{4.3}$$

Now, $(x-r, 1] = \phi_1\Big([0, (x-r)^{-1}-1)\Big)$, and consequently ($x \in (1/2, 1] \cap J$ implies that $1 \in I$)

$$m((x-r,1]) \ge \left(\frac{1}{1+0}\right)^{2h} m\Big([0,(x-r)^{-1}-1)\Big) = m\Big([0,(x-r)^{-1}-1)\Big). \tag{4.4}$$

Since it is straightforward to verify that $x + R - 1 \le (x - r)^{-1} - 1$ ($r < \kappa < 1/3$), combining (4.3) and (4.4), we get

$$m(\phi_n([1,x+R))) \le |\phi_n'(x)|^h m\Big([0,(x-r)^{-1}-1)\Big) \le |\phi_n'(x)|^h m((x-r,1]) = |\phi_n'(x)|^h m(B(x,r,R)).$$

Combining in turn this along with (4.2) and (4.1), we obtain

$$m(\phi_n(B(x,r,R))) \le 3|\phi_n'(x)|^h m(B(x,r,R)). \tag{4.5}$$

Before moving to the next case, observe first that, due to our assumptions,

$$Q := \max\left\{1, \sup_{j\ge 1}\left\{\frac{m([0,j^{-1}))}{m([0,(j+1)^{-1}))}\right\}\right\} < \infty.$$

Suppose in turn that $x \in [0, 1/2]$. We may assume without loss of generality that $x - r < 0$. Then, for every $n \in I$

$$\phi_n\Big(B(x,r,R)\Big) = \phi_n\Big((x-r,0]\Big) \cup \phi_n\Big((0,x+R)\Big). \tag{4.6}$$

Now,

$$\begin{aligned} m\Big(\phi_n\Big((0,x+R)\Big)\Big) &\le \left(\frac{1}{n}\right)^{2h} m\Big((0,x+R)\Big) = \left(\frac{n+x}{n}\right)^{2h}\left(\frac{1}{n+x}\right)^{2h} m\Big((0,x+R)\Big) \\ &= \left(1+\frac{x}{n}\right)^{2h} |\phi_n'(x)|^h m\Big((0,x+R)\Big) \le 2^h|\phi_n'(x)|^h m\Big((0,x+R)\Big) \\ &\le 2|\phi_n'(x)|^h m\Big(B(x,r,R)\Big). \end{aligned} \tag{4.7}$$

Suppose first that $n=1$. Then $\phi_1((x-r,0]) = [1,(1+x-r)^{-1}) \subset [1,\infty)$. Thus, $m(\phi_1((x-r,0])) = 0$, and combining this along with (4.6) and (4.7), we obtain

$$m(\phi_1(B(x,r,R))) \le 2|\phi_1'(x)|^h m(B(x,r,R)). \tag{4.8}$$

So, assume that $n \ge 2$. Then

$$\phi_n((x-r,0]) = \left[\frac{1}{n},\frac{1}{n+x-r}\right) = \phi_{n-1}\Big((x-r+1,1]\Big),$$

and therefore (note that if $n-1 \notin I$, then $m(\phi_n((x-r,0]) = m\Big(\phi_{n-1}((x-r+1,1])\Big) = 0)$

$$\begin{aligned} m\Big(\phi_n((x-r,0])\Big) &\le \left(\frac{1}{n-1+(x-r+1)}\right)^{2h} m\Big((x-r+1,1]\Big) \\ &= \left(\frac{n+x}{n+x-r}\right)^{2h}\left(\frac{1}{n+x}\right)^{2h} m\Big((x-r+1,1]\Big) \\ &= \left(1+\frac{r}{n+x-r}\right)|\phi_n'(x)|^h m\Big((x-r+1,1]\Big) \\ &\le 2|\phi_n'(x)|^h m\Big((x-r+1,1]\Big). \end{aligned} \tag{4.9}$$

Since $(x-r+1,1] = \phi_1\Big([0,(x-r+1)^{-1}-1)\Big)$, we get (as above if $1 \notin I$, then $m((x-r+1,1]) = 0$)

$$m\Big((x-r+1,1]\Big) \le \left(\frac{1}{1+0}\right)^{2h} m\Big([0,(x-r+1)^{-1}-1)\Big) = m\Big([0,(x-r+1)^{-1}-1)\Big). \tag{4.10}$$

