FRACTAL MEASURES AND ERGODIC THEORY OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

JANINA KOTUS AND MARIUSZ URBAŃSKI

Dedicated to the memory of Professor I.N. Baker

ABSTRACT. We discuss Hausdorff and packing measures of some significant subsets of Julia sets of large classes of transcendental entire and meromorphic functions. In particular, the class of hyperbolic entire and meromorphic functions satisfying some mild derivative growth conditions is explored in Section 5. This class contains hyperbolic exponential and elliptic functions dealt also with respectively in Sections 2 and 3. A substantial fraction of Section 2 and the whole Section 3 treat functions which are not hyperbolic or not necessarily hyperbolic. Also Walters expanding and non-expanding conformal maps are discussed in length. Other classes of transcendental maps are also touched. Frequently there appear statements about invariant measures, Hausdorff (especially its real analyticity), box and packing dimensions. A special attention is given to the methods developing thermodynamics formalism and conformal measures. The issues concerning the Lebesgue measure of various subsets of Julia sets are addressed in Section 7. In Section 3.3 a positive answer to Question 1 posed in [65] is given.

CONTENTS

2000*Mathematics Subject Classification.* Primary 37F35. Secondary 37F10, 30D05.

The research of the first author was supported in part by the the Polish KBN Grant No 2 PO3A 034 25 and the Warsaw University of Technology Grant no 504G 11200043000.
The second author was supported in part by the NSF Grant DMS 0400481. The research of both authors was supported in part by the NSF/PAN grant INT-0306004.

1. Introduction (presentation of topics to be dealt with)

The main goal of this survey is to provide an overview of methods and results which have been used in the past five years (plus Barański's paper [7]) to analyse in detail the finer fractal structure of Julia sets of some classes of transcendental entire and meromorphic functions. By finer fractal structure we, roughly speaking, mean any knowledge about vanishing, positivity, finiteness and infiniteness of Hausdorff and packing measures of some significant subsets of Julia sets, which themselves are not excluded. The methods used take as a starting point the development of appropriate versions of thermodynamic formalism, conformal measures and (infinite) iterated function systems. Apart from discussing Hausdorff and packing measures, we frequently mention the closely related concepts of Hausdorff, box and packing dimension, as well as invariant measures equivalent to either Hausdorff or packing measure. Our primary interest is in the classes of hyperbolic entire and meromorphic functions satisfying some mild derivative growth conditions (see Section 5), exponential ($\lambda \to \lambda \exp(z)$), elliptic and Walters expanding conformal maps, although we also discuss other classes in Section 2.5, Section 3.3, and separately in Section 4, where various subclasses of Walters expanding conformal maps are defined and explored. We devote one separated chapter (Section 7) to address the issues related to the Lebesgue measure of the Julia sets and some of their subsets, for instance the set of points escaping to ∞. Results concerning Lebesgue measure are also scattered in previous chapters; in Chapter 5 however they are treated more systematically and with bigger generality. As an immediate application of results proved in [42] and stated in Section 5, we give in Section 3.3 a positive answer to Question 1 posed in [65]. For the background concerning the topological dynamics of transcendental entire and meromorphic functions the reader is referred for example to [1]-[5], [8] and [55]. We would like to make it clear that this article has been writing over a relatively long period of time. Throughout new ideas and directions of development have been emerging. Intending to include them, we have tried to present the material respecting the historical development of the field and existing grouping of classes of entire and meromorphic functions.

Acknowledgement. We thank the referee whose valuable remarks and suggestions influenced the final form of our paper.

2. Exponential family

2.1. **Preliminaries.** In this chapter, except for the last two sections, we deal with the family $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbb{C}\setminus\{0\}}$ of entire maps $\mathcal{E}_\lambda : \mathbb{C} \mapsto \mathbb{C}$ given by the formula

$$\mathcal{E}_\lambda(z) = \lambda e^z.$$

All these maps are called exponential and $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbb{C}\setminus\{0\}}$ is called the exponential family. This family was investigated for the first time from the dynamical point of view by P. Fatou [27]. He conjectured that the Julia set of the map $\mathcal{E}_1$ is the entire complex plane $\mathbb{C}$. M. Misiurewicz proved this conjecture about sixty years later in [54] by ingenious but rather elementary methods. It was this event which revived anew the work on exponential family. Two other pioneering works in this area are due to M. Rees [59] and M. Lyubich [46] who proved that the map $\mathcal{E}_1$ is not ergodic with respect to the Lebesgue measure and the ω-limit set of Lebesgue almost every point coincides with the orbit of zero, $\{\mathcal{E}_1(0)\}_{n=0}^\infty$. The third one is due to R. Devaney and M. Krych who introduced in [23] a symbolic representation of exponential maps. This approach was the key point to clarify the topological picture of the Julia sets of hyperbolic exponential maps (see [19]-[24]), and was developed in the paper [1] by classifying hyperbolic Julia sets as Cantor bouquets and straight brushes. One should also mention at this point the paper [6] by I. Baker and P. Rippon who provided the first proof of Sullivan's Non-Wandering Theorem in the class of exponential functions. We should also mention the work of the work of C. McMullen [53], who proved that the Hausdorff dimension of the Julia set of each exponential map is 2. In fact he proved more, that the Hausdorff dimension of the set of points escaping to infinity is equal to 2. Although this was a good and interesting result, it was not the end of the story since the set of points escaping to infinity is actually dynamically insignificant: for example it cannot support any invariant Borel probability measure, although it does exhibit some interesting geometrical features (see [31], [32] and [33]). This situation made M. Urbański and A. Zdunik (see [74]-[77]) ask whether the complement of points escaping to infinity is dynamically and geometrically more interesting. The answer to this question is provided in all sections of this chapter. The basic construction, working in all these sections, first time introduced in a slightly different form in [7] (see also Section 3.2 and 4.3) whose analogs are applied to all maps considered in this article, is this. Let $\sim$ be the equivalence relation on $\mathbb{C}$ defined by declaring that $w \sim z$ if and only if $w - z \in 2\pi i\mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. Let $Q = \mathbb{C}/\sim$ be the quotient space of $\mathbb{C}$ by the relation $\sim$. Q is an infinite cylinder, conformally equivalent to the punctured plane $\mathbb{C}\setminus\{0\}$. Let $\Pi : \mathbb{C} \to Q$ be the canonical projection. Since each map $\mathcal{E}_\lambda : \mathbb{C} \to \mathbb{C}$ is constant on equivalence classes of the relation $\sim$, it can be treated as a map from Q to $\mathbb{C}$. The object we are after here, the map $E_\lambda : Q \to Q$ is defined as follows

$$E_\lambda = \Pi \circ \mathcal{E}_\lambda.$$

Notice that $E_\lambda \circ \Pi = \Pi \circ \mathcal{E}_\lambda$, that is the following diagram commutes.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathcal{E}_\lambda"] \arrow[d, "\Pi"'] & \mathbb{C} \arrow[d, "\Pi"] \\ Q \arrow[r, "E_\lambda"] & Q \end{tikzcd} \tag{2.1}$$

So E_λ is a factor, via a conformal, locally isometric semiconjugacy Π. It is clear that all local geometric fractal features of $\mathcal{E}_\lambda$ are the same as those of E_λ, and the same refers to most of the dynamical features. We would like to notice that, since Π is a local isometry,

$$J(E_\lambda) = \Pi(J(\mathcal{E}_\lambda)) \quad \text{and} \quad J(\mathcal{E}_\lambda) = \Pi^{-1}(J(E_\lambda)).$$

Let $J_{bd}(\mathcal{E}_\lambda)$ be the set of all points $z \in J(\mathcal{E}_\lambda)$, whose orbit $\{\mathcal{E}_\lambda^n(z)\}_{n=0}^\infty$ is bounded. The following general result, needed in many places of the metric theory of exponential maps, has been proved in [74] as

Proposition 2.1. *For every* $\lambda \in \mathbb{C}\setminus\{0\}$, $\mathrm{HD}(J_{bd}(\mathcal{E}_\lambda)) > 1$.

2.2. **Hyperbolic Maps: Measures and Dimensions.** An exponential map $\mathcal{E}_\lambda$ is called hyperbolic if it has an attracting periodic orbit. It equivalently means that (see [75]) that there are two constants $c > 0$ and $\kappa > 1$ such that

$$|(\mathcal{E}_\lambda^n)'(z)| \ge c\kappa^n$$

for all $z \in J(\mathcal{E}_\lambda)$ and all $n \ge 0$. From now onwards, unless otherwise stated, fix a hyperbolic exponential map $\mathcal{E}_\lambda$, denote it by $\mathcal{E}$, and denote E_λ by E. The subfamily of all hyperbolic exponential maps is very big, in particular it contains all parameters $\lambda \in (0, 1/e) \subset \mathbb{R}$. The reader may wish to consider only the case in which the map E corresponds to a parameter from this segment $(0, 1/e)$. In this subsection we describe the results and methods worked out in [74] and [75], although we closer follow the more matured exposition from [44]. The main objects of our focus are the radial Julia sets $J_r(\mathcal{E})$ and $J_r(E)$ introduced in [74]. The first one is defined as the complement of points escaping to infinity, that is the set of points $z \in J(\mathcal{E})$ that have a finite ω-limit point. Since $|\mathcal{E}(z)| = |\lambda|e^{\mathrm{Re}(z)}$, $J_r(\mathcal{E})$ is the set of points $z \in J(\mathcal{E})$ such that the sequence $\{\mathrm{Re}(\mathcal{E}^n(z))\}_{n=0}^\infty$ has a finite accumulation point. The set $J_r(E)$ is also defined as the set of points in the Julia set $J(E)$ that do not escape to infinity under the action of iterates of E. A point $z \in J_r(E)$ if and only if $z \in J(E)$ and there exists a point $y \in J(E)$ and an unbounded increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$y = \lim_{k\to\infty} E^{n_k}(z) \quad \text{and} \quad E^{n_k}(z) \in B(y, \delta/4)$$

for all $k \geq 1$, where

$$\delta = \frac{1}{4}\text{dist}(J(E), \{E^n(\Pi(0))\}_{n=1}^{\infty}) \tag{2.2}$$

is positive since the map E is hyperbolic. The most important feature of points z from $J_r(E)$ is that for every $k \geq 1$ there exists a unique holomorphic branch $E_z^{-n_k} : B(y, 2\delta) \to Q$ of E^{-n_k} sending the point $E^{n_k}(z)$ to z. What distinguishes here the points from $J_r(E)$ from other points escaping to ∞ is that the inverse branches $E_z^{-n_k}$ are all defined on the same ball.

2.2.1. **Pressure, Perron-Frobenius operators and generalised conformal measures.** In this subsection we gather material which will be needed to formulate and to sketch the proof of Bowen's formula, to get our hands on geometric measures and to discuss the proof of real-analytic dependence of the Hausdorff dimension of the Julia sets $J_r(\mathcal{E}_\lambda)$ on λ. We begin with the notion of topological pressure. The trouble is here that the phase space $J(E)$ is not any longer compact and the classical approach using covers, (n, ϵ)-separated sets or (n, ϵ)-spanning sets fails in the context of exponential functions. We therefore adapt the pointwise approach, equivalent in the case of open expanding maps to the classical ones (see [57]), which for exponential maps works very well. Given $t \geq 0$, the topological pressure of the potential $-t \log |E'|$ is given by the formula

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in E^{-n}(z)} |(E^n)'(z)|^{-t},$$

where z is an arbitrary point in $Q \setminus PC(E)$ and

$$PC(E) = \overline{\bigcup_{n\geq 0} E^n(\Pi(0))}.$$

It is not obvious at all that the limit defining topological pressure $\mathrm{P}(t)$ must exist. It can be proved using finer properties of the corresponding Perron-Frobenius operator. Independence of z is much easier; it almost immediately follows from Koebe's Distortion Theorem. In order to get closer to the meaning of this topological pressure $\mathrm{P}(t)$, let us look at the term corresponding to $n = 1$.

$$\mathrm{P}_z(1, t) = \sum_{x \in E^{-1}(z)} |E'(x)|^t = \sum_{n \in \mathbb{Z}} |E'(z_n)|^{-t} = \sum_{n \in \mathbb{Z}} |E(z_n)|^{-t} = \sum_{n \in \mathbb{Z}} |\tilde{z} + 2\pi i n|^{-t},$$

where $\tilde{z}$ is an arbitrary point from $\Pi^{-1}(z)$ and $z_n \in Q$ is the only point such that $\lambda e^{z_n} = \tilde{z} + 2\pi i n$. The series $\mathrm{P}_z(1, t)$ converges if and only if $t > 1$. Before moving on we would like to make two remarks. Firstly, it is straightforward to see that without projecting the dynamics to the cylinder Q, the series involved in the definition of the topological pressure would not converge already for $n = 1$. And this is the main reason to project $\mathcal{E}$ down to E. A different method to resolve this 'divergence' difficulty is provided in Section 5. Secondly, the dynamical significance of the pressure function and other concept produced by thermodynamic formalism is evident from Subsection 2.2.2. Its geometrical significance comes from Subsection 2.2.3, notably Theorem 2.11 (Bowen's formula).

The basic properties of the topological pressure $\mathrm{P}(t)$ are summarised in the following.

Lemma 2.2. *The function $t \mapsto \mathrm{P}(t)$, $t \geq 0$, has the following properties.*

(a) *There exists $t \in (0,1)$ such that $0 \leq \mathrm{P}(t) < +\infty$.*
(b) *$\mathrm{P}(t) < +\infty$ for all $t > 1$.*
(c) *The function $\mathrm{P}(t)$ restricted to the interval $(1,+\infty)$ is convex, continuous and strictly decreasing.*
(d) *$\lim_{t\to+\infty} \mathrm{P}(t) = -\infty$.*
(e) *There exists exactly one $t > 1$ such that $\mathrm{P}(t) = 0$.*

The basic concept of any version of thermodynamic formalism is the Perron-Frobenius operator and eigenvector of its dual operator, called in the sequel a generalised conformal measure. Let $C_b(J(E))$ be the Banach space of all bounded complex-valued continuous functions defined on $J(E)$. Given $t > 1$, the (bounded) linear operator $\mathcal{L}_t : C_b(J(E)) \to C_b(J(E))$ is given by the formula

$$\mathcal{L}_t(g)(z) = \sum_{x\in E^{-1}(z)} |E'(x)|^{-t} g(x) = \sum_{n\in\mathbb{Z}} |\tilde{z} + 2\pi in|^{-t} g(z_n).$$

Let $\mathcal{L}_t^* : C_b^*(J(E)) \to C_b^*(J(E))$ be the operator dual to $\mathcal{L}_t$, i.e.

$$\mathcal{L}_t^*(\nu)(g) = \nu(\mathcal{L}_t(g)).$$

Following the classical thermodynamic formalism, and what is extremely important for a geometrical purposes, one would like to find an eigenvector of the operator $\mathcal{L}_t^*$ corresponding to a positive eigenvalue. The classical approach (see [12], comp. [57]) is to consider the map

$$\nu \to \frac{\mathcal{L}_t^*(\nu)}{\mathcal{L}_t^*(\nu)(1\!\!1)} \tag{2.3}$$

defined on the convex space of Borel probability measures on $J(E)$. This mapping is easily seen to be continuous and in the classical case of open distance expanding maps, ones applies Schauder-Tichonov fixed point theorem, to get a fixed point of the map (2.3), and consequently, the demanded eigenvector of the operator $\mathcal{L}_t^*$. In our case however this method fails since the space of all Borel probability measures on $J(E)$ is not compact. So, one must proceed in a different way. A Borel probability measure m_t is called (t,α_t)-conformal (with $t > 1$ and $\alpha_t \geq 0$) if $m_t(J(E)) = 1$ and for any Borel set $A \subset Q$ restricted to which E is injective, one has

$$m_t(E(A)) = \int_A \alpha_t |E'|^t dm_t. \tag{2.4}$$

Note that a measure is (t, α_t)-conformal if and only if it is an eigenmeasure of the dual operator $\mathcal{L}_t^*$ corresponding to the eigenvalue α_t. One proceeds to construct an m_t measure by taking fruits of $K(V)$-methods described in Appendix 1. Namely, for every $n \geq 1$ put

$$K_n = \bigcap_{j=0}^{\infty} E^{-j}(\{z \in J(E) : \mathrm{Re} z \leq n\}).$$

Then each set K_n is a compact subset of the cylinder Q and $E(K_n) \subset K_n$. So, Lemma 8.1 applies to produce for every $t \geq 0$, a number $\mathrm{P}_n(t)$ and a measure m_n supported on K_n with the following two properties. If $A \subset K_n$ is a Borel set and $E|_A$ is one-to-one, then

$$m_n(E(A)) \geq \int_A e^{\mathrm{P}_n(t)} |E'|^t m_n. \tag{2.5}$$

If in addition $A \cap \{z \in Q \cap \mathrm{Re} z = n\} = \emptyset$, then

$$m_n(E(A)) = \int_A e^{\mathrm{P}_n(t)} |E'|^t m_n. \tag{2.6}$$

It is easy to see that in our context one can replace the inclusion $A \subset K_n$ by $A \subset \{z \in Q : \mathrm{Re} z \leq n\}$ and the two above formulae still hold. It follows from the definition of the numbers $\mathrm{P}_n(t)$ that $\mathrm{P}_n(t) \leq \mathrm{P}(t)$. Making use of the flexibility involved in this definition, one can arrange for the sequence $\{P_n(t)\}_{n=1}^{\infty}$ to be non-decreasing ($\mathrm{P}_n(t) \leq \mathrm{P}_{n+1}(t)$). If the cylinder Q were compact, the next step would be obvious and rather straightforward. Take as a candidate for a (t, α_t)-conformal measure any weak limit of the sequence $\{m_n\}_{n=1}^{\infty}$. Since however the cylinder Q and the Julia set $J(E)$ are not compact, one needs to show that the sequence $\{m_n\}_{n=1}^{\infty}$ is tight, what in our setting means that the measures of this sequence do not accumulate at $+\infty$. And this has been done in [74] and [75]. Now, taking any weakly converging subsequence of the sequence $\{m_n\}_{n=1}^{\infty}$ and using (2.5) along with (2.6), one relatively easily checks that its weak-limit m_t satisfies formula (2.4) with $\alpha_t = \exp(\lim_{n\to\infty} \mathrm{P}_n(t))$. This means that m_t is (t, α_t)-conformal and $\mathcal{L}_t^*(m_t) = \alpha_t m_t$. Its basic property, obtained without making use of the properties of the Perron-Frobenius operator is the following. Put

$$I_\infty(f) = \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}.$$

Then we have the following.

Proposition 2.3. *For every $t > 1$ there exists $M > 0$ such that for m_t-a.e. $x \in J(E)$*

$$\liminf_{n\to\infty} \mathrm{Re}(E^n(x)) \leq M.$$

In particular, $m_t(I_\infty(E)) = 0$ or equivalently $m_t(J_r(E)) = 1$.

This is the first signal that $J_r(E)$ is the right object to deal with. There will be more. In order to study the Perron-Frobenius operator $\mathcal{L}_t$, it is convenient to consider its normalised version $\hat{\mathcal{L}}_t = \alpha_t^{-1} \mathcal{L}_t$. As relatively soft facts (although obtained with some non-obvious tricks) one proves there the following.

Lemma 2.4. $\sup_{n\geq 1}\{||\hat{\mathcal{L}}_t^n(1\!\!1)||_\infty\} < \infty$.

and

Lemma 2.5. $\inf_{n\geq 1}\{\inf\{\hat{\mathcal{L}}_t^n(1\!\!1)(z) : z \in \{w \in Q : \mathrm{Re}w \leq x\}\}\} > 0$ *for every* $x \geq 0$ *large enough.*

These two facts are the main ingredients in the proof linking the topological pressure $\mathrm{P}(t)$ and the eigenvalue α_t. Namely

$$\alpha_t = e^{\mathrm{P}(t)}.$$

To obtain further and harder properties of the Perron-Frobenius operator $\hat{\mathcal{L}}_t$ we have to define the Banach space H_α of locally α-Hölder bounded continuous functions. Fix $\alpha \in (0,1]$. Given $g : J(E) \to \mathbb{C}$, let

$$v_\alpha(g) = \inf\{L \geq 0 : |g(y) - g(x)| \leq L|y-x|^\alpha \ \text{ for all } \ x, y \in J(E) \ \text{ with } |y-x| \leq \delta\}$$

be the α-variation of the function g, where $\delta > 0$ was defined in formula (2.2). Any function with bounded α-variation will be called α-Hölder or simply Hölder continuous if we do not want to specify the exponent of Hölder continuity. Let $C_b = C_b(J(E))$ be the space of all bounded continuous complex valued functions defined on $J(E)$ and let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty.$$

Clearly the space

$$\mathrm{H}_\alpha = \mathrm{H}_\alpha(J(E)) = \{g \in J(E) : ||g||_\alpha < \infty\}$$

endowed with the norm $||\cdot||_\alpha$ is a Banach space densely contained in C_b with respect to the $||\cdot||_\infty$ norm. Any member of the space H_α will be called a bounded Hölder continuous function with exponent α and any member of the space $\bigcup_{\alpha>0}\mathrm{H}_\alpha$ will be simply called a bounded Hölder continuous function. Proving the inequality $||\hat{\mathcal{L}}_t^n||_\alpha \leq \frac{1}{2}v_\alpha(g) + c||g||_\infty$ (for all $n \geq 1$ large enough), noting that images of bounded subsets in H_α under $\hat{\mathcal{L}}_t$ are relatively compact as subsets of $C_b(J(E))$ and applying the Ionescu-Tulcea and Marinescu Theorem, the final (at least for our purposes) properties of the Perron-Frobenius operator are collected in the following.

