The class of pseudo non-recurrent elliptic functions; geometry and dynamics

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Mathematics Subject Classification: *Primary 37F35, Secondary 30D05*

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1.1 Introduction

The understanding of the dynamics and geometry of elliptic functions rapidly develops since the papers [19], [20] and [15] have been published. Although these functions are relatively 'regular', they manifest such unexpected features as the fact that the Hausdorff dimension of their Julia set is always larger than 1 (see [19]) or, in the non-recurrent case, that the corresponding Hausdorff measure always vanishes whereas the packing measure, in the absence of parabolic points, is finite and positive.

In this manuscript we provide a systematic exposition of the geometric measure theory and ergodic theory of regular pseudo-nonrecurrent elliptic functions. In spite of possible associations steaming from the name, this is not a narrow class of functions. Just the opposite, it contains for example all hyperbolic and critically finite elliptic functions, and many more. In contrast to [20] we now allow critical points to land at poles and to escape to infinity. Unlike to [20], we treat in the present manuscript the conjugacy problem resulting in the complete rigidity theorem (Theorem 6.1) primarily saying that a Lipschitz conjugacy on Julia sets always extends to an affine conjugacy on $\mathbb{C}$. Its proof, although sharing some general features with the case of rational functions and conformal expanding repellers, is more subtle and involved. As one of the steps in its proof we establish in Theorem 6.3 real-analyticity of the Radon-Nikodym derivative of the of the invariant measure μ with respect to the conformal measure m.

As has been said our exposition is more systematic and elaborated than that in [20]. It deals with a larger class of functions, contains new material, new proofs, and improves many arguments used in [20].

We have already mentioned geometry (Hausdorff and packing measures) and rigidity. The third theme of the manuscript is the measurable dynamics with respect to the h-conformal measure m, where h is the Hausdorff dimension of the Julia set. The concept of conformal measure, essentially purely dynamical, and its weaker version like semi-conformality and almost conformality form the main technical tool employed in our manuscript. In particular we provide a refined proof of the existence, uniqueness and continuity of an h-conformal measure. This measure plays an essential role in showing that $\mathrm{H}^h(J(f)) = 0$ and, in the absence of parabolic periodic points, this measure turns out to coincide with the packing measure Π^h up to a multiplicative constant, i.e. although dynamically defined it gets purely geometrical characterization.

As we have said, the third theme of our manuscript is the measurable dynamics with respect to the h-conformal measure m. We prove the existence of an ergodic conservative σ-finite measure μ equivalent to m. Developing this direction, we study points of finite and infinite condensation of the measure μ, the concepts introduced in [35]. After collecting some basic facts about these points we show in Chapter 5 that ∞ is always a point of finite condensation, perhaps the most interesting fact about the measure μ. In the next section we relate points of infinite condensation with the set $\Omega(f)$ of rationally indifferent periodic points, providing in particular some sufficient conditions ($\Omega(f) = \emptyset$) for the invariant measure μ to be finite. At the end of this chapter we deal with parabolic points themselves.

1.2 General preliminaries

All the points (numbers) appearing in this paper are complex unless it is clear from the context that they are real. In particular x and y are always assumed to be complex numbers and not the real and imaginary parts of a complex number. Given a set $A \subset \mathbb{C}$ and $r > 0$, the symbol $B_e(A, r)$ denotes the Euclidean open r-neighborhood of the set A. Throughout the entire paper f^*, diam_s and $B_s(A, r)$ denote respectively the derivatives, diameters and open r-neighborhoods of the set A defined by means of the spherical metric whereas f' and diam_e are considered in the Euclidean sense. The spherical distance between any two points x and y in $\overline{\mathbb{C}}$ is denoted by $|x - y|_*$. We emphasize that when calculating f^*, we consider the spherical metric in the domain and in the codomain.

Definition 1.1 . *If $H : D \to \mathbb{C}$ is an analytic map, $z \in \mathbb{C}$, and $r > 0$, then by*

$$\mathrm{Comp}(z, H(z), H, r)$$

we denote the connected component of $H^{-1}(B_e(H(z), r))$ that contains z.

Suppose now that c is a critical point of an analytic map $H : D \to \mathbb{C}$. Then there exists $R = R(H, c) > 0$ and $A = A(H, c) \geq 1$ such that

$$A^{-1}|z-c|^{p_c} \leq |H(z) - H(c)| \leq A|z-c|^{p_c}$$

and

$$A^{-1}|z-c|^{p_c-1} \leq |H'(z)| \leq A|z-c|^{p_c-1}$$

for every $z \in \mathrm{Comp}(c, H(c), H, R)$, and that

$$H(\mathrm{Comp}(c, H(c), H, R)) = B_e(H(c), R),$$

where $p_c = p(H, c)$ is the order of H at the critical point c. In particular

$$\mathrm{Comp}(c, H(c), H, R) \subset B_e(c, (R/A)^{1/p_c}).$$

Moreover, by taking $R > 0$ sufficiently small, we can ensure that the two above inequalities hold for every $z \in B_e(c, (R/A)^{1/p_c})$ and the ball $B_e(c, (R/A)^{1/p_c})$ can be expressed as a union of p_c closed topological disks with smooth boundaries and mutually disjoint interiors such that the map H restricted to each of these interiors, is injective.

Koebe's $\frac{1}{4}$-Theorem. *If $z \in \mathbb{C}$, $r > 0$ and $H : B_e(z, r) \to \mathbb{C}$ is an arbitrary univalent analytic function, then $H(B_e(z, r)) \supset B_e(H(z), 4^{-1}|H'(z)|r)$.*

Koebe's Distortion Theorem, I (Euclidean version). *There exists a function $k : [0, 1) \to [1, \infty)$ such that for any $z \in \mathbb{C}$, $r > 0$, $t \in [0, 1)$ and any univalent analytic function $H : B_e(z, r) \to \mathbb{C}$ we have that*

$$\sup\{|H'(w)| : w \in B_e(z, tr)\} \leq k(t) \inf\{|H'(w)| : w \in B_e(z, tr)\}.$$

We put $K = k(1/2)$.

Koebe's Distortion Theorem, I (spherical version). *Given a number $s > 0$ there exists a function $k_s : [0, 1) \to [1, \infty)$ such that for any $z \in \overline{\mathbb{C}}$, $r > 0$, $t \in [0, 1)$ and any univalent analytic function $H : B_n(z, r) \to \overline{\mathbb{C}}$ such* *that the complement $\overline{\mathbb{C}} \setminus H(B_n(z,r))$ contains a spherical ball of radius s we have*

$$\sup\{|H^*(w)| : w \in B_n(z,tr)\} \le k_s(t) \inf\{|H^*(w)| : w \in B_n(z,tr)\}.$$

B_n stands in here for either Euclidean or spherical ball (if $z \neq \infty$).

The following is a straightforward consequence of these two distortion theorems.

Lemma 1.2 *Suppose that $D \subset \mathbb{C}$ is an open set, $z \in D$ and $H : D \to \mathbb{C}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B_e(H(z), 2R)$ for some $R > 0$. Then for every $0 \le r \le R$*

$$B_e(z, K^{-1}r|H'(z)|^{-1}) \subset H_z^{-1}(B_e(H(z), r)) \subset B_e(z, Kr|H'(z)|^{-1}).$$

Lemma 1.3 *Suppose that $D \subset \overline{\mathbb{C}}$ is an open set, $z \in D$ and $H : D \to \overline{\mathbb{C}}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B_s(H(z), 2R)$ for some $R > 0$ avoiding a spherical ball of some radius s. Then for every $0 \le r \le R$*

$$B_s(z, k_s^{-1}(1/2)r|H^*(z)|^{-1}) \subset H_z^{-1}(B_s(H(z), r)) \subset B_s(z, k_s(1/2)r|H^*(z)|^{-1}).$$

We also use the following more geometric versions of Koebe's Distortion Theorems involving moduli of annuli.

Koebe's Distortion Theorem, II (Euclidean version). *There exists a function $w : (0, +\infty) \to [1, \infty)$ such that for any two open topological disks $Q_1 \subset Q_2$ with Mod$(Q_2 \setminus Q_1) \ge t$ and any univalent analytic function $H : Q_2 \to \overline{\mathbb{C}}$ we have*

$$\sup\{|H'(\xi)| : \xi \in Q_1\} \le w(t) \inf\{|H'(\xi)| : \xi \in Q_1\}.$$

Koebe's Distortion Theorem, II (spherical version). *Given a number $s > 0$ there exists a function $w_s : (0, +\infty) \to [1, \infty)$ such that for any two open topological disks $Q_1 \subset Q_2$ with Mod$(Q_2 \setminus Q_1) \ge t$ and any univalent analytic function $H : Q_2 \to \overline{\mathbb{C}}$ such that the complement $\overline{\mathbb{C}} \setminus H(Q_2)$ contains a ball of radius s we have*

$$\sup\{|H^*(\xi)| : \xi \in Q_1\} \le w_s(t) \inf\{|H^*(\xi)| : \xi \in Q_1\}.$$

Given an analytic function H defined throughout a region $D \subset \mathbb{C}$, we put

$$\mathrm{Crit}(H) = \{z \in D : H'(z) = 0\}.$$

In the sequel we require the following technical lemma proven in [34] as Lemma 2.11.

