THE SCENERY FLOW FOR HYPERBOLIC JULIA SETS

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Abstract. We define the scenery flow space at a point z in the Julia set J of a hyperbolic rational map $T : \mathbb{C} \to \mathbb{C}$ with degree ≥ 2, and more generally for T a conformal mixing repellor.

We prove that, for hyperbolic rational maps, except for a few exceptional cases listed below, the scenery flow is ergodic. We also prove ergodicity for almost all conformal mixing repellors; here the statement is that the scenery flow is ergodic for the repellors which are not linear nor contained in a finite union of real-analytic curves, and furthermore that for the collection of such maps based on a fixed open set U, the ergodic cases form a dense open subset of that collection. Scenery flow ergodicity implies that one generates the same scenery flow by zooming down toward a.e. z with respect to the Hausdorff measure H^d, where $d =$ dimension (J), and that the flow has a unique measure of maximal entropy.

For all conformal mixing repellors, the flow is loosely Bernoulli and has topological entropy $\leq d$. Moreover the flow at a.e. point is the same up to a rotation, so as a corollary, one has an analogue of the Lebesgue density theorem for the fractal set, giving a different proof of a theorem of Falconer.

§1. Introduction.

Fractal sets often come equipped with a discrete dynamics, like the map $T(z) = z^2 + c$ on its Julia set J. Since this map is conformal (infinitesimally orientation- and angle-preserving) whenever the derivative DT is a non-zero complex number, one can use the nonlinear scaling given by the map itself to study the geometry of J. Thus for instance assuming for a rational map T the additional hypotheses that degree $T \geq 2$ and that T

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is **hyperbolic**, i.e. that changing in a conformal fashion the Riemannian metric around $J(T)$ there exists $\alpha > 1$ such that for every $z \in J$

$$|DT^n(z)| > \alpha$$

or more generally for a *conformal mixing repellor* (see §4.3, also [28] or [25], [26] and [24] for the definition and further properties), one can prove that the set is quasi self-similar, that the Hausdorff dimension is strictly between 0 and 2, that the Hausdorff measure at this dimension d is positive and finite, and moreover that $\mu = H^d\mid_J$ is **geometric** in the sense that $\exists c_1, c_2 > 0$ with

$$c_1\varepsilon^d < \mu(B_\varepsilon(z)) < c_2\varepsilon^d \tag{1.1}$$

for all sufficiently small ε (work of Bowen, Ruelle, Sullivan: [7], [28], [33] (see also [24]); for general background on Julia sets see also [9], and [19]).

In this paper we will study a linear, continuous-time dynamics which is constructed directly from the geometry of the set J. We imagine zooming toward some chosen point $z \in J$. Now for a fractal object like J, one will see a continuously changing scenery. This suggests the question which motivated this paper. Can one, at least for certain well-behaved fractal sets, model that process with the continuous dynamics of a measure-preserving ergodic flow?

To approach this question, we begin with some definitions. We will change the scale at a constant exponential rate and call $\{e^s(J-z) : s \in \mathbb{R}\}$ the **scenery** at the point z. This collection of sets is an orbit of a flow (i.e. additive $\mathbb{R}$-action) on the Borel subsets of the tangent space $\mathbb{C}$. The **scenery flow space** at z will be simply the collection of limit points as $s \to +\infty$ (i.e. the omega-limit set) of $(J-z)$, the Julia set translated so as to be centered at z, in an appropriate topology to be defined in a moment.

We want to think of the scenery flow at z as some sort of derivative or tangent object to the set; this interpretation will be made precise below. Note that for a differentiable manifold embedded in $\mathbb{R}^n$, this does agree with the usual notion, since the scenery flow will then consist of a single point (the tangent space at z).

By contrast, for a fractal set the scenery keeps changing. As we will see, for some points $z \in J$ the scenery is periodic or almost periodic; however for almost every z (with respect to the Hausdorff measure $\mu =$ restriction of H^d to J), one gets a random set-valued process. In fact this flow is loosely Bernoulli, has entropy $\leq d$ and is, up to a rotation, the same flow for μ – a.e. z. Furthermore, for almost all the repellors, in a strong sense explained below, the scenery flow is rotation- invariant, and is therefore exactly the same, for a.e. z.

Our first task is to construct the scenery flow. We topologise the collection of Borel sets of $\mathbb{C}$ by two natural pseudo-metrics, which will become metrics when restricted to the subclasses of sets of interest here. Limits will be proved to exist in both metrics. The topology for the **measure metric** $\rho(E,F)$ is defined by associating to $E \subseteq \mathbb{C}$ the restriction μ_E of H^d to E, and then testing against continuous real-valued functions with compact support, i.e. for the corresponding measures this is the weak-$*$ topology in $\mathcal{C}_c^*(\mathbb{C})$. To define the **local Hausdorff metric** $\hat{\rho}(E,F)$ on the closed subsets of $\mathbb{C}$ we fix a conformal map (the inverse of a stereographic projection) from $\mathbb{C}$ onto the Riemann sphere $S^2\backslash\{\infty\}$, add the point ∞ to the images of both sets, and then use the Hausdorff metric coming from the Euclidean metric on S^2 (see §1).

Now the idea for the construction of the scenery flow is (in retrospect!) extremely easy. We form the shift space $\prod_{-\infty}^{\infty} J$, with the product topology and left shift σ, and restrict to the subset $\hat{J} = \{\underline{z} = (\dots z_{-1}z_0z_1\dots) : T(z_j) = z_{j+1}\}$. Note that choice of a string $\underline{z}$ corresponds to choice of an initial point z_0 (which of course determines uniquely the "future" $z_1, z_2, \dots$) together with an infinite branch of preimages $z_{-1}, z_{-2}, \dots$. Now for each choice of z_0 and branch of pasts, we will define a Borel set $L_{\underline{z}} \subseteq \mathbb{C}$. This will simply be the limit (in either metric) of the Julia set centered at each z_{-n} and then expanded and rotated by that derivative:

$$L_{\underline{z}} \equiv \lim_{n\to\infty} DT^n(z_{-n}) \cdot (J - z_{-n}).$$

Convergence will be proved from a strong form of the Bounded Distortion Property (Theorem 2.8). We call $L_{\underline{z}}$ a **scene** or **limit set**. It is a countable "holomorphic" cover of the Julia set, and as such is analogous to the imaginary axis wrapping infinitely many times around the circle via the exponential map. Indeed, for the map $T(z) = z^2$ the Julia set J is the circle, the scenery of $J - 1$ at the point 0 is its tangent line, the imaginary axis, and the above limiting procedure yields the covering map $z \mapsto e^z - 1$ (see Theorem 2.10).

The limit set will be forward asymptotic to the scenery at z_0, in the sense that, for any choice of pasts, dist $(e^s L_{\underline{z}}, e^s(J - z_0))$ will converge to 0 (in either metric) as $s \to \infty$. But what we have done by constructing $L_{\underline{z}}$ is to define a point in the scenery flow itself. To see this we have to understand the relationships between the limit sets as the initial point changes by an application of the map T. To explain this, note that the limit sets (as one sees immediately from the construction) $L_{\underline{z}}$ and $L_{\sigma\underline{z}}$ are related by

$$L_{\sigma\underline{z}} = DT(z_0)L_{\underline{z}}.$$

Now, defining $r : J \to (0, \infty)$ and $\eta : J \to S^1 = \{w \in \mathbb{C} : |w| = 1\}$ by $r(z) = \log|DT(z)|$ and $\eta(z) = DT(z)/|DT(z)|$, and taking $z = z_0$, we can write the right-hand side as:

$$DT(z_0)L_{\underline{z}} = \eta(z)e^r(z)L_{\underline{z}}.$$

That is, up to a rotation, the orbit of the scenery flow at $\underline{z}$ returns after time $r(z_0)$ to the shifted coordinates $\sigma\underline{z}$. This return map is modeled by a skew product over the shift $(\hat{J}, \sigma)$ with circle fiber S^1 and skewing function $\varphi(\underline{z}) = \eta(z_0)$. We write this transformation as $(\hat{J}_1, \hat{\sigma})$, where $\hat{J}_1 = \hat{J} \times S^1$ and $\hat{\sigma}(\underline{z}, w) = (\sigma(\underline{z}), \varphi(\underline{z})w)$. (The space $\hat{J}_1$ can be thought of as the unit tangent bundle of the natural extension of (J, T).) We build the special flow with this base and with return time $r((\underline{z}, w)) \equiv r(z_0)$. This is the **model scenery flow** for the full scenery flow of J; the scenery flow at a point z_0, which was defined above, will then be shown to be a closed invariant set which is exactly the image of the orbit closure of the point $(\underline{z}, 0)$ under the continuous map defined by $(\underline{z}, 0) \mapsto L_{\underline{z}}$.

What now works out beautifully is that the version of the flow with base $(\hat{J}, \sigma)$ and return height r (i.e. where we forget about angles), has as a natural invariant measure ν (indeed as its unique measure of maximal entropy) a measure which on the cross-section is equivalent to the Hausdorff measure μ on J. This is, in fact, exactly the Sinai-Ruelle-Bowen Gibbs state for the function $-dr$. The uniqueness of this measure passes to the skew product, by some analysis based on a method of Furstenberg. This is described in the next paragraph. The fact that μ-a.e. scenery flow is (up to rotation) the same now follows from ergodicity of $(\hat{J}_1, \hat{\sigma})$ with respect to this measure. For the same reason (now with no need to worry about rotations) one has, by the Birkhoff ergodic theorem, an analogue of the Lebesgue Density Theorem for the set J (see [3], [4], [5], [12] and [13]): there is a constant $c > 0$ (the **average density**, also previously called the *order-two density*) such that for μ – a.e. $z \in J$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mu B(z, e^{-t})}{e^{-td}} \mathrm{dt}$$

exists and equals c (see Theorem 4.2). This is an average density of the Hausdorff measure, sandwiched between the upper and lower bounds c_1 and c_2 mentioned before for the geometric measure μ. Finally, Bowen's well-known formula for dimension (see [7], comp. [28]) now implies the upper bound of $\dim(J)$ for the topological entropy of the scenery flow.

When we analyze measures of maximal entropy for the scenery flow, it is sufficient to consider ergodic measures on $\hat{J}_1$ which project to the Gibbs state ν on $\hat{J}$, since the circle fibers add no entropy. Now there is a dichotomy: either $\nu \times m$ is ergodic (where m = Lebesgue measure on S^1), or the ergodic measure sits on k copies of the graph of a measurable function from $\hat{J}$ to S^1, for some k in $\mathbb{Z}$, and is unique up to rotation. Equivalently, $k\varphi$ is measurably cohomologous to zero. This is proved using a Fourier series method of Furstenberg [15], see Theorem 4.4.

In this second case, using the fact that the map T, being a conformal mixing repellor, is hyperbolic, we can, by two theorems of Liv sic (see [17] and [25]) reduce the analysis to a study of the periodic points. The first theorem (when reproved for the present case of circle fibers) allows one to show the measurable cohomology is actually continuous. In other words, the measure sits on the graphs of k continuous functions. By the second theorem, this is equivalent to having $\arg DT^n(z_0)$ be some multiple of $2\pi/k$, for each z_0 of period n.