Now, taking $\kappa > 0$ small enough (and consequently r and $r-x$ small enough), we see that $(x-r+1)^{-1}-1 = (1-(r-x))^{-1}-1 \le r-x+2(r-x)^2 \le r-x+2r^2$. Picking an integer $k \ge 1$ such that

$$\frac{1}{k+1} < x+r \le \frac{1}{k}, \tag{4.11}$$

we therefore get

$$\frac{1}{x-r+1}-1-\frac{1}{k} \le \frac{1}{x-r+1}-1-(x+r) \le r-x+2r^2-x-r = 2r^2-2x \le 2r^2 \le \frac{2}{k^2}.$$

If $k \geq 3$, this implies that $(x-r+1)^{-1}-1 < (k-2)^{-1}$. Therefore, using (4.11), we get

$$
\begin{aligned}
\frac{m\Big([0,(x-r+1)^{-1}-1)\Big)}{m([0,x+r))} &\leq \frac{m\Big([0,(k-2)^{-1})\Big)}{m\Big([0,(k+1)^{-1})\Big)} \\
&= \frac{m\Big([0,(k-2)^{-1})\Big)}{m\Big([0,(k-1)^{-1})\Big)} \frac{m\Big([0,(k-1)^{-1})\Big)}{m\Big([0,k^{-1})\Big)} \frac{m\Big([0,k^{-1})\Big)}{m\Big([0,(k+1)^{-1})\Big)} \qquad (4.12)\\
&\leq Q^3.
\end{aligned}
$$

If $k \leq 2$, then similarly

$$
\frac{m\Big([0,(x-r+1)^{-1}-1)\Big)}{m([0,x+r))} \leq \frac{m([0,1))}{m([0,1/3))} \leq Q^2.
$$

Combining this along with (4.12), (4.10) and (4.9), we obtain

$$
\begin{aligned}
m\Big(\phi_n\Big((x-r,0]\Big)\Big) &\leq 2Q^3|\phi_n'(x)|^h m([0,x+r)) \leq 2Q^3|\phi_n'(x)|^h m([0,x+R)) \\
&= 2Q^3|\phi_n'(x)|^h m\Big(B(x,r,R)\Big).
\end{aligned}
$$

Looking at (4.7) and (4.6), we therefore get

$$
m\Big(\phi_n(B(x,r,R)\Big) \leq 2(1+Q^3)|\phi_n'(x)|^h m\Big(B(x,r,R)\Big). \tag{4.13}
$$

Combining this estimate with (4.8) and (4.5), we see (note that $Q \geq 1$) that (4.13) is true for all $n \geq 1$, all $\kappa \in (0,1/3)$ small enough, all $r \in (0,\kappa)$, all $R \in [r,4r]$ and all $x \in J$. In particular

$$
m\Big(\phi_n(B(x,r)\Big) \leq 2(1+Q^3)|\phi_n'(x)|^h m(B(x,r). \tag{4.14}
$$

Keep now $x \in J$ and $r \in (0,\kappa)$. Take any two numbers $k,n \in I$. Since $\phi_n(B(x,r)) = B(\phi_n(x),r_n,R_n)$, where $r_n \geq r(n+1)^{-2}$ and $R_n \leq rn^{-2}$ (so $R_n/r_n \leq \left(\frac{n+1}{n}\right)^2 \leq 4$), the formula (4.13) applies, and we get

$$
m\Big(\phi_{kn}(B(x,r))\Big) = m\Big(\phi_k(B(\phi_n(x),r_n,R_n))\Big) \leq 2(1+Q^3)|\phi_k'(\phi_n(x))|^h m(B(\phi_n(x),r_n,R_n)).
$$

Applying (4.13) once more, we obtain that

$$
m\Big(B(\phi_n(x),r_n,R_n)\Big) = m\Big(\phi_n(B(x,r))\Big) \leq 2(1+Q^3)|\phi_n'(x)|^h m(B(x,r)).
$$