Lemma 1.4 *Suppose that an analytic map $Q \circ H : D \to \mathbb{C}$, a radius $R > 0$ and a point $z \in D$ are such that*

(a)

$$\mathrm{Comp}(H(z), Q(H(z)), Q, 2R) \cap \mathrm{Crit}(Q) = \emptyset$$

and

$$\mathrm{Comp}(z, Q \circ H(z), Q \circ H, R) \cap \mathrm{Crit}(H) \neq \emptyset.$$

(b) If c belongs to the last intersection and

$$\mathrm{diam}_e\big(\mathrm{Comp}(z, Q \circ H(z), Q \circ H, R)\big) \leq (AR(H,c))^{1/p_c}$$

then

$$|z - c| \leq KA^2 |(Q \circ H)'(z)|^{-1} R.$$

Proof. In view of Lemma 1.2

$$\mathrm{Comp}(H(z), Q(H(z)), Q, R) \subset B_e(H(z), KR|Q'(H(z))|^{-1}).$$

So, since $H(c) \in \mathrm{Comp}(H(z), Q(H(z)), Q, R)$, we get

$$H(c) \in B_e(H(z), KR|Q'(H(z))|^{-1}).$$

Thus, using this and (b) we obtain

$$\begin{aligned} A^{-1}|z - c|^{p_c} &\leq |H(z) - H(c)| \\ &\leq KR|Q'(H(z))|^{-1} \\ &= KR|(Q \circ H)'(z)|^{-1}|H'(z)| \\ &\leq KR|(Q \circ H)'(z)|^{-1} A|z - c|^{p_c - 1}. \end{aligned}$$

So, $|z - c| \leq KA^2 |(Q \circ H)'(z)|^{-1} R$. ∎

Lemma 1.5 *Let μ and ν be Borel probability measures on Y, a bounded subset of a Euclidean space. Suppose that there is a constant $M > 0$ and for every point $x \in Y$ there is a decreasing to zero sequence $\{r_j(x) : j \geq 1\}$ of positive radii such that for all $j \geq 1$ and all $x \in Y$*

$$\mu(B_e(x, r_j(x)) \leq M\nu(B_e(x, r_j(x)).$$

Then the measure μ is absolutely continuous with respect to ν and the Radon-Nikodym derivative $d\mu/d\nu \leq CM$, where C is a universal constant depending only on the dimension of the Euclidean space under consideration.

Proof. Consider a Borel set $E \subset Y$ and fix $\varepsilon > 0$. Since $\lim_{j\to\infty} r_j(x) = 0$ and since ν is regular, for every $x \in E$ there exists a radius $r(x)$ being of the form $r_j(x)$ such that $\nu(\bigcup_{x\in E} B_e(x, r(x)) \setminus E) < \varepsilon$. Now, by the Besicovič theorem (see [14]) we can choose a countable subcover $\{B_e(x_i, r_i(x))\}_{i=1}^{\infty}$ from the cover $\{B_e(x_i, r_i(x))\}_{x\in E}$ of E, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore, we obtain

$$\begin{aligned}\mu(E) &\leq \sum_{i=1}^{\infty} \mu(B_e(x_i, r_i(x))) \\ &\leq M \sum_{i=1}^{\infty} \nu(B_e(x_i, r_i(x))) \\ &\leq MC\nu(\bigcup_{i=1}^{\infty} B_e(x_i, r_i(x))) \\ &\leq MC(\varepsilon + \nu(E)).\end{aligned}$$

Letting $\varepsilon \searrow 0$ we obtain $\mu(E) \leq MC\nu(E)$. Thus μ is absolutely continuous with respect to ν with the Radon-Nikodym derivative bounded by MC. ■

Frequently in order to denote that a Borel measure μ is absolutely continuous with respect to ν we write $\mu \prec \nu$. We do not use any special symbol to record equivalence of measures (mutual absolute continuity).

Given a σ-finite measure space $(X, \mathcal{F}, \mu)$ a measurable almost everywhere defined, transformation $T : X \to X$ i said to be ergodic with respect to μ, or μ is said to be ergodic with respect to T, if and only if $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ whenever the measurable set A is T-invariant, meaning that $T^{-1}(A) = A$. The measure μ is said to be conservative with respect to T or T conservative with respect to μ if and only if for every measurable set A

with $\mu(A) > 0$,

$$\mu(\{z \in X : \sum_{n=0}^{\infty} 1_A \circ T^n(z) < +\infty\}) = 0.$$

Finally, the measure μ is said to be T-invariant, or T is said to preserve the measure μ if and only if $\mu \circ T^{-1} = \mu$. It follows from Birkhoff's Ergodic Theorem that every finite ergodic T-invariant measure μ is conservative, for infinite measures this is not longer true. Finally, two ergodic invariant measures defined on the same σ-algebra are either singular or they coincide up to a multiplicative constant.

By writing $A \preceq B$ we mean that there exists a positive constant C such that $A \leq CB$ for all A and B under consideration. Then $A \succeq B$ means that $B \preceq A$, and $A \asymp B$ says that $A \preceq B$ and $B \preceq A$.

1.3 Preliminaries concerning iteration of meromorphic functions

The Fatou set $F(f)$ of a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z \in \mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\mathbb{C}$. Thus, $F(f)$ is open; $J(f)$ is closed; $F(f)$ is completely invariant while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f)) = J(f) \cup \{\infty\}$. For a general description of the dynamics of meromorphic functions see e.g. [6]. We note that it easily follows from Montel's criterion of normality that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ has at least one pole which is not an omitted value, then

$$J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}.$$

In further chapters we will be dealing with the points

$$I_\infty(f) = \{z \in \mathbb{C} : z \in \bigcup_{n \geq 0} f^{-n}(\infty) \text{ or } \lim_{n \to \infty} f^n(z) = \infty\}$$

escaping to ∞ under iterates of f. Let us now provide two related concepts, which play the central role in the approach undertaken in this paper. If $t \geq 0$, then a measure m_s supported on $J(f)$ is said to be a spherical semi t-conformal for $f : \mathbb{C} \to \overline{\mathbb{C}}$, if

$$m_s(f(A)) \geq \int_A |f^*|^t \, dm_s \tag{1.1}$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is injective and m_s is said to be a spherical t-conformal for $f : \mathbb{C} \to \overline{\mathbb{C}}$ if

$$m_s(f(A)) = \int_A |f^*|^t \, dm_s \tag{1.2}$$

for these sets A. Notice that the σ-finite measure m_e determined by the requirement that

$$\frac{dm_e}{dm_s}(z) = (1 + |z|^2)^t \tag{1.3}$$

has the property that

$$m_e(f(A)) = \int_A |f'|^t dm_e$$

for this set A as above. It will be called an Euclidean t-conformal measure. In particular for every $w \in \Lambda$ we get that

$$\int_A |f'|^t dm_e = m_e(f(A)) = m_e(f(A + w)) = \int_{A+w} |f'|^t dm_e.$$

Since the derivative f' is periodic with respect to the lattice Λ, we thus get the following.

Proposition 1.6 *The Euclidean t-conformal measure m_e is T_w-invariant for every $w \in \Lambda$, where $T_w : \mathbb{C} \mapsto \mathbb{C}$ is the translation about the vector w given by the formula $T_w(z) = z + w$.*

As an immediate consequence of this proposition, we get the following.

Corollary 1.7 *For every $r > 0$,*

$$M(t, r) = \inf\{m_e(B_e(z, r)) : z \in J(f)\} > 0.$$

In the sequel we respect the convention that the spherical conformal measure (or their weaker versions) are labeled with the subscript 's' whereas Euclidean conformal measures (and their weaker versions) are labeled with the subscript 'e'. If no subscript is used, the conformal measure under consideration can be spherical as well as Euclidean.