Now this last condition allows us to analyze how the rotational symmetries of the scenery flow change as T varies in an appropriate space of conformal mixing repellors. By some basic complex analysis (the implicit function theorem and open mapping theorem for several complex variables) and Theorem 3.1 from [20] the case with discrete symmetry can at most happen for the repellors which are linear or are contained in a finite union of real-analytic curves. Moreover, these exceptional cases are negligible in the sense of Baire category, as their complement forms a dense open subset with respect to appropriate topology for the repellors; see Theorem 4.8.

For the case of hyperbolic rational functions, in view of results from [14], we have more specific information. Some examples of rational maps with discrete symmetries which easily come to mind are: $z \mapsto z^2$ and more general Blaschke products (for which the line field tangent to the circle is invariant); the Lattés map $z \mapsto (z^2+1)^2/4z(z^2-1)$, which is covered by the conformal Anosov endomorphism of the torus $z \mapsto (1+i)z$, hence its Julia set is the sphere and the image of any constant line field is invariant, the map $f(z) = z^2 - 2$, whose Julia set is the interval $[-2, 2]$, and $f(z) = z^2 - 3$ whose Julia set is a Cantor subset of $I\!R$; in these last two cases any constant line field is invariant. As we show in Theorem 1 from [14] (comp. [21]), this is essentially all that can happen in the general case: the discrete symmetry forces the map into a limited number of exceptional classes corresponding to these examples (plus one more for which we know of no concrete example). In our hyperbolic case the exceptional classes are even fewer in number (see Theorem 4.9).

So in conclusion, for both hyperbolic rational maps and conformal mixing repellors, for all T not in the corresponding exceptional set, we know that for $\nu-$ (hence $\mu-$) a.e. z_0 and w_0, their scenery flows are identical. For all T, including the exceptional cases, we know the following: the scenery flows are the same up to a fixed rotation. Moreover $\mu-$ almost surely the scenery flow at z_0 has a unique measure of maximal entropy, bounded above by $\dim(J)$. And finally, applying theorems of Rudolph [27], this flow is loosely Bernoulli (has a measure-theoretically Bernoulli cross-section).

We conjecture that in the case when the Hausdorff dimension of J is not equal to 1, one always has equality of dimension and entropy, and moreover, that the continuous semiconjugacy between the model scenery flow to the full scenery flow is at most finite-to one.

One can also construct a scenery flow at a point x in e.g. a Brownian zero set [3], the middle-third set and more generally a hyperbolic $\mathcal{C}^{1+\gamma}$ Cantor set [4, 3], and in a Fuchsian or Kleinian limit set [13]. Scenery flows of certain families of circle diffeomorphisms are studied in [2]. (To make the transition from [3], [12] to the present perspective, note that the scaling flow on local times with local uniform topology corresponds exactly to the scenery flow on sets with the measure topology).

For the present example of a hyperbolic Julia set, the dynamics of the map $T : J \to J$ is used in studying the scenery flow, as we have described. The same is the case for hyperbolic Cantor sets. We mention that for the Fuchsian case, one can take a similar approach, using the discrete dynamics of the group action on the limit set. However in this setting there is also a natural continuous-time dynamics (the geodesic flow on the unit tangent bundle) and it is much simpler to use this directly to study the scenery flow. By contrast, for the Brownian example, the scenery flow makes sense even though there is no natural dynamics on the zero set itself.

Meanwhile, Tan Lei [35], also see [19, Appendix A] has also studied rigorously the scaling structure near a point in certain fractal sets. Her theorem states, in the language of the present paper, that for a Misiurewicz point c, the scenery flows at c in J_c (for the map $z^2 + c$) and at c in ∂M (the boundary of the Mandelbrot set) are identical, and are topologically conjugate to either a single periodic orbit (if the multiplier is a root of unity) or an irrational flow on a torus (if the multiplier is $e^{i\theta}$ for θ irrational mod 2π, as then we have the suspension flow of an irrational circle rotation). Misiurewicz points form a countable dense subset of ∂M. Thus a general point $z \in \partial M$ is approximated by points whose scenery flow is known to exist and to be periodic or almost periodic, with $T-$periods going to infinity. In light of this observation, combined with the fact that the measure theory of the Julia sets corresponding to these points z is still far from worked out, it is an intriguing problem to try to understand the scenery flow for these points.

For a beautiful application of our limiting construction of the scenery flow see [18].

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§2. Convergence to limit sets.

Note. We assume until §4.3 that $T : \mathbb{C} \to \mathbb{C}$ is a hyperbolic rational map; aside from Theorem 2.10, all constructions, statements and proofs are valid without essential change for conformal mixing repellors, except for those statements which refer to parameter space, which is discussed in that section. From now on $d = \mathrm{HD}(J)$ is the Hausdorff dimension of the Julia set $J = J(T)$ and $\mu = H^d \mid_J$ is the d-dimensional Hausdorff measure restricted to J. The measure μ is d-**conformal measure**, i.e.

$$\mu(T(E)) = \int_E |DT(z)|^d d\mu(z)$$

for every Borel set $E \subseteq \mathbb{C}$ such that $T : E \to \mathbb{C}$ is $1-1$, In the next section we will describe the ergodic theory of μ (or rather of the Gibbs state ν, its T-invariant version) but at present all we need are the facts (originally proved, by Bowen and Ruelle, from that ergodic theory) that $0 < d < 2$ and that μ is a geometric measure.

Topologies. We first define a topology $\mathcal{T}$ on the collection Ω of closed subsets of $\mathbb{C}$, the **conformal map topology**. Convergence here will imply convergence in the two weaker topologies mentioned in the Introduction, which will be given by metrics when restricted to relevant subcollections of sets. This topology is also a *uniformity* (see the statements preceding Lemma 2.2), so we can e.g. speak of the uniform convergence of functions from another uniform space to this one.

Let $B_R(z_0)$ denote the open disk about z_0 of radius R. A neighborhood base for our topology will be indexed by R, $\varepsilon > 0$, with smaller neighborhoods given by large R and small $R\varepsilon$. For $E \subseteq \mathbb{C}$, we say $F \subseteq \mathbb{C}$ is (R, ε)-close to E if $\exists\, f : B_R(0) \to \mathbb{C}$, $(1-1$ and) conformal, such that

$$\|f(z) - z\|_\infty \le R\varepsilon$$

(i.e. f is uniformly close to the identity map) and

$$f(B_R(0) \cap E) = (f(B_R(0))) \cap F.$$

Sometimes we will instead need $\mathcal{C}^1$-closeness to the identity. The next lemma shows this follows from uniform closeness of either f or its derivative. We note that for $f(z)$ defined on B_R, to be (R, ε)-close to the identity is scaling invariant, in the sense that if we conjugated f by $g(z) = kz$, then the resulting function $g^{-1} \circ f \circ g$ would be $(R/k, \varepsilon)$-close to the identity.

Lemma 2.1. *Let $f : B_R(0) \to \mathbb{C}$ be holomorphic. Then:*

(i) *If $|f(z) - z| < R\varepsilon$ for all $z \in B_R$, then*

$$|Df(z) - 1| < 4\varepsilon \quad \textit{for all} \quad z \in B_{R/2}.$$

(ii) *If $f(0) = 0$ and $|Df(z) - 1| < \varepsilon \qquad \forall z \in B_R(0)$ then we have $|f(z) - z| < R\varepsilon$.*

Proof.
(i) We write $g(z) = f(z) - z$. By the Cauchy Integral Formula,

$$Dg(z) = \frac{1}{2\pi i}\int_{\partial B_R} \frac{g(z)}{(w-z)^2}dw.$$

For $w \in \partial B_R$ and $|z| < R/2$, $\quad |w-z| > R/2$. Therefore $|Dg(z)| \le \frac{1}{2\pi}\frac{R\varepsilon 4}{R^2}2\pi R = 4\varepsilon$.
(ii) Since the domain $B_R(0)$ is simply connected, the path integral is well-defined and we have:

$$|f(z) - z| = |\int_0^z Df(\zeta) - 1 d\zeta| \le \int_0^z |Df - 1| d\zeta \le \varepsilon|z| \le R\varepsilon.$$

□

Although (R, ε)-closeness does not quite define a metric, one does have the following:
(a) (approximate symmetry): If F is (R, ε)-close to E, then E is $((1-\varepsilon)R, \varepsilon)$-close to F.
(b) (approximate triangle inequality): If F is (R, ε_1)-close to E, and G is (R, ε_2)-close to F, then G is $((1-\varepsilon_1)R, \varepsilon_1 + \varepsilon_2)$-close to E.
It is also easy to check (recalling that these sets are assumed to be closed):
(c) If E is (R, ε)-close to F for all $R, \varepsilon > 0$, then $E = F$.
(d) $\mathcal{T}$ is a uniformity, using (R, ε) closeness.

Furthermore we have the following natural analogue of completeness, i.e. that "Cauchy sequences" converge:

Lemma 2.2. *Let E_i be a sequence of closed subsets of $\mathbb{C}$ satisfying: for each $R, \varepsilon > 0$, $\exists n$ such that for all $m \ge n$, E_m is (R, ε)-close to E_n. Then there exists a unique closed set $E \subseteq \mathbb{C}$ such that $E_n \to E$ in $\mathcal{T}$. Moreover for all $m \ge n$, E is (R, ε)-close to E_m.*

Proof. Given $R, \varepsilon > 0$, by hypothesis there exists n such that for all $m \ge n$, there is a $1-1$ holomorphic function $f_m : B_R \equiv B_R(0) \to \mathbb{C}$ with $\|f_m(z) - z\|_\infty \le \varepsilon R$ and $f_m(B_R \cap E_n) = (f(B_R)) \cap E_m$. Now since $\{f_m\}_{m=0}^\infty$ is uniformly bounded and hence is a normal family, there exists a subsequence f_{m_k} and a holomorphic function $f : B_R \to \mathbb{C}$ such that for any given $\bar\varepsilon > 0$, $\|f_{m_k} - f\|_\infty \le \bar\varepsilon R$ for all k large. By part (i) of Lemma 2.1, $|Df| > 0$ hence f is $1-1$. Therefore so is $f \circ f_{m_k}^{-1}$, which is defined on $B_{(1-\varepsilon)R}(0)$. Writing $w = f_{m_k}(z)$, we have for any $w \in B_{(1-\varepsilon)R}$,

$$|f \circ f_{m_k}^{-1}(w) - w| = |f(z) - f_{m_k}(z)| < \bar\varepsilon R.$$

Therefore defining $E(R) = f(B_R \cap E_n)$, we have shown that given $\bar\varepsilon > 0, \exists k$ such that $E(R)$ is $((1-\varepsilon)R, \bar\varepsilon)$-close to E_{m_k}. Let M_k also be large enough that, by the Cauchy property, for all $j > m_k$ we have that E_j is $(R, \bar\varepsilon)$-close to E_{m_k}. Then by (a) and (b) above, $E(R)$ is $((1-\varepsilon)(1-\bar\varepsilon)R, 2\bar\varepsilon)$-close to E_j. Therefore by (a), (b) and (c), the (closed) set $E(R)$ is uniquely defined in $B_{(1-\varepsilon)R}$ independent of the initial choice of n or the subsequence m_k. For the same reason, for $R' > R$ the sets $E(R')$ and $E(R)$ agree in $B_{(1-\varepsilon)R}$. Now since $\|f_{m_k}(z) - z\|_\infty \le \varepsilon R$ and $\|f(z) - f_{m_k}(z)\|_\infty < \bar\varepsilon R$ for $z \in B_R$ and all $\bar\varepsilon > 0$, we have that