Combining the last two estimates together, we finally get

$$
m\Big(\phi_{kn}(B(x,r))\Big) \leq 4(1+Q^3)^2|\phi_{kn}'(x)m(B(x,r)).
$$

This and (4.14) complete the proof. ■

Let us now recall from [7] that given $\delta \in [0,1]$ and $g > 1$, we call an infinite subset I of $I\!N$, (δ, g)-evenly distributed provided that there exist constants $A \geq 1$ and $p \geq 1$ such that for all real $n \geq 1$,

$$\#(I \cap [n, gn]) \leq An^{\delta}, \tag{4.15}$$

and, if in addition $n \in I \cap [p, \infty)$, then

$$\min\{\#(I \cap [n, gn], \#(I \cap [g^{-1}n, n])\} \geq A^{-1}n^{\delta} \geq 2. \tag{4.16}$$

If we do not want (do not need) to specify δ and g, we simply say that I is evenly distributed. A simple straightforward but useful property of (δ, g)-evenly distributed sets is contained in the following.

Lemma 4.3. *If $I \subset I\!N$ is a (δ, g)-evenly distributed set, then for all $n \in I\!N$ large enough $I \cap [n, gn] \neq \emptyset$ and $I \cap [g^{-1}n, n] \neq \emptyset$, and consequently*

$$\min\{\#(I \cap [n, g^2n], \#(I \cap [g^{-2}n, n])\} \geq A^{-1}g^{-2\delta}n^{\delta}.$$

We say that an infinite subset I of $I\!N$ is rapidly growing if for every $\xi > 0$ there exist $u \geq p$ and $\sigma > 1$ such that for every $n \in I \cap [u, \infty)$ and every $k \in [1, n/7]$

$$\#\Big(I \cap \big([n - \sigma k, n - k] \cup [n + k, n + \sigma k]\big)\Big) \geq \xi \#\Big(I \cap [n - k, n + k]\Big). \tag{4.17}$$

We have proved in [7] the following result as Proposition 3.3.

Proposition 4.4. *If $I \subset I\!N$ is a (δ, g)-evenly distributed set, then the system S_I is cofinitely (hereditarily) regular and $\theta_S = \delta/2$. Consequently $h > \delta/2$.*

The main result of this section is the following.

Theorem 4.5. *If $I \subset I\!N$ is a rapidly growing evenly distributed set, then the corresponding conformal measure m satisfies the (efd) property, and is consequently extremal due to Theorem 3.2.*

Proof. Our general strategy is to verify the assumptions of of Theorem 3.7. Observe that if $n \notin I$, then

$$m\Big([1/(n+1), 1/n]\Big) = 0, \tag{4.18}$$

and if $n \in I$, then

$$m\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right) \leq n^{-2h}. \tag{4.19}$$

If in addition $n \ge p$, then by (4.16) there is $k \in I \cap [n+1, gn)$. We then have

$$m\left(\left(0, \frac{1}{n+1}\right]\right) \ge m\left(\left[\frac{1}{k+1}, \frac{1}{k}\right]\right) \ge (k+1)^{-2h} \ge (2gn)^{-2h} = (2g)^{-2h} n^{-2h}.$$

Combining this with (4.18) and (4.19), we see that for all $n \in I \cap [p, \infty)$

$$\frac{m\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right)}{m\left(\left(0, \frac{1}{n+1}\right]\right)} \le (2g)^{2h}.$$

Thus, the assumptions of Lemma 4.2 are satisfied, and applying it, we see that due to Theorem 3.7, it suffices to verify the following condition: For every $\alpha \in (0,1)$ there exist $\beta > 1$ and a finite set $F \subset I$ such that for all $n \in I \setminus F$ and all $r \in (1/n^2, 1/3)$, we have

$$m(B(1/n, r)) \le \alpha m(B(1/n, \beta r)). \tag{4.20}$$

For each $r > 0$ put

$$I_-(r) = I \cap \left\{k \le n : \frac{1}{k} - \frac{1}{n} < r\right\} = I \cap \left[\frac{n}{1+rn}, n\right].$$