2

The dynamics of pseudo non-recurrent elliptic functions

2.1 Preliminary results concerning elliptic functions

As indicated in the introduction, throughout this paper $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a non-constant elliptic function. Every such function is doubly periodic and meromorphic. In particular, there exist two vectors w_1, w_2, $\mathrm{Im}(\frac{w_1}{w_2}) \neq 0$, such that for every $z \in \mathbb{C}$ and $n, m \in \mathbb{Z}$,

$$f(z) = f(z + mw_1 + nw_2).$$

The set

$$\Lambda = \{mw_1 + nw_2 : \ m, n \in \mathbb{Z} \}$$

is called the lattice of the elliptic function f. This object is independent of the choice of its generators w_1 and w_2. We call two points z and w equivalent and we write $z \sim w$ if $w - z \in \Lambda$, the lattice associated with the elliptic function f. Let

$$\mathcal{R} = \{t_1 w_1 + t_2 w_2 : 0 \leq t_1, t_2 \leq 1\}$$

be the basic fundamental parallelogram of f. It follows from the periodicity of f that $f(\mathbb{C}) = f(\mathcal{R})$. Therefore $f(\mathbb{C})$, as a closed and open subset of the connected set $\overline{\mathbb{C}}$ is equal to $\overline{\mathbb{C}}$. This means that each elliptic function is surjective. It also follows from the periodicity of f that

$$f^{-1}(\infty) = \bigcup_{m,n \in \mathbb{Z}} \big(\mathcal{R} \cap f^{-1}(\infty) + mw_1 + nw_2\big).$$

For every pole b of f let q_b denote its multiplicity. We define

$$q := \sup\{q_b : b \in f^{-1}(\infty)\} = \max\{q_b : b \in f^{-1}(\infty) \cap \mathcal{R}\}.$$

For every $r > 0$ let $B_r = \{z \in \overline{\mathbb{C}} : |z| > r\}$. Given $b \in f^{-1}(\infty)$ let $B_b(r)$ be the connected component of $f^{-1}(B_r)$ containing b. More generally, given

$k \geq 1$ and $b \in f^{-k}(\infty)$, let $B_r^k(b)$ be the connected component of $f^{-k}(B_r)$ containing b. Let $T \geq 1$ be so large that all components $B_b(T)$, $b \in f^{-1}(\infty)$, are mutually disjoint.

Recall that $\mathrm{Crit}(f)$ is the set of critical points of f i.e.

$$\mathrm{Crit}(f) = \{z : \ f'(z) = 0\}.$$

Its image, $f(\mathrm{Crit}(f))$, is called the set of critical values of f. Since $\mathcal{R} \cap \mathrm{Crit}(f)$ is finite and since $f(\mathrm{Crit}(f)) = f(\mathcal{R} \cap \mathrm{Crit}(f))$, the set of critical values $f(\mathrm{Crit}(f))$ is also finite. Thus, if $R > T$ is large enough, say $R \geq R_0$, then B_R contains no critical values of f, all sets $B_b(R)$ are simply connected, mutually disjoint, and there exists $A_1 = A_1(f, b) \geq 1$ such that for $z \in B_b(R)$

$$A_1^{-1}|z - b|^{-q_b} \leq |f(z)| \leq A_1|z - b|^{-q_b}. \tag{2.1}$$

If $U \subset B_R \setminus \{\infty\}$ is an open simply connected set, then all the holomorphic inverse branches $f_{b,U,1}^{-1}, \ldots, f_{b,U,q_b}^{-1}$ of f are well-defined on U, there exists $A_2 = A_2(f, b) \geq 1$ such that for every $1 \leq j \leq q_b$ and all $z \in U$ we have

$$A_2^{-1}|z|^{-\frac{q_b+1}{q_b}} \leq |(f_{b,U,j}^{-1})'(z)| \leq A_2|z|^{-\frac{q_b+1}{q_b}}. \tag{2.2}$$

Therefore (cf. [19]),

$$\begin{aligned} (2A_2)^{-1}\frac{|z|^{\frac{q_b-1}{q_b}}}{|b|^2} &\leq (2A_2)^{-1}\frac{|z|^{\frac{q_b-1}{q_b}}}{1 + |b|^2} \leq |(f_{b,U,j}^{-1})^*(z)| \\ &\leq 2A_2\frac{|z|^{\frac{q_b-1}{q_b}}}{1 + |b|^2} \leq 2A_2\frac{|z|^{\frac{q_b-1}{q_b}}}{|b|^2}, \end{aligned} \tag{2.3}$$

where the first left and the second right inequality sign we wrote assuming in addition that $|b|$ is large enough, say $|b| \geq R_1 > R_0$. We denote

$$A(f, b) = \max\{A_1(f, b),\ A_2(f, b)\} \tag{2.4}$$

for $b \in f^{-1}(\infty)$. A straightforward observation from the local behavior around poles is that for every $k \geq 1$ there exist constants $L_k \geq 1$ and $R_k > 0$ such that for all $b \in f^{-k}(\infty)$ and all $R \geq R_k$, we have

$$\begin{aligned} L_k^{-1}R_k^{-\frac{1}{q_b}} &\leq \mathrm{diam}_e(B_b^k(R)) \leq L_k R_k^{-\frac{1}{q_b}}, \\ L_k^{-1}R_k^{-\frac{1}{q_b}}(1 + |b|^2)^{-1} &\leq \mathrm{diam}_s(B_b^k(R)) \leq L_k R_k^{-\frac{1}{q_b}}(1 + |b|^2)^{-1}. \end{aligned} \tag{2.5}$$

Frequently, we will write L for L_1.

By $\mathrm{HD}(X)$ we denote the Hausdorff dimension of the set X. Its formal definition and some relevant properties are given in Chapter 3. We make the frequent use of the following fact, proven in [19].

Theorem 2.1 *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an arbitrary elliptic function, then*

$$\mathrm{HD}(J(f)) > \frac{2q}{q+1} \geq 1,$$

where, we recall, $q = \sup\{q_b : b \in f^{-1}(\infty)\} = \max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\}$.

Since the proof uses the theory of infinite conformal iterated function systems, it will not be repeated here.

2.2 Local behavior around parabolic fixed points

In this section $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an arbitrary elliptic function of degree ≥ 2; in fact all the results stated here are of local character and are true for all meromorphic functions. In particular, the map f is not assumed yet to be critically pseudo non-recurrent. In what follows we basically summarize the results concerning local behavior around parabolic fixed points which have been proved in [1], [10], and [11]. Although they were formulated and proved in the context of parabolic rational maps, that is, assuming that the Julia set contains no critical points, nevertheless they and their proofs are of local character and, in particular, extend to the class of all elliptic functions. Let $\Omega(f)$ denote the set of rationally indifferent periodic points of f. Throughout this section ω is a simple parabolic fixed point of f, that is $f(\omega) = \omega$ and $f'(\omega) = 1$.

First note that on a sufficiently small open neighborhood V of ω a holomorphic inverse branch $f_\omega^{-1} : V \to \overline{\mathbb{C}}$ of f which sends ω to ω is well defined. Moreover, V can be taken so small that on V the transformation f_ω^{-1} can be expressed in the form

$$f_\omega^{-1}(z) = z - a(z-\omega)^{p+1} + a_2(z-\omega)^{p+2} + a_3(z-\omega)^{p+3} + \dots \tag{2.6}$$

where $a \neq 0$ and $p = p(\omega)$ is a positive integer. Thus

$$f_\omega^{-1}(z) - \omega = z - \omega - a(z-\omega)^{p+1} + a_2(z-\omega)^{p+2} + a_3(z-\omega)^{p+3} + \dots$$

Consider the set $\{z : a(z-\omega)^p \in \mathbb{R} \text{ and } a(z-\omega)^p > 0\}$. This set is the union of p rays beginning in ω and forming angles which are integer multiples of $2\pi/p$. Denote these rays by $L_1, L_2, \dots, L_p$. For $1 \leq j \leq p$, $0 < r \leq \infty$

and $0 \le \alpha < 2\pi$, let $S_j(r,\alpha) \subset V$ be the set of those points z lying in the open ball $B_e(\omega, r)$ for which the angle between the rays L_j and the interval which joins the points ω and z does not exceed α. Using (2.6), an easy computation shows that there are $\alpha > 0$ and $\theta(\omega) > 0$ such that

$$|f_\omega^{-1}(z) - \omega| < |z - \omega| \quad \text{and} \quad |(f_\omega^{-1})'(z)| < 1 \tag{2.7}$$

for every $\omega \ne z \in S_1(\theta(\omega),\alpha) \cup \ldots \cup S_p(\theta(\omega),\alpha)$. The following version of Fatou's flower theorem, (see [1], [5], [27]) shows that the Julia set $J(f)$ approaches the fixed point ω tangentially to the lines $L_1, L_2, \ldots, L_p$. This can be precisely formulated as follows.