$\|f(z)-z\|_\infty \le \varepsilon R$ and hence $E(R)$ is (R,ε)-close to E_n. Therefore defining $E = \bigcup_{R>0} E(R)$, also E is (R,ε)-close to E_n. We conclude that $E_n \to E$ in $\mathcal{T}$; note that E is a closed set. By (c), the limit is unique. Finally, repeating the entire argument for each $m \ge n$, we have that E is (R,ε) close to E_m, as claimed. □

Next, the **measure topology** is defined on the collection of Borel sets $E \subseteq \mathbb{C}$ such that H^d is locally finite, i.e. for every $R > 0$, $H^d(E \cap B_R(0)) < \infty$. In that case $\mu_E \equiv H^d|_E$ defines a continuous linear functional on $C_c(\mathbb{C})$, the continuous real-valued functions with compact support; indeed, by the Riesz representation theorem, $\mathcal{C}_c^*$ is exactly the locally finite signed measures. The measure topology on sets will then be that given by the weak-$*$ topology for $\mathcal{C}_c^*$, on the corresponding measures.

To get a pseudo-metric, however, we need to further restrict the class of sets. First, for fixed $b > 0$, let $\mathcal{B}_b$ denote the collection of all Borel subsets E of $\mathbb{C}$ such that for all $R > 0$,

$$\mu_E(B_R(0)) < bR^d.$$

Proposition 2.3. *There is a bounded pseudo-metric ρ on $\mathcal{B}_b$ which is equivalent to the measure topology.*

Proof. For $f \in C_c$, we first define the pseudometric ρ_f by

$$\rho_f(E,F) = \left| \int f d\mu_E - \int f d\mu_F \right|.$$

Since C_c has a countable dense subset $\{f_i\}_{i=1}^\infty$ (in the sup norm), we can (because of the bound on measure) find weights $a_i > 0$ such that for $F \ne \emptyset$ and for any $E \in \mathcal{B}_b$ we have

$$\rho(E,F) \equiv \sum_{i=1}^\infty a_i \rho_{f_i}(E,F) < K$$

for some $K < \infty$. Now we define $\rho(E,F)$ by that sum for all $E, F \in \mathcal{B}_b$. This is clearly symmetric and satisfies the triangle inequality. By the triangle inequality we have for any $E, F \in \mathcal{B}_b$

$$\rho(E,F) < 2K < \infty,$$

so ρ is a bounded pseudometric, as claimed. □

Next, let $J = J(T)$ be the Julia set of the map T. Define $\mathcal{D} = \{a(J-z) : z \in J, a > 0\}$ and let $\overline{\mathcal{D}}$ denote the closure in the measure topology. We will now see that $\overline{\mathcal{D}} \subseteq \mathcal{B}_b$:

Lemma 2.4. *There exists $b > 0$ such that for every $E \in \overline{\mathcal{D}}$,*

$$\mu_E(B_R(0)) < bR^d.$$

Proof. A basic fact is that J is compact. Therefore there exists $b > 0$ such that for every $z \in J$, and all $\varepsilon > 0$ (not just ε sufficiently small), for $\mu \equiv \mu_J$ we have, since μ is a geometric measure,

$$\mu(B_\varepsilon(z)) < b\varepsilon^d.$$

Now if $E \in \mathcal{D}$ and so by definition can be written $E = a(J - z)$, then

$$\mu(E \cap B_R(0)) = \mu(a((J - z) \cap B_{R/a}(0))) \leq a^d(b(R/a)^d) = bR^d.$$

This inequality passes immediately over to E in the closure $\overline{\mathcal{D}}$. □

To prove the next lemma, we need the $\mathcal{C}^1$-closeness to the identity proved in (i) of Lemma 2.1. For $g : B_R(z_0) \to \mathbb{C}$, we write

$$\|g\|_{\mathcal{C}^1} \equiv \max\{\|g\|_\infty, \|Dg\|_\infty\}.$$

Lemma 2.5. *Let $E_i \in \overline{\mathcal{D}}$ be closed sets such that $E_i \to E$ in the conformal map topology. Then $E \in \overline{\mathcal{D}}$ and $E_i \to E$ in the measure metric.*

Proof. We first claim that for each $f \in C_c$, the sequence converges in the pseudometric ρ_f. We want to show that given $\varepsilon > 0$ then, for all n large enough,

$$\left| \int_{E_n} f dH^d - \int_E f dH^d \right| < \varepsilon.$$

Now $\exists R > 0$ such that $\mathrm{supp}(f) \subseteq B_R(0)$, and in fact such that $\mathrm{supp}(f) \subseteq (1-\varepsilon)B_R(0) \equiv B_{(1-\varepsilon)R}(0)$. By the hypothesis, for any $\delta > 0$ we have for all n sufficiently large that there exists $\varphi_n : B_R(0) \to \mathbb{C}$ with $\|\varphi_n(z) - z\|_{\mathcal{C}^1} < \delta$ and such that $\varphi_n(E_n \cap B_R(0)) = \varphi_n(B_R(0)) \cap E$. In particular, $\|\varphi_n(z) - z\|_\infty < \delta$ and so $\varphi_n(B_R(0)) \supseteq (1-\varepsilon)B_R(0)$. Therefore, using the integral form of the conformal transformation property for H^d,

$$\int_E f dH^d = \int_{\varphi_n(E_n)} f dH^d = \int_{E_n} f \circ \varphi_n |D\varphi_n|^d dH^d.$$

With δ chosen small enough that $|w - z| < \delta \Rightarrow |f(w) - f(z)| < \bar{\varepsilon}$, for the difference we have a bound of $\bar{\varepsilon}(1+\delta)^d H^d(E_n \cap B_R(0))$. So with $\bar{\varepsilon}$ chosen appropriately, this is less than ε, as we claimed. The same estimate shows that $E \in \overline{\mathcal{D}}$, which finishes the proof. □

Next, we define the **local Hausdorff metric** $\bar{\rho}$ on the collection of all closed subsets of $\mathbb{C}$ as follows. We fix an invertible holomorphic map P from $\mathbb{C}$ onto $S^2 \backslash \{\infty\}$, the Riemann sphere minus a point. (In other words, P^{-1} is a stereographic projection.) Then we define $\bar{\rho}(E, F)$ to be the distance between $P(E) \cup \{\infty\}$ and $P(F) \cup \{\infty\}$, in the Hausdorff metric on closed subsets of S^2 determined by its usual sphere metric.

Lemma 2.6. *Let $E_i \subseteq \mathbb{C}$ be closed sets such that $E_i \to E$ in the conformal map topology. Then E is closed, and $E_i \to E$ in the local Hausdorff metric.*

Proof. This is obvious from the definitions, since to get E_i to be inside an ε-neighborhood of $P(E)$, if $\overline{P(E)}$ contains ∞ we can add to all sets an ε-ball $B_\varepsilon(\infty)$ around ∞ in S^2, and in $\mathbb{C}$ let R be such that the map $\varphi : B_R(0) \to \mathbb{C}$ satisfies that φ is ε-close to the identity and $P((1-\varepsilon)B_R(0)) \supseteq S^2 \setminus B_\varepsilon(\infty)$. Note that all we need is $\|\varphi - z\|_\infty \leq \varepsilon$; control of $D\varphi$ is not necessary here. □

Bounded Distortion Property.

In this subsection we recall our main technical tool, the full version of Koebe's distortion theorem (see [16]), and we derive one of its geometrical consequences in the context of hyperbolic rational functions.

Theorem 2.7. *(Koebe's distortion theorem). For every $\varepsilon > 0$ there exists $0 < \eta \leq 1$ such that if $f : B_r(w) \to \mathbb{C}$ is a univalent conformal map defined on a an arbitrary ball $B_r(w)$, then*

$$\left| \frac{Df(z)}{Df(\xi)} - 1 \right| < \varepsilon.$$

for all $z, \xi \in B_{\eta r}(w)$.

Let us come back to $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. If $T^n(z_{-n}) = z_0$ (i.e. a choice of the n^{th} inverse image is understood), then we write $\widetilde{DT}^n$ for the affine map on $\mathbb{C}$ defined by:

$$z \mapsto \big((DT^n(z_{-n}))(z - z_{-n})\big) + z_0.$$

We want to emphasize that $\widetilde{DT}^n$ depends on both z_0 and z_{-n}.

Theorem 2.8. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $z_0 \in J(T)$, $n \geq 0$, and $T^n(z_{-n}) = z_0$, then the formula*

$$\widetilde{DT}^n \circ T^{-n} \quad \textit{defines a function on} \quad B_\delta(z_0)$$

which is (δ, ε)-close to the identity.

Proof. Let $r > 0$ be so small that for all $n \geq 1$ all holomorphic inverse branches of T^n are well-defined on all balls with radii r and centers in the Julia set $J(T)$. We write T^{-n} for such a branch, with the choice of $z_{-n} \in T^{-n}(\{z_0\})$ understood.

Put $f(z) \equiv \widetilde{DT}^n \circ T^{-n} : B_\delta(z_0) \to \mathbb{C}$. Then

$$|Df(z) - 1| = \left| \frac{DT^{-n}(z)}{DT^{-n}(z_0)} - 1 \right| < \varepsilon$$

for every $z \in B_{\eta r}$, where η ascribed to ε comes from Theorem 2.7. Hence by Lemma 2.1, $\|f(z) - z\|_\infty \leq \varepsilon\eta r$ and we are done with $\delta = \eta r$. $\square$

Limit Sets. Now for our hyperbolic Julia set (J, T), let the shift space $(\hat{J}, \sigma)$ be defined as in the Introduction. We note that $\hat{J}$ is compact, and that the left shift σ is a homeomorphism of $\hat{J}$; this is topologically the natural invertible version of (J, T).

Given $\underline{z} = (\dots z_{-1} z_0 z_1 \dots) \in \hat{J}$, we define

$$L_{\underline{z},n} \equiv DT^n(z_{-n}) \cdot (J - z_{-n}).$$

We defined above the affine map $\widetilde{DT}^n : \mathbb{C} \to \mathbb{C}$. The definition depended implicitly on a choice of $\underline{z} \in \hat{J}$; in the next proof we will need to indicate this choice, and so will write below $\widetilde{DT}^n_{\underline{z}}$ for the map

$$w \mapsto DT^n(z_{-n})(w - z_{-n}) + z_0.$$

Here is the theorem we have been leading up to.