For every $r \in (0, 1/n)$ put

$$I_+(r) = I \cap \left\{k \ge n : \frac{1}{n} - \frac{1}{k} < r\right\} = I \cap \left[n, \frac{n}{1-rn}\right]$$

and for $r \ge 1/n$ put

$$I_+(r) = I \cap [n, +\infty).$$

Set also

$$I(r) = I_+(r) \cup I_-(r).$$

Since the symmetric difference of $J \cap B(x,r)$ and $\bigcup_{k \in I(r)} \phi_k(J)$ is contained in $\phi_{\inf I_-(r)-1}(J) \cup \phi_{\sup I_+(r)}(J)$, we easily deduce from Lemma 4.3 that

$$m\Big(B(1/n, r)\Big) \asymp \sum_{k \in I(r)} m(\phi_k(J)) \asymp \sum_{k \in I(r)} k^{-2h} \asymp \sum_{k \in I_+(r)} k^{-2h} + \sum_{k \in I_-(r)} k^{-2h}. \tag{4.21}$$

Take now an arbitrary $\xi > 0$ and fix $u \ge p \ge 2$ and $\sigma \ge 3$ so that the condition (4.17) is satisfied. The assumptions of our theorem imply that I is (δ, g)-evenly distributed with some $\delta \in (0,1]$ and some $g > 1$. Fix $n \in I \cap [u, \infty)$ and consider

Case 1^0: $r \in \left[\frac{1}{4(\sigma+1)n}, \frac{1}{3}\right]$.

Using Lemma 4.3, we get for all $n \in I \cap [u, \infty)$ sufficiently large that

$$
\begin{aligned}
\sum_{k \in I_-(r)} k^{-2h} &= \sum_{k=\frac{n}{1+rn}}^{n} k^{-2h} \asymp \sum_{j=0}^{\log_{g^2}(1+rn)} (ng^{-2j})^{-2h+\delta} = n^{\delta-2h} \sum_{j=0}^{\log_{g^2}(1+rn)} g^{2(2h-\delta)j} \\
&\asymp n^{\delta-2h} \frac{g^{2(2h-\delta)\log_{g^2}(1+rn)} - 1}{g^{2(2h-\delta)} - 1} \asymp n^{\delta-2h} g^{2(2h-\delta)\log_{g^2}(1+rn)} \\
&= n^{\delta-2h}(1+rn)^{2h-\delta} \in [n^{\delta-2h}(rn)^{2h-\delta}, n^{\delta-2h}\Big((4\sigma+5)rn\Big)^{2h-\delta}] \\
&= \left[r^{2h-\delta}, \left(\frac{4\sigma+5}{4(\sigma+1)}\right)^{2h-\delta} r^{2h-\delta}\right] \subset [r^{2h-\delta}, 4r^{2h-\delta}].
\end{aligned}
\tag{4.22}
$$

Therefore, using (4.21), we get that

$$
m\Big(B(1/n, \sigma^3 r)\Big) \geq \sum_{k \in I_-(\sigma^3 r)} k^{-2h} \succeq \sigma^{3(2h-\delta)} r^{2h-\delta}. \tag{4.23}
$$

Also

$$
\begin{aligned}
\sum_{k=n}^{\infty} k^{-2h} &\asymp \sum_{j=0}^{\infty} (ng^{2j})^{\delta-2h} = n^{\delta-2h} \sum_{j=0}^{\infty} g^{(\delta-2h)j} \asymp n^{\delta-2h} \frac{1}{1-g^{\delta-2h}} \asymp \left(\frac{1}{n}\right)^{2h-\delta} \\
&\leq \Big(4(\sigma+1)\Big)^{2h-\delta} r^{2h-\delta}.
\end{aligned}
$$

Combining this, (4.22) and (4.21), we obtain

$$
m\Big(B(1/n, r)\Big) \preceq 4^{2h-\delta} r^{2h-\delta} + \Big(4(\sigma+1)\Big)^{2h-\delta} r^{2h-\delta} \leq \Big(4(\sigma+2)\Big)^{2h-\delta} r^{2h-\delta}.
$$

Combining this with (4.23), we get

$$
m\Big(B(1/n, \sigma^3 r)\Big) \succeq \sigma^{2h-\delta} m\Big(B(1/n, r)\Big) \tag{4.24}
$$

for all $\sigma \geq 3$ large enough. Consider now

Case 2^0: $r \in \left[\frac{1}{n^2}, \frac{1}{4(\sigma+1)n}\right]$.
Then

$$
1 \geq \frac{n}{1+rn} \cdot \frac{1}{n} = \frac{1}{1+rn} \geq \frac{1}{1+\frac{1}{4(\sigma+1)}} \geq \frac{1}{2}.
$$