Theorem 2.6. *If* $t > 1$ *then we have the following.*

(a) *The number* 1 *is a simple isolated eigenvalue of the operator* $\hat{\mathcal{L}}_t : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$.

(b) *The eigenspace of the eigenvalue* 1 *is generated by nowhere vanishing function* $\psi_t \in \mathrm{H}_\alpha$ *such that* $\int \psi_t dm_t = 1$ *and* $\lim_{\mathrm{Re}z\to+\infty}\psi_t(z) = 0$.

(c) *The number* 1 *is the only eigenvalue of modulus* 1.

(d) *There exists* $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ *such that*

$$\hat{\mathcal{L}}_t = Q_1 + S,$$

where $Q_1 : \mathrm{H}_\alpha \to \mathbb{C}\psi_t$ *is a projection on the eigenspace* $\mathbb{C}\psi_t$, $Q_1 \circ S = S \circ Q_1 = 0$ *and*

$$||S^n||_\alpha \leq C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \geq 1$.

2.2.2. **Invariant measures.** Mostly as a consequence of Theorem 2.6 we get the following.

Theorem 2.7. *If $t > 1$, then the measure $\mu = \mu_t = \psi_t m_t$ is E-invariant, ergodic with respect to each iterate of E and equivalent to the measure m_t. In particular $\mu(J_r(E)) = 1$.*

Due to Theorem 2.6 the E-invariant measure μ has much finer stochastic properties than ergodicity of all iterates of E. Here these follow.

Theorem 2.8. *The dynamical system (E, μ_t) is metrically exact i.e., its Rokhlin natural extension is a K-system.*

The proof of this fact is the same as the proof of Corollary 3.7 in [17]. The next two theorems are standard consequences of Theorem 2.6 (see [15] and [57] for example). Let g_1 and g_2 be real square-μ_t integrable functions on $J_r(E)$. For every positive integer n the n-th correlation of the pair g_1, g_2, is the number

$$C_n(g_1, g_2) := \int g_1 \cdot (g_2 \circ E^n)\, d\mu_t - \int g_1\, d\mu_t \int g_2\, d\mu_t,$$

provided the above integrals exist. Notice that due to the E-invariance of μ we can also write

$$C_n(g_1, g_2) = \int (g_1 - \int g_1 d\mu_t)\Big((g_2 - \int g_2 d\mu_t) \circ E^n\Big)\, d\mu_t.$$

We have the following.

Theorem 2.9. *There exist $C \geq 1$ and $\rho < 1$ such that for all $g_1 \in \mathrm{H}_\alpha(Q), g_2 \in L^1(\mu_t)$*

$$C_n(g_1, g_2) \leq C\rho^n \|g_1 - \int g_1 d\mu_t\|_\alpha \|g_2 - \int g_2 d\mu_t\|_{L^1}.$$

Let $g : J_r(E) \to I\!R$ be a square-integrable function. The limit

$$\sigma^2(g) = \lim_{n\to\infty} \frac{1}{n} \int \Big(\sum_{j=0}^{n-1} g \circ E^j - n \int g d\mu_t \Big)^2 d\mu_t$$

is called the asymptotic variance or dispersion of g, provided it exists.

Theorem 2.10. *If $g \in \mathrm{H}_\alpha(Q)$, $\alpha \in (0,1)$, then $\sigma^2(g)$ exists and, if $\sigma^2(g) > 0$, then the sequence of random variables $\{g \circ E^n\}_{n=0}^\infty$ with respect to the probability measure μ_t satisfies the Central Limit Theorem, i.e.*

$$\mu_t\left(\left\{x \in J_r(E) : \frac{\sum_{j=0}^{n-1} g \circ E^j - n\int g d\mu_t}{\sqrt{n}} < r\right\}\right) \to \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{r} e^{-t^2/2\sigma^2}\, dt.$$

2.2.3. **Bowen's Formula, Hausdorff and packing measures.** It was stated in Lemma 2.2 that there exists exactly one value $t > 1$ such that $\mathrm{P}(t) = 0$, and we call this value h. Then the measure m_h is h-conformal in the sense of Sullivan meaning that

$$m_h(E(A)) = \int_A |E'|^h dm_h \tag{2.7}$$

for every Borel set $A \subset J(E)$ such that $E|_A$ is one-to-one. By the analogy with the classical situation of hyperbolic rational functions (see [73]), one may guess that h is the Hausdorff dimension of $J_r(E)$ and m_h, up to a multiplicative constant, the Hausdorff and packing measure on $J_r(E)$. The first guess turns out to be entirely true, the later one only partially.

Theorem 2.11. *(Bowen's formula) The number h, the unique zero of the pressure function $t \to \mathrm{P}(t), t > 1$, is equal to* $\mathrm{HD}(J_r(E))$.

Since the definition of the pressure function $\mathrm{P}(t)$ has a priori nothing to do with the set $J_r(E)$, this theorem in particular indicates that $J_r(E)$ is the right object to deal with.

Theorem 2.12. $0 < \mathrm{H}^h(J_r(E)) < \infty$.

This indicates that the h-dimensional Hausdorff measure H^h on $J_r(E)$ is the right geometric measure. In particular, the purely dynamically defined, h-conformal measure m_h gets its geometrical interpretation as the normalised Hausdorff measure.

Concerning the h-dimensional packing measure P^h, the situation seems to be much worse. We have the following.

Proposition 2.13. *We have $\mathrm{P}^h(J_r(E)) = \infty$. In fact $\mathrm{P}^h(G) = \infty$ for every open nonempty subset of $J_r(E)$.*

It might seem that a measure with such strange properties is completely useless. However since the 2-dimensional packing measure on $\mathbb{C}$ is proportional to the 2-dimensional Lebesgue measure and this latter one is not locally infinite, we immediately get from Proposition 2.13 the following.

Corollary 2.14. *It holds that* $h = \mathrm{HD}(J_r(E)) < 2$.

Since $J_r(E) \supset J_{bd}(E)$, as an immediate consequence of Proposition 2.1, we get the following.

Corollary 2.15. *It holds that* $h = \mathrm{HD}(J_r(E)) > 1$.

Note that in our case every compact E-invariant set is hyperbolic. So, the Shishikura's Hausdorff dimension, being by definition the supremum over all hyperbolic sets, is equal to the supremum over all E-invariant compact sets. Since any E-invariant Borel probability measure μ on $J(E)$ is supported on $J_r(E)$ ($\mu(J_r(E) = 1)$, and since each compact E-invariant subset of $J(E)$ is contained in $J_r(E)$, we thus get the following

Theorem 2.16. *The hyperbolic dimension of* $J(E)$ *and the supremum of Hausdorff dimensions of all Borel probability* E*-measures, both being bounded above by* $\mathrm{HD}(J_r(E))$*, are strictly less than* $\mathrm{HD}(J(E)) = 2$.

The two suprema appearing in Theorem 2.16 are in fact equal (see [57]). Note that it is still an open question whether there exists a rational function with the hyperbolic dimension smaller than the Hausdorff dimension of the Julia set.

2.2.4. **Real-analyticity of the Hausdorff dimension function.** Since this article primarily concerns measures, and dimensions are treated more briefly, we shall provide a very short description of how to prove that the Hausdorff dimension of the radial Julia set $J_r(E_\lambda)$ depends in real-analytic fashion on hyperbolic parameters λ. The full proof, technically rather complicated can be found in [75] and [44]. One starts of with the trivial observation that we can restrict our attention to an arbitrary fixed component W of the set of all hyperbolic parameters. Fix one parameter $\lambda_0 \in W$. It is known (see[26], comp.[76]) that each function $E_\lambda : \mathbb{C} \to \mathbb{C}$ is topologically conjugate to E_{λ_0} via a quasi-conformal homeomorphism $h_\lambda : \mathbb{C} \to \mathbb{C}$. This allows us to define new Perron-Frobenius operators $\mathcal{L}^0_{\lambda,t} : \mathrm{H}_\alpha(J(E_{\lambda_0})) \to H_\alpha(J(E_{\lambda_0}))$, $(\lambda, t) \in W \times (1, +\infty)$ as follows

$$\mathcal{L}^0_{\lambda,t} g(z) = \sum_{x \in E^{-1}_{\lambda_0,t}(z)} |E'_\lambda(h_\lambda(x))|^{-t} g(x).$$

The most technically involved task is now to demonstrate that given $t_0 > 1$, there exists a polydisc $\mathbb{D}((\lambda_0, t_0); R) \subset \mathbb{C}^2 \times \mathbb{C}$ such that the operator-valued function $(\lambda, t) \mapsto \mathcal{L}^0_{\lambda,t} \in L(\mathrm{H}_\alpha(J(E_{\lambda_0})))$, $(\lambda, t) \in W \times (1, +\infty)$ has a holomorphic extension to this polydisc $\mathbb{D}((\lambda_0, t_0), R)$. The rest is the right combination of Theorem 2.6 (that $e^{P(t)}$ is a simple isolated eigenvalue of $\mathcal{L}^0_{\lambda,t}$), the perturbation theory for linear operators, Theorem 2.11 (Bowen's formula) and the inverse function theorem.

2.3. **Hyperbolic Maps: Thermodynamic Formalism and Multifractal Analysis.** In this section our intention is to provide a relatively compact, nevertheless comprehensive summary of the article [72], which provides a systematic account of the thermodynamic formalism and multifractal analysis of hyperbolic exponential maps and, natural in this context, 1^+-tame Hölder continuous potentials. We keep the notation and terminology from the previous section.

2.3.1. **Pressures, potentials, transfer operators and conformal measures.**
Given $n \geq 0$ let

$$S_n\phi = \phi + \phi \circ E + \cdots + \phi \circ E^{n-1}.$$

The following simple distortion fact is necessary for nearly all the results presented below.

Lemma 2.17. *For every $\alpha > 0$ there exists $L_\alpha > 0$ such that if $\phi : J(E) \to \mathbb{C}$ is an α-Hölder function, then*

$$|S_n\phi(E_v^{-n}(y)) - S_n\phi(E_v^{-n}(x))| \leq L_\alpha v_\alpha(\phi)|y-x|^\alpha$$

for all $n \geq 1$, all $x, y \in J(E)$ with $|x-y| \leq \delta$ and all $v \in E^{-n}(x)$.

Put $Q_+^c = \{z \in Q : \mathrm{Re} z \geq \inf\{\mathrm{Re}(J(E))\}\}$. Given $\kappa \in \mathbb{R}$ a Hölder continuous function $\phi : Q_+^c \to \mathbb{R}$ is called κ-tame (and we put $\kappa = \kappa(\phi)$) if

$$A_\phi = \sup\{|\phi(z) + \kappa \mathrm{Re} z| : z \in Q_+^c\} < +\infty.$$

A Hölder continuous function $\phi : Q_+^c \to \mathbb{R}$ is called κ^+-tame if it is a-tame with some $a > \kappa$. Especially important will turn out to be the 1^+-tame functions. Any κ-tame function with any $\kappa \in \mathbb{R}$ is called a tame function. The above introduced tameness notions refer to any Hölder continuous complex-valued function if its real part satisfies the respective conditions. Note that iff $\phi : J(f) \to \mathbb{C}$ is a 0^+-tame or 0-tame function, then e^ϕ is a bounded Hölder continuous function with the same exponent.

The topological pressure of the tame potentials is defined by the pointwise method. It can be proved that for $\phi : J(E) \to \mathbb{R}$, a 0^+-tame potential, the following limit

$$P_z(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in E^{-n}(z)} \exp(S_n\phi(x))$$

exists and is independent of $z \in J(E)$. Its common value is denoted by $\mathrm{P}(\phi)$ and is called the topological pressure of the potential ϕ. Assume now ϕ to be a 1^+-tame potential. The Perron-Frobenius operator $\mathcal{L}_\phi : C_b \to C_b$ is defined by the formula

$$\mathcal{L}_\phi g(z) = \sum_{x \in E^{-1}(z)} \exp(\phi(x))g(x) = \sum_{k=-\infty}^{+\infty} \exp(\phi(z_k))g(z_k), \tag{2.8}$$

where z_k is the only point of the singleton $\pi\Big(\mathcal{E}^{-1}(\tilde{z}+2\pi ik)\Big)$ and $\tilde{z}$ is an arbitrary point from $\pi^{-1}(z)$ and Q. The dual operator $\mathcal{L}_\phi^*: C_b^* \to C_b^*$ is given by the formula $\mathcal{L}_\phi^*\mu(g) = \mu(\mathcal{L}_\phi g)$. Applying the $K(V)$ method from [17] (see Appendix 1) and the tightness argument, one can show that there exists m_ϕ, a unique Borel probability measure on $J(E)$ that is an eigenmeasure of $\mathcal{L}_\phi^*$. The corresponding eigenvalue is equal to $e^{\mathrm{P}(\phi)}$. This equivalently means that

$$m_\phi(E(A)) = \int_A \exp(\mathrm{P}(\phi) - \phi)dm_\phi$$

for every Borel set $A \subset Q$ restricted to which E is injective. The measure m_ϕ is called ϕ-conformal. Applying the famous Tulcea-Ionescu and Marinescu Theorem, using Lemma 2.17 and (2.8), one eventually ends up with the following properties of the Perron-Frobenius operator $\mathcal{L}_\phi$.

Theorem 2.18. *If $\phi : J(E) \to (0,\infty)$ is a 1^+-tame potential, then we have the following.*

(a) *The number 1 is a simple isolated eigenvalue of the operator $e^{-\mathrm{P}(\phi)}\hat{\mathcal{L}}_\phi : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$.*
(b) *The eigenspace of the eigenvalue 1 is generated by a nowhere vanishing function $g_\phi \in \mathrm{H}_\alpha$ such that $\int g_\phi dm_\phi = 1$ and $\lim_{\mathrm{Re}z\to+\infty} g_\phi(z) = 0$.*
(c) *The number 1 is the only eigenvalue of modulus 1.*
(d) *With $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ as in Theorem 2.6, we have*
$$e^{-\mathrm{P}(\phi)}\hat{\mathcal{L}}_\phi = Q_1 + S,$$
where $Q_1 : \mathrm{H}_\alpha \to \mathbb{C}g_\phi$ is a projector on the eigenspace $\mathbb{C}g_\phi$, $Q_1 \circ S = S \circ Q_1 = 0$ and
$$||S^n||_\alpha \le C\xi^n$$
for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \ge 1$.

2.3.2. **Ergodic theory of invariant measures.** As a fairly straightforward consequence of of Theorem 2.18, one links conformal measures with ergodic theory. More precisely:

Theorem 2.19. *If $\phi : J(E) \to (0,\infty)$ is a 1^+-tame potential, then the measure $\mu = \mu_\phi = g_\phi m_\phi$ is E-invariant, ergodic with respect to each iterate of E and equivalent to the measure m_ϕ. In particular $\mu(J_r(E)) = 1$. In addition the dynamical system (E,μ_ϕ) is metrically exact i.e., its Rokhlin natural extension is a K-system.*

The measure μ_ϕ is called the invariant Gibbs state of the potential ϕ. Due to Theorem 2.18 the E-invariant measure μ_ϕ has additional stochastic properties than listed in the theorem above. Namely, the correlations of Hölder continuous bounded potentials decay (with respect to the dynamical system (E,μ_ϕ) exponentially fast, and for every $g \in \mathrm{H}_\alpha$, the sequence $\{g \circ E^n\}_{n=1}^\infty$ of identically distributed random variables (with respect to the measure μ_ϕ) converges in distribution to a Gauss distribution provided that the asymptotic variance (dispersion) of g

is positive. As it was proved in [78] (and the proof is repeated in [72]), the Gibbs state μ_ϕ is an equilibrium state for the potential ϕ. As a matter of fact, the following, nearly classical version of the Variation Principle holds.

Theorem 2.20. *If $\mathcal{E} : \mathbb{C} \to \mathbb{C}$ is hyperbolic and $\phi : J(E) \to \mathbb{C}$ is a 1^+-tame potential, then the invariant measure μ_ϕ is an equilibrium state of the potential ϕ, that is*

$$\mathrm{P}(\phi) = \sup\{h_\mu(E) + \int \phi d\mu\},$$

where the supremum is taken over all Borel probability E-invariant ergodic measures μ with $\int \phi d\mu > -\infty$, and

$$\mathrm{P}(\phi) = h_{\mu_\phi} + \int \phi d\mu_\phi.$$

2.3.3. **Analytic properties of the pressure function.** The sections 7 and 8 of [72] contain a detailed analysis of differentiability properties of the topological pressure function. Let $\phi, \psi : J(E) \to \mathbb{R}$ be arbitrary two tame functions. Consider the sets

$$\Sigma_1(\phi,\psi) := \{q \in \mathbb{C} : \mathrm{Re}(q)\kappa(\phi) + \kappa(\psi) > 1\}$$

and

$$\Sigma_2(\phi,\psi) := \{(q,t) \in \mathbb{C} \times \mathbb{C} : \mathrm{Re}(q)\kappa + \mathrm{Re}(t)\gamma > 1\}.$$

The key ingredient to all further analytic properties of various 'thermodynamical objects' is the following.

Proposition 2.21. *If $\phi, \psi : J(E) \to \mathbb{R}$ are arbitrary two tame functions, then the function $q \mapsto \mathcal{L}_{q\phi+\psi} \in L(H_\alpha)$, $q \in \Sigma_1(\phi,\psi)$, is holomorphic.*

Using the perturbation theory for linear operators (Kato-Rellich Theorem, see Theorem XII.8 in [58]), as fairly straightforward consequence of Proposition 2.21, we get the following.

Lemma 2.22. *If ϕ and ψ are arbitrary tame functions, then the function $q \mapsto \mathrm{P}(q\phi+\psi)$, $q \in \Sigma_1(\phi,\psi) \cap \mathbb{R}$, is real-analytic, and likewise, the function $(q,t) \mapsto \mathrm{P}(q\phi + t\psi)$, $(q,t) \in \Sigma_2(\phi,\psi)$, is real-analytic.*

In the classical theory of distance expanding maps (see [57]) the first and the second derivative of the pressure function are calculable although the calculations leading to the formula for the second derivative are rather tedious. Even more tedious calculations, performed in Section 8 of [72] resulted in the following.

Theorem 2.23. *Suppose that $\phi : J(E) \to I\!R$ is a 1^+-tame function and $\psi : J(E) \to I\!R$ is a tame function. Then*

$$\frac{d}{dt}\Big|_{t=0}\mathrm{P}(\phi + t\psi) = \int \psi d\mu_\phi.$$

and

Theorem 2.24. *Suppose that $\phi : J(E) \to I\!R$ is a 1^+-tame function and $\psi, \zeta : J(E) \to I\!R$ are tame functions. Then*

$$\frac{\partial^2}{\partial s \partial t}\Big|_{(0,0)}\mathrm{P}(\phi + s\psi + t\zeta) = \sigma^2(\psi, \zeta),$$

where

$$\begin{aligned}\sigma^2(\psi,\zeta) &= \lim_{n\to\infty}\frac{1}{n}\int S_n\Big(\psi - \mu_\phi(\psi)\Big)S_n\Big(\zeta - \mu_\phi(\zeta)\Big)d\mu_\phi \\ &= \int(\psi - \mu_\phi(\psi))(\zeta - \mu_\phi(\zeta))d\mu_\phi + \sum_{k=1}^\infty \int(\psi - \mu_\phi(\psi))(\zeta - \mu_\phi(\zeta))\circ E^k d\mu_\phi + \\ &\qquad \sum_{k=1}^\infty \int(\zeta - \mu_\phi(\zeta))(\psi - \mu_\phi(\psi))\circ E^k d\mu_\phi\end{aligned}$$

(if $\psi = \zeta$ we simply write $\sigma^2(\psi)$ for $\sigma^2(\psi,\psi)$)

2.3.4. **Cohomologies.** Of course the natural question arises of when the two 1^+-tame potentials have the same Gibbs (equilibrium) states. The problem is completely solved in Theorem 9.1 from [72], and although the solution is the same as in the classical case, some of its constituents, especially item (3) are rather unexpected. Let $\mathcal{F}$ be any class of real-valued functions defined on $J(E)$. Two functions $\phi, \psi : J(E) \to I\!R$ are said to be cohomologous in the class of function $\mathcal{F}$ if there exists a function $u \in \mathcal{F}$ such that

$$\phi - \psi = u - u \circ E.$$

Our solution is this.

Theorem 2.25. *If $\phi, \psi : J(E) \to I\!R$ are two arbitrary 1^+-tame functions, then the following conditions are equivalent:*

(1) $\mu_\phi = \mu_\psi$.
(2) *There exists a constant R such that for each $n \geq 1$, if $E^n(z) = z$ ($z \in J(E)$), then*

$$S_n\phi(z) - S_n\psi(z) = nR.$$

(3) *The difference $\psi - \phi$ is cohomologous to a constant R in the class of bounded Hölder continuous functions.*
(4) *The difference $\psi - \phi$ is cohomologous to a constant in the class of bounded continuous functions.*

(5) *The difference $\psi - \phi$ is cohomologous to a constant in the class of all functions defined everywhere in $J(E)$.*
(6) *There exist constants S and T such that for every $z \in J(E)$ and every $n \geq 1$*

$$|S_n\phi(z) - S_n\psi(z)| \leq T.$$

If these conditions are satisfied, then $R = S = \mathrm{P}(\phi) - \mathrm{P}(\psi)$.

As its complement we list the following result whose proof uses Rokhlin's natural extension and the concept of canonical conditional measures.