Lemma 2.2 *(Fatou's flower theorem) For every $\alpha > 0$ there exists $0 < r_1(\omega,\alpha) \le \theta(\omega)$ such that*

$$J(f) \cap B_e(\omega, r_1(\omega,\alpha)) \subset S_1(r_1(\omega,\alpha),\alpha) \cup \ldots \cup S_p(r_1(\omega,\alpha),\alpha).$$

Since the Julia set $J(f)$ is fully invariant ($f^{-1}(J(f)) = J(f)$ and $f(J(f)) = J(f) \cup \{\infty\}$), we conclude from this lemma and (2.7) that for every $0 < \theta_1(\omega) \le \theta(\omega)$, we have

$$f_\omega^{-1}(J(f) \cap B_e(\omega, \theta_1(\omega))) \subset J(f) \cap B_e(\omega, \theta_1(\omega)). \tag{2.8}$$

Thus all iterates $f_\omega^{-n} : J(f) \cap B_e(\omega,\theta_1(\omega)) \to J(f) \cap B_e(\omega,\theta_1(\omega))$, $n = 0, 1, 2, \ldots$ are well defined. From Lemma 2.2 and (2.8) we obtain

$$\begin{gathered} \forall \alpha > 0 \ \exists r_2(\omega,\alpha) > 0 \ \forall 1 \le j \le p \\ f_\omega^{-1}(S_j(r_2(\omega,\alpha),\alpha) \cap J(f)) \subset S_j(r_2(\omega,\alpha),\alpha). \end{gathered} \tag{2.9}$$

Put

$$\theta = \theta(f,\omega) = \min\{\theta_1(\omega), r_1(\omega,\alpha), r_2(\omega,\alpha) : \omega \in \Omega(f)\}. \tag{2.10}$$

Then, it follows from (2.7) and Lemma 2.2 that for every $z \in J(f) \cap B_e(\omega,\theta)$,

$$\lim_{n\to\infty} f_\omega^{-n}(z) = \omega. \tag{2.11}$$

In fact, it can be proved that this convergence is uniform on compact subsets of $B_e(\omega,\theta) \cap J(f) \setminus \{\omega\}$. See (2.13) for even stronger result. By precise computations one can prove the following.

Lemma 2.3 *For every $\tau > 0$ sufficiently small and every $z \in J(f) \cap B_e(\omega,\theta)$*

$$f_\omega^{-1}(B_e(z, \tau|z-\omega|)) \subset B_e(f_\omega^{-1}(z), \tau|f_\omega^{-1}(z) - \omega|).$$

This lemma immediately leads to the following.

Lemma 2.4 *For every $\tau > 0$ sufficiently small, every $z \in J(f) \cap B_e(\omega, \theta)$ and every $n \geq 0$ there exists a unique holomorphic inverse branch*

$$f_\omega^{-n} : B_e(z, 2\tau|z-\omega|) \to B_e(f_\omega^{-n}(z), 2\tau|f_\omega^{-n}(z) - \omega|)$$

of f^n which sends z to $f_\omega^{-n}(z)$.

The following two results were proved in [1] (cf. Proposition 8.3 and Theorem 8.4) and in [10] (cf. Lemma 1).

Lemma 2.5 *For every $z \in J(f) \cap B_e(\omega, \theta)$ there exists $C(z) \geq 1$ such that for every $n \geq 1$*

$$C(z) n^{-\frac{p+1}{p}} \leq |(f_\omega^{-n})'(z)|,\ |(f_\omega^{-n})^*(z)| \leq C(z) n^{-\frac{p+1}{p}} \tag{2.12}$$

and

$$\lim_{n\to\infty} |f_\omega^{-n}(z) - \omega| n^{1/p} = (|a|p)^{-1/p}. \tag{2.13}$$

Lemma 2.6 *For every $z \in J(f) \cap B_e(\omega, \theta)$ there exists constants $C(z) \geq 1$ such that for every $n \geq 1$*

$$\lim_{n\to\infty} |f_\omega^{-n}(z) - \omega| n^{1/p} = (|a|p)^{-1/p} \tag{2.14}$$

uniformly on compact subsets of $B_e(\omega, \theta) \cap J(f) \setminus \{\omega\}$.

Lemma 2.7 *Let m be a semi t-conformal measure for f. Then for every $R > 0$ there exists a constant $C = C(t, \omega, R) \geq 1$ such that for every $0 < r \leq R$*

$$\frac{m(B_e(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}},\ \frac{m(B_s(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}} \leq C,$$

where $\alpha_t(\omega) = t + p(\omega)(t-1)$. If m is t-conformal, then in addition

$$\frac{m(B_e(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}},\ \frac{m(B_s(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}} \geq C^{-1}.$$

Proof. In view of (1.3) it suffices to prove this lemma for the Euclidean measure m_e. Take $R > 0$ so small that $\overline{B_e(\omega, R)} \subset B_e(\omega, \theta)$ and let

$$P = J(f) \cap \{z : R(2\|f'\|)^{-1} \leq |z - \omega| \leq R\},$$

where $\|f'\| = \sup\{|f'(z)|\}$ (the supremum is taken over a compact neighborhood $\overline{B_e(\omega, \theta)} \cap J(f)$). Let

$$\delta = \tau \inf\{|z - \omega| : z \in P\} > 0.$$

Since P is compact, there are finitely many points $z^1, \dots, z^q$ in P, such that

$$P \subset J(f) \cap \big(B_e(z^1, \delta) \cup \ldots \cup B_e(z^q, \delta)\big),$$

and we may assume that δ is so small that

$$f_\omega^{-n}(B_e(z^i, \delta)) \cap B_e(z^i, \delta) = \emptyset$$

for $i = 1, \dots, q$ and $n = 1, 2, \dots$. Fix a constant $c_1 > \max\{(|a|p)^{-1/p}, (|a|p)^{1/p}\}$ and for every $n \geq 1$ define

$$P_n = \{z \in B_e(\omega, \theta) \cap J(f) : c_1^{-1} n^{-1/p} \leq |f_\omega^{-n}(z) - \omega| \leq c_1 n^{-1/p}\}.$$

By the local behavior of f around a parabolic point we conclude that for every

$$z \in (B_e(\omega, R) \setminus \{\omega\}) \cap J(f)$$

there exists $l \geq 0$ such that

$$R(2\|f'\|)^{-1} < |f^l(z) - \omega| < R,$$

i.e. $f^l(z) \in P$. Therefore, the set

$$J(f) \cap \{z : R(2\|f'\|)^{-1} < |z - \omega| < R\}$$

is non-empty ($B_e(\omega, R) \cap (J(f) \setminus \{\omega\})$ is non-empty since $J(f)$ is perfect). Moreover, since it is open in $J(f)$, we deduce that for some $1 \leq j \leq q$ the set $B_e(z^j, \delta) \cap P$ has non-empty interior in $J(f)$. Hence

$$M = m_e(B_e(z^j, \delta) \cap P) > 0.$$

By Lemma 2.6 there is $n_0 \geq 1$ such that $f_\omega^{-n}(z) \in P_n$ for every $n \geq n_0$ and $z \in P$. In other words this means that $P_n \supset f_\omega^{-n}(P)$ for $n \geq n_0$. Thus

$$B_e(\omega, c_1 n^{-1/p}) \supset \bigcup_{k=n}^{\infty} P_k \supset \bigcup_{k=n}^{\infty} f_\omega^{-k}(P) \supset \bigcup_{k=n}^{\infty} \bigcup_{i=1}^{q} f_\omega^{-n}(B_e(z^i, \delta) \cap P).$$

On the other hand, for any $z \in B_e(\omega, R) \setminus \{\omega\}$ let $l(z) \geq 0$ be the smallest integer such that $f^l(z) \in P$. Take $n_1 \geq n_0$ so large that if $z \in B_e(\omega, c_1 n_1^{-1/p})$, then $l(z) \ge n_0$. Consider now any $z \in J(f) \cap B_e(\omega, c_1 n^{-1/p}) \setminus \{\omega\}$ with $n \ge n_1$. Since $l(z) \ge n_0$ and $f^{l(z)}(z) \in P$ we conclude that $z = f_\omega^{-l(z)}(f^{l(z)}(z)) \in P_{l(z)}$. Therefore $c_1^{-1} l(z)^{-1/p} \le c_1 n^{-1/p}$ and consequently $l(z) \ge c_1^{-2p} n$. Hence

$$J(f) \cap B_e(\omega, c_1 n^{-1/p}) \subset \{\omega\} \cup \bigcup_{l \ge c_1^{-2p} n} f_\omega^{-l}(P) = \{\omega\} \cup \bigcup_{i=1}^{q} \bigcup_{l \ge c_1^{-2p} n} f_\omega^{-l}(B_e(z^i, \delta)).$$