Theorem 2.9. *For each* $\underline{z} \in \hat{J}$*, there exists a unique closed set* $L_{\underline{z}} \subseteq \mathbb{C}$ *such that* $L_{\underline{z},n} \to L_{\underline{z}}$ *in the conformal map topology. The convergence is uniform in* $\underline{z}$*, and the function* $\underline{z} \mapsto L_{\underline{z}}$ *is continuous. In addition* $DT(z_0)L_{\underline{z}} = L_{\sigma\underline{z}}$.

Proof. By Lemma 2.2 it will be enough, for proving convergence, to show the sequence is Cauchy in the sense given there. We are to show that for this $\underline{z}$, given $R, \varepsilon > 0$, there exists n such that for all $m \geq n$, $L_{\underline{z},m}$ is (R,ε)-close to $L_{\underline{z},n}$. Let δ be good in Theorem 2.8 for error $\tilde{\varepsilon}$ where $\tilde{\varepsilon} = \varepsilon/||DT||_\infty$, i.e. so as to have $(\delta, \tilde{\varepsilon})$ closeness to the identity (on the $\delta-$ball centered at *any* z_0). Let n be the least integer such that $\overline{R} \equiv DT^n(z_{-n}) \cdot \delta > R$. This implies that

$$R < \overline{R} < ||DT||_\infty R.$$

Now define the affine map g from $B_\delta(z_{-n})$ to $B_{\overline{R}}(0)$:

$$g(z) = DT^n(z_{-n}) \cdot (z - z_{-n}).$$

Writing $k = m - n$, the map $\varphi \equiv \widetilde{DT}^k_{\sigma^{-n}(\underline{z})} \circ T^{-k} : B_\delta(z_n) \to \mathbb{C}$ satisfies

$$\varphi(w) = \big((DT^k(z_{-m})) \cdot (T^{-k}(w) - z_{-m})\big) + z_m$$

so from Theorem 2.8, $\varphi : B_\delta(z_{-n}) \to \mathbb{C}$ is $(\delta, \tilde{\varepsilon})$-close to the identity. Therefore by the scaling property of this distance, the conjugate

$$\Phi \equiv g \circ \varphi \circ g^{-1} : B_{\overline{R}}(0) \to \mathbb{C}$$

is $(\overline{R}, \tilde{\varepsilon})$-close to the identity. The restriction of Φ to $B_R \subseteq B_{\overline{R}}$ thus has sup norm distance to the identity bounded above by $\overline{R}\varepsilon/||DT||_\infty < R\varepsilon$. Hence we have (R,ε)-closeness, as we wanted. Now from the definitions,

$$\Phi(L_{\underline{z},n} \cap B_R(0)) = L_{\underline{z},m} \cap \Phi(B_R(0)).$$

So $L_{\underline{z},m}$ is (R,ε)-close to $L_{\underline{z},n}$, as claimed. Hence by Lemma 2.2 there exists a unique closed set $L_{\underline{z}}$ such that $L_{\underline{z},n} \to L_{\underline{z}}$ in $\mathcal{T}$.

To prove this convergence is uniform (in the sense of (R,ε)-closeness), we do the above argument for each $\underline{z} \in \hat{J}$. That is, we choose δ from Theorem 2.11 for error equal to $\varepsilon/||DT||_\infty$, and define $n = n(\underline{z})$ to be the least integer such that $|DT^n(z_{-n})|\delta > R$. Now since $\alpha < |DT|$, this is bounded above by some n_0. By the last sentence in the statement of Lemma 2.2, not only is $L_{\underline{z}}(R,\varepsilon)$-close to $L_{\underline{z},n(\underline{z})}$, but we have that for all $\underline{z}$, $L_{\underline{z}}$ is (R,ε)-close to $L_{\underline{z},n_0}$. This proves uniform convergence.

Finally, to prove $L_{\underline{z}}$ is a continuous function of $\underline{z}$ (from the product topology on $\hat{J} \subseteq \prod_{-\infty}^{\infty} \mathbb{C}$ to $\mathcal{T}$), note that for each fixed n, the function $\underline{z} \mapsto L_{\underline{z},n}$ is continuous in $\underline{z}$. Since a uniform limit of continuous functions is continuous, the function $\underline{z} \mapsto L_{\underline{z}}$ is continuous. The formula $DT(z_0)L_{\underline{z}} = L_{\sigma\underline{z}}$ is now an immediate consequence of the definition of the sets $L_{\underline{z}}$. We are done. $\square$

We have constructed the scenery $L_{\underline{z}}$ as a limit of the sets $L_{\underline{z},n}$, showing these are a Cauchy sequence by considering maps defined on successively larger balls $B(R,0)$. These maps are individually holomorphic but were defined only locally, because of the necessity to choose branches, so they do not in themselves form a Cauchy sequence of maps on $\mathbb{C}$. This method has the advantage of working equally well for the conformal repellors.

Once we have constructed $L_{\underline{z}}$ in this way, we can however (for the rational map case) reverse the point of view, studying a sequence of maps which is inverse to those above but now globally defined.

For fixed $\underline{z}$, we define for $n \geq 0$ a map $\Phi_{\underline{z},n} : \mathbb{C} \to \mathbb{C}$ by:

$$\Phi_{\underline{z},n}(w) = T^n\big(z_{-n} + (DT^n(z_{-n}))^{-1} \cdot w\big) - z_0.$$

Theorem 2.10. *For each n, $\Phi_{\underline{z},n}(L_{\underline{z},n}) = (J - z_0)$. This is a holomorphic cover of degree (degree $T)^n$. The limit $\Phi_{\underline{z}} = \lim_{n\to\infty} \Phi_{\underline{z},n}$ converges uniformly on compact sets and $\Phi_{\underline{z}} : \mathbb{C} \to \overline{\mathbb{C}}$ is a meromorphic function, which restricted to $L_{\underline{z}}$ is a countable cover of $(J - z_0)$. The map $\Phi_{\underline{z}}$ has a countable infinity of critical points, except for the cases $T(z) = z^q$, $q \in \mathbb{Z}$, where it has none. The following two diagrams (written as one) commute:*

$$\begin{array}{ccc}
(\mathbb{C}, L_{\underline{z}}) & \xrightarrow{w\mapsto DT(z_0)\cdot w} & (\mathbb{C}, L_{\sigma(\underline{z})}) \\
{\scriptstyle \Phi_{\underline{z}}+z_0}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{\sigma\underline{z}}+z_1} \\
(\mathbb{C}, J) & \xrightarrow{w\mapsto T(w)} & (\mathbb{C}, J)
\end{array}$$

Proof. Note that from the definition, $\Phi_{\underline{z},n}$ has degree $(\deg(T))^n$. A point $w \in L_{\underline{z},n}$ can be expressed as $w = DT^n(z_{-n})(x - z_{-n})$ for some point $x \in J$. Applying the above definition, it follows that $\Phi_{\underline{z},n}(w) = T^n(x) - z_0 \in J - z_0$, as claimed.

To prove convergence, we will first show that the inverses $\Phi^{-1}_{\underline{z},n}$, when restricted to a small ball about 0, form a Cauchy sequence. This easily implies the same fact for the $\Phi_{\underline{z},n}$. Then we transport this result to large balls.

By Theorem 2.8, applied with $\varepsilon = 1$, there exists $\tilde{\delta} > 0$ such that for all $\underline{z} \in \hat{J}$ and all $n \geq 0$, the holomorphic inverse branch $T^{-n} : B_{z_0}(\tilde{\delta}) \to \mathbb{C}$ of T^{-n} sending z_0 to z_n is well-defined and the map $\widetilde{DT}^n \circ T^{-n} : B_{\tilde{\delta}}(z_0) \to \mathbb{C}$ is $(1,\tilde{\delta})$-close to the identity in sup norm on that ball. Given $\varepsilon > 0$, we will first find $n \geq 0$ such that for all $m \geq n$, and for all $w \in B_{\tilde{\delta}}(0)$,

$$|\Phi^{-1}_{\underline{z},n}(w) - \Phi^{-1}_{\underline{z},m}(w)| < 2||DT||_\infty \tilde{\delta}\varepsilon.$$

As the first step notice that by the our choice of $\tilde{\delta}$,

$$\widetilde{DT}^n \circ T^{-n}(B_{\tilde{\delta}}(z_0)) \subset B_{(1+1)\tilde{\delta}}(z_0) = B_{2\tilde{\delta}}(z_0).$$

Now up to conjugation by a translation, the map $\Phi^{-1}_{\underline{z},n}$ is equal to $\widetilde{DT}^n \circ T^{-n}$. That is, $\Phi^{-1}_{\underline{z},n}(w) = DT^n(z_{-n}) \cdot (T^{-n}(w + z_0) - z_{-n})$, and consequently

$$Phi^{-1}_{\underline{z},n}(w) = (\widetilde{DT}^n \circ T^{-n})(w + z_0) - z_0. \tag{2.1}$$

Thus $\Phi_{\underline{z},n}^{-1}(B_{\tilde{\delta}}(0)) \subseteq B_{2\tilde{\delta}}(0)$.

Let us write A_n for the map $A_n(x) = DT^n(z_{-n}) \cdot x$. We note that for $m = n + k$ with $k \geq 0$,

$$\Phi_{\underline{z},m}^{-1} = \left(A_n \circ \Phi_{\sigma^{-n}(\underline{z}),k}^{-1} \circ A_n^{-1}\right) \circ \Phi_{\underline{z},n}^{-1}.$$

We have that $A_n^{-1} \circ \Phi_{\underline{z},n}^{-1}((B_{\tilde{\delta}}(0)) \subseteq B_{\hat{\delta}_n}(0)$, where $\hat{\delta}_n = (DT^n(z_{-n}))^{-1}2\tilde{\delta}$. We have been given $\varepsilon > 0$. Let $\delta > 0$ be the number ascribed to ε in Theorem 2.8. By hyperbolicity, $\lim_{n\to\infty} \hat{\delta}_n = 0$. Let $n \geq 1$ be the minimal integer such that $\hat{\delta}_n \leq \delta$. Then $\hat{\delta}_n \geq \delta/||DT||_\infty$. Thus for w in the ball $B_{\tilde{\delta}}(0)$, for all $k \geq 0$ and $m = n + k$,

$$\begin{aligned}|\Phi_{\underline{z},m}^{-1}(w) - \Phi_{\underline{z},n}^{-1}(w)| &= |A_n \circ \Phi_{\sigma^{-n}(\underline{z}),k}^{-1} \circ A_n^{-1} \circ \Phi_{\underline{z},n}^{-1}(w) - \Phi_{\underline{z},n}^{-1}(w)| \\ &\leq |DT^n(z_{-n})| \cdot \varepsilon\delta \leq |DT^n(z_{-n})| \cdot \varepsilon ||DT||_\infty \hat{\delta}_n = 2||DT||_\infty \tilde{\delta}\varepsilon.\end{aligned}$$