Proposition 2.26. *If $\phi : Q_+^c \to I\!R$ is a 1^+-tame function and $\psi : Q_+^c \to I\!R$ is a tame function, then $\sigma^2_{\mu_\phi}(\psi) = 0$ if and only if ψ is cohomologous to a constant function in the class of bounded Hölder continuous functions on $J(E)$.*

2.3.5. **Hausdorff dimension of Gibbs states.**
Given a Borel probability measure μ on a metric space, we define for a point x in this space, the number (called the local dimension at the point x)

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

provided that the limit exists. Recall that $\mathrm{HD}(\mu)$, the Hausdorff dimension of the measures μ is the infimum of the numbers $\mathrm{HD}(Y)$ taken over all Borel sets Y such that $\mu(Y) = 1$. If $d_\mu(x)$ is constant a.e. (denote it by d_μ), then $\mathrm{HD}(\mu) = d_\mu$. If μ is a Borel probability E-invariant measure on $J(E)$, then the number

$$\chi_\mu = \int \log |E'| d\mu$$

is called the Lyapunov exponent of the map E with respect to the measure μ. Note that this exponent is always positive. The proof of the following only result stated in this section appeared first time [71] and was repeated in [72].

Theorem 2.27. *If $\mathcal{E} : \mathbb{C} \to \mathbb{C}$ is a hyperbolic exponential map and $\phi : J(E) \to \mathbb{C}$ is a 1^+-tame potential, then the local dimension $d_\mu(x)$ exists for μ_ϕ-a.e. $z \in J(E)$, and is equal to $h_{\mu_\phi}/\chi_{\mu_\phi}$. In particular*

$$\mathrm{HD}(\mu_\phi) = \frac{h_{\mu_\phi}}{\chi_{\mu_\phi}}.$$

2.3.6. **Multifractal analysis of 1^+-tame functions.** Fix a 1^+-tame potential $\phi : Q_+^c \to I\!R$. Subtracting $\mathrm{P}(\phi)$ from ϕ, we can assume without loss of generality that $\mathrm{P}(\phi) = 0$ and call ϕ normalised. Consider the two-parameter family of potentials $\phi_{q,t} : P_+ \to I\!R$, $q, t \in I\!R$, defined as follows.

$$\phi_{q,t} = -t \log |E'| + q\phi.$$

Note that $\phi_{q,t}$ is a $t+q\kappa(\phi)$-tame function. We have the following.

Lemma 2.28. *For every $q \in I\!R$ there exists a unique $t = T(q) \in I\!R$ such that $\mathrm{P}(\phi_{q,t}) = 0$. In addition, $(q, T(q)) \in \Sigma_2(\phi, -\log|E'|)$.*

The mapping $T : I\!R \to I\!R$ is called the temperature function. The multifractal analysis of a Borel probability measure μ scrutinise the level sets of the pointwise dimension function d_μ. Although defined in an entirely "fractal manner", this function exhibits frequently very regular features. This is also the case in our situation, and we will discuss them now in the context of Gibbs (equilibrium) states μ_ϕ. For technical reasons our analysis will be performed on a subset $J_{rr}(E)$ of $J_r(E)$, which will also turn out to be fairly large, and whose, somewhat technical definition, we provide now. Given an integer $s > \inf\Big(\mathrm{Re}J(E)\Big)$ and a point $z \in J(E)$ let $\{t_s^n(z)\}_{n=1}^\infty$ denote the sequence of consecutive visits of the point z to $Q_s = \{z \in J(E) : \mathrm{Re}z \le s\}$ under the action of E, i.e. this sequence is strictly increasing (perhaps finite, perhaps empty), $E^{t_s^n(z)}(z) \in Q_q$ for all $n \ge 1$ and $E^j(z) \notin Q_s$ for all $t_s^n(z) < j < t_s^{n+1}(z)$. Now, define the sets

$$M_s = \left\{ z \in J_r(E) : \lim_{n\to\infty} \frac{\log\left|\left(E^{t_s^{n+1}(z)-t_s^n(z)}\right)'\left(E^{t_s^n(z)}(z)\right)\right|}{|\log\left|\left(E^{t_s^n(z)}\right)'(z)\right|} = 0 \ \text{ and } \ \lim_{n\to\infty}\frac{t_s^{n+1}(z)}{t_s^n(z)} = 1 \right\}$$

and

$$J_{rr}(E) = \bigcup_s M_s,$$

where the union is taken over all integers $s > \inf\Big(\mathrm{Re}J(E)\Big)$. The robustness of the set $J_{rr}(E)$ is reflected in the fact that $\mu(J_{rr}(E)) = 1$ for every Borel probability E-invariant ergodic measure on $J(E)$ with finite Lyapunov exponent, in particular if $\mu = \mu_\phi$, where ϕ is a 1^+-potential).

Given a 1^+-tame function $\phi : Q_+^c \to I\!R$, and a real number $\alpha \ge 0$, we define the following two sets.

$$\mathcal{K}_\phi(\alpha) = \left\{ z \in J_r(E) : \lim_{n\to\infty} \frac{\mathrm{P}(\phi)n - S_n\phi(z)}{\log|(E^n)'(z)|} = \alpha \right\}$$

and

$$D_\phi(\alpha) = \left\{ z \in J_{rr}(E) : \lim_{r\to 0} \frac{\log\mu_\phi(B(z,r))}{\log r} = \alpha \right\}.$$

The firs relation between the sets $\mathcal{K}_\phi(\alpha)$ and $D_\phi(\alpha)$ is that

$$D_\phi(\alpha) = \mathcal{K}_\phi(\alpha) \cap J_{rr}(E). \tag{2.9}$$

We set

$$k_{\mu_\phi}^+ = \mathrm{HD}(\mathcal{K}_\phi(\alpha)) \ \text{ and } \ k_{\mu_\phi}^- = \mathrm{HD}(D_\phi(\alpha)).$$

By (2.9), we have
$$k^-_{\mu_\phi} \le k^+_{\mu_\phi}.$$
The first result of this section establishes, as the most transparent feature, real analyticity of the functions $k^-_{\mu_\phi}$ and $k^+_{\mu_\phi}$. As a byproduct a closer relations between the sets $\mathcal{K}_\phi(\alpha)$ and $D_\phi(\alpha)$ is obtained.

Theorem 2.29. *Suppose that $\phi : Q_+^c \to I\!R$ is a 1^+-tame function. Then the functions $k^-_{\mu_\phi}$ and $k^+_{\mu_\phi}$ coincide in their natural domain (α_1, α_2) produced in item (c). Denote their common value by k_{μ_ϕ}. Then the following statements are true.*

(a) *The pointwise dimension $d_{\mu_\phi}(x)$ exists for μ_ϕ-almost every $x \in J(E)$ and*
$$d_{\mu_\phi}(x) = \frac{\mathrm{P}(\phi) - \int \phi d\mu_\phi}{\int \log|E'| d\mu_\phi}.$$
(b) *The function $q \mapsto T(q)$, $q \in I\!R$ is real-analytic, $T(0) = \mathrm{HD}(J_r(E))$, $T'(q) < 0$ and $T''(q) \ge 0$. In addition T'' vanishes in one point if and only if it vanishes at all points, if and only if $\mu_\phi = \mu_{-h\log|E'|}$, if and only if ϕ and $-h\log|E'|$ are cohomologous modulo constant in the class of all (bounded) Hölder continuous functions.*
(c) *The function $-T'(q)$ attains values in an interval (α_1, α_2) where $0 \le \alpha_1 \le \alpha_2 < \infty$.*
(d) *For every $q \in I\!R$, $k_{\mu_\phi}(-T'(q)) = T(q) - qT'(q)$.*
(f) *The function k_{μ_ϕ} is real-analytic throughout its whole domain (α_1, α_2).*
(f) *If $\mu_\phi \ne \mu_{-h\log|E'|}$, then the functions $k_{\mu_\phi}(\alpha)$ and $T(q)$ form a Legendre transform pair.*

This theorem has been established assuming only that ϕ is Hölder continuous (and 1^+-tame). Assuming more, that ϕ is harmonic on a half-cylinder containing the Julia set $J(E)$, and performing a more involved analysis of analytic properties of the relevant Perron-Frobenius operators, we were able to show that the function k_{μ_ϕ} also depends in a real-analytic fashion on the parameter λ. This can be regarded as an extension of the real analyticity of the Hausdorff dimension of the Julia set $J_r(E)$ discussed in Section 2.2.4. Let us now explain in greater detail what we mean when saying that k_{μ_ϕ} depends real-analytically on λ. Fix $a \in I\!R$ and set
$$\mathrm{Hyp}(a) = \left\{\lambda \in \mathrm{Hyp} : \mathrm{Re}\Big(\inf\{J(\mathcal{E}_\lambda)\}\Big) > a\right\}.$$
Clearly $\mathrm{Hyp}(a)$ is an open set. Put
$$U_{a,\phi} := \bigcup_{\lambda \in \mathrm{Hyp}(a)} \{\lambda\} \times (\alpha_1(\lambda), \alpha_2(\lambda)),$$
where $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ are the numbers coming from items (c) and (f) of Theorem 2.29. The set $U_{a,\phi}$ is open and vertically connected, the latter meaning that for every $\lambda \in \mathrm{Hyp}(a)$, the set $(\{\lambda\} \times I\!R) \cap U_{a,\phi}$ is connected. Given $\lambda \in \mathrm{Hyp}(a)$, let μ_ϕ^λ be the Gibbs state corresponding to

the potential ϕ and the dynamical system $E_\lambda : J(E_\lambda) \to J(E_\lambda)$. The function $k_\phi : U_{a,\phi} \to [0,2]$ is defined by the formula

$$k_\phi(\lambda, \alpha) = k_{\mu_\phi^\lambda}(\alpha).$$

The second main theorem of this section is the following.

Theorem 2.30. *If $\phi : Q_a^c \to I\!R$ is a harmonic 1^+-tame potential, then the function $k_\phi : U_{a,\phi} \to I\!R$ is real-analytic.*

In view of item (b) of Theorem 2.29 the function k_ϕ is non-degenerate provided that the function ϕ is not cohomologous to $-\mathrm{HD}(J_r(E_\lambda)) \log |E_\lambda'|$ for any $\lambda \in \mathrm{Hyp}(a)$ in the class of all (bounded) Hölder continuous functions. The potential ϕ is then called essential. The last result proved in [72] provides an easy sufficient condition for a harmonic 1^+-tame potential to be essential. One requires that $\kappa(\phi) \geq 2$.

2.4. **$\mathcal{E}_{1/e}$ and Parabolic Implosion.** In this section we describe the results obtained in [76] and [79]. We first, summarising [76], deal with the map $\mathcal{E}_{1/e} : \mathbb{C} \to \mathbb{C}$ alone, and then, describing [79], we look what happens if λ converges to $1/e$ from the left and from the right along the real axis. It turns out that in the former case the Hausdorff dimension of the radial Julia set $J_r(E_\lambda)$ behaves continuously at the point $1/e$, and in the latter case it behaves 'highly' discontinuously at the parameter $\lambda = 1/e$.

2.4.1. **The map $\mathcal{E}_{1/e}$ alone.** We again work with the map $E_{1/e} : Q \to Q$. The set $J_r(E_{1/e})$ must be however now slightly modified. First notice that $E_{1/e}(1) = 1, E_{1/e}'(1) = 1$ and $E_{1/e}''(1) = 1 \neq 0$ so that 1 is a parabolic fixed point with one petal. Since the Julia set near the fixed point 1 lies entirely in the repelling petal, there exists $\theta > 0$ such that if $z \in J(E_{1/e}) \cap B(1,\theta) \setminus \{1\}$ then $E_{1/e}^n(z) \notin B(1,\theta)$ for some $n \geq 1$. Since in addition $E_{1/e}^n(0)$ converges to 1 through the attracting petal at 1, we see that there exists $\delta > 0$ such that

$$\bigcup_{z \in J(E_{1/e}) \setminus B(1,\theta)} B(z, 4\delta) \cap \{E_{1/e}^n(0)\}_{n=0}^\infty = \emptyset. \tag{2.10}$$

Now, almost as in the case of hyperbolic parameters, a point $z \in J_r(E_{1/e})$ is said to belong to $J_r(E)$ if and only if there exist $y \in J_r(E_{1/e}) \setminus B(1,\theta)$ (note here the difference with the hyperbolic case) and an unbounded increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y = \lim_{k\to\infty} E_{1/e}^{n_k}(z)$ and $E_{1/e}^{n_k}(z) \in B(y, 4\delta)$ for all $k \geq 1$. Note that because of (2.10), for every $k \geq 1$ there exists a unique holomorphic branch $E_{1/e}^{-n_k} : B(y, 2\delta) \to Q$ of $E_{1/e}^{-n_k}$ sending $E_{1/e}^{n_k}(z)$ to z. Note that also $J_r(E_{1/e})$ coincides with the radial set defined exactly as in the hyperbolic case after removing the inverse image of 1, i.e. the set $\bigcup_{n\geq 0} E_{1/e}^{-n_k}(1)$. Given $t \geq 0$, a t-conformal measure for $E_{1/e} : J_r(E_{1/e}) \to J_r(E_{1/e})$ is defined in the same way as in the hyperbolic case (see beginning of Section 2.2.3) with h replaced by t in (2.7). Let $h = \mathrm{HD}(J_r(E_{1/e}))$. Some properties of t-conformal measures are described by the following theorem.

Theorem 2.31. *The following hold.*

(1) *h is the unique t for which an atomless conformal measure exists.*
(2) *There exists a unique h-conformal measure m for $E_{1/e} : J(E_{1/e}) \to J(E_{1/e})$. The measure m is atomless.*
(3) *The h-conformal measure m is ergodic and conservative.*
(4) *If ν is a t-conformal measure for $E_{1/e}$ and $t > 1$, $t \neq h$, then $t > h$ and $\nu\Big(\bigcup_{n\geq 0} E_{1/e}^{-n}(1)\Big) = 1$.*

The geometric, Hausdorff, and packing measures have features analogous to the hyperbolic case (although the proofs about Hausdorff measures are even more involved).

Theorem 2.32. $0 < \mathrm{H}^h(J_r(E_{1/e})) < \infty$.

Proposition 2.33. *We have $\mathrm{P}^h(J_r(E_{1/e})) = \infty$. In fact $\mathrm{P}^h(G) = \infty$ for every open nonempty subset of $J_r(E_{1/e})$.*

Based on this proposition and Proposition 2.1, the same arguments as in the hyperbolic case give the following.

Corollary 2.34. $1 < \mathrm{HD}(J_r(E_{1/e})) < 2$.

2.4.2. **Left-hand side continuity.** In this section we present the idea and single out the basic steps of the proof of the following.

Theorem 2.35. $\lim_{\lambda\nearrow 1/e} \mathrm{HD}(J_r(E_\lambda)) = \mathrm{HD}(J_r(E_{1/e}))$.

To start the short discussion of the proof take an arbitrary sequence $\{\lambda_n\}_{n=1}^\infty$ converging to $1/e$ from the left. Let $h_n = \mathrm{HD}(J_r(E_{\lambda_n}))$ and let m_n be the h_n conformal measure for E_{λ_n}. The first thing is to prove that the sequence $\{m_n\}_{n=1}^\infty$ is tight. Let u be an arbitrary accumulation point of the sequence $\{h_n\}_{n=1}^\infty$. We know that $u \in [1, 2]$. Passing to a subsequence we may assume without loss of generality that $u = \lim_{n\to\infty} h_n$. Passing to yet another subsequence we may assume (due to tightness of $\{m_n\}_{n=1}^\infty$) that the sequence $\{m_n\}_{n=1}^\infty$ converges weakly to a measure m. It is not too difficult to show that m is u-conformal and $m(J_r(E_{1/e})) = 1$. In fact, inspecting carefully the proof of Proposition 2.1, we see that there exists $s > 1$ such that

$\mathrm{HD}(J_r(E_\lambda)) \geq s$ for all $\lambda \in (0, 1/e)$ sufficiently close to $1/e$. Consequently $1 < s \leq u \leq 2$. It now follows from Theorem 2.31 that $u \geq \mathrm{HD}(J_r(E_{1/e}))$. It also follows from this theorem that in order to conclude the proof, that is to show that $u \leq \mathrm{HD}(J_r(E_{1/e}))$, it suffices to demonstrate that the u-conformal measure m is atomless. Conjugating the map E_λ by affine transformations so that their attracting fixed point becomes 0 (so, independent of parameter λ) one carefully checks that the measures m_n (actually their images under conjugacies) do not charge too much arbitrarily small neighbourhoods of zero.

2.4.3. **Parabolic implosion for $\mathbf{E_{1/e}}$.** The question we want to discuss in this short section is what happens if $\lambda \searrow 1/e$ along the real axis. Since the Julia set $J(\mathcal{E}_{1/e})$ is a nowhere dense Cantor bouquet and since $J(\mathcal{E}_\lambda) = \mathbb{C}$ for all $\lambda > 1$ (this follows from the fact that the trajectory of 0, $\{\mathcal{E}_\lambda^n(0)\}_{n=0}^\infty$, escapes to infinity (see [6]), we have obvious discontinuity of the Julia set of $\mathcal{E}_\lambda$ if $\lambda \searrow 1/e$ but no discontinuity of Hausdorff dimension, all these sets are known (see[53]) to have the Hausdorff dimension 2. In order to observe more interesting phenomena on the level of the Hausdorff dimensions, one considers the dynamically more significant sets $J_r(E_\lambda)$. So far, we have defined these sets only in hyperbolic and parabolic cases. Now, the time comes to do it in full generality. So, we bring down here Definition 3.7 from [77].

Definition 2.36. *$J_r(\mathcal{E}) \subset \mathbb{C}$ is the set of those points $z \in \mathbb{C}$ for which there exists an unbounded sequence $\{n_k(z)\}_{k=1}^\infty$ such that*

$$\mathrm{dist}\Big(\{\mathcal{E}^{n_k(z)}(z)\}_{k=1}^\infty, \{\mathcal{E}_\lambda^n(0)\}_{n=0}^\infty\Big) > 0$$

and the set $\mathrm{Re}\Big(\{\mathcal{E}^{n_k(z)}(z)\}_{k=1}^\infty\Big)$ is bounded. The set $J_r = J_r(E) \subset Q$ is defined to be $\Pi(J_r(\mathcal{E}))$.

Note that if the trajectory of 0, $\{\mathcal{E}_\lambda^n(0)\}_{n=0}^\infty$ is dense, then $J_r(E) = \emptyset$. Note also that, as in hyperbolic and parabolic case, all points in $J_r(E)$ allow univalent holomorphic pull backs from a bounded region. Precisely, it means that if $z \in J_r(E)$ then there exist $y \in J_r(E)$, $\delta(z) > 0$ and unbounded increasing sequence $\{s_k\}_{k=1}^\infty$ (which could be extracted from the sequence $\{n_k(\tilde{z})\}_{k=1}^\infty$ guaranteed by Definition 2.36 with some $\tilde{z} \in \Pi^{-1}(z)$) such that

$$\lim_{k\to\infty} E^{s_k}(z) = y, \quad |E^{s_k}(z) - y| < \frac{1}{4}\delta(z), \quad k = 1, 2, \dots,$$

and for every $k \geq 1$ there exists a unique holomorphic branch $E_z^{-s_k} : B(E^{s_k}(z), 2\delta(z)) \to Q$ of E^{-s_k} sending $E^{s_k}(z)$ to z. Finally note that the radial Julia set defined in Definition 2.36 coincides in the hyperbolic and parabolic case with the respective sets defined in Section 2.2 and 2.2.3. The ultimate results, which will be called global dimensionwise parabolic implosion phenomena, obtained in [79] are these.

Theorem 2.37. *For every $\sigma \in \mathbb{R}$ there exists a sequence $\{\epsilon_k\}_{k=1}^\infty$ of positive reals converging to 0 such that $(-\pi/\sqrt{\epsilon_k}) \to \sigma(mod\ 1)$ as $k \to \infty$, and*

$$\liminf_{k\to\infty} \mathrm{HD}(J_r(E_{1/e+\epsilon_k})) > \mathrm{HD}(J_r(E_{1/e})).$$

and its consequence:

Corollary 2.38. *We have that*

$$\limsup_{\lambda\searrow 1/e} \mathrm{HD}(J_r(E_\lambda)) > \mathrm{HD}(J_r(E_{1/e})).$$

The structure of proof of Theorem 2.37 is this. First, it was proved in [77], (compare Section 2.5) that for every $\lambda > 1/e$ there exists a unique $\mathrm{HD}(J_r(E_\lambda))$-conformal measure m_λ for the map E_λ. Let u be the value of the lower limit appearing in Theorem 2.37. Using the concept, characteristic for any parabolic implosion, of Lavaurs maps, one shows as the first step that the sequence $\{m_{\lambda_k}\}_{k=1}^\infty$, $\lambda_k = \frac{1}{e} + \epsilon_k$, is tight, and any of its weak-limits is a measure, u-conformal for both $E_{1/e} : \mathbb{C} \to \mathbb{C}$ and $g_\sigma : \mathbb{C} \setminus J(E_{1/e}) \to \mathbb{C}$ where g_σ is the Lavaurs map corresponding to the parameter σ. The second step is to prove that the series

$$\sum_{n=1}^\infty \sum_{x\in E_{1/e}^{-n}(1)\setminus\bigcup_{k=0}^{n-1} E_{1/e}^{-k}(1)} |(E_{1/e}^n)'(x)|^{-u}$$

converges. Since, by Theorem 2.31(2), the same series, with u replaced by $\mathrm{HD}(J_r(E_{1/e}))$ diverges (otherwise, we would get apparently atomic $\mathrm{HD}(J_r(E_{1/e}))$-conformal measure supported on $\bigcup_{n\ge 0} E_{1/e}^{-n}(1) \subset J(E_{1/e})$), we conclude that $u > \mathrm{HD}(J_r(E_{1/e}))$. So, we are done.