Since the sets $\{f_\omega^{-n}(J(f) \cap B_e(z^j, \delta))\}$, $n = 1, 2, \ldots$, are mutually disjoint, it follows from Koebe's Distortion Theorem, I (Euclidean version) and the semi-conformality of the measure m that

$$\begin{aligned} m_e\big(B_e(\omega, c_1 n^{-1/p}) \setminus \{\omega\}\big) &\le m_e\Big(\bigcup_{i=1}^{q} \bigcup_{l \ge c_1^{-2p} n} f_{\omega,i}^{-l}(B_e(z^i, \delta))\Big) \\ &\le q K^t C_0^t \sum_{l \ge c_1^{-2p} n} l^{-\frac{p+1}{p} t} \le C'(n^{-1/p})^{\alpha_t(\omega)} \end{aligned}$$

where $C' > 0$ denotes some constant and where $C_0 = \max\{C(z^1), \ldots, C(z^q)\}$. If, in addition, m_e is t-conformal we have

$$\begin{aligned} m_e\big(B_e(\omega, c_1 n^{-1/p}) \setminus \{\omega\}\big) &\ge \sum_{k=n}^{\infty} m_e(f_\omega^{-k}(B_e(z^j, \delta) \cap P) \\ &\ge \sum_{k=n}^{\infty} K^{-t} C(z^j)^{-t} (k^{-\frac{p+1}{p}})^t M \\ &\ge M K^{-t} C(z^j)^{-t} \sum_{k=n}^{2n} (k^{-\frac{p+1}{p}})^t \\ &\ge M K^{-t} C(z^j)^{-t} n((2n)^{-\frac{p+1}{p} t}) \\ &= 2^{-\frac{p+1}{p} t} M K^{-h} C(z^j)^{-t} (n^{-1/p})^{\alpha_t(\omega)}. \end{aligned}$$

The proof is finished observing that $\lim_{n\to\infty} \frac{(n+1)^{-1/p}}{n^{-1/p}} = 1$. ■

2.3 Basic properties of critically pseudo non-recurrent elliptic functions

We say that the elliptic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is (critically) pseudo non-recurrent if the following conditions are satisfied.

(1) If $c \in \mathrm{Crit}(f) \cap J(f)$, then either

(1a) ω-limit set $\omega(c)$ is a compact subset of $\mathbb{C}$ (i.e. $\infty \notin \omega(c)$) and $c \notin \omega(c)$ or

(1b) $c \in \bigcup_{n\geq 1} f^{-n}(\infty)$, or

(1c) $\lim_{n\to\infty} f^n(c) = \infty$,

and

(2) if $c \in \mathrm{Crit}(f) \cap F(f)$, then there exists either an attracting periodic point ω of f or a rationally indifferent periodic point ω of f such that

$$\omega(c) \subset \{f^n(w),\, n \geq 0\}.$$

We will frequently write critically pseudo non-recurrent as well as pseudo non-recurrent, and in each case this will refer to functions defined above.

Definition 2.8 *The set of critical points captured respectively by (1a), (1b) and (1c) will be refereed to as* $\mathrm{Crit}_c(f)$, $\mathrm{Crit}_p(f)$ *and* $\mathrm{Crit}_\infty(f)$.

For every $c \in \mathrm{Crit}_p(f)$ let $n(c) \geq 2$ be the only integer such $f^j(c)$ is well defined for all $0 \leq j \leq n(c)$ and $f^n(c) = \infty$. Set

$$\begin{aligned}
\mathrm{PC}_c(f) &= \bigcup_{c\in\mathrm{Crit}_c(f)} \{f^j(c) : j \geq 1\} \\
\mathrm{PC}_c^0(f) &= \mathrm{Crit}_c(f) \cup \mathrm{PC}_c(f), \\
\mathrm{PC}_p(f) &= \bigcup_{c\in\mathrm{Crit}_p(f)} \{f^j(c) : 1 \leq j \leq n(c)-1\}, \\
\mathrm{PC}_p^0(f) &= \mathrm{Crit}_p(f) \cup \mathrm{PC}_p(f), \\
\mathrm{PC}_\infty(f) &= \bigcup_{c\in\mathrm{Crit}_\infty(f)} \{f^j(c) : j \geq 1\}, \\
\mathrm{PC}_\infty^0(f) &= \mathrm{Crit}_\infty(f) \cup \mathrm{PC}_\infty(f), \\
\mathrm{PC}(f) &:= \mathrm{PC}_c(f) \cup \mathrm{PC}_p(f) \cup \mathrm{PC}_\infty(f), \\
\mathrm{PC}^0(f) &:= \mathrm{PC}_c^0(f) \cup \mathrm{PC}_p^0(f) \cup \mathrm{PC}_\infty^0(f).
\end{aligned} \tag{2.15}$$

It immediately follows from this definition that $c \notin \omega(c)$ for all $c \in Crit(f) \cap J(f)$. For every $c \in \mathrm{Crit}_\infty(f)$ let

$$q_c = \limsup_{n\to\infty} q_{b_n}, \tag{2.16}$$

where $f^n(c)$ is near the pole b_n and, we recall, q_{b_n} is its multiplicity, p_c was defined just after Definition 1.1. Let

$$l_\infty = \max\{p_c q_c : c \in \mathrm{Crit}_\infty(f)\}$$

(if $\mathrm{Crit}_\infty(f) = \emptyset$, $l_\infty = 0$). Our main hypothesis is that

$$h > \frac{2l_\infty}{l_\infty + 1} \tag{2.17}$$

and then the pseudo non-recurrent function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is called *regular.*

This assumption is needed in order to show that the h-conformal measure constructed in Lemma 4.3 is atomless. This is a prerequisite for, essentially all, our considerations concerning geometric measures (Hausdorff and packing) and measurable dynamics with respect to the measure class generated by the conformal measure m. The place in the paper from which we do need regularity is the proof of Lemma 4.26.

Therefore, for every $c \in \mathrm{Crit}_\infty(f)$, $h > \frac{2p_c q_c}{p_c q_c + 1}$. Hence

$$\frac{p_c - 1}{p_c} h < (q_c + 1)h - 2q_c.$$

So there exists $h_- \in (1, h)$ such that

$$\frac{p_c - 1}{p_c} h_- < (q_c + 1)h_- - 2q_c, \tag{2.18}$$

and therefore there exists $\kappa_c > 0$ such that

$$\frac{p_c - 1}{p_c} h_- < \kappa_c < (q_c + 1)h_- - 2q_c. \tag{2.19}$$

The right-hand side of this formula is equivalent to the following

$$\left(\frac{h_- - \kappa_c}{2 - \kappa_c}\right)\left(\frac{q_c + 1}{q_c}\right) > 1. \tag{2.20}$$

For any set $A \subset \mathbb{C}$, let

$$O_+(A) = \bigcup_{n \geq 0} f^n(A).$$

Similarly as in the paper [19] the basic technical tool for our approach in this paper is formed by an appropriate version of Mañe's Theorem. Let us recall that we proved in [20] the following theorem.

Theorem 2.9 *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic function and $\Omega(f)$ denote the set of rationally indifferent periodic points of f. If a point $x \in J(f) \setminus \Omega(f)$ is not contained in the ω-limit set of a recurrent critical point, then for every $\varepsilon > 0$ there exists a neighborhood U of x such that*

(a) For all $n \geq 0$, every connected component of $f^{-n}(U)$ has Euclidean diameter $\leq \varepsilon$;

(b) There exists $N > 0$ such that for all $n \geq 0$ and every connected component V of $f^{-n}(U)$, the degree of $f^n_{|V}$ is $\leq N$.

We first prove a version of Przytycki's lemma from [29] for the sake of completeness, since it forms the first step in the proof of Theorem 2.9 and since there are places, where one has to proceed more subtly than in the case of rational functions.

Lemma 2.10 *For every integer $K \geq 0$ and every $0 < \lambda < 1$, the following holds. For every $\varepsilon > 0$ and every $\kappa > 0$, there exists $\delta_0 = \delta_0(K, \epsilon, \lambda, \kappa) > 0$ such that for every $\delta \leq \delta_0$ and every $x \in \mathbb{C}$ at the distance at least κ away from the set of parabolic points and attracting points, for every $n \geq 0$ and every connected component $W = \mathrm{Comp}(f^{-n}(B_e(x, \delta)))$ such that $f^n_{|W}$ has at most K critical points counted with multiplicities, for every component $W' = \mathrm{Comp}(f^{-n}(B'))$ in W, for the disc $B' = B_e(x, \lambda\delta)$ we have*

$$\mathrm{diam}_e(W') \leq \varepsilon$$

and $\mathrm{diam}_e(W') \to 0$ for $n \to \infty$ uniformly (i.e., independently of the choices of B and W').