This proves the sequence of maps $\Phi_{\underline{z},n}^{-1}$ is Cauchy on the ball $B_{\tilde{\delta}}(0)$ and using Theorem 2.8 it is not difficult to deduce that the sequence $\Phi_{\underline{z},n}$ is also Cauchy on a ball about 0 of some small fixed size $\check{\delta}$. The estimate holds uniformly for all $\underline{z} \in \hat{J}$. Next we note that

$$\Phi_{\underline{z},m}(w) = T^n \circ \left(\Phi_{\sigma^{-n}(\underline{z}),k}(A_n^{-1}(w)) + z_n)\right) - z_0.$$

Fixing n while letting k approach ∞, what we have just shown implies that this is Cauchy on the ball of radius $R = DT^n(z_{-n}) \cdot \check{\delta}$. By taking n large enough, this ball is arbitrarily big. This is what we aimed to show. Hence the limit exists, it is a meromorphic function defined and holomorphic on all of $\mathbb{C}$ and the diagrams involved in the formulation of our theorem commute. Each $\Phi_{\underline{z},n}$ restricted to $L_{\underline{z},n}$ gives a degree (degree $T)^n$-"cover" of $J - z_0$; it follows that $\Phi_{\underline{z}}$ restricted to $L_{\underline{z}}$ is a countably infinite "holomorphic" cover of $(J - z_0)$. From the commutative diagram, the lift by $\Phi_{\underline{z}} + z_0$ of a critical point c for T which happens to be in the image of $\Phi_{\underline{z}} + z_0$, is a critical point for $\Phi_{\sigma\underline{z}} + z_1$. Now since all our rational maps with the exception of the maps of the form $z \mapsto z^q$, $q \in \mathbb{Z}$, have at least two distinct critical points, and since by Picard's theorem the image of the entire function $\Phi_{\underline{z}}$ misses at most one point in $\mathbb{C}$, the image must contain at least one such point c. Since $\Phi_{\underline{z}} + z_0$ has countable degree, $\Phi_{\sigma\underline{z}} + z_1$ and therefore also $\Phi_{\underline{z}} + z_0$ has a countable infinity of critical points, except for the cases $T : z \mapsto z^q$. This is what we claimed. □

Example. Let z_0 be a fixed point for T^n. Let $\underline{z}$ be the infinite concatenation (in the negative direction) of the word $T(z_0), T^2(z_0), \ldots, T^{n-1}(z_0), z_0$. Then writing $\lambda = DT^n(z_0)$ (the *multiplier* at that periodic point), the commutative diagram above simplifies, since for $\Phi \equiv \Phi_{\underline{z}} = \Phi_{\sigma^n\underline{z}}$, and we have $T^n \circ \Phi(w) = \Phi(\lambda \cdot w)$. This is a globalization (to all of $\mathbb{C}$) of Koenig's classical linearization theorem (see [19]); thus the construction of scenery can be seen as giving a type of Koenig's theorem for general (non-periodic) points.

Consider the particular exceptional case $T(z) = z^2$, and choose $z_{-n} = z_0 = 1$ for all n. The scenery at the fixed point 1 is the tangent line to the circle at that point, which wraps around the circle via $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. So our countable holomorphic cover should be the exponential map $z \mapsto e^z$, shifted over to have as image the circle centered at -1. And this indeed is the case: the formula yields

$$\Phi_{\underline{z},n}(w) + 1 = \left(1 + w \cdot 2^{-n}\right)^{2^n};$$

for real w by taking logs, we can verify the classical formula $(1+w/n)^n \to e^w$; since we know $\Phi_{\underline{z},n} \to \Phi_{\underline{z}}$ which is holomorphic, this equality extends to $\mathbb{C}$. Therefore $\Phi_{\underline{z}}(w) = e^w - 1$, as it should be.

We remark that the covering maps for other Julia sets are therfore a kind of generalization of the exponential map, and so may be of interest in their own right.

§3. The model scenery flow and the scenery flow.

As in §2, we write Ω for the collection of all closed subsets of $\mathbb{C}$, topologized by the conformal map topology $\mathcal{T}$. We define the **magnification flow** τ_t on Ω to be the flow (action of the additive reals) given by dilation by the real exponential factor e^t. That is, $\tau_t(E) = e^t \cdot E$ for $E \in \Omega$. Note that the flow is (jointly) continuous. For $z \in J$ we write Ω_z for the τ_t omega-limit set of the translated Julia set $J - z$ (thought of as a *point* in the space Ω). That is,

$$\Omega_z \equiv \bigcap_{t=0}^{+\infty} cl\left(\bigcup_{s=t}^{+\infty} \tau_s(J-z)\right)$$

where cl denotes $\mathcal{T}$-closure. This set is closed and is flow-invariant. We write

$$\Omega_J = \bigcup_{z\in J} \Omega_z.$$

Definition. *(Ω_J, τ_t) is the* **scenery flow of** *J, and (Ω_z, τ_t) is the* **scenery flow of** *J* **at** *z. The* **full** *scenery flow of J, denoted by $(\widehat{\Omega}_J, \tau_t)$, is τ_t acting on the the space which is defined to be the image of Ω_J under all rotations, $E \mapsto w \cdot E$, $w \in S^1$.*

The model scenery flow.

We will next define a **special flow** a we will show that the full scenery flow of J is its topological factor.

We move to an abstract setting, recalling the standard definitions for a general map. A special flow is constructed from an invertible transformation on a set called the **base** of the flow, and a strictly positive function defined on the base. For the resulting flow, the base will be a **Poincaré cross-section**, and the function will give the time of return to the base. One says the flow is **built under** the function and **over** the base map.

Our base will be a compact topological space X, with base map a homeomorphism F and with continuous **return time function** r. The **flow space** $X_{F,r}$ is the compact topological space which is the quotient of the space $X \times \mathbb{R}$ by the equivalence relation generated by the identification

$$(x, s + r(x)) \sim (F(x), s).$$

Finally the flow τ_t on $X_{F,r}$ is defined by

$$\tau_t(x, s) = (x, s + t).$$

Remark.

(1) We mention that to a topologist, what we are calling the base of the flow will instead be a fiber of a fiber bundle. The topological conventions are as follows.

In the special case where $r \equiv 1$, one calls $(X_{F,r}, \tau_t)$ the **suspension flow**. If one interested in only the topology of the flow space, one might as well make this assumption since the topology is the same for all r positive. The suspension can be thought of as a fiber bundle over the circle $\mathbb{R}/\mathbb{Z}$ with fiber X; note that then the circle is the base of the fiber bundle (so $X_{F,r}$ "fibers over the circle") while X is the base of the flow. (We will follow the dynamical rather than the topological usage of "base").

(2) We note that the usual picture of the special flow depicts a fundamental domain for a group action. The group is $\mathbb{Z}$, acting on $X \times \mathbb{R}$, with the natural action generated by the identification.

It is important to note that the identification space, and the flow, make sense for arbitrary functions r. This function is a return time to a cross-section exactly when it is positive. In our particular case, the positivity of r will be a consequence of (indeed equivalent to) the strict hyperbolicity of the rational map T.

The map T is however not invertible. It is made invertible in a canonical way known in topology as the projective (inverse) limit and in ergodic theory as the natural extension. In order to describe this construction let $f : X \to X$ be a surjective map. Then the space

$$\hat{X} = \left\{ (x_n)_{n=-\infty}^{\infty} \in \prod_{n=-\infty}^{\infty} X : x_{n+1} = f(x_n) \right\}$$

and the map $\hat{f} : \hat{X} \to \hat{X}$ given by the formula

$$\hat{f}(x_n)_{n=-\infty}) = (f(x_n))_{n=-\infty}^{\infty} = (x_{n+1})_{n=-\infty}^{\infty}$$

are called natural extensions (inverse limits) respectively of the space X and them $\hat{f}$.

We remark that (as is easy to show) the natural extension satisfies the following universal property: it is the smallest homeomorphism which factors onto the map, in the sense that any other such homeomorphism factors through it.

Now we return to our rational map T. The base transformation for the special flow $\widehat{T}$ can be defined geometrically, as follows. The normalized derivative map $DT(z)/|DT(z)|$ acts naturally on the unit tangent bundle of J; $\widehat{T}$ denotes the natural extension of this map, i.e. its (unique) invertible version given by the above Lemma. We write $\hat{J}_1 = \hat{J} \times S^1$, and setting $\phi(\underline{z}) = DT(z_0)/|DT(z_0)|$, we define the homeomorphism $\hat{\sigma}$ on $\hat{J}_1$ to be the skew product transformation with shift base and skewing function ϕ, i.e. $\hat{\sigma}(\underline{z}, w) = (\hat{T}(\underline{z}), \phi(\underline{z})w)$. It is clear that the map $\hat{\sigma}$ is naturally conjugate to the natural extension of the map $\sigma : J \times S^1 \to J \times S^1$ given by the formula $\sigma(z, w) = (T(z), \phi(z)w)$.

Definition. *The* **model scenery flow** *of J is the special flow (Σ, S_t) with base map $(\hat{J}_1, \hat{\sigma})$ and return time function $r(\underline{z}) = \log |DT(z_0)|$.*

We define a map Ψ from $\hat{J}_1 \times \mathbb{R} = \hat{J} \times S^1 \times \mathbb{R}$ to Ω (the closed subsets of $\mathbb{C}$) by $(\underline{z}, w, t) \mapsto w\, e^t\, L_{\underline{z}}$.

Theorem 3.1. *The full scenery flow (Ω_J, τ_t) is a topological factor of the model scenery flow (Σ, S_t), with factor map given by Ψ.*

Proof. By the last claim of Theorem 2.9, Ψ is well-defined as a map from Σ to Ω, i.e. it respects the identifications. Continuity of Ψ with respect to the first coordinate (in $\hat{J}$) has been proved in Theorem 2.13. The continuity of Ψ with respect to the other two coordinates is obvious. For the rest of the proof, i.e. that $\Psi(\Sigma) = \Omega_J$ we need the following lemma.