2.5. **Non-Hyperbolic Exponential Maps.** In [77] a class of exponential maps was considered whose members all had Julia sets equal to the whole complex plane. In particular, 0 belongs then to the Julia set and the maps are not hyperbolic (see beginning of Section 2.2). The lack of hyperbolicity was to some extent compensated by extremely fast convergence to ∞ of forward iterates of zero. Namely, a parameter $\lambda \in \mathbb{C} \setminus \{0\}$ is called a super-growing parameter if $\lim_{n\to\infty} \mathrm{Re}(\mathcal{E}^n(0)) = +\infty$, and there exists $c > 0$ such that for all $n \ge 1$,

$$|\mathrm{Re}\mathcal{E}_\lambda^{n+1}(0)| \ge c\exp(\mathrm{Re}(\mathcal{E}_\lambda^n(0)) = \frac{c}{|\lambda|}|\mathcal{E}_\lambda^{n+1}(0)|.$$

Now fix an arbitrary super-growing parameter λ. Put $\mathcal{E} = \mathcal{E}_\lambda$ and $E = E_\lambda$. We have observed at the beginning of Subsection 2.2.3 that $J(E) = \mathbb{C}$. The radial Julia set $J_r(E)$ defined in Definition 2.36 turns out to be as 'nice' as in the hyperbolic and parabolic case. Indeed, although the Perron-Frobenius operator method does not seem to be naturally applicable for a map whose Julia set contains 0, using $K(V)$-methods (see Appendix 1) and the tightness of the sequence of semi-conformal measures this method produces, it was possible to prove the existence and uniqueness of an h-conformal measure m for the map $E : Q \to Q$, where

$h = \mathrm{HD}(J_r(E))$. In addition this measure was proved to be ergodic, conservative (meaning that $\sum_{n\geq 0} 1\!\!1_F \circ E^n(z) = \infty$ for m-a.e. $z \in \mathbb{C}$) and $m(J_r(E)) = 1$. As in the hyperbolic case the conformal measure turns out to be a normalised version of the h-dimensional Hausdorff measure H^h, and in fact we have this.

Theorem 2.39. $0 < \mathrm{H}^h(J_r(E)) < \infty$.

Proposition 2.40. *We have* $\mathrm{P}^h(J_r(E)) = \infty$. *In fact* $\mathrm{P}^h(G) = \infty$ *for every open nonempty subset of* $J_r(E)$.

This proposition in the same way as in Section 2.4.3 implies that $h < 2$. Since $J_r(E)$ contains $J_{bd}(E)$, using Proposition 2.1, we have therefore the following.

Theorem 2.41. $1 < \mathrm{HD}(J_r(E)) < 2$.

This theorem has the following rather unexpected corollary, proved for the first time in [46] by a different method.

Corollary 2.42. *If* λ *is a super-growing parameter then for Lebesgue almost every point* $z \in \mathbb{C}$, $\omega(z) = \{\mathcal{E}_\lambda^n(0)\}_{n=0}^\infty \cup \{\infty\}$.

We show how to prove Corollary 2.42. Let $\mathcal{E} = \mathcal{E}_\lambda$. Since $\mathrm{HD}(J_r(\mathcal{E})) < 2$, the complement of $J_r(\mathcal{E})$ is a set of full measure. Fix a point $z \notin J_r(\mathcal{E})$. By the definition of $J_r(\mathcal{E})$, this implies that $\omega(z) \subset \{\mathcal{E}_\lambda^n(0)\}_{n=0}^\infty \cup \{\infty\}$. We only have to check that, actually, the equality holds for almost every point. So, assume that $\omega(z) = \infty$. The set of such points has Lebesgue measure 0; actually, this is true for a large class of maps, see e.g. [53] or [26]. Next, assume that $\omega(z) = \{\infty\} \cup \{\mathcal{E}_\lambda^n(0)\}_{n=k}^\infty$ for some $k > 0$. Then, there exists an infinite sequence of integers s_i such that $\mathcal{E}^{s_i}(z) \to \mathcal{E}_\lambda^k(0)$. Then, denoting $n_i = s_i - 1$, we see that $\mathrm{Re}\mathcal{E}^{n_i}(z) \to \mathrm{Re}\mathcal{E}_\lambda^k(0)$ and, moreover, $\mathrm{dist}(\mathcal{E}^{n_i}(z), \{\mathcal{E}_\lambda^n(0)\}_{n=k}^\infty) > 0$. Consequently, $z \in J_r(\mathcal{E})$, a contradiction.

By the tightness argument the following fact was also established in [77].

Theorem 2.43. *The function* $\lambda \to \mathrm{HD}(J_r(E_\lambda)), \lambda \in (1/e, +\infty)$ *is continuous.*

An open problem is whether this function is real-analytic.

Leaving geometric measures and dimensions, let us have a closer look at the dynamical properties of the h-conformal measure m for the (super-growing) map $E : \mathbb{C} \to \mathbb{C}$. Ergodicity and conservativity of the measure m along with the fact that the orbit of zero escapes to ∞, make it possible to apply M. Martens' method (see Appendix 2) to construct (up to the multiplicative constant) a σ-finite E-invariant Borel measure μ equivalent to m. The method of M. Martens leaves open a natural procedure of checking whether the measure is finite or infinite. One must carefully control the distortion when going down to singularities, the orbit of zero, $+\infty$ and $-\infty$. By fairly technical and rather complicated arguments this was done in Theorem 4.6 in [77]. The result is this.

Theorem 2.44. *There exists a unique Borel probability E-invariant measure μ absolutely continuous with respect to the* $\mathrm{HD}(J_r(E))$*-conformal measure m. In addition μ is ergodic and equivalent to m.*

2.6. **Fatou Functions, Sine and Cosine Families and Further.** In the paper [44] the family $f_\lambda(z) = e^{-z} + z + \lambda, \mathrm{Re}\lambda \geq 1$, of Fatou's functions was investigated in great detail. Although all of them have a Baker domain at ∞, this turned out not to preclude the possibility of analysing those maps from geometrical (fractal) and (measure-theoretical) dynamical point of view. The paper [44] provides a uniform treatment of issues dealt with in [74] and [75] in the setting of the technically more complicated Fatou's functions f_λ. The results here are the same as those discussed in Section 2.2. Also the family of functions $f_\lambda(z) = \lambda(1 - e^{2z})^{-1}$ consisting of transcendental meromorphic functions was studied in detail with its own methods in [43]. The appropriate results about conformal, Hausdorff, packing and invariant measures have been obtained there.

Also I. Coiculescu and B. Skorulski undertook in [13] and [14] the issues signalised in [74] and [75], by extending the results proved there to the case of the family $\mathcal{H}$ of hyperbolic maps of the form $\sum_{j=0}^{n} a_j e^{(j-k)z}$, where $0 < k < n$. The Julia sets of these maps contain Cantor bouquets. Note that this family includes exponential $\lambda\exp(z)$, sine $(\lambda\sin(z))$ and cosine $(\lambda\cos(z))$ families.

3. Elliptic functions

3.1. **General Facts.** If $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ are such that $\mathrm{Im}(\omega_2/\omega_1) \neq 0$, then the set

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$$

is called the lattice generated by the numbers ω_1 and ω_2. A non-constant meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is called elliptic (with respect to the lattice Λ) if and only if

$$f(z + \omega_1) = f(z + \omega_2) = f(z)$$

for all $z \in \mathbb{C}$. This equivalently means that f is Λ-invariant, $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$ and all $\omega \in \Lambda$. The Fatou set of a meromorphic function consists of points $z \in \mathbb{C}$ which admit neighbourhoods restricted to which all iterates of our meromorphic functions are well-defined and form a normal family. The Julia set is by definition the complement (in $\mathbb{C}$) of the Fatou set. Since $f^{-1}(\infty)$, the set of poles is an infinite set, $J(f)$ is the closure of all prepoles $\bigcup_{n\geq 0} f^{-n}(\infty)$. For every pole b of f let $q_b \geq 1$ denote its multiplicity. Define

$$q = \max\{q_b : b \in f^{-1}(\infty)\} \geq 1.$$

Associating with the elliptic function f an infinite conformal iterated function system in the sense of [52] and determining its θ number introduced in [52], we were able to prove in [41] the following.

Theorem 3.1. *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function, then*

$$\mathrm{HD}(J(f)) > \frac{2q}{1+q} \geq 1.$$

The obvious consequence of Theorem 3.1 is that if f has poles of large multiplicities, then the Hausdorff dimension of the Julia set of f is close to 2. It may suggest that this dimension is always equal to 2. In section 3.3 we will describe a large class of examples with Julia set of dimension less than 2. Here we will give a simple construction showing that for each lattice Λ there exists an elliptic Λ-invariant function whose Julia set is not the entire complex plane $\mathbb{C}$. Indeed, let Λ be a lattice and let $g : \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic Λ-invariant function for which some zero, call it b, is not a critical point of g. Consider the family of functions $\{g_\lambda = \lambda g + b\}_{\lambda \in \mathbb{C}\setminus\{0\}}$. This family consists of Λ-invariant elliptic functions and $g_\lambda(b) = \lambda g(b) + b = 0 + b = b,\ g'_\lambda(b) = \lambda g'(b)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. So, if $|\lambda| < 1/|g'(b)|$, then b is an attracting fixed point of g. The, non-empty, basin of attraction to b under g_λ is contained in the Fatou set, and consequently $J(g_\lambda) \neq \mathbb{C}$.

We would like to mention that using the same methods as in [41] it was shown in [49] that the Hausdorff dimension of the Julia set of any function of the form $\exp \circ f$, where f is elliptic, is equal to 2. These functions are doubly periodic but have essential singularities. We would also like to mention that in the papers [28]-[29] elliptic functions, actually Weierstrass elliptic $\wp$-functions, were found with Julia sets of various topological types.

As the reader may recall, we dealt in Section 2 with the set of points not escaping to ∞, and despite of the fact that set of points escaping to ∞ was large (of Hausdorff dimension 2), its complement turned out to be geometrically and dynamically sound. For elliptic functions, the situation is in some sense better. We have the following.

Theorem 3.2. *If $f:\mathbb{C}\to\overline{\mathbb{C}}$ is an elliptic function, then*

$$\mathrm{HD}(I_\infty(f))\le\frac{2q}{1+q}.$$

And as its immediate consequence, we get the following.

Corollary 3.3. *If $f:\mathbb{C}\to\overline{\mathbb{C}}$ is an elliptic function, $h:=\mathrm{HD}(J(f))$, then $\mathrm{H}^h(I_\infty(f))=0$, and consequently $l_2(I_\infty(f))=0$.*

3.2. **Gibbs and Equilibrium States.** Keeping in this section the assumption that $f:\mathbb{C}\to\overline{\mathbb{C}}$ is an arbitrary elliptic function, our aim is to discuss the results concerning Gibbs and equilibrium states obtained in [50]. Let $T=\mathbb{C}/\Lambda$ be the torus generated by the lattice Λ and let $\Pi:\mathbb{C}\to\mathbb{C}/\Lambda$ be the canonical projection. Let $\mathcal{P}=\Pi(f^{-1}(\infty))$ be the set of 'poles' on the torus T. The map $f:\mathbb{C}\setminus f^{-1}(\infty)\to\mathbb{C}$ uniquely projects down to the holomorphic map $F:T\setminus\mathcal{P}\to T$ so that $F\circ\Pi=\Pi\circ f$, i.e. the following diagram commutes

$$\begin{array}{ccc}
\mathbb{C}\setminus f^{-1}(\infty) & \xrightarrow{f} & \mathbb{C}\\
\Pi\big\downarrow & & \big\downarrow\Pi\\
T\setminus\mathcal{P} & \xrightarrow{F} & T.
\end{array}\tag{3.1}$$

Notice that we have a little bit more. Since the function f is constant on fibres of Π, there exists a unique holomorphic map $\hat f:T\to\overline{\mathbb{C}}$ such that $\hat f(\Pi(z))=f(z)$ for all $z\in\mathbb{C}$. Analogously as in Section 2, the dynamical system $F:T\setminus\mathcal{P}\to T$ will be our our primary object of interest in this section.

Following the classical case of subshift of finite type [12] (more generally the case of open distance expanding maps) or, more appropriately in this context, the approach initiated in [18], one is tempted to develop the theory of Gibbs and equilibrium states for Hölder continuous potentials $\phi:T\to\mathbb{R}$. To be really general suppose that $\phi:T\to\mathbb{R}$ is an arbitrary function; no other assumptions. The basic tool of any known version of thermodynamic formalism is an appropriate Perron-Frobenius operator which in our context would take on the form

$$\mathcal{L}_\phi g(x)=\sum_{y\in F^{-1}(x)}e^{\phi(y)}g(y).$$

Notice that the series defining the Perron-Frobenius operator $\mathcal{L}_\phi$ is infinite and in order to make it well-defined and bounded on the Banach space $C(T)$ of continuous functions on T, one should demand that with a universal constant $C > 0$

$$\mathcal{L}_\phi(1\!\!1) = \sum_{y \in F^{-1}(x)} e^{\phi(y)} \le C$$

for all $x \in T$. Let us examine what this requirement really means. First of all we immediately see that ϕ cannot be uniformly bounded from below. To get a deeper insight fix $\tilde{x} \in \Pi^{-1}(x)$. Then $y \in F^{-1}(x)$ if and only if there exists $\omega \in \Lambda$ such that $\hat{f}(y) = \tilde{x} + \omega$. Therefore

$$\mathcal{L}_\phi(1\!\!1) = \sum_{\omega \in \Lambda} \sum_{y \in \hat{f}^{-1}(\tilde{x}+\omega)} e^{\phi(y)}.$$

If $|\omega|$ is big, then $y \in \hat{f}^{-1}(\tilde{x} + \omega)$ is near the pole b of $\hat{f} : T \to \mathbb{C}$, where we can write that

$$\tilde{x} + \omega = \hat{f}(y) = \frac{G_b(y)}{(y-b)^{q_b}}$$

with G_b, a holomorphic function defined near b, such that $G_b(b) \ne 0$ and where, let us recall, $q_b \ge 1$ is the multiplicity of the pole b. Since the set of poles $\mathcal{P} \subset T$ is finite the series

$$\sum_{b \in \mathcal{P}} \sum_{\omega \in \Lambda} |\tilde{x} + \omega|^{-(2+\epsilon_b)}$$

converges with arbitrarily chosen $\epsilon_b > 0$. Trying to apply the comparison test, we would therefore require that with some constant $L > 0$

$$\exp(\phi(y)) \le L|\tilde{x} + \omega|^{-(2+\epsilon_b)} = L\left(\frac{|y-b|^{q_b}}{|G_b(y)|}\right)^{2+\epsilon_b}$$

for all poles $b \in \mathcal{P}$ and all $y \in \hat{f}^{-1}(\tilde{x} + \omega)$ being close to b. Or equivalently

$$\phi(y) \le \log L - (2+\epsilon_b)\log|G_b(y)| + (2+\epsilon_b)q_b \log|y-b|$$

near b. This inequality suggests us that we deal with the class of potentials $\phi : T \to \mathbb{C}$, called in the sequel summable, satisfying the following two conditions.

(a) For any open set V containing $\mathcal{P}$, ϕ is Hölder continuous on $T \setminus V$.
(b) For every pole $b \in \mathcal{P}$ there are $\epsilon_b > 0$ and Hölder continuous function H_b such that $\phi(z) = H_b(z) + (2+\epsilon_b)q_b \log|z-b|$ on a sufficiently small neighbourhood of b.

Of course the most significant potentials of the form $-t\log|F'|, (t > 0)$. If the map F is hyperbolic then these potentials are summable and Bowen's formula holds. If however F is not hyperbolic, in particular if the Julia set contains critical points, then the potentials $-t\log|F'|$ are not any longer summable, and as in the case of rational functions, the theory described in this subsection does not apply to them. However, as long as the critical points are not recurrent a lot can be said about conformal, invariant and geometric measures associated to the potential $-h\log|F'|$ (see Subsection 3.3).

Given a measurable function $\psi : T \to [0, +\infty]$, a Borel probability measure m on T is said to be ψ-conformal if and only if $m(J(F)) = 1$ and

$$m(F(A)) = \int_A \psi dm$$

for every Borel set $A \subset T$ such that $F_{|A}$ is one-to-one. Unlike the case of hyperbolic exponential functions, due to compactness of the torus T, it is much easier here to construct (generalised) conformal measures. Namely, the map

$$\nu \to \frac{\mathcal{L}_\phi^* \nu}{\mathcal{L}_\phi^* \nu(1\!\!1)}$$

discussed in Subsection 2.2.1 is in our space continuous on the compact convex set of Borel probability measures on T. So, the Schauder-Tichonov theorem applies and we obtain a $\kappa e^{-\phi}$-conformal measure with some constant $\kappa > 0$. The problem of defining pressure is however the same as in Subsection 2.2.1 and we resolve it in the same way as there by employing the pointwise definition.

$$P(\phi, x) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y \in F^{-n}(x)} \exp\left(\sum_{j=0}^{n-1} \phi \circ F^j(y)\right) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\phi^n(1\!\!1)(x).$$

The key point to obtain all the results discussed below is a very detailed analysis of the behaviour of the normalised Perron-Frobenius operator $\hat{\mathcal{L}}_\phi = \kappa^{-1}\mathcal{L}_\phi$. Apart from (a) and (b) the third general assumption is that

$$\sup\{P(\phi, x) : x \in T\} > \sup(\phi).$$

Concerning the operator $\hat{\mathcal{L}}_\phi$ itself, it turns out to be almost periodic and admits a continuous, everywhere positive function $\rho : T \to I\!\!R$ such that

$$\hat{\mathcal{L}}_\phi(\rho) = \rho.$$

As a result of an extensive analysis of its behaviour, one gets the following.

Lemma 3.4. *For every $x \in T, P(\phi, x)$ is the same, and the common value $P(\phi)$, called the topological pressure of ϕ, is given by the following formula*

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\phi^n(x) = \lim_{n\to\infty} \frac{1}{n} \log ||\mathcal{L}_\phi^n(1\!\!1)||_\infty.$$

Theorem 3.5. *There exists a unique $\exp(P(\phi)-\phi)$-conformal measure m_ϕ on T and a unique Borel probability F-invariant measure μ_ϕ absolutely continuous with respect to the measure m_ϕ. The measure μ_ϕ is in fact equivalent to m, and $\frac{d\mu_\phi}{dm_\phi} = \rho$, the fixed point of the Perron-Frobenius operator $\hat{\mathcal{L}}_\phi$, normalised so that $\int \rho d\mu_\phi = 1$. The measure μ_ϕ is called the Gibbs state of the summable potential ϕ.*

Passing to equilibrium states, denote by M_ϕ (M_ϕ^e) the space of all F-invariant (ergodic) Borel probability measures on $J(F)$ for which $\int \phi d\mu > -\infty$. Let $\chi_\mu = \int \log|F'|d\mu$ be the characteristic Lyapunov exponent of the measure μ. A simple observation (based on similar behaviour of ϕ and $\log|F'|$ near poles) is that if $\mu \in M_\phi^e$ then $\chi_\mu < +\infty$. A more involved argument (more difficult than in the case of rational functions since $F : T \setminus \mathcal{P} \to T$ is not Lipschitz continuous) leads to the following

Theorem 3.6. *(Ruelle's inequality) If $\mu \in M_\phi^e$, then $h_\mu(F) \le 2\max\{0, \chi_\mu\}$. In particular if $h_\mu(F) > 0$, then $\chi_\mu > 0$.*

Another technical fact needed to establish the variational principle and to identify all equilibrium states (which are defined just below) of ϕ, is this.

Proposition 3.7. *If $\mu \in M_\phi^e$ and $\chi_\mu > 0$, then there exists a countable generating partition $\alpha\,(mod\mu)$ such that its entropy $\mathrm{H}_\mu(\alpha)$ is finite. In particular, $h_\mu(F) = \int \log J_\mu d\mu$, where J_μ is the Jacobian $\left(\frac{d\mu\circ F}{d\mu}\right)$ of F with respect to the measure μ, well-defined on the complement of a set of measure zero.*

Armed with these last two results and Theorem 3.5, one proves the following

Theorem 3.8. *(Variational Principle) We have*

$$P(\phi) = \sup\{h_\mu(\phi) + \int \phi d\mu : \mu \in M_\phi\}.$$

A measure $\mu \in M_\phi$ is called an equilibrium state of ϕ if $h_\mu(\phi) + \int \phi d\mu = P(\phi)$. The following result can be therefore treated as a completion of Theorem 3.8.

Theorem 3.9. *The Gibbs state μ_ϕ (proved to exist in Theorem 3.5) is the unique equilibrium state of the potential ϕ.*

Addressing referee's questions we would like to remark that in Theorem 3.6 and Proposition 3.7 the assumption $\mu \in M_\phi$ can be relaxed and as, a consequence, one gets similarly as in the case of rational functions, the following

Theorem 3.10. *If μ is a Borel probability F-invariant ergodic measure with positive Lyapunov exponent, then*

$$\mathrm{HD}(\mu) = \frac{h_\mu}{\xi_\mu},$$

and moreover,

$$\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}=\mathrm{HD}(\mu)$$

for μ_ϕ – a.e. $x\in J(F)$.