Proof. Suppose, on the contrary, that there exist a sequence $\{x_n\}_{n=1}^{\infty}$ of points in a distance at least κ apart from the set of parabolic points and attracting points, a sequence $\delta_n \searrow 0$, a sequence of components $W_n = \mathrm{Comp}(f^{-k_n}(B_e(x_n, \delta_n)))$ with $k_n \to \infty$, as $n \to \infty$ such that the number of critical points of each map f^{k_n} on W_n is bounded by K and W'_n, the sequence associate to W_n as in the statement of the lemma, such that $\lim_{n\to\infty} \mathrm{diam}_e(W'_n) \neq 0$. Then for each n, there exists $L = L(n), 0 \leq L \leq K$, such that there is no critical value of $f^{k_n}_{|W_n}$ in

$$P(n) := B_e\left(x_n, \delta_n\left(\lambda + (1-\lambda)\frac{L+1}{K+1}\right)\right) \setminus B_e\left(x_n, \delta_n\left(\lambda + (1-\lambda)\frac{L}{K+1}\right)\right).$$

Without loss of generality, we may assume that all the components W'_n intersect the fundamental region $\mathcal{R}$. Put

$$W_n^{(1)} := \mathrm{Comp}\left(f^{-k_n}\left(B_e\left(x_n, \delta_n\left(\lambda + (1-\lambda)\frac{L(n)}{K+1}\right)\right)\right)\right)$$

$$W_n^{(2)} := \mathrm{Comp}\left(f^{-k_n}\left(B_e\left(x_n, \delta_n\left(\lambda + (1-\lambda)\frac{L(n)+1}{K+1}\right)\right)\right)\right)$$

the components containing W'_n,

$$P_n := W_n^{(2)} \setminus W_n^{(1)}$$

and for every $0 \le m \le k_n$, $i = 1, 2$,

$$W_{n,m}^{(i)} = f^{k_n - m}(W_n^{(i)}), \quad P_{n,m} := f^{k_n - m}(P_n) = W_{n,m}^{(2)} \setminus W_{n,m}^{(1)}.$$

For each n, let $m = m(n) \le k_n$ be the least integer such that

$$\mathrm{diam}_e(W_{n,m}^{(1)}) \ge \inf\{\mathrm{dist}_e(c_1, c_2);\ c_1, c_2 \in \mathrm{Crit}(f),\ c_1 \ne c_2\}.$$

So for every $0 \le t < m(n)$, the set $P_{n,t}$ is a topological annulus. That is so because at each step back by f^{-1} from $P_{n,t-1}$ to $P_{n,t}$ there is at most one branch point for f^{-1} from $W_{n,t-1}^{(i)}$ to $W_{n,t}^{(i)}$, $i = 1, 2$. Now, all the annuli $P_{n,m(n)-1}$'s have moduli bounded below by $2^{-K}(1-\lambda)\frac{1}{K+1}$. Since in addition all the components W'_n intersect the fundamental region $\mathcal{R}$, it follows from Montel's Theorem that there exists a topological (maybe not geometric) annulus P contained in all $P_{n,m(n)-1}$'s for a subsequences n_s, which bounds a topological disk D. So $D \subset W_{n_s,m(n_s)-1}^{(2)}$. Hence

$$f^{m(n_s)-1}(D) \subset B_e(x_n, \delta_n).$$

Passing yet to another subsequence, we may assume that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $y \in \mathbb{C}$ at distance at least κ apart from the set of parabolic points and attracting points. Thus the family of functions $\{f^{m(n_s)-1}(D) \to \mathbb{C}\}_{s=1}^{\infty}$ is equicontinuous and consequently D cannot intersect the Julia set $J(f)$. If they were contained in a preimage of a Siegel disk or a Herman ring, the limit of diameters of iterates $f^{m(n_s)-1}(D)$ would be positive. Thus D is contained in the basin of attraction to an attracting periodic orbit or a parabolic periodic orbit. In either case, the limit of the sets $f^{m(n_s)-1}(D)$ would be contained in either an attracting periodic orbit or a parabolic periodic orbit. Since this limit would coincide with y, we get a contradiction. The proof is complete. ■

Remark 2.11 *Obviously this lemma remains true (with the proof requiring only minor modifications) if, instead of the disk $B_e(x, \delta)$, one takes the square centered at x and with edges of length δ. This is the version we will need in the next theorem.*

Proof of Theorem 2.9. The core of the theorem is (a), from which the property (b) follows easily. Given an open set $U \subset \overline{\mathbb{C}}$, denote $c(U, n)$ the set of connected components of $f^{-n}(U)$. Observe that $V \in c(U, n)$ implies $f^j(V) \in c(U, n-j)$ for all $0 \le j \le n$. If $V \in c(U, n)$ define

$$\Delta(V, n) = \#\{\xi \in V; (f^n)'(\xi) = 0\},$$

counted with algebraic multiplicity. A square is the set S of the form

$$S = \{z \in \mathbb{C} : \ |\mathrm{Re}(z-p)| < \delta, \ |\mathrm{Im}(z-p)| < \delta\}.$$

The point p is the center and δ is its radius. Let S be a square with center p and radius δ, given $k > 0$, denote by S^k the square with center p and radius $k\delta$.

(a) If S is a square with radius δ, denote by $\mathcal{L}(S)$ the family of squares contained in $S^{3/2} - S$ and having radius $\delta/4$. Denote by $\mathcal{L}^*(S)$ the family of squares $S_0^{3/2}$ with $S_0 \in \mathcal{L}(S)$. Suppose that x is not a parabolic point and does not belong to the ω-limit set of recurrent critical point. Then there exists $\delta_0 > 0$ such that

(1) there is no critical point c of f such that there exists $0 \le n_1 \le n_2$ satisfying

$$|f^{n_1}(c) - c| < \delta_0 \quad \text{and} \quad |f^{n_2}(c) - x| < \delta_0;$$

(2) $|x - p| > 10\delta_0$ for every parabolic or attracting periodic point p.

Given $\varepsilon > 0$, take $\varepsilon_1 > 0$ satisfying

(3) $0 < \varepsilon_1 < \min\{\varepsilon/10, \delta_0/10\}$;
(4) if U is an open connected set with $\mathrm{diam}_e(U) \le 2\varepsilon_1$, then $\mathrm{diam}_e(W) \le \delta_0$ for all $W \in c(U, 1)$.

Let N_0 be the number of equivalence classes of the relation $\sim$ between critical points of f. Take $N_1 > 2$ such that

(5) If S is a square and $V \in c(S, n)$ satisfies $\Delta(V, n) \le N_0 + 1$, then the number of connected components of $f^{-n}(S^{2/3})$ contained in V is $\le N_1$.

Finally, let δ be given by

(6) $\delta = \min\{\delta_0/10, \varepsilon_1/10, \delta(2N_0, \varepsilon_1/20N_1, 2/3, \delta_0)\}$, where

$$\delta(2N_0, \varepsilon_1/20N_1, 2/3, \delta_0)$$

was produced in Lemma 2.10.

Let S_0 be the square of center x and radius δ. Suppose that Theorem 2.9(a) fails for $U = S_0$. Then there exists $n > 0$ and $V \in c(S_0, n)$ with $\mathrm{diam_e(V)} \geq \varepsilon \geq 10\varepsilon_1$. On the other hand, by (6), $\mathrm{diam}_e(S_0) = 2\sqrt{2}\delta < 3\delta < \varepsilon_1$. Hence there exists an integer $n_0 \geq 0$ such that there exists $V_0 \in c(S_0^{3/2}, n_0)$ satisfying

(7) $\mathrm{diam}_e(f^{-(n_0-i)}(S_0) \cap f^i(V_0)) \leq \varepsilon_1$ for all $1 \leq i \leq n_0$, and
(8) $\mathrm{diam}_e(f^{-n_0}(S_0) \cap V_0) > \varepsilon_1$.

Since $\mathrm{diam}_e(S_0) < \varepsilon_1$ it follows that $n_0 > 0$. Now, starting with S_0 we shall construct a sequence of squares $S_0, S_1, S_2, \ldots$ and strictly positive integers $n_0 \geq n_1 \geq n_2 \ldots$ satisfying

(9) $S_{j+1} \in \mathcal{L}^*(S_j)$ and
(10) there exists $V_j \in c(S_j^{3/2}, n_j)$ such that

$$\mathrm{diam}_e(f^{(-n_j-i)}(S_j) \cap f^i(V_j)) \leq \varepsilon_1$$

for all $1 \leq i \leq n_j$ and

$$\mathrm{diam}_e(f^{-n_j}(S_j) \cap V_j) > \varepsilon_1.$$

From (7) and (8), it follows that S_0 satisfies (10). If we construct such a sequence of squares and integers, then Theorem 2.9 will be proved by contradiction because the condition $n_0 \geq n_1 \geq n_2 \ldots \geq n_m \geq \ldots > 0$ implies that $n_j = n_i$ for all $\geq i$ for a certain i. But (a) implies that the radius of S_j is $(\frac{3}{8})^j\delta$; in particular $\mathrm{diam}(S_j) \to 0$ when $j \to +\infty$. But by (10),

$$\varepsilon_1 < \mathrm{diam}_e(f^{-n_j}(S_j) \cap V_j) = \mathrm{diam}_e(f^{-n_i}(S_j) \cap V_j),$$

$$V_j \in c(S_j^{3/2}, n_j) = c(S_j^{3/2}, n_i).$$

Taking $j \to +\infty$, and recalling that i is fixed and $\lim_{j\to\infty} \mathrm{diam}_e(S_j) = 0$, we conclude that the inequality above cannot hold.