Lemma 3.2. *For any z_0 in J and all $\underline{z}$ in $\hat{J}$ with that zeroth coordinate, $(J - z_0)$ is forward asymptotic to $L_{\underline{z}}$ under the magnification flow τ_t on Ω (the closed subsets of $\mathbb{C}$), in the topology $\mathcal{T}$.*

Proof. From the uniform convergence of Theorem 2.9, given $R, \varepsilon > 0$ there exists n_0 such that for all $\underline{z}$, for all $n > n_0$, $L_{\underline{z},n}$ is (R, ε)-close to $L_{\underline{z}}$. Therefore, writing "$\sim$" for (R, ε)-closeness, we get $DT^n(z_0) \cdot (J - z_0) \sim L_{\sigma^n(\underline{z})} = DT^n(z_0)\, L_{\underline{z}}$. Since $|DT^n(z_0)| = \log|DT^n(z_0)|$, this means that $\tau_t(J - z_0) \sim \tau_t\, L_{\underline{z}}$ and we are done. $\square$

Now we finish the proof of the Theorem 3.1. By Lemma 3.2, the omega-limit set of $(J - z_0)$ coincides with the omega-limit set of $L_{\underline{z}}$. This shows that $\widehat{\Omega}_J \subseteq \Psi(\Sigma)$. For the reverse containment we are due to lemma 3.2 to show that given $(\underline{z}, \beta, s) \in \hat{J}_1 \times S^1 \times \mathbb{R})$, there exists $\underline{\omega} \in \hat{J}$ and $w \in S^1$ such that $w\, \tau_t(L_{\underline{\omega}})$ comes infinitely often arbitrarily close to $\beta\, e^s\, L_{\underline{z}}$ as $t \to +\infty$. It will be enough to show this for $\beta = 0$ and $s = 0$. Now since there exists a point with a dense T-orbit in the Julia set, the same is true for the shift map σ on $\hat{J}$. Let $\underline{\omega}$ be such a point. Then, in particular, there exists an increasing to infinity sequence $\{n_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \sigma^{n_k}(\underline{\omega}) = \underline{z}$. By the last claim of Theorem 2.9 for the sequences $t_n = \log|DT^n(\omega_0)|$ and $w_n = DT^n(\omega_0)/|DT^n(\omega_0)|$, we have $w_n \tau_{t_n} L_{\underline{\omega}} = L_{\sigma^n \underline{\omega}}$ and therefore $w_{n_k} \tau_{t_{n_k}} L_{\underline{\omega}}$ converges to $L_{\underline{z}}$. Since the circle S^1 is compact, passing to a subsequence, we may assume that $w_{n_k} \to w$ for some $w \in S^1$. Then also $w \tau_{t_{n_k}} L_{\underline{\omega}}$ converges to $L_{\underline{z}}$ and we are done. $\square$

Remark. Since the model scenery space Σ is a compact space and the map from there to the closed sets Ω is continuous, the image is compact. Thus, the set of all scenes is compact in the topology $\mathcal{T}$, hence is a compact metric space in the local Hausdorff and measure metrics.

§4 Ergodic theory and rotational behavior.

§4.1 Gibbs states and the projected flow.

We begin by considering the ergodic theory of the special flow built over the base $(\hat{J}, \sigma)$ with return time function $r(\underline{z}) = \log|DT(z_0)|$. That is, we for now are ignoring all angular information. This **projected flow** $(\overline{\Sigma}, \bar{S}_t)$ is a factor of the model scenery flow via the projection $(\underline{z}, w, s) \mapsto (\underline{z}, s)$. For an expanding $\mathcal{C}^{1+\alpha}$ map on a Cantor set in $[0, 1]$ the same flow played a key role in the analysis of density properties of the Cantor set, see §3 of [3]. Indeed it was a close reanalysis of the convergence proof given there which led us to the "linearization" construction of the scenery used in §2 above and in [5]. The existence of average density now follows as a corollary (see Theorem 4.2).

The following theorem, describing the ergodic theory of the projected flow, follows from the fundamental work of Bowen, Ruelle and Sinai. Some of the main points in the development of the "SRB theory" relevant here are: Lemma 10 of [7] (for **Bowen's formula** for Hausdorff dimension); §8 of [30] and Proposition 3.1 of [8] (for the relationship between measures of maximal entropy for flows and Gibbs states on a cross-section).

For completeness we include the proof. In summary, two separate results from the SRB theory (the relationship between Gibbs states and the Hausdorff measures on the one hand, and a Gibbs state on the cross-section and the measure of maximal entropy for a flow on the other) are brought together, linked by the scenery flow.

For further details and background see [3-5], [12] and [13]. For exposition on Bowen's formula see [24]. See [6], [7], [22], [24] and [29] for treatments of the theory of Gibbs states.

Theorem 4.1. *Let μ be Hausdorff d-dimensional measure H^d restricted to the hyperbolic Julia set J. Then writing ν for the (unique) T-invariant probability measure which is absolutely continuous with respect to μ and extending to the invertible map $(\hat{J}, \sigma)$, the product $\bar{\nu}$ of ν with Lebesgue measure on $\mathbb{R}$ (and normalized) gives the unique measure of maximal entropy for the projected flow $(\overline{\Sigma}, \bar{\nu}, \bar{\tau}_t)$. This flow is ergodic and its topological entropy equals $HD(J)$.*

Proof. Bowen's theorem [7] states that there exists a unique positive number d such that the topological pressure $P(\varphi) = 0$ for $\varphi(z) = -d \log |DT(z)|$, that d is the Hausdorff dimension of J and that d-dimensional Hausdorff measure μ is equal to the Ruelle eigenmeasure (times a constant). This measure is ergodic for the shift map. Hence the flow built over it is also ergodic. Now using the fact that $P(\varphi) = 0$, it follows from the variational principle [6], [29] (see also [24]) that $\sup \left\{ h(m) + \int_J \varphi \, dm \right\}$ where the sup is taken over invariant probability measures m on J. The sup is achieved uniquely by the Gibbs state ν (which is the unique invariant measure equivalent to the eigenmeasure μ). Let $\hat{\phi}(z) = \log |DT(z_0)|$ and let $\hat{m}$ and $\hat{\nu}$ be Rokhlin's natural extensions of the measures m and ν respectively (see for ex. Chapter 1 of [24] for details of this construction). Then $\int \phi d\nu = \int \hat{\phi} d\nu$ and therefore the equality $\frac{h(v)}{\int -\varphi \, dv} = 1$ can be rewritten in the form

$$\frac{h(\hat{\nu})}{\int_{\hat{J}} \log |DT| d\hat{\nu}} = d.$$

Now Abramov's formula [1] states that the measure theoretic entropy for a special flow is equal to the entropy of the base transformation divided by expected return time. This is exactly the left-hand side of the equation. Hence by the variational principle for flows $h_{\overline{S}} \geq d$, where $h_{\overline{S}}$ is the topological entropy of the model scenery flow $\overline{S}_t$. Let now $\hat{m}$ be an arbitrary shift-invariant measure on $\hat{J}$ and let m be its projection on J. Then m is T-invariant and $h(m) - d \int \hat{\phi} d\hat{m} = h(m) - d \int \phi \, dm \leq 0$. Thus

$$\frac{h(\hat{m})}{\int_{\hat{J}} \log |DT| d\hat{m}} \leq d$$

and invoking the variational principle for flows again, we conclude that $h_{\overline{S}_t} \le d$. The proof is complete. $\square$

The next theorem now follows as a corollary. Part (iii) is due, with a different proof, to Falconer [11].

Theorem 4.2. *For a rational map T with hyperbolic Julia set (or, more generally, for a conformal mixing repellor),*

(i) for $\nu-$a.e. $\underline{z} \in \hat{J}$, the average density at $0 \in \mathbb{C}$ of H^d restricted to $L_{\underline{z}}$ exists, and this value is a.s. constant on $\hat{J}$.

(ii) If for some $\underline{z} \in \hat{J}$, the average density at 0 of $L_{\underline{z}}$ exists, then the same holds for J at z, with equal value, where $z = z_0$ is the zeroth coordinate of $\underline{z}$.

(iii) For μ-a.e. point $z \in J$, the average density exists, and the value is a.s. constant.

(iv) for all $\underline{z} \in \hat{J}$, the average density exists and is the same at H^d-a.e. point $z \in L_{\underline{z}}$.

Proof. The basic idea is simple, though there is a subtle analysis point we remark on later, so we will be careful with the details. Part (i) is a consequence of the Birkhoff ergodic theorem applied to the projected flow (which is ergodic); (ii) holds because the two sets are forward asymptotic under scaling; (iii) follows from (i) together with (ii), and (iv) will be proved from (iii). By the scaling (conformal) property of Hausdorff measures

$$f(\underline{z}, t) := \frac{H^d\big(B(0, e^{-t}) \cap L_{\underline{z}}\big)}{e^{-td}} = H^d(B(0,1) \cap e^t \cdot L_{\underline{z}}).$$

Thus

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\underline{z}, t)\mathrm{d}t$$

is the average density at $0 \in \mathbb{C}$ of $L_{\underline{z}}$ since, by the last claim of Theorem 2.9, the function f is well-defined on the projected flow space $\overline{\Sigma}$. We claim $f \in L^1(\overline{\Sigma}, \bar{\nu})$. Indeed, it is bounded away from 0 and ∞: from (1.1) $c_1\varepsilon^d \le H^d(B(0,\varepsilon) \le c_2\varepsilon^d$ (i.e. $H^d|_J$ is geometric) and from Theorem 2.10, $L_{\underline{z}}$ is locally a conformal image of a piece of J, hence $H^d|_{L_{\underline{z}}}$ is also a geometric measure, which implies boundedness. Alternatively, the bounds for $L_{\underline{z}}$ follow directly from those for J by the estimates to follow. By Theorem 4.1 the projected scenery model flow is ergodic. Hence, by the Birkhoff ergodic theorem

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\underline{z}, t+s)\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tau_t(\underline{z}, s))\mathrm{d}t = \int_{\overline{\Sigma}} f(\underline{z}, s)\mathrm{d}\overline{\nu}. \tag{4.1}$$

Taking, in particular, $s = 0$, we see that the average density of $L_{\underline{z}}$ at 0 is equal to $\int_{\overline{\Sigma}} f(\underline{w}, s)\mathrm{d}\overline{\nu}(w)$ for ν-a.e. $\underline{z}$, proving (i).

Next, suppose we are given that the above limit exists for some $L_{\underline{z}}$ and some $\underline{z}$. Fixing $z = z_0$, the zeroth coordinate of $\underline{z}$, define $A_t = e^t(J - z) = \tau_t(J - z)$, writing this time

$$\tilde{f}(\underline{z}, t) := \frac{H^d(J \cap B(z, e^{-t}))}{e^{-dt}} = H^d(B(0,1) \cap A_t).$$

We wish to show the time average for $\tilde{f}$ exists and takes on the same value as for $f(\underline{z}, t)$. For $\delta > 0$, let φ_δ be a continuous function from $\mathbb{C}$ to $[0, 1]$ such that $\varphi_\delta = 1$ on $B(0, 1)$ and $= 0$ outside of $B(0, e^\delta)$. Write $\tilde{f}_\delta(t) = \int_{A_t} \varphi_\delta(w) \mathrm{d}H^d$ and $f_\delta(\underline{z}, t) = \int_{e^t L_{\underline{z}}} \varphi_\delta(w) \mathrm{d}H^d$. Note that $\varphi_\delta(w \cdot e^\delta) \le \chi_B(w) \le \varphi_\delta(w)$ for all $w \in \mathbb{C}$. Hence, again by the conformal transformation property of H^d, $e^{-d\delta} f_\delta(\underline{z}, t + \delta) \le f(\underline{z}, t) \le f_\delta(\underline{z}, t)$ for all t, δ; the analogous inequalities hold for $\tilde{f}_\delta, \tilde{f}$. By (4.1) and the Birkhoff ergodic theorem we have

$$
\begin{aligned}
e^{-d\delta} \int_{\overline{\Sigma}} f_\delta \mathrm{d}\overline{\nu} &= e^{-d\delta} \lim_{T\to\infty} \frac{1}{T} \int_0^T f_\delta(\underline{z}, t) \mathrm{d}t = e^{-d\delta} \lim_{T\to\infty} \frac{1}{T} \int_0^T f_\delta(\underline{z}, t + \delta) \mathrm{d}t \\
&\le \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\underline{z}, t) \mathrm{d}t \le \lim_{T\to\infty} \frac{1}{T} \int_0^T f_\delta(\underline{z}, t) \mathrm{d}t. = \int_{\overline{\Sigma}} f_\delta \mathrm{d}\overline{\nu}
\end{aligned} \tag{4.2}
$$