3.3. **Critically Non-Recurrent Elliptic Functions.**

3.3.1. **Preliminaries.** Throughout this whole section except at the very end, we discuss the content of [42]. In the previous section an elliptic function was exclusively treated as a dynamical system. Trying to say something finer than in Section 3.1 about the fractal geometry of the Julia set of an elliptic function, some restrictions on the class of functions to be analysed are needed. In [42] this class was defined by analogy with the case of rational functions of the Riemann sphere treated in [69], [70]; comp. [73].

Definition 3.11. *An elliptic function $f:\mathbb{C}\to\overline{\mathbb{C}}$ is called critically non-recurrent if and only if the following three conditions are satisfied*

(a) *$c\notin\omega(c)$ for every critical point c of f lying in $J(f)$*
(b) *$\omega(c)$ is a compact set for every critical point c of f lying in $J(f)$*
(c) *every critical point c in the Fatou set belongs to the basin of attraction of either an attracting or rationally indifferent periodic point*

Denote by $\Omega(f)$ the set of rationally indifferent periodic points of f. Let

$$\mathrm{Crit}(J(f))$$

be the set of all critical points of f that are contained in the Julia set $J(f)$. The result which permits us to start off and to continue our analysis of critically non-recurrent elliptic function is the following result bringing Mañé's theorem from [47] to the context of elliptic functions.

Theorem 3.12. *Let $f:\mathbb{C}\to\overline{\mathbb{C}}$ be a non-recurrent elliptic function. If $X\subset J(f)\setminus\Omega(f)$ is a closed subset of $\mathbb{C}$, then for every $\epsilon>0$ there exists $\delta>0$ such that for every $x\in X$ and every $n\geq 0$, all the connected components of $f^{-n}(B(x,\delta))$ have Euclidean diameters $\leq\epsilon$.*

Let

$$\mathrm{Sing}^-(f)=\bigcup_{n\geq 0}f^{-n}\Big(\Omega(f)\cup\mathrm{Crit}(J(f))\cup f^{-1}(\infty)\Big).$$

The following proposition is a consequences of Theorem 3.12. It is important for further considerations in [42], and is interesting in itself as it gives a very precise description of radial points.

Proposition 3.13. *If $z \in J(f) \setminus \mathrm{Sing}^-(f)$, then there exist a positive number $\eta(z)$, an increasing sequence of positive integers $\{n_j\}_{j\geq 1}$, and a point*

$$x = x(z) \in \omega(z) \setminus (\Omega(f) \cup \omega(\mathrm{Crit}(z)))$$

such that $x \neq \infty$ if $z \notin I_\infty(f)$, $\lim_{j\to\infty} f^{n_j}(z) = x$ and

$$\mathrm{Comp}(z, f^{n_j}(z), f^{n_j}, \eta(z)) \cap \mathrm{Crit}(f^{n_j}) = \emptyset$$

for every $j \geq 0$, where $\mathrm{Comp}(z, f^{n_j}(z), f^{n_j}, \eta(z))$ *denotes the connected component of $f^{-n_j}(B(f^{n_j}(z), \eta(z)))$ that contains z and* $\mathrm{Crit}(z) = \{c \in \mathrm{Crit}(J(f)) : c \in \omega(z)\}$.

3.3.2. **Hausdorff, packing and conformal measures.** As in Section 2 the link between dynamics and geometry of elliptic functions is provided by Sullivan's conformal measures. Recall that given $t \geq 0$, a Borel probability measure m_s on $J(f)$ is called t-conformal if and only if

$$m_s(f(A)) = \int_A |f^*|^t dm_s,$$

whenever $A \subset J(f)$ is a Borel set such that $f_{|A}$ is one-to-one and f^* is the derivative of f with respect to the spherical metric on $\mathbb{C}$. If we give up the finiteness assumption of m_s but we replace f^* by the standard Euclidean derivative f' of f, we denote the resulting conformal measure by m_e and call it the Euclidean t-conformal measure for f. The relation between these two measures is that

$$\frac{dm_e}{dm_s}(z) = (1 + |z|^2)^t.$$

In fact, several weaker versions of conformal measures are needed in [42], but we will not discuss them here. Let $h = \mathrm{HD}(J(f))$ be the Hausdorff dimension of the Julia set of f. The h-conformal measure for f is constructed by the $K(V)$-method described in Appendix 1. The bad set here, called crossing in [42], is any finite set

$$Y \subset \{\infty\} \cup \Omega(f) \cup \bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(J(f))$$

such that the following conditions are satisfied

(y1) $\infty \in Y$.
(y2) $Y \cap \{f^n(x) : n \geq 1\}$ is a singleton for all $x \in \mathrm{Crit}(J(f))$.
(y3) $Y \cap \mathrm{Crit}(f) = \emptyset$.
(y4) $\Omega(f) \subset Y$.

Now, we can choose any sequence $\{r_n\}_{n=1}^{\infty}$ of positive reals converging down to zero and such that $\partial B_s(Y, r_n) \cap \mathrm{Crit}(f) = \emptyset$, where the subscript 's' indicates that the ball is considered with respect to the spherical metric on $\mathbb{C}$. Following the general scheme outlined in Appendix 1, one defines now the compact forward invariant sets

$$K_n = \bigcap_{j=0}^{\infty} J(f) \setminus f^{-n}(B(Y, r_n)), \quad n \geq 1,$$

and one applies Lemma 8.2 to get semi-conformal measures. This is the beginning of the route. Introducing a special order in $\mathrm{Crit}(J(f))$, a stratification of the closure of the postcritical set, fighting with rationally indifferent periodic points, critical points and poles of f, which are potential candidates for atoms of the described conformal measure, we eventually arrive at this.

Theorem 3.14. *There exists a unique h-conformal measure m for $f : \mathbb{C} \to \overline{\mathbb{C}}$. This measure is atomless.*

Using this conformal measure m to gain information about Hausdorff and packing measure, after lengthy and fairly technical considerations, we end up with the following.

Theorem 3.15. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically non-recurrent elliptic function. If $h = \mathrm{HD}(J(f)) = 2$, then $J(f) = \mathbb{C}$. So suppose that $h < 2$. Then*
(a) $\mathrm{H}^h(J(f)) = 0$,
(b) $\mathrm{P}^h(J(f)) > 0$,
(c) $\mathrm{P}^h(J(f)) = \infty$ *if and only if* $\Omega(f) \neq \emptyset$,

where H^h and P^h are defined by the means of spherical metric on $\overline{\mathbb{C}}$.

So, if $\mathrm{HD}(J(f)) < 2$, then the Hausdorff measure always vanishes, whereas packing measures turns out to be the right geometric measure exactly when there are no parabolic periodic points. If $\Omega(f) \neq \emptyset$ then $\mathrm{H}^h(J(f)) = 0, \mathrm{P}^h(J(f)) = \infty$ (even locally) and no geometric interpretation of the h-conformal measure has been so far found in this case.

Note that a similar phenomenon has been observed by D. Sullivan in the context of geometrically finite Kleinian groups withs cusps of different ranks (see [67]) and in [52] in the case of conformal irregular infinite iterated function systems.

3.3.3. **Invariant measures equivalent to h-conformal measure.** Since it is not difficult to show that the h-conformal measure m of the postcritical set vanishes, the method of M. Martens (see Appendix 2) applies and leads to the following.

Theorem 3.16. *There exists a σ-finite f-invariant measure μ that is absolutely continuous with respect to the h-conformal measure m. In addition, μ is ergodic and conservative.*

The most intriguing problem here is to determine whether the σ-finite invariant measure μ is finite or infinite. In order to deal with this problem, it is useful to recall from [70] the concepts of finite and infinite condensations. Namely, a point z is of finite condensation with respect to a Borel measure ν if there is an open neighbourhood U of z such that $\nu(U) < \infty$; otherwise z is said to be of infinite condensation of measure ν. Our strategy to cope with the problem of finiteness of the measure μ was to identify the points of its finite and infinite condensation. To our surprise, careful estimates permitted us to prove the following.

Theorem 3.17. *∞ is a point of finite condensation of the measure μ.*

We were able to go further to establish the following.

Theorem 3.18. *The set of points of infinite condensation of μ is contained in the set of parabolic points $\Omega(f)$.*

As an immediate consequence of this theorem, we get the following.

Corollary 3.19. *If $\Omega = \emptyset$, then there exists an f-invariant probability measure μ equivalent to m.*

Since the case $J(f) = \mathbb{C}$ rules out parabolic points, as an immediate consequence of this corollary we get

Corollary 3.20. *If $J(f) = \mathbb{C}$, then there is a unique probability measure μ equivalent to the Lebesgue measure on $\mathbb{C}$.*

It follows from the above that in order to understand the problem of finiteness of the σ-finite f-invariant measure μ, one must analyse in detail the parabolic points. Such analysis has been done in [70] and moves unchanged to the case of critically non-recurrent elliptic functions to give the following.

Proposition 3.21. *If $\omega \in \Omega \setminus \overline{O_+(\mathrm{Crit}(J(f)))}$, then μ has infinite condensation at ω if and only if $h \le \frac{2p(\omega)}{p(\omega)+1}$.*

As an immediate consequence of this proposition and Theorem 3.1, we get the following remarkable corollary.

Corollary 3.22. *If* $\Omega \cap \overline{O_+(\mathrm{Crit}(J(f)))} = \emptyset$ *and*

$$\max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\} \geq \max\{p(\omega) : \omega \in \Omega(f)\},$$

then the invariant measure μ *is finite.*

Proposition 3.23. *If* $\omega \in \Omega$ *and* $h \leq \frac{2p(\omega)}{p(\omega)+1}$, *then* μ *has infinite condensation at* ω.

Theorem 3.24. *If* $c \in J(f)$ *is a critical point of* f *of order* s, $\omega = f(c) \in \Omega$, *and* $h \leq \frac{2sp(\omega)}{p(\omega)+1}$, *then* μ *has infinite condensation at* ω.

4. Walters expanding conformal maps

In this chapter we present the theory developed in [39]. The chapter begins with a very general setting and then we gradually narrow it down to applications to very concrete meromorphic functions.

4.1. **Basic Facts and Definitions.** We first define Walters expanding mappings and collect their selected properties needed in the sequel. For a full account of Walters theory see [80].

So, let X_0 be an open and dense subset of a compact metric space X endowed with a metric ρ. We call a continuous map $T : X_0 \to X$ Walters expanding provided that the following conditions are satisfied:

(1a) The set $T^{-1}(x)$ is at most countable for each $x \in X$.
(1b) There exists $\delta > 0$ such that for every $x \in X$ and every $n \geq 0$, $T^{-n}(B(x, 2\delta))$ can be written uniquely as a disjoint union of open sets $\{B_y(x)\}_{y \in T^{-n}(x)}$ such that $y \in B_y(x)$ and $T^n : B_y(x) \to B(x, 2\delta)$ is a homeomorphism from $B_y(x)$ onto $B(x, 2\delta)$. The corresponding inverse map from $B(x, 2\delta)$ to $B_y(x)$, $y \in T^{-n}(x)$, will be denoted by T_y^{-n}.
(1c) There exist $\lambda > 1$ and $n \geq 1$ such that for every $x \in X$, every $y \in T^{-n}(x)$ and all $z_1, z_2 \in B_y(x)$

$$d(T^n(z_1), T^n(z_2)) \geq \lambda d(z_1, z_2).$$

(1d) $\forall \epsilon > 0\, \exists s \geq 1\, \forall x \in X$ $T^{-s}(x)$ is ϵ-dense in X.

Recall that a function $g : Y \to I\!R$, where (Y, ρ) is a metric space, is Hölder continuous if there exist $\beta > 0$ and $L > 0$ such that for all $y_1, y_2 \in Y$, $|g(y_1) - g(y_2)| \leq L\rho(y_1, y_2)^\beta$. The parameter β is called the Hölder exponent of the function g and L is called its Hölder constant. A function $\phi : X_0 \to I\!R$ is called dynamically Hölder if there exists $\beta > 0$ and $L > 0$ such that for every $n \geq 1$, every $x \in X$ and every $y \in T^{-n}(x)$, the restriction $\phi|_{T_y^{-n}(B(x,\delta))}$ is Hölder continuous with exponent $\beta > 0$ and constant L. For every $n \geq 1$ put

$$S_n\phi(x) = \sum_{j=0}^{n-1} \phi \circ T^j(x).$$

Using (1c), the standard argument in thermodynamic formalism shows that there exists a constant $C>0$ such that

$$\forall x\in X,\ \ \forall y,z\in B(x,\delta),\ \ \forall n\geq 0,\ \ \forall u\in T^{-n}(x)$$
$$|S_n\phi(T_u^{-n}(y))-S_n\phi(T_u^{-n}(z))|\leq Cd(y,z)^\beta. \tag{4.1}$$

The function $\phi: X_0\to X$ is called summable if

$$\sup_{x\in X}\left\{\sum_{y\in T^{-1}(x)}\exp(\phi(y))\right\}<\infty.$$

Given $x\in X$, similarly as in the case of elliptic functions, we set

$$P_x(\phi)=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{y\in T^{-n}(x)}\exp(S_n(\phi(y))).$$

It is not difficult to prove that if $\phi: X_0\to I\!R$ is dynamically Hölder, then $P_x(\phi)=P_y(\phi)$ for all $x,y\in X$. The common value is called the topological pressure of ϕ with respect to T and is denoted by $P(\phi)$. We should notice that $P(\phi)<\infty$ if and only if ϕ is summable. From the results of P. Walters in [80] one can extract the following.

Theorem 4.1. *If $T: X_0\to X$ is a Walters expanding map and $\phi: X_0\to X$ is a dynamically Hölder summable function, then there exist m_ϕ and μ_ϕ, Borel probability measures on X such that*

(a) $\forall n\geq 1,\ \forall x\in X,\ \forall y\in T^{-n}(x)$ *and for every Borel set* $A\subset T_y^{-n}(B(x,\delta))$

$$m_\phi(T^n(A))=\int_A e^{P(\phi)-S_n(\phi)}dm_\phi$$

(b) *μ_ϕ is T-invariant which means that $\mu_\phi\circ T^{-1}=\mu_\phi$, ergodic and equivalent to m_ϕ with continuous Radon-Nikodym derivative bounded away from zero and infinity.*

The reader familiar with thermodynamic formalism, for example with Sections 2.2.1 or 3.2, may notice that the property (a) means that the measure m_ϕ is an eigenmeasure of the operator dual to the appropriate Perron-Frobenius operator with eigenvalue $e^{P(\phi)}$. Many additional stochastic properties of the dynamical system (T,μ_ϕ) can be found in [80].

A Walters expanding map $F: X_0\to X$ is called conformal if $X\subset \mathbb{C}$ and if for every $x\in X$, every $n\geq 1$ and every $y\in F^{-n}(x)$ the inverse map $F_y^{-n}: B_X(x,2\delta)\to X_0$ has a (unique) holomorphic extension to the ball $B_{\mathbb{C}}(x,2\delta)$. This extension will be denoted by the same symbol F_y^{-n}. From now and throughout this entire section we assume that F is a Walters expanding conformal map. Of special importance will be the following functions $g_t: X_0\to I\!R, t\geq 0$ given by the formulae

$$g_t(x)=-t\log|F'(x)|.$$

It immediately follows from Koebe's distortion theorem that each function g_t is dynamically Hölder with the Hölder exponent $1/3$. Following [52] we define θ_F to be the infimum of all $t \geq 0$ for which the function g_t is summable. Due to Proposition 2.4 in [39],

$$\theta_F = \inf\{t \geq 0 : P(g_t) < \infty\}.$$

The following proposition is a straightforward standard consequence of the definition of pressure and property (1c).

Proposition 4.2. *The function $P : (\theta_F, \infty) \to I\!R$ is convex, continuous, strictly decreasing and $\lim_{t\to+\infty} P(t) = -\infty$.*

We define

$$h_F = h = \inf\{t : P(t) \leq 0\}.$$

Obviously $h_F \geq \theta(F)$. Following the terminology of [52] we call the map F regular if $P(h) = 0$, strongly regular if there exists $t \geq 0$ such that $0 < P(t) < \infty$ and hereditarily regular if $P(\theta_F) = \infty$ (which due to (1c) and (1d) implies that $\lim_{t\nearrow\theta_F} P(t) = +\infty$). In view of Proposition 4.2 each strongly regular map is regular and each hereditarily regular map is strongly regular. If F is regular, then $m = m_{g_h}$ is called the h-conformal measure for F. Its F-invariant version will be denoted by μ. Let

$$X_\infty = \bigcap_{n\geq 0} F^{-n}(X_0).$$

The following statement is an immediate consequence of Theorem 4.1

Theorem 4.3. *If F is a regular Walters expanding conformal map, then there exists a unique F-invariant Borel probability measure μ_h absolutely continuous with respect to the h-conformal measure m_h. The measure μ_h is ergodic and the Radon-Nikodym derivative is bounded away from zero and ∞.*

4.2. **Hausdorff and Box Dimensions, Hausdorff and Packing Measures.** In this entire section we will be primarily interested in the dynamical system $F : X_\infty \to X_\infty$ and geometry of the set X_∞. The first result in this direction, a version of Bowen's formula is this.

Theorem 4.4. *If $F : X_0 \to X$ is a Walters expanding conformal map, then $\mathrm{HD}(X_\infty) \leq h$. If, in addition, F is strongly regular, then $\mathrm{HD}(X_\infty) = h$ and, in particular, $\mathrm{HD}(X_\infty) > \theta_F$.*

Passing to the upper ball-counting dimension (occasionally called box-counting or Minkowski dimension), we let X be an arbitrary metric space and A an arbitrary subset of X. We denote by $N_r(A)$ the minimal number of balls with centres in the set A and of radius $r > 0$ needed to cover A. The upper ball-counting dimension of A is defined to be

$$\mathrm{BD}(A) = \limsup_{r\to 0} \frac{\log N_r(A)}{-\log r}.$$

The formula for the upper ball-dimension of the set X is given by the following.

Theorem 4.5. *If F is a regular Walters expanding conformal map and if W is a finite δ-net of X, then*

$$\begin{aligned}\mathrm{BD}(X) &= \max\{\mathrm{HD}(X_\infty), \mathrm{BD}(F^{-1}(W))\} \\ &= \max\{\mathrm{HD}(X_\infty), \max\{\mathrm{BD}(F^{-1}(w)) : w \in W\}\}.\end{aligned}$$

We would like to bring the reader's attention to the fact that although the set $F^{-1}(W)$ is countable, its box dimension can be positive. We would also like to emphasise the fact that in Theorem 4.5 only the first inverse iterate $F^{-1}(W)$ is involved and higher inverse iterates are not needed. The problem of determining whether Hausdorff (H^h) and packing (P^h) measures of the set X_∞ are finite and positive is a more delicate issue. However the following general result holds.

Theorem 4.6. *If F is a regular Walters expanding conformal map, then $\mathrm{H}^h(X_\infty) < \infty$ and $\mathrm{P}^h(X_\infty) > 0$. In addition $\mathrm{H}^h \ll m_h$ and $m_h \ll \mathrm{P}^h$.*

The general tools, applied for example to jump-like conformal maps, to deal with the problem whether $H^h(X_\infty)$ is positive or vanishes or $P^h(X_\infty)$ is finite or infinite are collected in the following theorems.

Theorem 4.7. *Suppose F is a regular Walters expanding conformal map. Assume there exist $\gamma \geq 1$ and $L > 0$ such that for every $x \in X_0$ and for every r satisfying the condition $r \geq \gamma \mathrm{diam}(\mathrm{F_x^{-1}}(\mathrm{B}(\mathrm{F}(\mathrm{x}), \delta)))$, we have $m(B(x), r)) \leq Lr^h$. Then $\mathrm{H}^h(X_\infty) > 0$.*

Theorem 4.8. *Suppose F is a regular Walters expanding conformal map. If there exist a sequence of points $z_j \in X, j \geq 1$, and a sequence of positive reals $\{r_j\}_{=1}^{\infty}$ such that $r_j \leq \delta/2$ and*

$$\overline{\lim}_{j\to\infty} \frac{m_h(B(z_j, r_j))}{r_j^h} = \infty,$$

then $\mathrm{H}^h(X_\infty) = 0$.

Theorem 4.9. *Suppose that F is a regular Walters expanding conformal map. Assume that there exist $\gamma \geq 1$ and $0 < \xi \leq \delta$ such that for every $x \in X_0$ and for every r satisfying the condition $\gamma \mathrm{diam}(\mathrm{F_x^{-1}}(\mathrm{B}(\mathrm{F}(\mathrm{x}), \delta)))) \leq \mathrm{r} \leq \xi$, we have $m_h(B(x, r)) \geq Lr^h$. Then $\mathrm{P}^h(X_\infty) < \infty$.*

Theorem 4.10. *Suppose F is a regular Walters expanding conformal map. If there exists a sequence of points $z_j \in X, j \geq 1$ and a sequence of positive reals $\{r_j\}_{=1}^{\infty}$ such that*

$$\underline{\lim}_{j\to\infty} \frac{m_h(B(z_j, r_j))}{r_j^h} = 0,$$

then $\mathrm{P}^h(X_\infty) = \infty$.

Let Asymp(f) denote from now on the set of asymptotic values of the function f under consideration. Concluding this rather abstract section we would like to write that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a meromorphic mapping for which

$$J(f) \cap \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{Crit}_G(f) \cup \mathrm{Asymp}(f))} = \emptyset,$$

where $J(f)$ is the Julia set of f, $\mathrm{Crit}_G(f)$ is the set of general critical points (*i.e.* the set of critical points or multiple poles of f), then it is not difficult to prove that if $M : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a Möbius transformation such that $M(\infty) \notin J(f)$, then the Julia set of the map $\tilde{f} = M^{-1} \circ f \circ M : \overline{\mathbb{C}} \setminus M^{-1}(\infty) \to \overline{\mathbb{C}}$ is a compact subset of $\mathbb{C}$ and $\tilde{f}$ restricted to its Julia set is a conformal Walters expanding map. In particular all the theorems proved in this section apply to $\tilde{f}$.