The sequence $\{S_j\}$ and $\{n_j\}$ will be constructed by induction starting with S_0. Suppose S_i and n_i are constructed for $0 \leq i \leq j$. To find S_{j+1} and

n_{j+1}, we begin by observing that from (a) it follows that if $p \in S \in \mathcal{L}^*(S_j)$, then, by the contraction of the squares S_i,

$$|p - x| \le \sum_{i=0}^{j} \operatorname{diam}_e(S_i) = \sum_{i=0}^{j+1} \left(\frac{3}{8}\right)^i \operatorname{diam}_e(S_0) = 2\sqrt{2} \sum_{i=0}^{j+1} \left(\frac{3}{8}\right)^i \delta \le 4\sqrt{2}\delta.$$

Hence, if a point q satisfies $\operatorname{dist}_e(q, S) \le \delta_0$, we have

$$|q - x| \le 4\sqrt{2}\delta + \delta_0 \le 2\delta_0.$$

By (2), this means that

(11) $\operatorname{dist}_e(q, S) > \delta_0$ for all $S \in \mathcal{L}^*(S_j)$ and all parabolic or attracting periodic point q.

For the induction step (i.e., the construction of S_{j+1} and n_{j+1}), we shall use the following lemma.

Lemma 2.12 *If $U \subset \mathbb{C}$ is an open neighborhood of x and $V \in c(U, n)$ satisfies*

$$\operatorname{diam}_e(f^i(V)) \le \delta_0, \quad 0 \le i \le n,$$

then

$$\Delta(V, n) \le N_0.$$

Proof. If $\Delta(V, n) \ge N_0 + 1$, there exists $N_0 + 1$ different points x_i, $1 \le i \le N_0+1$, in V such that $(f^n)'(x_j) = 0$. This means that for each $1 \le i \le N_0+1$, there exist $1 \le m_i < n$, such that $f^{m_i}(x_i)$ is a critical point. Recalling that N_0 is the number of the equivalence classes of the equivalence relation $\sim$, we see that there exist two different points in the set $\{x_i;\ 1 \le i \le N_0 + 1\}$, which we denote by x_1, x_2, and two critical points c_1 and c_2 in the same equivalence class of the equivalence relation $\sim$, such that $f^{m_1}(x_1) = c_1$ and $f^{m_2}(x_2) = c_2$. Assume without loss of generality that $0 \le m_1 \le m_2$. Then by the choice of δ_0, $m_1 < m_2$ and

$$\begin{aligned} |f^{m_2-m_1}(c_2) - c_2| &= |f^{m_2-m_1}(c_1) - c_2| = |f^{m_2}(x_1) - f^{m_2}(x_2)| \\ &\le \operatorname{diam}_e(f^{m_2}(V)) \le \delta_0 \end{aligned}$$

and

$$|f^{n-m_2}(c_2) - x| = |f^{n-m_2}(f^{m_2}(x_2)) - x| = |f^n(x_2) - x| \le \delta_0,$$

contradicting property (1) of δ_0. ■

Now, to find S_{j+1} and n_{j+1} we first claim that there exists $S \in \mathcal{L}(S_j)$ which for some $0 < n \leq n_j$ has $V \in c(S,n)$ with $\mathrm{diam}_e(V) \geq \varepsilon_1/10N_1$. Suppose that the claim is false. Then, for all $1 \leq i \leq n_j$,

$$\begin{aligned}\mathrm{diam}_e(f^i(V_j)) &\leq \mathrm{diam}_e\big(f^{-(n_j-i)}(S_j) \cap f^i(V_j)\big) \\ &\quad + \sup\{\mathrm{diam}_e(W);\ W \in c(S, n_j - i), S \in \mathcal{L}(S_j)\} \\ &\leq \varepsilon_1 + \varepsilon_1/10N_1 \leq 2\varepsilon_1.\end{aligned}$$

From this inequality applied to $i = 1$ and property (4), we have

$$\mathrm{diam}_e(V_j) \leq \delta_0.$$

Moreover, since $2\varepsilon_1 \leq \delta_0$ (by (3)),

$$\mathrm{diam}_e(f^i(V_j)) \leq \delta_0$$

for all $1 \leq i \leq n_j$, hence for all $0 \leq i \leq n_j$. By Lemma 2.10, this proves that $\Delta(V_j, n_j) \leq N_0$. Then, since $V_j \in c(S_j^{2/3}, n_j)$ it follows from (5), (11) and Lemma 2.10 that

$$[W \in c(S_j, n_j), W \subset V_j] \Rightarrow \mathrm{diam}_e(W) \leq \varepsilon_1/10N_1.$$

Moreover, by the way N_1 was chosen, we have

$$\#\{W \in c(S_j, n_j); W \subset V_j\} \leq N_1$$

and we assume that

$$[S \in \mathcal{L}(S_j), G \in c(S, n_j)] \Rightarrow \mathrm{diam}_e(G) \leq \varepsilon/10N_1.$$

Now observe that V_j is the union of sets $G \in c(S, n_j), G \subset V_j, S \in \mathcal{L}(S_j)$ and the sets $W \in c(S_j, n_j), W \subset V_j$. Moreover, for any two sets W', W'' in this family there exist $W' = W_0, W_1, \ldots, W_k = W''$ in $c(S_j, n_j)$ and contained in V_j such that for all $0 \leq i < k$ there exist $S_i \in \mathcal{L}(S_j)$ and $U_i \in c(S_i, n_j)$ such that $\overline{U}_i \cap \overline{W}_i \neq \emptyset$ and $\overline{U}_i \cap \overline{W}_{i+1} \neq \emptyset$. Then

$$\mathrm{diam}_e(V_j) \leq N_1 \left(\varepsilon_1/10N_1 + \varepsilon_1/10N_1\right) = \varepsilon_1/5,$$

contradicting the last inequality in condition (10). This completes the proof of the claim. Now we can take $S \in \mathcal{L}(S_j)$ such that $\mathrm{diam}_e(V) \geq \varepsilon/10N_1$ for some $V \in c(S,n), 0 \leq n \leq n_j$. Take $\tilde{V} \in c(S^{3/2}, n)$ containing V. Suppose that $\Delta(\tilde{V}, n) \leq N_0$. Then by Lemma 2.10 and condition (6)

$$\mathrm{diam}_e(V) \leq \varepsilon_1/20N_1,$$

since $V \in c((S^{3/2})^{2/3}, n)$ and is contained in $\tilde{V}$. This contradicts the fact that

$$\operatorname{diam}_e(V) \geq \varepsilon_1/10N_1$$

and proves $\Delta(\tilde{V}, n) \geq N_0 + 1$. From Lemma 2.12, it follows that

$$\operatorname{diam}_e(f^i(\tilde{V})) > \delta_0$$

for some $0 \leq i \leq n$. Now we define $S_{j+1} = S^{3/2}$. Then $f^i(\tilde{V}) \in c(S^{3/2}, n-i)$ and $\operatorname{diam}_e(f^i(\tilde{V})) > \delta_0 \geq 10\varepsilon_1$. Moreover, $\operatorname{diam}_e(S_{j+1}) \leq 2\delta < \varepsilon_1$. Then there exists $0 \leq n_{j+1} \leq n - i \leq n_j - i$ and $V_{j+1} \in c(S_{j+1}, n_{j+1})$ such that

$$\operatorname{diam}_e(f^{-n_{j+1}}(S_{j+1}) \cap V_{j+1}) > \varepsilon_1$$

and

$$\operatorname{diam}_e(f^{-n_{j+1}+i}(S_{j+1}) \cap f^i(V_{j+1})) \leq \varepsilon_1.$$

Observe that $n_{j+1} > 0$ since $\mathrm{diam_e}(\mathrm{S_{j+1}}) < 2\delta < \varepsilon_1$. This completes the construction of the sequence $\{S_j\}$ and $\{n_j\}$ and the proof of part (a) of Theorem 2.9. Property (b) of Theorem 2.9 follows from (a) and Lemma 2.12. ■

As its rather straightforward consequence, in exactly the same way as Theorem 2.7 from [20] one can prove the following.

Theorem 2.13 *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically pseudo non-recurrent elliptic function. If $X \subset J(f) \setminus \Omega(f)$ is a closed subset of $\mathbb{C}$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ and every $n \geq 0$, all the connected components of $f^{-n}(B_e(x, \delta))$ have Euclidean diameters $\leq \epsilon$.*

Since this theorem forms an extremely important tool in our paper and promptly distinguishes the class of critically pseudo non-recurrent elliptic functions from among all other elliptic functions, we would like to provide a few words of comment. First, Mañe's original most general result is this.