Therefore,

$$
\sum_{j \in I_-(r)} j^{-2h} = \sum_{j=\frac{n}{1+rn}}^{n} j^{-2h} \asymp n^{-2h} \# \left(I \cap \left[\frac{n}{1+rn}, n\right]\right) \tag{4.25}
$$

and

$$
\sum_{j \in I_-(2\sigma r) \setminus I_-(r)} j^{-2h} = \sum_{j=\frac{n}{1+2\sigma rn}}^{\frac{n}{1+rn}} j^{-2h} \succeq n^{-2h} \# \left(I \cap \left[\frac{n}{1+2\sigma rn}, \frac{n}{1+rn}\right]\right). \tag{4.26}
$$

Since $rn \le \frac{1}{4(\sigma+1)}$, we get

$$1 \le \frac{n}{1-rn} \cdot \frac{1}{n} \le \frac{1}{1-\frac{1}{4(\sigma+1)}} \le 2$$

and therefore

$$\sum_{j\in I_+(r)} j^{-2h} = \sum_{j=n}^{\frac{n}{1-rn}} \asymp n^{-2h}\#\left(I \cap \left[n, \frac{n}{1-rn}\right]\right). \tag{4.27}$$

Similarly

$$1 \le \frac{n}{1-2\sigma rn} \cdot \frac{1}{n} \le \frac{1}{1-\frac{2\sigma}{4(\sigma+1)}} = \frac{1}{1-\frac{\sigma}{2(\sigma+1)}} = \frac{2(\sigma+1)}{\sigma+2} \le 2,$$

and therefore

$$\sum_{j\in I_+(2\sigma r)\setminus I_+(r)} j^{-2h} = \sum_{j=\frac{n}{1-rn}}^{\frac{n}{1-2\sigma rn}} j^{-2h} \asymp n^{-2h}\#\left(I \cap \left[\frac{n}{1-rn}, \frac{n}{1-2\sigma rn}\right]\right).$$

Combining this and (4.26), we get

$$\sum_{j\in I(2\sigma r)\setminus I(r)} j^{-2h} \succeq n^{-2h}\#\left(I \cap \left(\left[\frac{n}{1+2\sigma rn}, \frac{n}{1+rn}\right] \cup \left[\frac{n}{1-rn}, \frac{n}{1-2\sigma rn}\right]\right)\right), \tag{4.28}$$

and combining (4.25) with (4.27), we obtain

$$\sum_{j\in I(r)} j^{-2h} \asymp n^{-2h}\#\left(I \cap \left[\frac{n}{1+rn}, \frac{n}{1-rn}\right]\right). \tag{4.29}$$

Put now

$$k = \frac{n}{1-rn} - n = \frac{rn^2}{1-rn}.$$

Since $r \ge 1/n^2$, we get

$$k \ge \frac{n}{1-\frac{1}{n}} - n = \frac{n}{n-1} > 1. \tag{4.30}$$

Since $rn \le \frac{1}{4(\sigma+1)}$, we obtain

$$k \le \frac{n}{1-\frac{1}{4(\sigma+1)}} - n = n\frac{1}{4\sigma+3} \le \frac{n}{7}. \tag{4.31}$$

We want now to check that

$$n + \sigma k \le \frac{n}{1-2\sigma rn}. \tag{4.32}$$

This equivalently means that $-2\sigma rn^2 + \sigma k(1-2\sigma rn) \le 0$ or

$$\frac{rn^2}{1-rn}(1-2\sigma rn) \le 2rn^2 \Leftrightarrow \frac{1-2\sigma rn}{1-rn} \le 2,$$

and since this last inequality is obviously true, so is (4.32). Hence

$$I \cap [n, n+\sigma k] \subset I \cap \left[n, \frac{n}{1-2\sigma rn}\right]. \tag{4.33}$$