Now by Lemma 3.2 and Lemma 2.5, the closed sets $(J - z)$ and $L_{\underline{z}}$ are forward asymptotic in the magnification flow τ_t with respect to the measure metric. Since this allows sampling against continuous functions with compact support, it applies to φ_δ. By our assumption the time average of $f_\delta(\underline{z}, t)$ exists, so this implies the average for $\tilde{f}_\delta$ also exists, with the same value, equal to $\int_{\overline{\Sigma}} f_\delta \mathrm{d}\overline{\nu}$. Thus

$$
\begin{aligned}
e^{-d\delta} \int_{\overline{\Sigma}} f_\delta \mathrm{d}\overline{\nu} &= e^{-d\delta} \lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{f}_\delta(t) \le \liminf_{T\to\infty} \frac{1}{T} \int_0^T \tilde{f}(\underline{z}, t) \\
&\le \limsup_{T\to\infty} \frac{1}{T} \int_0^T \tilde{f}(\underline{z}, t) \le \lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{f}_\delta(t) = \int_{\overline{\Sigma}} f_\delta \mathrm{d}\overline{\nu}.
\end{aligned} \tag{4.3}
$$

Since this is true for every $\delta > 0$, combining (4.2) and (4.3) the limit for $\tilde{f}$ exists and equals that for f. This proves (ii). Part (iii) follows immediately. Finally, for (iv), when the set $L_{\underline{z}}$ is shifted to be centered at a point $\xi \in L_{\underline{z}}$ by $L_{\underline{z}} - \xi$, then from the definitions, $L_{\underline{z}} - \xi$ is $wL_{\underline{\xi}}$ for some $w \in S^1$ and choice of preimages for ξ. Thus since the average density exists at 0 for $L_{\underline{z}}$ for a.e. $\underline{z}$, for *every* $\underline{z}$ it will exist H^d-almost surely in that scene $L_{\underline{z}}$. □

We remark that part (ii) also follows from (i), by local conformal invariance of average density together with Theorem 2.10.

Remark. The subtle analysis point we referred to above is this. Since the function $\chi_{B(0,1)}$ is not continuous, we cannot apply directly convergence in the measure metric. One's first attempt is probably to argue that the boundary of the disk is however a negligible set. So one covers it by n balls of radius $1/n$, estimating their measure from the geometric measure property. And indeed, this method will work fine for $d =$dimension$(J) \ge 1$; but for $d \in (0, 1)$ the estimate blows up for large n. Our way out is to create bounds by dilating; since this corresponds to shifting the time, it does not matter after the time average is taken.

§4.2 Rotational behavior: Furstenberg's lemma.

The main result of this and the next section will be that the scenery flow is ergodic for the repellors which are not linear nor contained in a finite union of real-analytic curves and which form a dense open subset. The same will be proved for all hyperbolic rational maps, except for a few exceptional cases listed in Theorem 4.9. Furthermore, the full scenery flow is equal to the scenery flow at μ-almost every point in J.

Ergodicity of a special flow is equivalent to ergodicity of a cross-section map, since invariant subsets correspond. In this sub-section we will study equivalent conditions for ergodicity of the base map $(\hat{J}, \hat{\sigma})$. To prove this we make use of two methods, one due originally to Furstenberg and one to Livsic. These form part of the general developing theory of group-valued cocycles for group actions. For completeness, we give full proofs of what we need.

We begin with a general skew product with circle fiber. Let F be a (not necessarily invertible) measure-preserving map of a measure space X with invariant ergodic probability measure ρ. We assume that the **skewing function** $\varphi : X \to S^1$, is measurable, and that X is a compact metric space. We write $\tilde{X} = X \times S^1$ and define $\tilde{F}$ on $\tilde{X}$ by $\tilde{F}(x, w) = (F(x), \varphi(x)w)$. Lebesgue measure on S^1 will be denoted by m.

Proposition. *The product measure $\tilde{\rho} = \rho \times m$ is invariant for $\tilde{F}$.*

Proof. The idea is that since fibers are rotated by φ and then simply exchanged according to the measure-preserving map F, the skew product should preserve $\tilde{\rho}$ by Fubini's theorem. To make this precise, following the proof of Lemma 2.1 in [15], test against a fuction f in $L^1(\tilde{\rho})$; interchanging the order of order of integration proves invariance. □

We will write $\mathcal{M}_\rho$ for the collection of $\tilde{F}$-invariant measures with **marginal** ρ (i.e. which project to ρ). We are interested in finding equivalent conditions such that there is only one such measure (i.e. $\mathcal{M}_\rho$ is the singleton $\{\tilde{\rho}\}$), in which case we will say $\tilde{F}$ is ρ-**uniquely ergodic**. First we discuss some other forms that an invariant measure can take. Suppose there exists a measurable function $u : X \to S^1$ such that

$$\varphi(x) = \frac{u \circ F(x)}{u(x)} \qquad \text{(for } \rho\text{-a.e. } x). \tag{4.3}$$

In this case one says φ is a (multiplicative) **coboundary**, or is **cohomologous to zero**. Now notice that the graph of u is a $\tilde{F}$-invariant subset of $\tilde{X}$. Hence there exists an invariant measure which is not $\tilde{\rho}$: just lift ρ to a measure supported on that graph (or more generally, add parallel bands of mass).

We mention another interpretation of equation (4.3). Via the map $(x, u(x)w) \mapsto (x, w)$, $\tilde{F}$ is isomorphic to $F \times$ (identity), i.e. to a skew product which does nothing in the fibers. This isomorphism can be viewed as a fiber-preserving change of coordinates on $\tilde{X}$, given by choosing a new origin for each circle; the new origin is $u(w)$. Conversely, a fiber-preserving isomorphism defines such a function u.

So far we have, therefore:

Proposition 4.3. *The following are equivalent, for $\varphi :\ X \to S^1$ measurable:*
(a) there exists u measurable with $\varphi(x) = u \circ F(x)/u(x)$

(b) there exists a fiber-preserving isomorphism from $\tilde{F}$ to $F \times$ (identity)
(c) there exists a measurable function from X to S^1 whose graph is invariant. □

It is clear that the example described above of an invariant measure different from $\tilde{\rho}$ will generalize to that of a map which instead of fixing the circle, permutes k equal intervals. In fact conversely, as we will now see, this is all that can happen.

This represents a generalization of Lemma 2.1 from [15]. (Furstenberg considers the case where the base transformation itself is uniquely ergodic).

Theorem 4.4. *If in addition to the above $\tilde{X}$ is a compact metric space, then the following are equivalent.*
(a) $\tilde{\rho}$ is not ergodic.
(b) $\tilde{F}$ is not ρ-uniquely ergodic.
(c) There exist $k \in \mathbb{Z}$ and $u : X \to S^1$ measurable such that

$$\varphi^k = \frac{u \circ F}{u}.$$

Proof. (b) $\implies$ (a). We will show that if $\hat{\rho}$ is ergodic, then $\mathcal{M}_\rho = \{\tilde{\rho}\}$. We learned this argument from Eli Glasner; another nice argument can be given using generic points, following [15]. We recall that since $\tilde{X}$ is a compact metric space, the $\tilde{F}$-invariant measures form a weak*-compact convex set, with the ergodic measures as the extreme points. Let $\tilde{\rho} \in \mathcal{M}_\rho$. Now if we average the measure $\tilde{\rho}$ along each circle fiber by Lebesgue measure on S^1, we get the measure $\tilde{\rho}$. Therefore, $\tilde{\rho}$ is a convex combination of the rotated (and invariant) measures $R_w\,\tilde{\rho}$. Hence if $\tilde{\rho}$ is ergodic, then $\tilde{\rho} = \tilde{\rho}$.

(a) $\implies$ (b). Supposing $\tilde{\rho}$ is not ergodic, it can be written as a convex combination of two invariant measures, $\tilde{\rho}_1$ and $\tilde{\rho}_2$. The only thing to check (to contradict ρ-unique ergodicity) is that they project to ρ. But whatever measure they project to must be absolutely continuous with respect to the projection of $\tilde{\rho}$, i.e. ρ; hence by ergodicity of ρ it equals ρ.

(a) $\implies$ (c). We cannot improve on Furstenberg's beautiful little argument, which we include for completeness. Assume that $\tilde{\rho}$ is not ergodic for $\tilde{F}$. Then (see e.g. [36]) there exists a non-constant (real or complex)-valued function G in $L^2(\tilde{X}, \tilde{\rho})$. By Fubini's theorem, for ρ-a.e. circle fiber, G is in L^2 of that fiber. So there are Fourier coefficients $a_n(w)$ for the fiber over w, with $G(x, z) = \sum_{-\infty}^{\infty} a_n(x)\, z^n$. Now calculating $G \circ \tilde{F}$, uniqueness of the Fourier coefficients implies that

$$a_n(\xi) = a_n(F\xi)\,\varphi^n(\xi)$$

for each n. By ergodicity of F, the modulus of each $a_n(\xi)$ is ρ-a.s. constant. Since G is assumed non-constant, for some $k \neq 0$, $|a_k| \neq 0$ (ρ-almost surely). So for that k, we can normalize a_k in the equation above. Then, defining $u(\xi)$ by $u(\xi) = |a_k|/a_k(\xi)$, the equation becomes $\varphi^k = u \circ F/u$, proving (c).

Finally it is clear that (c) $\Longrightarrow$ (a), from the previous Proposition, by putting mass on the graph of u. Or, we note that the function $H_k(\xi, z) \equiv a_k(\xi) \cdot z^k$ is $\tilde{F}$-invariant, which contradicts ergodicity. $\square$

Remark 4.5. The set of k such that (c) holds is an ideal in $\mathbb{Z}$. If, say, ℓ generates this principal ideal, we can describe the collection of all $\tilde{F}$-invariant functions: they can be expressed as some combination of the H_k just defined, for all multiples k of ℓ.

§4.3 Ergodicity of the scenery flow.

First we consider the case of a **conformal mixing repellor** (X, T). We recall the definition [28], [25], [26], [24]: let $X \subseteq V \subseteq U \subseteq M$ where M is a Riemann surface, V and U are open, X is compact, and we have a conformal map $T : U \to M$ satisfying:
- $T(X) = X$;
- T is hyperbolic on X;
- $\cap (T|_V)^{-n}(V) = X$;
- for any W open in X, there exists n with $T^n(W) \supseteq X$.

We recall that (X, T) is called **real-analytic** if X is contained in a finite union of real-analytic curves. Recall also from [34], [23], and [20] that a conformal mixing repellor (X, T) is said to be **linear** if the conformal structure on X admits a conformal linear refinement so that f is linear, that is, if there exists an atlas $\{\varphi_t\}$ that is a family of conformal injections $\phi_t : U_t \to \mathbb{C}$, where $\bigcup_t U_t \supset X$ such that all the maps $\phi_t \phi_s^{-1}$ and $\phi_t f \phi_s^{-1}$ are affine Möbius transformations.