4.3. **Barański and Post-Barański Maps, I.** In this section based on Section 4 from [39] we give a brief account of the class of Barański and post-Barański maps. The latter are all Walters expanding conformal maps and provide a good motivation for dealing in the next section with the larger class of jump-like conformal maps. We consider a class of transcendental meromorphic functions of the form

$$f(z) = H(\exp(Q(z))) \quad \text{and} \quad \tilde{f}(z) = \exp(Q(H(z))),$$

where Q and H are non-constant rational functions. Let $Q^{-1}(\infty) = \{d_j : j = 1, \dots, m\}$ be the set of poles of Q. Then

$$f(z) = H(\exp(Q(z))) : \overline{\mathbb{C}} \setminus \{d_j ; j = 1, \dots, m\} \to \overline{\mathbb{C}}$$

and

$$\tilde{f}(z) = \exp(Q(H(z))) : \bar{\mathbb{C}} \setminus H^{-1}(\{d_j : j = 1, \dots, m\}) \to \bar{\mathbb{C}} \setminus \{0, \infty\}.$$

We additionally assume that there is at least one pole d_i of Q such that $d_i \neq H(0), H(\infty)$. We may assume without losing generality that $d_i = d_1$. Then the set

$$Ess_\infty(f) := \bigcup_{n=0}^{\infty} f^{-n}(\{d_j : i = 1, \dots, m\})$$

contains infinitely many points. Since $\{0, \infty\} \cap H^{-1}(d_1) = \emptyset$, the set

$$Ess_\infty(\tilde{f}) := \bigcup_{n=0}^{\infty} \tilde{f}^{-n}(H^{-1}(\{d_j := 1, \dots, m\}))$$

contains infinitely many points. Using Montel's criterion it can be easily proved that

$$J(f) = \overline{Ess_\infty(f)} \quad \text{and} \quad J(\tilde{f}) = \overline{Ess_\infty(\tilde{f})}.$$

Notice that $\mathrm{Asymp}(f) = \{H(0), H(\infty)\}$ and $\mathrm{Asymp}(\tilde{f}) = \{0, \infty\}$ are respectively the sets of asymptotic values f and $\tilde{f}$. We say that f is a Barański map if the following conditions are satisfied

(1) $J(f) \cap \overline{\bigcup_{n=0}^{\infty} f^n \left(Crit(f) \cup \mathrm{Asymp}(f)\right)} = \emptyset$,
(2) if $a \in Crit(Q)$, then $\exp(Q(a))$ is not a pole of H,
(3) if H has a multiple pole, then $Q(\infty) \neq \infty$.

The map $\tilde{f}$ is then called a post-Barański map. Barański himself in his pioneering paper [7] considered the case where Q is the identity map. The maps f and $\tilde{f}$ are closely related, namely as the following two formulae show, one is a factor of the other.

$$f \circ H(z) = H \circ \tilde{f}(z) \qquad z \notin Ess(\tilde{f}). \tag{4.2}$$

$$\exp(Q) \circ f = \tilde{f} \circ \exp(Q) \qquad z \notin Ess(f). \tag{4.3}$$

These relations allow us to deduce lots of valuable dynamical and geometrical properties of the map f from the corresponding properties of the map $\tilde{f}$. This is why the rest of this section is devoted to the post-Barański map $\tilde{f}$. The first observation is that there is a $\kappa \in (0, +\infty)$ such that

$$J(\tilde{f}) \subset \{z : e^{-\kappa} < |z| < e^{\kappa}\}.$$

The second one is that

$$J(\tilde{f}) \cap \overline{\bigcup_{n=0}^{\infty} \tilde{f}^n \left(\mathrm{Crit}_G(\tilde{f}) \cup \mathrm{Asymp}(\tilde{f})\right)} = \emptyset.$$

Armed with these two observations, one can prove the following first basic results about post-Barański maps.

Theorem 4.11. *$\tilde{f} : J(\tilde{f}) \setminus \{b_j : j = 1, \dots, p\} \to J(\tilde{f})$ is a Walters expanding conformal map.*

Here

$$\{b_j : j = 1, 2, \dots, p\} = (Q \circ H)^{-1}(\infty).$$

For every $j = 1, \dots, p$ let $q_j \geq 1$ be the order of b_j treated as a pole of $Q \circ H$. Note that for every $z \in J(\tilde{f})$, each holomorphic branch of $\tilde{f}^{-1}$ defined on the ball $B(z, 2\delta)$ can be expressed in the form

$$\tilde{f}^{-1}_{j,a,n}(w) = (Q \circ H)^{-1}_{j,a}(\log(w) + 2\pi i n), \tag{4.4}$$

where $j = 1, \dots, p$, $a = 1, \dots, q_j$, $\log w$ is the value of the logarithm of w lying in the rectangle $[-\kappa, \kappa] \times [0, 2\pi]$ and $(Q \circ H)^{-1}_{j,a}$ is a local holomorphic inverse branch of $Q \circ H$. For n with sufficiently large modulus each such inverse branch can be interpreted as a branch of $(Q \circ H)^{-1}$ defined on some vertical strip either of the form $[-\kappa, \kappa] \times [T, +\infty]$ or $[-\kappa, \kappa] \times [-\infty, -T]$,

$T >> 1$, depending up on whether n is positive or negative and sending ∞ to the pole b_j of $Q \circ H$. Let (X, d) be a compact metric space. For every $A, B \subset X$ define

$$\mathrm{dist}(A, B) := \inf\{d(a, b) : a \in A,\ b \in B\}$$

and

$$\mathrm{Dist}(A, B) := \sup\{d(a, b) : a \in A,\ b \in B\}.$$

Tedious technical calculations lead to the following basic facts about post-Barański map.

Theorem 4.12. *The map $\tilde f_{|} J(\tilde f)$ is a jump-like conformal map, i.e. there exist $C \geq 1$ and $A \geq 2$ such that the following conditions are satisfied:*

(2a) $\{b_j : j = 1, \ldots, b_p\} \cap \tilde f^{-1}(J(\tilde f)) = \emptyset$.

(2b) *For every $x \in J(\tilde f)$ the set $\tilde f^{-1}(x)$ can be uniquely represented as*

$$\{x_{j,a,n} :\ n \in \mathbb{Z},\ 1 \leq j \leq p,\ 1 \leq a \leq q_j\}.$$

(2c) $\max_{1 \leq j \leq p} \max_{1 \leq a \leq q_j} \sup_{x \in J(\tilde f)}\{\lim_{n\to\infty} \mathrm{Dist}(b_j, \tilde f^{-1}_{j,a,n}(B(x, \delta)))\} = 0.$

(2d) $\forall z \in J(\tilde f),\ \ \forall 1 \leq j \leq p,\ \forall 1 \leq a \leq q_j,\ \ \forall n \in \mathbb{Z},\ \ |n| \geq A$

$$C^{-1}|n|^{-\frac{q_j+1}{q_j}} \leq |(\tilde f^{-1}_{j,a,n})'(z)| \leq C|n|^{-\frac{q_j+1}{q_j}}.$$

(2e) $\forall w, z \in J(\tilde f),\ \ \forall 1 \leq j \leq p,\ \ \forall a, b \in \{1, \ldots, q_j\},\ \ \forall k, n \in \mathbb{Z},\ ||k| - |n|| \geq A,$
$|n| \geq A,\ |k| \geq A$

$$\mathrm{dist}(\tilde f^{-1}_{j,a,k}(B(w, \delta)), \tilde f^{-1}_{j,b,n}(B(z, \delta))) \geq C^{-1}\left||k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}}\right|.$$

(2f) $\forall w, z \in J(\tilde f),\ \ \forall 1 \leq j \leq p,\ \ \forall a \in \{1, \ldots, q_j\},\ \ \ \forall k, n \in \mathbb{Z},\ kn > 0,\ \ ||k| - |n|| \geq A,$
$|n| \geq A,\ |k| \geq A$

$$\mathrm{Dist}(\tilde f^{-1}_{j,a,k}(B(w, \delta)), \tilde f^{-1}_{j,b,n}(B(z, \delta))) \leq C\left||k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}}\right|.$$

As an immediate consequence of (2d), with a bigger constant C perhaps, we get the following

$$\forall x \in J(\tilde f),\ \ \forall_{1 \leq j \leq p},\ \ \forall 1 \leq a \leq q_j,\ \ \forall n \in \mathbb{Z},\ \ |n| \geq A$$

$$C^{-1}|n|^{-\frac{q_j+1}{q_j}} \leq diam(\tilde f^{-1}_{j,a,n}(B(x, \delta))) \leq C|n|^{-\frac{q_j+1}{q_j}}. \tag{4.5}$$

Letting $k \to \infty$, it immediately follows from (2c), (2e) and (2f) that

$$\forall 1 \leq j \leq p,\ \ \forall_{x \in J(\tilde f)},\ \ \forall 1 \leq a \leq q_j,\ \ \forall |n| \geq 2A$$

$$C^{-1}|n|^{-\frac{1}{q_j}} \leq \mathrm{dist}(b_j, \tilde f^{-1}_{j,a,n}(B(x, \delta))) \leq \mathrm{Dist}(b_j, \tilde f^{-1}_{j,a,n}(B(x, \delta))) \leq C|n|^{-\frac{1}{q_j}}. \tag{4.6}$$

We end this section with large classes of examples of Barański maps. The family $f_\lambda(z) = \lambda\tan(z)$, $0 < |z| < 1$ was explored in [7]. In fact the map $\lambda\tan(z)$ is a Barański map as long as both asymptotic values $\pm\lambda i$ lie in basins of attraction to attracting periodic cycles. If $Q(z)$ is a polynomial different than identity, we have a transcendental meromorphic function f of the complex plane with one essential singularity at ∞. The most transparent class of examples is provided by the following.

Example 1. Let

$$f(z) = \frac{A\exp(z^p) + B\exp(-z^p)}{C\exp(z^p) + D\exp(-z^p)}, \quad AD - BC \neq 0.$$

Thus $\mathrm{Crit}(f) = \{0\}$, $\mathrm{Crit}_G(f) = \mathrm{Crit}(f)$ and $\mathrm{Asymp}(f) = \{\frac{A}{C}, \frac{B}{D}\}$. If $\frac{A}{C}, \frac{B}{D} \neq \infty$, then f is not entire. Notice that conditions (2) and (3) of the definition of Barański maps are always satisfied for the map f. If in addition, condition (1) is satisfied, then f is a Barański map and all the results stated in the forthcoming Section 4.5 apply. If Q is not a polynomial, then f has more then one essential singularity. Let us analyse in detail the following concrete example of this type.

Example 2. Let $H(z) = z, Q(z) = \frac{z-1}{z+1}$. Then $f : \mathbb{C}\setminus\{-1\} \to \mathbb{C}\setminus\{0,\infty\}$,

$$f(z) = \exp\left(\frac{z-1}{z+1}\right).$$

and $\tilde{f} = f$. Since the pole of Q is not an omitted value of f, we see that $\bigcup_{n=0}^{\infty} f^{-n}(-1)$ contains infinitely many points and consequently

$$J(f) = \overline{\bigcup_{n=0}^{\infty} f^{-n}(-1)}.$$

Since $f^{-1}(S^1) \subset S^1$, we have $f^{-n}(-1) \in S^1$ for all $n \in \mathbb{N}$. Therefore $J(f) \subset S^1$. We shall prove that f is a Barański map and its Julia set $J(f)$ is a topological Cantor set. Note that $\mathrm{Crit}_G(f) = \emptyset$ and $\mathrm{Asymp}(f) = \{0, \infty\}$. One can check that $f(1) = 1$ and $f'(1) = 1/2$, so the number $= 1$ is an attracting fixed point of f. Thus $J(f)$ is a topological Cantor set contained in the circle S^1. In order to conclude the proof it is now sufficient to demonstrate that 1 attracts both asymptotic values 0 and ∞. Since $f'(x) > 0$ for $x \in \mathbb{R}\setminus\{-1\}$, the function f is strictly increasing on $(-\infty,-1)$ and $(-1,+\infty)$. Now, if $x \in (1,\infty)$, then $f(1) < f(x) < x$. This implies that $\lim_{n\to\infty} f^n(x) = 1$ for all $x \in (1,\infty)$. In particular $\lim_{n\to\infty} f^n(\infty) = 1$ since $f(\infty) = e \in (1;\infty)$. If $x \in (-1,1)$, then $x < f(x) < f(1) = 1$. This implies that $\lim_{n\to\infty} f^n(x) = 1$ for all $x \in (-1,1)$. In particular $\lim_{n\to\infty} f^n(0) = 1$ since $f(0) = 1/e \in (-1,1)$. We are done.

4.4. **Jump-like Conformal Maps.** It turns out that the properties established in Theorem 4.11 and Theorem 4.12 are themselves sufficient to provide a fairly complete description of dynamics and geometry of the maps appearing in these theorems. This motivated us to single them out and to introduce the class jump-like conformal maps. We call a Walters expanding conformal map $F : X_0 \to X$ jump-like if the following requirements are met. There exist $C \geq 1, p \geq 1, A \geq 2, b_j \in X$ and $q_j \geq 1$ for every $j = 1, \dots, p$ such that the following conditions are satisfied:

(3a) $\{b_1, \dots, b_p\} \cap F^{-1}(X) = \emptyset$.

(3b) For every $x \in X$, the set $F^{-1}(x)$ can be uniquely represented as

$$\{x_{j,a,n} : n \in \mathbb{Z},\ 1 \leq j \leq p,\ 1 \leq a \leq q_j\}.$$

(3c) $\max_{1 \leq j \leq p} \max_{1 \leq a \leq q_j} \sup_{x \in X} \{\lim_{n \to \infty} \mathrm{Dist}(b_j, F^{-1}_{x_{j,a,n}}(B(x, \delta)))\} = 0$.

(3d) $\forall x \in X,\ \ \forall 1 \leq j \leq p,\ \ \ \forall 1 \leq a \leq q_j,\ \ \forall n \in \mathbb{Z},\ \ |n| \geq A,$

$$C^{-1}|n|^{\frac{q_j+1}{q_j}} \leq |F'(x_{j,a,n})| \leq C|n|^{\frac{q_j+1}{q_j}}.$$

(3e) $\forall y, z \in X,\ \ \forall 1 \leq j \leq p,\ \ \forall a, b \in \{1, \dots, q_j\},\ \ \forall k, n \in \mathbb{Z},\ \ ||k| - |n|| \geq A,$
$|n| \geq A,\ \ |k| \geq A,$

$$\mathrm{dist}(F^{-1}_{y_{j,a,k}}(B(y, \delta)), F^{-1}_{z_{j,b,n}}(B(z, \delta))) \geq C^{-1} \left| |k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}} \right|.$$

(3f) $\forall y, z \in X,\ \ \forall 1 \leq j \leq p,\ \ \forall a \in \{1, \dots, q_j\},\ \ \forall k, n \in \mathbb{Z},\ kn > 0,\ \ ||k| - |n|| \geq A$
$|n| \geq A,\ \ |k| \geq A,$

$$\mathrm{Dist}(F^{-1}_{y_{j,a,k}}(B(y, \delta)), F^{-1}_{z_{j,a,n}}(B(z, \delta))) \leq C \left| |k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}} \right|.$$

The name 'jump-like' is motivated by a fairly strong formal similarity of those maps and the jump maps considered in the theory of parabolic rational functions. It is a matter of relatively simple calculation to prove the following

Proposition 4.13. *Suppose that $F : X_0 \to X$ is a jump-like conformal map and let $q = \max\{q_j : 1 \leq j \leq p\}$. Then the map F is hereditarily regular and $\theta_F = \frac{q}{q+1}$.*

As an immediate consequence of this proposition and Theorem 4.3, we get the following.

Theorem 4.14. *If F is a jump-like conformal mapping, then there exists a unique F-invariant Borel probability measure μ_h absolutely continuous with respect to the h-conformal measure m_h. The measure μ_h is ergodic and the Radon-Nikodym derivative is bounded away from zero and ∞.*

Since it is easy to see that if F is a jump-like conformal map, then for every $x \in X$, $\mathrm{BD}(F^{-1}(x)) = \frac{q}{q+1}$, where $q = \max\{q_j : 1 \leq j \leq p\}$, as an immediate consequence of Proposition 4.13 and Theorem 4.5 we get the following.

Theorem 4.15. *If $F : X_0 \to X$ is a jump-like conformal map, then*

$$\mathrm{BD}(X) = \mathrm{HD}(X_\infty).$$

Based on the assumptions (3a)-(3f) one can verify by very technical considerations the sufficient conditions established in Section 4.2. As the result we get the following complete description of geometric measures, and simultaneously, geometric characterisation of dynamically defined, conformal measure m_h.

Theorem 4.16. *Suppose that $F : X_0 \to X$ is a jump-like conformal map. Then*
(a) *If $h < 1$, then $0 < \mathrm{P}^h(X_\infty) < \infty$ and $\mathrm{H}^h(X_\infty) = 0$.*
(b) *If $h = 1$, then $0 < \mathrm{P}^h(X_\infty)$, $\mathrm{H}^h(X_\infty) < \infty$.*
(c) *If $h > 1$, then $0 < \mathrm{H}^h(X_\infty) < \infty$ and $\mathrm{P}^h(X_\infty) = \infty$.*

4.5. **Barański and Post-Barański Maps, II.** In view of Theorem 4.11 and Theorem 4.12 from Section 4.3, all post-Barański maps are jump-like. Therefore, their fractal and dynamical properties proved in [39] can be briefly comprised in the following.

Theorem 4.17. *Proposition 4.13, Theorem 4.15, Theorem 4.14 and Theorem 4.16 are true with jump-like conformal maps replaced by post-Barański maps.*

Since $H(J(\tilde{f})) = J(f)$, where f is a Barański map and H is the rational function involved in the formula defining it, one can deduce relatively easily the following.

Theorem 4.18. *If f is a Barański map, then:*
(a) *If $h < 1$, then $P^h(J(f)) > 0$ and $\mathrm{P}^h|_{J(f)}$ is σ-finite, while $\mathrm{H}^h(J(f)) = 0$.*
(b) *If $h = 1$, then $\mathrm{P}^h(J(f)) > 0, \mathrm{H}^h(J(f)) > 0$ and both measures restricted to $J(f)$ are σ-finite.*
(c) *If $h > 1$, then $\mathrm{H}^h(J(f)) > 0$ and $\mathrm{H}^h|_{J(f)}$ is σ-finite, while $\mathrm{P}^h(J(f)) = \infty$,*
where the Hausdorff measure and packing measure are defined by means of Euclidean metric.

Theorem 4.19. *If f is a Barański map, then:*
(a) *If $h < 1$, then $0 < \mathrm{P}^h(J(f)) < \infty$ and $\mathrm{H}^h(J(f)) = 0$.*
(b) *If $h = 1$, then $0 < \mathrm{P}^h(J(f))$, $\mathrm{H}^h(J(f)) < \infty$.*
(c) *If $h > 1$, then $0 < \mathrm{H}^h(J(f)) < \infty$ and $\mathrm{P}^h(J(f)) = \infty$,*
where the Hausdorff measure and packing measure are defined by means of spherical metric.

Call Hausdorff or packing measure on $J(f)$ geometric if it is finite and positive. Using the semi-conjugacy $H \circ \tilde{f} = f \circ H$ established in Section 4.3, the following result is an immediate consequence of Theorem 4.17.

Theorem 4.20. *If f is a Barański map then there exists a unique f-invariant probability measure equivalent to a conformal measure.*

5. Hyperbolic Meromorphic and Entire Functions

The paper [51], whose contents we briefly describe in this section, presents a new uniform approach to the theory of thermodynamic formalism for a very wide class of meromorphic functions of finite order. The key point is to associate to a given meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ a suitable Riemannian metric $d\sigma = \gamma |dz|$. One then uses Nevanlinna's theory to construct conformal measures for the potentials $-t \log |f'|_\sigma$ and control the corresponding Perron–Frobenius operator's. Here

$$|f'(z)|_\sigma = |f'(z)| \frac{\gamma \circ f(z)}{\gamma(z)}$$

is the norm of the derivative of f with respect to the metric $d\sigma$. With this tool in hand one is able to obtain geometric information on the Julia set $J(f)$ and on the radial (or conical) Julia set

$$J_r(f) = \{z \in J(f) \ : \ \liminf_{n\to\infty} |f^n(z)| < \infty\} \, .$$

In [51], in contrast to the works reported on in the previous sections, no periodicity is needed nor Walters expanding property is assumed to be satisfied. We now give a fairly precise description of the results obtained in [51].