We want now to check that

$$n - \sigma k \geq \frac{n}{1+2\sigma rn} \tag{4.34}$$

This equivalently means that $2\sigma rn^2 - \sigma k(1+2\sigma rn) \geq 0$ or (recall that $k = rn^2/(1-rn)$)

$$\frac{rn^2}{1-rn}(1+2\sigma rn) \leq 2rn^2 \Leftrightarrow \frac{1+2\sigma rn}{1-rn} \leq 2 \Leftrightarrow 1+2\sigma rn \leq 2-2rn \Leftrightarrow 2(\sigma+1)rn \leq 1$$

and this last inequality is true because of our definition of the Case 2^0. Hence (4.34) is satisfied, and we get

$$I \cap [n-\sigma k, n] \subset I \cap \left[\frac{n}{1+2\sigma rn}, n\right].$$

Along with (4.33) this gives that

$$I \cap [n-\sigma k, n+\sigma k] \subset I \cap \left[\frac{n}{1+2\sigma rn}, \frac{n}{1-2\sigma rn}\right]. \tag{4.35}$$

We also need to check that

$$n - k \leq \frac{n}{1+rn}. \tag{4.36}$$

But this equivalently means that $rn^2 - k(1+rn) \leq 0$ or

$$k \geq \frac{rn^2}{1+rn} \Leftrightarrow \frac{rn^2}{1-rn} \geq \frac{rn^2}{1+rn} \Leftrightarrow 1+rn \geq 1-rn.$$

This last inequality is obviously true and (4.36) is verified. Hence

$$I \cap [n-k, n+k] \supset I \cap \left[\frac{n}{1+rn}, \frac{n}{1-rn}\right].$$

As an immediate consequence of this and (4.35), we obtain

$$I \cap \left(\left[\frac{n}{1+2\sigma rn}, \frac{n}{1+rn}\right] \cup \left[\frac{n}{1-rn}, \frac{n}{1-2\sigma rn}\right]\right) \supset I \cap \Big([n-\sigma k, n-k] \cup [n+k, n+\sigma k]\Big). \tag{4.37}$$

Looking now at (4.28), (4.37), (4.17) (which applies due to (4.30)), (4.31)), (4.36), the definition of k, and (4.29), we get

$$\begin{aligned}
\sum_{j\in I(2\sigma r)} j^{-2h} &= \sum_{j\in I(r)} j^{-2h} + \sum_{j\in I(2\sigma r)\setminus I(r)} j^{-2h} \ge \sum_{j\in I(2\sigma r)\setminus I(r)} j^{-2h} \\
&\ge n^{-2h}\#\left(I\cap\left(\left[\frac{n}{1+2\sigma rn},\frac{n}{1+rn}\right]\cup\left[\frac{n}{1-rn}\frac{n}{1-2\sigma rn}\right]\right)\right) \\
&\ge n^{-2h}\#\Big(I\cap\big([n-\sigma k,n-k]\cup[n+k,n+\sigma k]\big)\Big) \ge \xi n^{-2h}\#\Big(I\cap\big([n-k,n+k]\big)\Big) \\
&\ge \xi n^{-2h}\#\left(I\cap\left[\frac{n}{1+rn},\frac{n}{1-rn}\right]\right) \asymp \xi\sum_{j\in I(r)} j^{-2h} \\
&\asymp \xi m(B(1/n,r)).
\end{aligned}$$

Thus for $\sigma>1$ large enough, we have

$$m\Big(B(1/n,\sigma^3 r)\Big) \ge m\Big(B(1/n,2\sigma r)\Big) \ge \xi m(B(1/n,r)).$$

Along with (4.24) this implies that for all ξ (and consequently σ) large enough, all $n\in I$ large enough, and for every $r\in[||\phi_n'||,1/3]$, we have

$$m\Big(B(1/n,\sigma^3 r)\Big) \ge \min\{\xi,\sigma^{2h-\delta}\}m(B(1/n,r)).$$

Letting $\xi\nearrow+\infty$ (and consequently $\sigma\nearrow+\infty$), we see that (4.20) is verified. We are done. ■

5. Real Continued Fractions; Examples

Represent any set $I\subset I\!\!N$ as a non-decreasing sequence $\{a_n\}_{n=1}^\infty$. Recall that I is said to have bounded gaps if $\sup_{n\ge1}\{a_{n+1}-a_n\}<\infty$. Our first result in this section is the following.