We mention that while conformal mixing repellors clearly include the hyperbolic rational maps, they are much more general. They contain for example local (in a neighbourhood of the Julia set) analytic perturbations of hyperbolic rational functions and the limit sets of Kleinian groups of Schottky type. Other examples, linear in the sense defined above, come from generalizing the map $x \mapsto 4x (\mathrm{mod}\ 1)$ mapping $[0, 1/4] \cup [1/2, 3/4] \to [0, 1]$. Notice that this particular example can be extended to be holomorphic on a neighborhood of the intervals, in such a way as to be biholomorphically conjugate to the map $f(z) = z^4$ restricted to some neighbourhood of the set $\{\mathrm{e}^{i\theta} : 0 \leq \theta \leq \pi/2\} \cup \{\mathrm{e}^{i\theta} : \pi \leq \theta \leq 3\pi/2\}$; and this map is linear in the sense defined above. A class of similar examples is provided by the maps $f(z) = z^n$ for $n \neq 0, 1, -1$.

The results from [23] and [20] contain the following.

Theorem 4.6. *Let (X, T) be a conformal mixing repellor. Then the following conditions are equivalent.*

(1) The repellor (X, T) is linear.

(2) The Jacobian of T with respect to the Gibbs measure μ equivalent to the Hausdorff measure H^d on X, is locally constant.

(3) There exists a cover $\{B_\lambda\}_{\lambda \in \Lambda}$ of X consisting of open disks, a family of continuous functions $\gamma_\lambda : B_\lambda \to \mathbb{R}$, $\lambda \in \Lambda$, and constants $c^{(1)}_{\lambda.\lambda'}, c^{(2)}_{\lambda.\lambda'}$ such that for all $\lambda, \lambda' \in \Lambda$

$$\gamma_\lambda - \gamma_{\lambda'} = c^{(1)}_{\lambda.\lambda'}$$

on $B_\lambda \cap B_{\lambda'}$ and

$$\arg_\lambda(DT) - \gamma_\lambda + \gamma_{\lambda'} \circ T = c^{(2)}_{\lambda.\lambda'}$$

on $B_\lambda \cap T^{-1}(B'_\lambda)$, where $\arg_\lambda(DT) : B_\lambda \to \mathbb{R}$ is a continuous branch of the argument of DT defined on the simply connected set B_λ.

We would like to warn the reader that the property (3) is not equivalent with multiplicative cohomology of the function $DT/|DT|$ to $\mathbb{1}$. In order to provide the reader with a more complete picture, which will be used in the proof of Theorem 4.8 we quote here Remark 2 from the proof of the implication $(ec) \Rightarrow (d2)$ of Theorem 3.1 of [20].

Remark 4.7. *In the case when X is not real-analytic, having equations in item (3) and functions γ_λ respectively holding and defined on balls of X itself would already be sufficient to prove item (1).*

And indeed, assuming that the two equations in (3) hold on the balls $B_\lambda \cap \overline{J}$ ($\overline{J}$ corresponds to X in the termonology from [20]) and proceedings exactly as in the proof of implication (ec)$\Rightarrow$(d2) from [20] with the only change that the balls would be understood relative to $\overline{J}$, we would obtain the formula

$$\gamma_{\lambda(t_0)}(z) = \gamma_{\lambda(t_0)}(v_{t_0}) + \sum_{k=1}^{\infty} \arg_{\lambda(t_k)}(\phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(z)) - \arg_{\lambda(t_k)} \phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(v_{t_0})). \tag{4.4}$$

with the notation taken from the proof of implication (ec)$\Rightarrow$(d2) in [20]. Since the formula (3.10) from [20] would be true on $B(v_{t_0}, \delta)$ understood as an open ball in $\mathbb{C}$, the right-hand side of (4.4) would provide a harmonic extension of $\gamma_{\lambda(t_0)}$ from $\overline{J} \cap B(v_{t_0}, \delta)$ to $B(v_{t_0}, \delta)$. Since $\overline{J}$ is not real-analytic (hence no open subset of $\overline{J}$ is real-analytic; see Lemma 2.1 in [20]), both equations appearing in (3) would hold on appropriaate sets involving balls understood as open subsets of $\mathbb{C}$. This would mean that the condition (eh) from [20] would be satisfied and our condition (1) would follow exactly as in [20].

Theorem 4.8. *Let U be an open set in $\mathbb{C}$. Let $\mathcal{R}_U$ denote the collection of all conformal mixing repellors $f : U \to \mathbb{C}$. Then the full scenery flow and the model scenery flow are ergodic for the repellors which are not linear nor contained in a finite union of real-analytic curves. Furthermore the ergodic maps form a dense open subset of $\mathcal{R}_U$, with respect to the $\mathcal{C}^1-$ topology on $\mathcal{R}_U$.*

Proof. In view of Theorem 3.1 it suffices to prove this theorem for the model scenery flow. It follows from Livsic theory [17], [25] that the following are equivalent, given a hyperbolic map f and a real-valued Hölder continuous observable φ:

(1) φ is cohomologous to zero in the class of Hölder continuous functions
(2) φ is cohomologous to zero in the class of measurable functions
(3) $S^n\varphi(x) = 0$ for each periodic point x, where n is the period of x.

In [171,2] this is proved for real-valued φ (in additive notation of course; this is why cohomologous to 0 replaces cohomologous to 1); one can show, using the methods from the proof of Theorem 1.28 (p.40) of [6] and from the proof of Lemma 1 (p.13) of [25] that for a circle-valued function the analogous statements hold: for some $k \in \mathbb{Z}$,

(1′) φ^k is cohomologous to 1 in the class of Hölder continuous functions
(2′) φ^k is cohomologous to 1 in the class of measurable functions
(3′) $\left(\prod_{j=0}^{n-1} \varphi \circ f^j(x)\right)^k = 1$ for each periodic point x where n is the period of x.

It follows from Theorem 4.4 that nonergodicity of the model scenery flow is equivalent to the following: that the function

$$\varphi(z) = \left(\frac{Df(z)}{|Df(z)|}\right)^k$$

is a measurable coboundary (in S^1) for some integer k. This is exactly condition (2′) above. Since this is equivalent with condition (1′), we conclude by Remark 4.7 that if the model scenery flow is not ergodic, then the repellor (f, X) is forced to be either linear or real-analytic. Suppose now that $\lim_{n\to\infty} f_n \to f$ where in the C^1-topology on $\mathcal{R}_U$. By the structural stability of f we may assume that all f_n are so close to f that there exist homeomorphisms $h_n : X_n \to X$ establishing topological conjugacy between $f_n : X_n \to X_t$ and $f : X \to X$ and $\lim_{n\to\infty} h_n = \mathrm{Id}$ in the C^0-topology. Suppose now that for every $n \geq 1$, $\left(\prod_{j=0}^{n-1} \varphi_n \circ f_n^j(x)\right)^k = 1$ for each f_n-periodic point x with period n, where $\varphi_n = Df(z)/|Df(z)|$. Let $z \in X$ be a periodic point of f with some period, say $n \geq 1$. Then $h_n^{-1}(z)$ is a periodic point of period n for every $t \in [0, 1)$. Since ϕ_n converges uniformly to ϕ and since $\lim_{t\to 1} h_n^{-1}(z) = z$, we conclude that

$$\left(\prod_{j=0}^{n-1} \varphi \circ f^j(z)\right)^k = \lim_{t\to 1}\left(\left(\prod_{j=0}^{n-1} \varphi_n \circ f_n^j(h_n^{-1}(z))\right)^k\right) = \lim_{t\to 1} 1^k = 1.$$

Thus the set of mixing repellors in $\mathcal{R}_U$ satisfying (3') is closed in the C^1-topology and therefore its complement is open.

If now $\left(\prod_{j=0}^{n-1} \varphi \circ f^j(x)\right)^k = 1$ for some f-periodic point with period, say n, then using the implicit function theorem, one can destroy this condition by an appropriate small perturbation of f (compare the proof of Corollary 4.15 in [6]), proving denseness. We are done. □

In the case of hyperbolic rational functions we shall prove the following.

Theorem 4.9. *The full scenery flow and the model scenery flow of a hyperbolic rational function are ergodic unless*

(1) The Julia set $J(T)$ is a geometric circle and T is biholomorphically conjugate to a finite Blaschke product.

(2) The Julia set $J(T)$ is totally disconnected and $J(T)$ is contained in a real-analytic curve with self-intersections (if any) lying outside of the Julia set.

Proof. In view of Theorem 3.1 it suffices to prove this theorem for the model scenery flow. Exactly as in the proof of the previous theorem, the model scenery flow is not ergodic if and only if the function

$$z \mapsto \left(\frac{DT(z)}{|DT(z)|} \right)^k$$

is a measurable coboundary (in S^1) for some integer k. But in [14] we have proved the following

Theorem 4.10. *Let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of of degree ≥ 2 and let $\phi : J(T) \to \mathbb{R}$ be a Hölder continuous potential satisfying $P(\phi) > \sup(\phi)$. Suppose that there exist $k \geq 1$ and a measurable function $u : J(T) \to S^1$ such that for μ_ϕ-a.e. $z \in J(T)$*

$$\left(\frac{DT(z)}{|DT(z)|} \right)^k = \frac{u(T(z))}{u(z)}.$$

Then the map T is critically finite and either
(a) T has a superattracting fixed point with a preimage at which T has a different degree.
(b) T is critically finite with parabolic orbifold.
(c) The Julia set $J(T)$ is a geometric circle and T is biholomorphically conjugate to a finite Blaschke product.
(d) The Julia set $J(T)$ is a closed segment of either a geometric circle or a straight line and T is biholomorphically conjugate to a Chebyshev polynomial.
(e) $J(T)$ is totally disconnected and $J(T)$ is contained in a real-analytic curve with self-intersections (if any) lying outside of the Julia set $J(T)$.

In fact it suffices in this theorem to assume that $P(S_n\phi) > \sup(S_n\phi)$ for some $n \geq 1$. In our case $\phi = -d \log |DT|$ and $P(\phi) = 0$. Since T is hyperbolic, $|DT^n| \geq \lambda > 1$ on the Julia set for some $n \geq 1$ and some $\lambda > 1$. This implies that $P(S_n\phi) = 0 > -d \log \lambda \geq \sup(S_n\phi)$. Thus to conclude the proof it suffices to rule out in our context the cases (a), (b) and (d). And indeed, the case (b) is ruled out since then the Julia set is equal to the entire sphere $\overline{\mathbb{C}}$ and the map T cannot be hyperbolic. Since the map T restricted to its Julia set is never 1-to-1, we conclude that in the case (d), the Julia set, as a real-analytic interval, would have to contain at least one critical point and therefore this case is also impossible. In order to exclude the case (a) it suffices to invoke V. Mayer's improvement (see [21]) of our Theorem 4.10 which says exactly the same as our theorem but with the case (a) excluded. We are done. □

We would like to mention that all known to us examples described in item (e) do not have self-intersections.

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