5.1. **Thermodynamical Formalism.** The main idea, which among others allows one to abandon periodicity, is to associate to a given meromorphic function f a Riemannian conformal metric $d\sigma = \gamma |dz|$ with respect to which the Perron-Frobenius-Ruelle (or transfer) operator

$$\mathcal{L}_t \phi(w) = \sum_{z \in f^{-1}(w)} |f'(z)|_\sigma^{-t} \phi(z) \tag{5.1}$$

is well defined and has all the required properties that make the thermodynamical formalism work. Such a good metric can be found for meromorphic functions $f : \mathbb{C} \to \overline{\mathbb{C}}$ that are of finite order ρ and do satisfy the following growth condition for the derivative:

Rapid derivative growth: There are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa > 0$ such that

$$|f'(z)| \geq \kappa^{-1}(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \tag{5.2}$$

for all finite $z \in J(f) \setminus f^{-1}(\infty)$. Throughout the entire paper we use the notation

$$\alpha = \alpha_1 + \alpha_2.$$

This condition is very general and forms the second main idea of [51]. It is comfortable to work with and relatively easy to verify for a large natural class of functions which include the entire exponential family λe^z, certain other periodic functions ($sin(az + b)$, $\lambda \tan(z)$, elliptic functions...), the cosine-root family $\cos(\sqrt{az+b})$ and the composition of these functions with arbitrary polynomials. The Riemannian metric σ we mentioned above is this.

$$d\sigma(z) = (1 + |z|^{\alpha_2})^{-1}|dz|.$$

The third and fourth basic ideas in [51] were to revive the old method of construction of conformal measures from [16] (which itself stemmed from the work of Sullivan [66], [67], [68] and Patterson [56]) and to employ results and methods coming from Nevanlinna's theory. These allow to perform the construction of conformal measures and to get a good control of the Perron-Frobenius-Ruelle operator, resulting in the following key result of [51].

Theorem 5.1. *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an arbitrary hyperbolic meromorphic function of finite order ρ that satisfies the rapid derivative growth condition (5.2), then for every $t > \frac{\rho}{\alpha}$ the following are true.*

1. *The topological pressure* $\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n(1\!\!1)(w)$ *exists and is independent of* $w \in J(f) \cap \mathbb{C}$.
2. *There exists a unique $\lambda|f'|_\sigma^t$-conformal measure m_t and necessarily $\lambda = e^{\mathrm{P}(t)}$. Also, there exists a unique Gibbs state μ_t, i.e. μ_t is f-invariant and equivalent to m_t. Moreover, both measures are ergodic and supported on the radial (or conical) Julia set.*
3. *The density $\psi = d\mu_t/dm_t$ is a continuous and bounded function on the Julia set $J(f)$.*

Note that for the existence of $e^{\mathrm{P}(t)}|f'|_\sigma^t$-conformal measures the assumption of hyperbolicity is not needed. Note also that even in the context of exponential functions (λe^z) and Walters expanding conformal maps, this result is strictly speaking new since it concerns the map f itself and not its projection onto infinite cylinder. An important case in Theorem 5.1 is when h is a zero of the pressure function $t \mapsto \mathrm{P}(t)$. In this situation, the corresponding measure m_h is $|f'|_\sigma^h$-conformal, also called simply h-conformal. Such a (unique) zero $h > \rho/\alpha$ exists provided the function f satisfies the following two additional conditions:

Divergence type: The series $\Sigma(t,w) = \sum_{z\in f^{-1}(w)} |z|^{-t}$ diverges at the critical exponent (which is the order of the function $t = \rho$; w is any non Picard exceptional value).

Balanced growth condition: There are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa > 0$ such that

$$\kappa^{-1}(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \le |f'(z)| \le \kappa(1 + |z|^{\alpha_1})(1 + |f(z)|^{\alpha_2}) \tag{5.3}$$

for all finite $z \in J(f) \setminus f^{-1}(\infty)$.

Indeed, we have the following.

Theorem 5.2. *(Bowen's formula) If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a hyperbolic meromorphic function that is of finite order $\rho > 0$, of divergence type and of balanced derivative growth with $\alpha_1 \ge 0$, then* *the pressure function* $\mathrm{P}(t)$ *has a unique zero* $h > \rho/\alpha$ *and*

$$\mathrm{HD}(J_r(f)) = h\ .$$

In addition, one easily proves that $\mathrm{HD}(J_r(f)) < 2$.

5.2. **Real Analyticity.** In the paper [51] the authors developed a new approach to the issue of real analyticity of the hyperbolic dimension of hyperbolic meromorphic functions. It allowed them to employ the method of holomorphic extensions of generalised Perron-Frobenius operators worked out in [75] (comp. [14], [44]). As the most transparent outcome of this work, the following theorem (extending the results from [74], [44] and [14]) has been proved.

Theorem 5.3. *Let* $f : \mathbb{C} \to \overline{\mathbb{C}}$ *be either the sine, tangent, exponential or the Weierstrass elliptic function and let* $f_\lambda(z) = f(\lambda_d z^d + \lambda_{d-1}z^{d-1} + ... + \lambda_0)$, $\lambda = (\lambda_d, \lambda_{d-1}, ..., \lambda_0) \in \mathbb{C}^* \times \mathbb{C}^d$. *Then the function*

$$\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$$

is real-analytic in a neighbourhood of each parameter λ^0 *giving rise to a hyperbolic function* f_{λ^0}.

This result is an example of an application of the general Theorem 5.4 that we present below. Recall that the Speiser class $\mathcal{S}$ is the set of meromorphic functions $f : \mathbb{C} \to \overline{\mathbb{C}}$ that have a finite set of singular values $\mathrm{Sing}(f^{-1})$. We will work in the subclass $\mathcal{S}_\prime$ which consists in the functions $f \in \mathcal{S}$ that have a strictly positive and finite order $\rho = \rho(f)$, and that are of divergence type. Fix Λ, an open subset of $\mathbb{C}^N$, $N \geq 1$. Let

$$\mathcal{M}_\Lambda = \{f_\lambda \in \mathcal{S}_0\,;\, \lambda \in \Lambda\}\ ,\ \ \Lambda \subset \mathbb{C}^N,$$

be a holomorphic family such that the singular points $sing(f_\lambda^{-1}) = \{a_{1,\lambda}, ..., a_{d,\lambda})$ depend continuously on $\lambda \in \Lambda$. Consider furthermore $\mathcal{H} \subset \mathcal{S}_\prime$, the set of hyperbolic functions from $\mathcal{S}_\prime$ and put

$$\mathcal{HM}_\Lambda = \mathcal{M}_\Lambda \cap \mathcal{H}.$$

We say that $\mathcal{M}_\Lambda$ *is of bounded deformation* if there is $M > 0$ such that for all $j = 1, ..., N$

$$\left|\frac{\partial f_\lambda(z)}{\partial \lambda_j}\right| \leq M|f'_\lambda(z)|\ \ ,\quad \lambda \in \Lambda\ \ \textit{and}\ \ z \in J(f_\lambda). \tag{5.4}$$

We also say that $\mathcal{M}_\Lambda$ is of *uniformly balanced growth* provided every $f \in \mathcal{M}_\Lambda$ satisfies the condition (5.3) with some fixed constants $\kappa, \alpha_1, \alpha_2$.

Theorem 5.4. *Suppose $\phi_{\lambda^0} \in \mathcal{HM}_\Lambda$ and that $U \subset \Lambda$ is an open neighbourhood of λ^0 such that $\mathcal{M}_\mathcal{U}$ is of uniformly balanced growth and of bounded deformation. Then the map*

$$\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$$

is real-analytic near λ^0.

6. Non-hyperbolic Barański maps

In this section we report on the class $\mathcal{R}$ of function of the form

$$f(z) = R \circ \exp(z), \tag{6.1}$$

where R is a non-constant rational function and the singular set $\mathrm{Sing}(f^{-1})$ of f^{-1} is allowed to intersect the Julia set. The results we described were obtained by Kotus and Skorulski in [36],[37], [38] [63]. Note that formally the above function f is of the form of Barański functions explored in the last three sections of Chapter 4, with $R = H$ and Q being the identity map. The singular set $\mathrm{Sing}(f^{-1})$ clearly consists of (finitely many) critical values of f and two asymptotic values $R(0), R(\infty)$. The class $\mathcal{R}$ is defined to consist of those functions given by (6.1) for which there exists an integer $q \geq 0$ such that

$$\infty \in f^q\Big(\{R(0), R(\infty)\}\Big).$$

The class $\mathcal{H}$ instead consists of those functions defined by (6.1) for which $\infty \notin \{R(0), R(\infty)\}$ and the forward orbit of the set $\{R(0), R(\infty)\}$ under iterations of f stays within a positive distance from the Julia set $J(f)$. In what follows, we will consider the classes $\mathcal{R}$ and $\mathcal{H}$ separately.

6.1. **The Class $\mathcal{R}$.** First, single out from $\mathcal{R}$ the family $\mathcal{P}$ of all its entire functions, $f \in \mathcal{P}$ if and only if $R^{-1}(\infty) \in \{0, \infty\}$. Set then $Q := \mathcal{R} \setminus \mathcal{P}$. Let $\mathcal{Q}_1$ be the family of all those functions in $\mathcal{Q}$ for which exactly one of the asymptotic values $R(0)$ or $R(\infty)$ is eventually mapped onto ∞. Put $\mathcal{Q}_2 := \mathcal{Q} \setminus \mathcal{Q}_1$, i.e. both $R(0)$ and $R(\infty)$ eventually land at ∞. Put

$$P_1(f) := \overline{\Theta^+(\mathrm{Sing}(f^{-1})) \setminus \Theta^+(\{R(\infty)\})} \tag{6.2}$$

and

$$P_2(f) := \overline{\Theta^+(\mathrm{Sing}(f^{-1})) \setminus \Theta^+(\{R(0), R(\infty)\})}. \tag{6.3}$$

Let χ denote the chordal metric on the Riemann sphere $\overline{\mathbb{C}}$. Set

$$\begin{aligned}
\mathcal{P}^* :=& \{f \in \mathcal{P} : \mathrm{dist}_\chi(P_1(f), J(f)) > 0\},\\
\mathcal{Q}_1^* :=& \{f \in \mathcal{Q}_1 : \mathrm{dist}_\chi(P_1(f), J(f)) > 0\},\\
\mathcal{Q}_2^* :=& \{f \in \mathcal{Q}_2 : \mathrm{dist}_\chi(P_2(f), J(f)) > 0\}.
\end{aligned}$$

and

$$\mathcal{R}^* := \mathcal{P}^* \cup \mathcal{Q}_1^* \cup \mathcal{Q}_2^*. \tag{6.4}$$

Exactly as in Section 2, the map f is projected down to the map F of the cylinder $\mathcal{C} = \mathbb{C}/\sim$ and $J_r(F)$ is defined to be composed of all those points $z \in J(F)$ whose ω-limit set $\omega(z)$ is not contained in the union of $\{-\infty, +\infty\}$ and the closure of the postsingular set of f. Using method of $K(V)$ set (see Appendix 1) the following two theorems have been proved in [63] (comp.[62]).

Theorem 6.1. *Let $f \in \mathcal{R}^*$. Then*

a) *There exists h-conformal measure m on $J(F)$ for F such that m is atomless and $m(J_r(F)) = 1$.*
b) *If m' is a probabilistic measure on $J(F)$ which is t-conformal for some $t > 1$, then $m' = m$.*
c) *F is ergodic with the respect to the measure m.*
d) *$1 < \mathrm{HD}(J_r(F)) = h < 2$.*

and

Theorem 6.2. *If $f \in \mathcal{R}^*$ and $t > 1$, then there exist a unique α_t and a unique (t,α_t)-conformal measure m_t for $F : J(F) \to J(F) \cup \{\infty\}$. In addition, $m_t(J_r(F)) = 1$ and the map F is ergodic with respect to the measure m_t.*

Restricting our attention to the class $\mathcal{Q}_2^* \subset \mathcal{R}^*$, we would like to report that J.Kotus, making use of Theorem 6.1, was able to prove the following.

Theorem 6.3. *If $f \in \mathcal{Q}_2^*$, then $0 < \mathrm{H}^h(J_r(F)) < \infty$, and, in particular, H^t and the conformal measure m (coming from Theorem 6.1) coincide up to a multiplicative constant.*

Theorem 6.4. *The packing measure P^h, restricted to $J_r(F)$, is locally infinite at every point of $J_r(F)$.*

Theorem 6.5. *For every function $f \in \mathcal{Q}_2^*$ there exists exactly one Borel probability F-invariant measure μ absolutely continuous with respect to the h-conformal measure m. Moreover μ is ergodic and equivalent to m.*

6.2. **The Class $\mathcal{H}$.** Let $B = \Pi(f^{-1}(\infty))$, where Π is the canonical projection from $\mathbb{C}$ onto the cylinder $\mathbb{C}/\sim$. Notice that if $f \in \mathcal{H}$, then there exists $\tilde{K} > 0$ such that

$$\tilde{K} < \mathrm{Re}J(F) < \tilde{K}. \tag{6.5}$$

Observe that every point in B is a discontinuity point of F. We call the points of B the poles of F. The set of critical points of F we denote by

$$\mathrm{Crit}(F) := \pi(\mathrm{Crit}(f)).$$

Let

$$J_{bd}(F) = \{z \in J(F) : \sup_{n\geq 1} |\mathrm{Re}(f^n(z))| < \tilde{K}, \quad \inf_{n\geq 1}\{|f^n(z) - b_i|\} > 0,\ b_i \in B\}.$$

Let c be a critical point of f eventually mapped onto ∞, i.e. there exists $k \geq 2$ and a pole $b \in B$ such that $f^{k-1}(c) = b$. Then there exist $A = A(f^{k-1}, c) \geq 1$ and $p \geq 2$ such that

$$A^{-1}|z - c|^p \leq |f^{k-1}(z) - f^{k-1}(c)| \leq A|z - c|^p. \tag{6.6}$$

Then $p = p(f^{k-1}, c)$ we call the order of f^{k-1} at the critical point c. Let q_b denote the multiplicity of the pole b. Define

$$p := \sup\{p_c : \ c \in \mathrm{Crit}(f) \ \text{ s.t. } f^k(c) = \infty \text{ for some } k \in I\!N\}$$
$$q := \sup\{q_b : \ b \in B \text{ s.t. } \exists c \in \mathrm{Crit}(f),\ \exists k \in I\!N \text{ and } f^k(c) = b\}.$$

Let $J_r(F)$ be the set of the points in $J(F)$ whose ω-limit set is not contained in B. The following results have been proved in [37].

Theorem 6.6. *If $f \in \mathcal{H}$ then* $\mathrm{HD}(J_{bd}(F)) > \frac{pq}{pq+1}$.

Theorem 6.7. *If $f \in \mathcal{H}$, then* $\mathrm{HD}(J_r(f)) = \mathrm{HD}(J_r(F)) = h \in (\frac{pq}{pq+1}, 2)$.

Theorem 6.8. *If $f \in \mathcal{H}$, then the h-conformal measure m is a unique t-conformal probability measure, with $t > \frac{pq}{pq+1}$, for $F : J(F) \to J(f) \cup \{\infty\}$. In addition, m is conservative and ergodic.*

Concerning Gibbs and equilibrium states, we would like to end this section by bringing reader's attention to the fact that, as it was shown in [36], with the same methods as those worked out in [50], one can prove all the same results for the functions in the class $\mathcal{H}$ and appropriate potentials, as those stated in Section 3.2 for elliptic functions.

7. Transcendental entire and meromorphic functions - the Lebesgue measure outlook

7.1. **The Lebesgue Measure of Points Escaping to ∞.** In the previous chapter we have explored in detail the fractal and dynamical properties of some significant classes of transcendental entire and meromorphic functions. We have frequently supplied the reader with the information concerning the Lebesgue measure of Julia set and points escaping to infinity. In this chapter we would like to deal with much bigger classes ($\mathcal{S}$ and $\mathcal{B}$) of transcendental functions and to discuss the Lebesgue measure of corresponding Julia sets and points escaping to ∞. Recall that

$$\mathcal{S} = \{f : \mathbb{C} \to \overline{\mathbb{C}}: \text{ transcendental meromorphic s.t. } Sing(f) \text{ is finite}\}$$

and

$$\mathcal{B} = \{f : \mathbb{C} \to \overline{\mathbb{C}}: \text{ transcendental meromorphic s.t. } Sing(f) \text{ is bounded}\},$$

where

$$Sing(f^{-1}) = \{z \in \mathbb{C}: z \text{ is a finite singularity of } f^{-1}\}.$$

Recall also that

$$P(f) = \{z \in \mathbb{C}: z \text{ is a finite singularity of } f^{-n} \text{ for some n} \geq 1\}$$

and for every $n \geq 1$ define

$$I_n(f) = \{z \in \mathbb{C}: \lim_{m \to \infty} f^{mn}(z) = \infty\}.$$

Let l_2 be the Lebesgue measure on $\mathbb{C}$. For a long time, it was expected in conformal dynamics that either $J(f) = \overline{\mathbb{C}}$ or $l_2(J(f)) = 0$. Whereas this is still an open problem in the class of all rational functions, for transcendental functions this dichotomy fails. This failure was already established in 1987 by C. McMullen who proved in [53] the following remarkable fact.

Theorem 7.1. *Let for all $a, b \in \mathbb{C}$, $f_{a,b}(z) = \sin(az + b)$. If $a \neq 0$, then $l_2(I_1(f_{a,b})) > 0$. Consequently $l_2(J(f_{a,b})) > 0$.*

We would like however to add that (see Section 2.5) I. Coiculescu and B. Skorulski proved in [13] that the set of points not escaping to infinity under the action of $f_{a,b}, a \neq 0$ has Hausdorff dimension less than 2. In the opposite direction A. Eremenko and M. Lyubich in [26] formulated a rather general sufficient condition for the set $I_1(f)$ to have zero Lebesgue measure. Given $r, R > 0$ let $\Theta_R(r, f)$ denote the linear measure of the set $\{\theta : |f(re^{i\theta})| < R\}$. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ satisfy the following condition.

$$E(f, R) := \liminf_{r \to \infty} \frac{1}{\ln r} \int_1^r \Theta_R(r, f) \frac{dt}{t} > 0. \tag{7.1}$$

The sufficient condition of A. Eremenko and M. Lyubich is this.

Theorem 7.2. *If $f \in \mathcal{B}$ is a transcendental entire function and $E(f, R) > 0$ for all $R > 0$, then $l_2(I_1(f)) = 0$.*

The assumption of this theorem has been checked for all entire functions of finite order which have at least one finite logarithmic singularity, in particular for all exponential maps $z \in \lambda \exp(z)$, $\lambda \in \mathbb{C} \setminus \{0\}$, studied in Chapter 2.

Theorem 7.2 was extended in [34] to the class of meromorphic functions (in $\mathcal{B}$) under some additional conditions on the orders of poles, their residual and principal parts. In particular, these assumptions are satisfied for all elliptic functions. However in [41] (see Section 3.1) we have proved for these maps much stronger result, that $\mathrm{HD}(I_1(f)) < 2$ with completely different methods. Another important class of meromorphic functions fulfilling the assumptions of Keen and Kotus from [34] is formed by the maps $\lambda\tan(z)$, $\lambda \in \mathbb{C}\setminus\{0\}$.

By other methods, results of a different kind were obtained by Bock in [11]. First, for every $n \geq 1$, he introduced the class

$$\mathcal{B}_n := \{f : \mathbb{C} \to \bar{\mathbb{C}} :\ \text{transcendental meromorphic s.t. } \bigcup_{i=0}^{n-1} f^i(Sing(f^{-1})) \cap \mathbb{C} \text{ is bounded}\}.$$

Notice that $\{\mathcal{B}_n\}_{n=1}^{\infty}$ is a descending sequence and $\mathcal{B}_1 = \mathcal{B}$. A plane set E is called *thin at* ∞ if its density is bounded away from 1 in all sufficiently large disks, that is, if there exist a positive R and ϵ such that for all complex z and every disc $B(z,r)$ of centre z and radius $r > R$

$$\text{density}(E, B(z,r))) = \frac{l_2(E \cap B(z,r))}{D(z,r)} < 1 - \epsilon.$$

Bock's first result is this.

Proposition 7.3. *Let $\omega \in \mathbb{C}\setminus\{0\}$ and let $f \in \mathcal{B}_n$ be periodic with period ω. If there is $r > 0$ such that the set $f^{-n}(B(0,r))$ is thin at ∞, then $l_2(I_n(f)) = 0$.*

It easily follows from this proposition that the Lebesgue measure of the set of points escaping to infinity under any fixed member of the exponential or tangent family is equal to zero.

Let us now formulate Bock's results going in the opposite direction. For any $s \in (-\pi,\pi)$, $\alpha > 0$ and $K > 0$ let

$$W_{\alpha,K}(s) := \{z \in D(0,1) : \exists v \in \{-1,0,1\} \quad |\arg(z) - s - 2v\pi| \leq K/(-\log(|z|))^\alpha\}.$$

The announced theorem is this.

Theorem 7.4. *Let $f \in \mathcal{B}_n$, $n \geq 1$. Suppose that there exist $\alpha > 0, t_0 > 0, R_0 > 1, N \geq 1$, and angles $s_0, \ldots, s_{N-1} \in [-\pi,\pi)$ such that for all $t > t_0$*

$$B(0,R_0) \setminus \bigcup\{W_{\alpha,t}(s_v) : \quad v \in \{0,\ldots,N-1\}\} \subset f^{-n}(B(0,e^t)).$$

Then $l_2(I_n(f)) > 0$.

This theorem along with Proposition 7.3 has rather unexpected consequences for the tangent family. Let $g_\lambda(z) = \lambda\tan(z), \lambda \in \mathbb{C}\setminus\{0\}$. For every $p \geq 1$ let

$$\mathcal{C}_p = \{\lambda \in \mathbb{C}\setminus\{0\}, \quad g_\lambda^p(\pm\lambda i) = \infty\}.$$

It follows from Theorem 7.4 that if $\lambda \in \mathcal{C}_p$, then $l_2(I_{p+1}(g_\lambda)) > 0$, whereas we already noted, that $l_2(I_1(g_\lambda)) = 0$ by Proposition 7.3.