Theorem 5.1. *If $I\subset I\!\!N$ has bounded gaps, then the continued fraction iterated function system S is regular and the corresponding conformal measure (which due to Corollary 5.9 from [6] is up to a multiplicative constant equal to the h-dimensional packing measure on J_S) satisfies the (efd) condition, and is consequently extremal due to Theorem 3.2.*

Proof. It is obvious that I is $(1,g)$-evenly distributed for all $g>1$ large enough. Since

$$\lim_{\sigma\to\infty}\left(\inf_{n\ge1}\inf_{k\ge1}\left\{\frac{\#\Big(I\cap[n+k,n+\sigma k]\Big)}{2k}\right\}\right)=\infty,$$

we see that the set I is rapidly growing. Thus, invoking Theorem 4.5 completes the proof. ■

Corollary 5.2. *If $I\subset I\!\!N$ is an arithmetic progression, then the corresponding conformal measure satisfies the (efd) condition and is extremal.*

As was noticed in Theorem 5.1, the h-conformal measure of any system with bounded gaps is up to a multiplicative constant equal to the h-dimensional packing measure on J_S. It follows from Corollary 4.5 in [6] that if in addition I is a proper infinite subset of $I\!N$, then the h-dimensional Hausdorff measure on J_S vanishes.

Lemma 5.3. *Suppose that $\{x_n\}_{n=1}^{\infty}$, an unbounded increasing sequence of positive integers and $\{d_n\}_{n=1}^{\infty}$, a non-decreasing sequence of positive integers satisfy the following conditions.*

(a)

$$x_n + d_n < \frac{6}{7}(x_{n+1} - d_{n+1}).$$

(b) *For every $j \geq 1$*

$$\liminf_{n\to\infty} \frac{d_n}{x_{n+j} - x_n + d_{n+j}} > 0$$

Then the set

$$I = \bigcup_{n\geq 1} [x_n - d_n, x_n + d_n]$$

is rapidly growing.

Proof. Fix an integer $\xi \geq 1$. In view of (b),

$$\theta = \theta_\xi := \inf_{n\geq 1} \left\{ \frac{d_n}{x_{n+\xi} - x_n + d_{n+\xi}} \right\} > 0. \tag{5.1}$$

Put

$$\sigma = (2\xi + 1)(1 + \theta^{-1}).$$

Take $n \geq 2$, $l \in [x_n - d_n, x_n + d_n]$ and $k \in [1, l/7]$. Consider separately two cases:

Case 1^0. $k \leq (2\xi + 1)^{-1} d_n$
and
Case 2^0. $k > (2\xi + 1)^{-1} d_n$.

Let us deal first with the Case 1^0. Suppose that $l - \sigma k \geq x_n - d_n$. Then $[l - \sigma k, l - k] \cap I = [l - \sigma k, l - k]$, and therefore

$$\#\Big(I \cap [l - \sigma k, l - k]\Big) = (\sigma - 1)k \geq \xi 2k \geq \xi\#\Big(I \cap [l - k, l + k]\Big).$$

So suppose that $l - \sigma k < x_n - d_n$ and $l \geq x_n$. Then $I \cap [l - \sigma k, l - k] \supset [x_n - d_n, l - k]$, and therefore

$$\#\Big(I\cap[l-\sigma k, l-k]\Big) \geq l-k-x_n+d_n \geq d_n-k \geq (2\xi+1)k-k = \xi 2k \geq \xi\#\Big(I\cap[l-k, l+k]\Big).$$

If $l \leq x_n$ and $l + \sigma k \leq x_n + d_n$, then $I \cap [l + k, l + \sigma k] = [l + k, l + \sigma k]$ and therefore $\#\Big(I\cap[l+k, l+\sigma k]\Big) = (\sigma-1)k \geq 2\xi k \geq \xi\#(I\cap[l-k, l+k]\Big)$. If $l \leq x_n$ and $l+\sigma k > x_n+d_n$, then $I \cap [l+k, l+\sigma k] \supset [l+k, x_n+d_n]$. Hence, $\#\big(I \cap [l+k, l+\sigma k]\big) \ge x_n + d_n - l - k \ge d_n - k \ge (2\xi+1)k - k = 2\xi k \ge \xi \#(I \cap [l-k, l+k]\big)$.

So, consider the Case 2^0. In view of (a) we have for every $n \ge 1$ that

$$l + k \le \frac{8}{7} l \le \frac{8}{7}(x_n + d_n) < \frac{7}{6}(x_n + d_n) < x_{n+1} - d_{n+1} \tag{5.2}$$

and

$$l - k \ge \frac{6}{7} l \ge \frac{6}{7}(x_n - d_n) > x_{n-1} + d_{n-1}. \tag{5.3}$$

Hence $I \cap [l-k, l+k] \subset [x_n - d_n, x_n + d_n]$, and consequently

$$\#(I \cap [l-k, l+k]\big) \le 2d_n. \tag{5.4}$$

Suppose now that $l + \sigma k \ge x_{n+\xi} + d_{n+\xi}$. It then follows from (5.2) that

$$I \cap [l+k, l+\sigma k] \supset I \cap [x_{n+1} - d_{n+1}, x_{n+\xi} + d_{n+\xi}] \supset \bigcup_{j=n+1}^{n+\xi} [x_j - d_j, x_j + d_j].$$

Hence

$$\#\big(I \cap [l+k, l+\sigma k]\big) \ge 2\sum_{j=1}^{\xi} d_{n+j} \ge 2\xi d_n.$$

Combining this along with (5.4), we see that

$$\#\big(I \cap [l+k, l+\sigma k]\big) \ge \xi \#(I \cap [l-k, l+k]\big)$$

for all $n \ge 1$. So, suppose in turn that

$$l + \sigma k < x_{n+\xi} + d_{n+\xi}. \tag{5.5}$$

Then $\sigma k < x_{n+\xi} + d_{n+\xi} - l \le x_{n+\xi} + d_{n+\xi} - x_n + d_n$, and since we are in the Case 2^0, $\sigma(2\xi+1)^{-1} d_n \le x_{n+\xi} - x_n + d_{n+\xi} + d_n$. Invoking the definition of σ, we therefore get $(1+\theta^{-1})d_n < x_{n+\xi} - x_n + d_{n+\xi} + d_n$. Equivalently, $d_n < \theta(x_{n+\xi} - x_n + d_{n+\xi})$, which contradicts (5.1) and finishes the proof. ■

Applying this lemma we can easily construct the sets I with unbounded gaps whose corresponding conformal measure satisfies the (efd) property. Indeed, we have the following.

Theorem 5.4. *There exists an infinite evenly distributed rapidly growing subset I of $I\!N$ with unbounded gaps. Consequently the corresponding conformal measure on J_I satisfies the (efd) property and is consequently extremal due to Theorem 3.2.*

Proof. Let $A \ge 4$ be an integral multiple of 4. For every $n \ge 4$ put

$$x_n = A^n \text{ and } d_n = \frac{1}{4} A^n.$$

Set

$$I = \bigcup_{n\geq 1} [x_n - d_n, x_n + d_n]$$

Obviously I is evenly distributed. In order to verify that I is rapidly growing, we shall check the assumptions of Lemma 5.3. Since $x_n + d_n = \frac{5}{4}A^n$ and since

$$\frac{6}{7}(x_{n+1} - d_{n+1}) = \frac{6}{7}\cdot\frac{3}{4}A^{n+1} = \frac{9}{14}AA^n \geq \frac{36}{14}A^n \geq 2A^n,$$

we see that condition (a) is satisfied. Since for all $n \geq 1$ and all $j \geq 1$,

$$x_{n+j} - x_n + d_{n+j} = A^{n+j} - A^n + \frac{1}{4}A^{n+j} = A^n\left(\frac{5}{4}A^j - 1\right),$$

we get

$$\frac{d_n}{x_{n+j} - x_n + d_{n+j}} = \frac{\frac{1}{4}A^n}{A^n\left(\frac{5}{4}A^j - 1\right)} = \frac{1}{5A^j - 4} > 0.$$

Thus condition (b) also holds, and all the assumptions of Lemma 5.3 have been verified. Hence, I is rapidly growing. It is therefore left to check that I has unbounded gaps. To see it notice that

$$x_{n+1} - d_{n+1} - (x_n + d_n) = A^{n+1} - \frac{1}{4}A^{n+1} - A^n - \frac{1}{4}A^n = A^n\left(\frac{3}{4}A - \frac{5}{4}\right)$$

converges to $+\infty$ when $n \to +\infty$. Applying now Theorem 4.5 finishes the proof. ■

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