7.2. **Milnor's Metric Attractors.** Let (M, ρ) be a compact Riemann manifold, let X be an arbitrary subset of M and let $T : X \to M$ be a continuous map. Put $X_\infty = \bigcap_{n=0}^\infty T^{-n}(M)$. Then $T(X_\infty) \subset X_\infty$. A closed set $A \subset M$ is called a Milnor's metric attractor of T provided that there exists a Borel set $B \subset X_\infty$ with the same Lebesgue measure as X_∞ and such that $\lim_{n\to\infty} \rho(T^n(z), A) = 0$ or equivalently $\omega(z) \subset A$ for all $z \in B$. Notice that any countable intersection of Milnor's attractors is a Milnor's attractor. A is called a minimal Milnor's attractor if it does not contain any proper subset which is a Milnor's attractor.

The structure of Milnor's attractors for transcendental meromorphic functions was described fairly completely in [11]. Let us formulate it here. We treat $\overline{\mathbb{C}}$ as a compact Riemannian manifold with the spherical metric. Bock's result is this.

Theorem 7.5. *Let f be a transcendental meromorphic function. Then at least one of the following statements holds:*

(a) *the set $\overline{P(f)} \cup \{\infty\}$ is a Milnor's attractor for $f : J(f) \to \overline{\mathbb{C}}$*
(b) *$J(f) = \overline{\mathbb{C}}$ and $\overline{\mathbb{C}}$ is the minimal Milnor's attractor for $f : J(f) \to \overline{\mathbb{C}}$. Furthermore, the map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is conservative with respect to the Lebesgue measure.*

We would like to note that an analogous result in the class of rational functions has been proved by M. Lyubich in [45]. In [40] we have provided an alternative proof of a part (b) of Bock's theorem. The precise formulation of our result is the following.

Proposition 7.6. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that $J(f) = \overline{\mathbb{C}}$. If $l_2(\{z : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}) = 0$, then f is ergodic and conservative with respect to the Lebesgue measure.*

As an immediate consequence of Theorem 7.5, we get the following result noted in Bock's paper [11].

Corollary 7.7. *If the set $Sing(f^{-1})$ is finite and each singularity of f^{-1} is preperiodic but not periodic then the statement (b) holds for f. In particular $\overline{\mathbb{C}}$ is the minimal attractor for f.*

For an entire transcendental function f it is impossible to satisfy the assumption of this criterion because in this case ∞ is a transcendental singularity which does not belong to the domain of f and therefore can not be a preperiodic point of f. However Bock proved in [10] the following.

Corollary 7.8. *Let f be non-constant entire function. Suppose that $Sing(f^{-1})$ is finite and each singularity of f^{-1} is preperiodic but not periodic. Then either ∞ or $\overline{\mathbb{C}}$ is a minimal Milnor's attractor for $f : \mathbb{C} \to \mathbb{C}$. In the later case it even holds that $\omega(z) = \mathbb{C}$ for Lebesgue a.e. $z \in \mathbb{C}$.*

The first example of a transcendental function for which $\overline{P(f)}$ is a metric attractor is due to M. Rees who proved in [59] that if $E(z) = e^z$, then $\{E^n(0)\}_{n=0}^{\infty} \cup \{\infty\}$ is a Milnor's attractor for f. She also proved that the exponential map is not recurrent, i.e. that there exists a Borel set $B \subset \overline{\mathbb{C}}$ with positive Lebesgue measure such that $B \cap f^n(B) = \emptyset$ for all $n \geq 1$. M. Lyubich has clarified the situation completely by proving in [46] the following (see also Section 2.1)

Theorem 7.9. *$\{E^n(0)\}_{n=0}^{\infty} \cup \{\infty\}$ is the minimal Milnor's attractor's for the exponential function $E(z) = e^z$. In addition, E is not ergodic with respect to the Lebesgue measure.*

J.M. Hemke has provided in [30] several sufficient conditions for the alternative (a) from Theorem 7.5 to hold. We present now some of his results. The most general of Hemke's sufficient conditions is this.

Theorem 7.10. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function, $A \subset \mathbb{C}$ finite and $G \subset \mathbb{C}$, such that*

(a) *there exists $\epsilon > 0$, such that the map*

$$\overline{s} : G \mapsto A \cup \{0\} : z \to \begin{cases} s & \text{if} \quad \exists s \in A : |f(z) - s| \leq \exp(|z|^{\epsilon}) \\ 0 & \text{if} \quad |f(z)| \geq \exp(|z|^{\epsilon}) \end{cases}$$

is well defined and there are $\delta_1, \delta_2 \in \mathbb{R}$, such that for all $z \in G$,

$$|z|^{\delta_1} \leq \left| \frac{f'(z)}{f(z) - \overline{s}(z)} \right| \leq |z|^{\delta_2};$$

(b) *there exists $B > 1$ and $\beta \in (-\infty, 1)$, such that for every measurable set $D \subset \{z : \text{dist}(z, \mathbb{C} \setminus G) \leq 2|z|^{-\delta_1}\}$,*

$$l_2(D) \leq B \text{diam}(D) \sup_{z \in D} |z|^{\beta};$$

(c) $\lim_{m \to \infty} f^m(s) = \infty$ *and $B(f^m(s), 2|f^m(s)|^{\tau}) \subset G$ for some $\tau > \beta$, almost all $m \in \mathbb{N}$ and all $s \in A$.*

Then the set $T(f) := \{z : \omega(z) \subset \overline{\Theta^+(A)}\}$ has positive measure. In particular, if $A = P(f)$, then $\overline{P(f)}$ is a Milnor's attractor for f.

A point $z \in \mathbb{C}$ is said to escape exponentially, if $\lim_{n\to\infty} f^n(z) = \infty$ and

$$|f^n(z)| \geq \exp(|f^{n-1}(z)|^\delta)$$

for some $\delta > 0$, and all integers $n \geq 1$. Theorem 7.10 was applied in [30] to the class of function of the form

$$f(z) = \int_0^z P(t)\exp(Q(t))dt + c \tag{7.2}$$

with polynomials P and Q and $c \in \mathbb{C}$, such that Q is non-constant and P not identically zero. These functions have at most $\deg(Q)$ finite asymptotic values and $\deg(P)$ critical points. Namely:

Theorem 7.11. *Let f be a meromorphic function of the form (7.2). Suppose that all finite asymptotic values escape exponentially. Then the Julia set $J(f)$ has positive Lebesgue measure and $\overline{P(f)}$ is a Milnor's attractor for $f : J(f) \to J(f) \cup \infty$. In addition, if $\deg(Q) \geq 3$, then $l_2(F(f)) < \infty$.*

The function $f(z) = \exp(z^3 + az + b)$, where $a = \left(\frac{27\pi^2}{16}\right)^{1/3}$ and $b = \log(\sqrt{a/3})$ was proved in [30] to satisfy all the assumptions of Theorem 7.10. Under additional assumption Hemke was able to identify the Milnor's minimal attractor by proving the following.

Theorem 7.12. *Let f satisfy the assumptions of Theorem 7.11. Suppose also that every critical point either escapes exponentially, is pre-periodic or is contained in an attractive Fatou-component. Then $\overline{\Theta^+(A)}$ is the minimal Milnor's attractor for f, where A denotes the set of finite asymptotic values of f.*

All the functions defined by (7.2) are entire and have a rational Schwarzian derivative. The asymptotic behaviour of such functions is understood fairly well. Theorem 7.11 continuous to be true for all such functions satisfying some additional natural assumptions (see [30]).

It is well-known and easy to see that if f is a transcendental meromorphic function and $\bigcup_{n\geq 0} f^{-n}(\infty)$ contains at least three distinct points, then

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

It may in particular happen that if all singular values of f belong to $J(f)$, then after finitely many iterates all these singular values land on poles. Such situation was thoroughly studied for functions of the form

$$f(z) = \frac{a\exp(z^p) + b\exp(-z^p)}{c\exp(z^p) + d\exp(-z^p)},$$

$p \in \mathbb{N}$, $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$. All these functions are said to form the class $\mathcal{F}$. Given $z \in \mathbb{C}$ we put

$$\Theta^+(z) = \{f^n(z)\}_{n=1}^\infty.$$

We say that $f \in \mathcal{F}$ satisfies condition (C1) if the following statements are satisfied

(i) The asymptotic values $\frac{a}{c}$ and $\frac{b}{d}$ are finite and eventually mapped onto ∞ i.e. there are $q_1, q_2 \in I\!N$ such that $f^{q_1-1}(\frac{a}{c}) = f^{q_2 1}(\frac{b}{d}) = \infty$

(ii) If zero is a critical point, then either the trajectory of zero is bounded or zero is eventually mapped onto ∞ and

$$\Theta^+(0) \cap \Big(\Theta^+\Big(\frac{a}{c}\Big) \cup \Theta^+\Big(\frac{b}{d}\Big)\Big) = \{\infty\}.$$

We say that a function $f \in \mathcal{F}$ satisfies condition (C2) if condition (C1) holds and, if zero is a critical point and it is not eventually mapped onto infinity, then

$$l_2(\{z \in J(f) : \omega(z) \subset \overline{\Theta^+(0)}\}) = 0.$$

B. Skorulski proved in [61] and [62] the following.

Theorem 7.13. *If f satisfies condition (C1), then*

(i) *there exists a set $E \subset J(f)$ with a positive Lebesgue measure such that $\omega(z) = P_{Asymp(f)} := \Theta^+(\frac{a}{c}) \cup \Theta^+(\frac{b}{d})$ for all $z \in E$, and the action of f is not ergodic on $J(f)$ with respect to the Lebesgue measure.*

(ii) *if $q_1 \neq q_2$ then $l_2(I_n(f)) = 0$ for every $n \geq 1$.*

Moreover, if condition (C2) is satisfied, then in particular $P_{Asymp(f)}$ is the Milnor's attractor of f.

Corollary 7.14. *Let $p > 1, k \in Z\!\!\!Z$ and*

$$f(z) = \sqrt[p]{\frac{\pi i}{2} + k\pi i}\,\frac{\exp(z^p) - \exp(-z^p)}{\exp(z^p) + \exp(-z^p)}.$$

Then the Fatou set $F(f)$ is nonempty since zero is a superattracting fixed point. The asymptotic values $\xi_1 = \sqrt[p]{\pi i/2 + k\pi i}$, $\xi_2 = -\sqrt[p]{\pi i/2 + k\pi i}$ are mapped onto ∞ i.e. $f(\xi_1) = f(\xi_2) = \infty$. The Julia set $J(f)$ has positive measure and for almost all $z \in J(f)$, $\omega(z) = P_{Asymp(f)}$. In particular, $P_{Asymp(f)}$ is the Milnor's attractor of f. Moreover, f is not ergodic with respect to the Lebesgue measure on $J(f)$.

Theorem 7.13 has been recently extended by Skorulski in [63] to the large class $\mathcal{R}$ thoroughly treated in Section 6.1. Sticking with the class of functions dealt with in Section 6.1, note that each function $f \in \mathcal{P}$ can be represented in the following form

$$f(z) = \sum_{j=-n_2}^{n_1} a_j e^{jz}.$$

So, if $n_1, n_2 > 0$, then f has no finite asymptotic values. Denote the class of those functions by $\mathcal{P}_2$ and its complement (in $\mathcal{P}$) by $\mathcal{P}_1$. Put also

$$\mathcal{R}_1 = \mathcal{Q}_1 \cup \mathcal{P}_1 \quad \text{and} \quad \mathcal{R}_2 = \mathcal{Q}_2 \cup \mathcal{P}_2.$$

For every $f \in \mathcal{R}$, put

$$P_{Asymp}(f) = \Theta^+(\{R(0), R(\infty)\}).$$

The combined results of Skorulski from [62] and [63] give the following.

Theorem 7.15. *If $f \in \mathcal{R}_2$, then $P_{Asymp}(f)$ is a Milnor's attractor for f. In particular, the Lebesgue measure of $J(f)$ is positive.*

Let

$$\mathcal{R}_2^* := \{f \in \mathcal{R}_2 : \mathrm{dist}_\chi(P_2(f), J(f)) > 0\}.$$

B. Skorulski has proved in [63] the following.

Theorem 7.16. 1. *If $f \in \mathcal{R}_2^*$, then $P_{\mathrm{Asymp}}(f)$ is the minimal Milnor's attractor for f.*

2. *If $q_1 \neq q_2$, then the Lebesgue measure of the set*

$$I_n(f) := \{z : \lim_{k \to \infty} f^{nk}(z) = \infty\}$$

is equal to zero for all $n \in I\!N$.

3. *If $f \in \mathcal{R}_2^* \cap \mathcal{Q}$, then there does not exist any f-invariant measure on $J(f)$ which is absolutely continuous with the respect to the Lebesgue measure and finite on all compact subsets of $J(f)$.*
4. *If $f \in \mathcal{R}_2^*$ and $n_1 = n_2$, then f is not ergodic on $J(f)$ with the respect to the Lebesgue measure.*

7.3. **The Lebesgue Measure of Julia Sets.** It is by now a standard fact (see [68] for the first proof) that the Julia set of an expanding rational function has Lebesgue measure zero. As shows Theorem 7.1 with $a, b \neq 0$ sufficiently small in moduli, this property already fails in the class of expanding sine maps. Let us however see what can be said assuming in addition that the Julia set is thin at ∞. In order to avoid any confusion let us introduce the following two classes of functions.

$$\mathcal{E} := \{f : \mathbb{C} \to \mathbb{C} : f \text{ is entire such that } \ dist(\overline{P(f)}, J(f)) > 0\}$$

and

$$\mathcal{E}_0 := \{f : \mathbb{C} \to \mathbb{C} : f \text{ is entire such that } \overline{P(f)} \text{ is compact, } \overline{P(f)} \cap J(f) = \emptyset\}.$$

Obviously $\mathcal{E}_0 \subset \mathcal{E}$. The following theorem was first proved by C. McMullen in [53] for the class $\mathcal{E}_0$ and by G. Stallard in [64] for the class $\mathcal{E}$.

Theorem 7.17. *If $f \in \mathcal{E}$ and B is a measurable completely invariant subset of $J(f)$ such that B is thin at ∞, then $l_2(B) = 0$. In particular if $J(f)$ is thin at ∞ then $l_2(J(f)) = 0$.*

Going beyond hyperbolicity but still keeping a rather general setting, we formulate the following two remarkable results, the first one proved by A. Eremenko and M. Lyubich in [26] and the second proved by B. Skorulski in [61]

Theorem 7.18. *Suppose that $f \in \mathcal{S}$ is a transcendental entire function such that $E(f,R) > 0$ for all $R > 0$. Assume that the orbit of every finite singularity of f^{-1} is either absorbed by a repelling cycle or converges to an attracting or to a neutral rational cycle. Then either $J(f) = \overline{\mathbb{C}}$ or $l_2(J(f)) = 0$.*

Theorem 7.19. *Let $f \in \mathcal{F}$. If one of the two asymptotic values of f is mapped onto ∞, while the second asymptotic value and the critical point of f are in the Fatou set, then $l_2(J(f)) = 0$.*

8. Appendix 1: (K(V) method of constructing semiconformal measures)

Given a continuous map $T : X \to X$ from a topological space X into itself, the map T is said to be non-open at the point $x \in X$ if and only if for every open neighbourhood V of x there exists an open set $U \subset V$ such that $T(U)$ is not open. The set of all points in X at which T is not open, is denoted by $NO(T)$. A point $c \in X$ is said to be critical of T if there is no open neighbourhood W of c such that the map $T_{|W}$ is one-to-one. The set of all critical points of T is denoted by $\mathrm{Crit}(T)$.

Now let X be a compact subset of the extended complex plane $\overline{\mathbb{C}}$. We say that $f \in \mathcal{A}(X)$ provided that $f : X \to X$ is a continuous map which can be meromorphically extended to a neighbourhood $U(f) = U(f,X)$ of X in $\overline{\mathbb{C}}$. Denote by $M_e^+(f)$ the set of all Borel probability ergodic f- invariant measures on X with positive entropy and for any Borel measure μ on X we denote $\mathrm{HD}(\mu)$ the Hausdorff dimension of the measure μ. Finally, define

$$\mathrm{DD}(X) = \sup\{\mathrm{HD}(\mu) : \mu \in M_e^+(f)\}.$$

Obviously $\mathrm{DD}(X) \le \mathrm{HD}(X)$. Proceeding as in [17] (comp. Chapter 10 of [57]) for more mature exposition) with the set $K(V)$ replaced by X, one can prove the following two useful auxiliary results.

Lemma 8.1. *Suppose that X is a compact subset of $\overline{\mathbb{C}}$, $f \in \mathcal{A}(X)$ and $f : X \to X$ has no critical points. Then for all $t \ge 0$ there exist $\mathrm{P}(t) \in \mathbb{R}$ and a Borel probability measure m_t on X with the following two properties:*

$$m_t(f(A)) \ge \int_A e^{\mathrm{P}(t)} |f'|^t m_t,$$

and, if in addition $A \subset U(f) \setminus NO(f)$, then

$$m_t(f(A)) = \int_A e^{\mathrm{P}(t)}|f'|^t m_t.$$

Lemma 8.2. *Suppose that X is a compact subset of $\overline{\mathbb{C}}$, $f \in \mathcal{A}(X)$ and $f : X \to X$ has no critical points. Then there exists $s(X) \in [0, DD(X)]$ and a Borel probability measure m called $s(X)$-semi-conformal on X with the following two properties:*

$$(a) \qquad m(f(A)) \geq \int_A |f'|^{s(X)} dm$$

for any Borel set $A \subset U(f)$ such that $f_{|A}$ is one-to-one

$$(b) \qquad m(f(A)) = \int_A |f'|^{s(X)} dm$$

for any Borel set $A \subset U(f) \setminus NO(f)$ such that $f_{|A}$ is one-to-one.

Given two compact sets $X \subset Y \subset \mathbb{C}$ and a function $f \in \mathcal{A}(Y)$, we say that X is a branchwise contained in Y provided that the following condition is satisfied. There is $\delta > 0$ such that for every $x \in X$ and for every $n \geq 0$ there exists a holomorphic inverse branch $f_x^{-n} : B(f^n(x), \delta) \to \mathbb{C}$ of f^n sending $f^n(x)$ to x and such that $f^j(f_x^{-n}(B(f^n(x), \delta)) \subset U(f, Y) \setminus NO(f)$ for all $j = 0, 1, \ldots, n$.

Proceeding similarly as in the proof of Lemma 3.2 in [77] with the same obvious modifications, one gets the following.

Lemma 8.3. *Suppose that Y is a compact subset of $\mathbb{C}$ and $f \in \mathcal{A}(Y)$. If f has no critical points in Y and X is a compact set branchwise contained in Y (notice that we do not assume X to be a forward invariant under f), then* $\mathrm{HD}(X) \leq s(Y)$.

These two lemmas are most frequently applied in the context when $f : S \to \overline{S}$ is a holomorphic map of a Riemann surface S (usually $\mathbb{C}$, $\overline{\mathbb{C}}$, $\overline{\mathbb{C}} \setminus \{0\}$) into Riemann surface $\overline{S}$ such that $\bigcup_{n \geq 0} f^{-n}(V) \supset (\overline{S} \setminus S) \cup \mathrm{Crit}(f)$. Next one defines the set

$$K(V) = \bigcap_{n=0}^{\infty} f^{-n}(\overline{S} \setminus V).$$

Then $f(K(V)) \subset K_V$ and if $\overline{S} \setminus V$ is compact, then so is the set $K(V)$. One can apply Lemma 8.1 and Lemma 8.2 with $X = K(V)$ and $U\big(f|_{K(V)}\big) = \overline{S} \setminus V$. Notice that $NO(f|_{K(V)}) \subset \partial V$. In order to get a conformal measure one lets V decrease to a set which is usually finite and one takes an arbitrary weak limit measure of semi-conformal measures produced in Lemma 8.1 or Lemma 8.2. It requires a separate proof (sometimes easy, sometimes difficult) to show that such a limit measure is conformal for $f : S \to \overline{S}$. Finally, one may control the exponent with the help of Lemma 8.2 and Lemma 8.3. However, the latter lemma is not always applicable and then one must undertake another approach in order to determine the exponent of the conformal measure produced as that weak limit.

9. Appendix 2: Martens' method of constructing σ-finite invariant measures

Suppose that X is a σ-compact metric space, ν is a Borel probability measure on X, positive on open sets, and that a measurable map $f : X \to X$ is given with respect to which measure ν is quasi-invariant, i.e. $\nu \circ f^{-1} << \nu$. Moreover, we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure ν. We also assume that $\nu(X \setminus \bigcup_{n\geq 0} A_n) = 0$, and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$\nu(f^{-k}(A_m) \cap A_n) > 0,$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [48] reads as follows.

Theorem 9.1. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure ν. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1} \frac{\nu(A)}{\nu(A_n)} \leq \frac{\nu(f^{-k}(A))}{\nu(f^{-k}(A_n))} \leq K_n \frac{\nu(A)}{\nu(A_n)},$$

then T has a σ-finite T-invariant measure μ that is absolutely continuous with respect to ν. In addition, μ is equivalent to ν, conservative and ergodic, and unique up to a multiplicative constant. Moreover, for every Borel set $A \subset X$

$$\mu(A) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \nu(f^{-k}(A))}{\sum_{k=0}^{n} \nu(f^{-k}(A_0))}.$$

This theorem is widely used in conformal dynamics in the context where ν is a conformal measure. The distortion assumption, the higher displayed formula above, is usually derived from Koebe's distortion theorem.

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Janina Kotus, Faculty of Mathematics and Information Sciences
Warsaw University of Technology
Warsaw 00-661, Poland
E-mail: janinak@impan.gov.pl

Mariusz Urbański, Department of Mathematics, University of North Texas, P.O. Box 311430, Denton, TX 76203-1430, USA
E-mail:urbanski@unt.edu
Web: http://www.math.unt.edu/~urbanski