Theorem 2.14 *Suppose that $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational function of degree $d \geq 2$. Suppose also that $x \in J(f)$ is not a rationally indifferent periodic point nor does x belong to the ω-limit set of any recurrent critical point. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $n \geq 0$ all the connected components of $f^{-n}(B_s(x, \delta))$ have spherical diameters $\leq \epsilon$.*

It is easy to see that in the context of rational functions Theorem 2.13 follows from Theorem 2.14 if $\mathbb{C}$ is replaced by $\overline{\mathbb{C}}$ and Euclidean diameters are replaced by spherical ones. We also observe that Theorem 2.13 follows easily from Theorem 2.9, the elliptic counterpart of Theorem 2.14, as long as the set $X \subset J(f) \setminus \Omega(f)$ is assumed to be a compact subset of the complex plane $\mathbb{C}$. The proof that Theorem 2.13 is also true for closed, not compact, subsets of $\mathbb{C}$ results from its "compact" part as follows. Suppose that $X \subset J(f) \setminus \Omega(f)$ is a closed subset of $\mathbb{C}$. Let

$$\Delta = \mathrm{dist}_e(\Omega(f), f^{-1}(\infty)) > 0.$$

In view of (2.2) and (2.5) there exists $R > 0$ so large that if $|f(z)| \geq R/2$, then for some $b \in f^{-1}(\infty)$, $z \in B_b(R/2)$

$$|f'(z)| \geq 2 \text{ and } \mathrm{diam}_e(B_b(R/2)) \leq \Delta/2. \tag{2.21}$$

Consider now the compact set

$$Y = X \cup (J(f) \setminus B_e(\Omega(f), \Delta/2)) \setminus B_R$$

and the corresponding number $0 < \delta \leq \min\{\epsilon, R/2\}$ ascribed to Y and the number $\min\{\epsilon, R/2\}$ according to the "compact" part of Theorem 2.13. In order to complete the proof it suffices to show that if $x \in B_R$, then the Euclidean diameter of each connected component $C_n(x)$ of $f^{-n}(B_e(x, \delta))$ does not exceed ϵ for every $\epsilon > 0$. Indeed, fix $w \in f^{-n}(x) \cap C_n(x)$ and let $1 \leq k \leq n$ be the least integer such that $f^{n-k}(w) \notin B_R$ provided it exists. Otherwise, set $k = n$. We shall show by mathematical induction that

$$\mathrm{diam}_e\big(f^{n-j}(C_n(x))\big) \leq \delta \leq \min\{\epsilon, R/2\} \tag{2.22}$$

for every $0 \leq j \leq k$. For $j = 0$, this formula is true since $f^n(C_n(x)) = B_e(x, \delta)$. Suppose that it is true for some $0 \leq j \leq k-1$. Since $f^{n-j}(w) \in B_R$ and since $\mathrm{diam}_e(f^{n-j}(C_n(x))) \leq R/2$, we conclude that

$$f^{n-j}(C_n(x)) \subset B_{R/2}. \tag{2.23}$$

It therefore follows from the first part of formula (2.21) that

$$\mathrm{diam}_e\big(f^{n-(j+1)}(C_n(x))\big) \leq \frac{1}{2}\mathrm{diam}_e\big(f^{n-j}(C_n(x))\big) \leq \delta.$$

This proves formula (2.22). In the case when $k = n$, the result follows from (2.22). Otherwise, it follows from (2.23) and the second part of formula (2.21) that

$$f^{n-k}(C_n(x)) \subset \mathbb{C} \setminus B_e(\Omega(f), \Delta/2).$$

Since we also know that $f^{n-k}(w) \notin B_R$, we conclude that $f^{n-k}(w) \in Y$, we see that $\mathrm{diam}_e(C_n(x)) \leq \min\{\epsilon, R/2\} \leq \epsilon$. We are done.

Now we shall collect all the other results from Section 2.3 of [20] formulated in the context of critically pseudo non-recurrent functions. The proofs requires no modifications. As a consequence of Theorem 2.13 we prove the following.

Corollary 2.15 *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically pseudo non-recurrent elliptic function. If $X \subset J(f) \cup \{\infty\} \setminus \Omega(f)$ is compact, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ and every $n \geq 0$, all connected components of $f^{-n}(B_s(x, \delta))$ have Euclidean diameters $\leq \epsilon$.*

Proof. Apply Theorem 2.13 for the set $f^{-1}(\infty)$ and given $\epsilon > 0$. This gives us the corresponding number $\delta_1 > 0$. Taking now $\xi > 0$ so small that each connected component of $f^{-n}(B_s(\infty, \xi))$ is contained in $B_e(b, \delta_1)$ for some pole $b \in f^{-1}(\infty)$ consider the set $Y = X \setminus B_s(\infty, \xi)$. Since Y is a compact subset of $\mathbb{C}$, it follows from Theorem 2.13 that there exists $\delta_2 > 0$ such that for every $x \in Y$ and every $n \geq 0$ all the connected components of $f^{-n}(B_s(x, \delta))$ have Euclidean diameters $\leq \epsilon$. Consider a finite cover

$$\{B_s(x_1, \delta_2), \ldots, B_s(x_k, \delta_2), B_s(\infty, \xi)\}$$

of X, where $x_j \in Y$ for all $j = 1, 2, \ldots, k$. Taking as δ half of the Lebesgue number of this cover (see [22]) finishes the proof. ■

Let

$$\theta = \theta(f) = \min\bigl\{\min\{\theta(f^a, \omega) : \omega \in \Omega(f)\}, \frac{1}{2}\mathrm{dist}_e(\Omega(f), \mathrm{Crit}(f))\bigr\} > 0, \tag{2.24}$$

where $a \geq 1$ is so large that all parabolic points of f^a are simple and the numbers $\theta(f^a, \omega)$ are defined in (2.10). We denote

$$A = A(f) = \max\{A(f, c),\ A(f, b) :\ c \in \mathrm{Crit}(f),\ b \in f^{-1}(\infty)\,\}, \tag{2.25}$$

where $A(f, c)$ was defined just after Definition 1.1, $A(f, b)$ was defined in (2.4). Recall from Chapter 2 that two points z and w are equivalent and write $z \sim w$ if $w - z \in \Lambda$, the lattice associated to the elliptic function f. Obviously, $z \sim w$ implies that $O_+(z) = O_+(w)$ and $\omega(z) = \omega(w)$. Since the set $\mathrm{Crit}(f) \cap \mathcal{R}$ is finite, our assumptions on the set of critical points $\mathrm{Crit}(f)$

imply that each of the following three numbers below is positive.

$$\min\{\text{dist}_e(c, O_+(f(c)) : c \in \text{Crit}(f)\}$$
$$\min\{(A(f,c)R(f,c))^{1/p_c} : c \in \text{Crit}(f)\}$$
$$\min\{|c - c'| : c, c' \in \text{Crit}(f) \text{ and } c \neq c'\},$$

where $p_c = p(f,c)$ is the order of the critical point c of f. Both, p_c and $R(f,c)$, were defined just after Definition 1.1. Fix a positive constant

$$\beta < \theta/2 \tag{2.26}$$

smaller than those three numbers. Since f contains no recurrent critical points, it follows from Theorem 2.13 that there exists $0 < \gamma < 1/4$ such that if $n \geq 0$ is an integer, $z \in J(f)$ and $f^n(z) \notin B_e(\Omega(f), \theta)$, then

$$\text{diam}_e\big(\text{Comp}(z, f^n(z), f^n, 2\gamma)\big) < \beta. \tag{2.27}$$

From now on fix also $0 < \tau < \theta^{-1}\min\{\beta, 2\gamma\}$ so small as required in Lemma 2.4 for every $\omega \in \Omega(f)$ and so small that for every $z \in J(f)$

$$\text{diam}_e\big(\text{Comp}(z, f(z), f, \theta\tau)\big) < \min\{\beta, 2\gamma\}. \tag{2.28}$$

Lemma 2.16 *If $n \geq 0$ is an integer, $\eta > 0$, $z \in J(f)$ and for every $k \in \{0, 1, \ldots, n\}$*

$$\text{diam}_e\big(\text{Comp}(f^k(z), f^n(z), f^{n-k}, \eta)\big) \leq \beta,$$

then each connected component $\text{Comp}(f^k(z), f^n(z), f^{n-k}, \eta)$ *contains at most one critical point of f and the equivalence class of each critical point intersects at most one of these components.*

Proof. The first part is obvious by the choice of β. In order to prove the second part suppose that

$$c_1 \in \text{Crit}(f) \cap \text{Comp}(f^{k_1}(z), f^n(z), f^{n-k_1}, \eta),$$
$$c_2 \in \text{Comp}(f^{k_2}(z), f^n(z), f^{n-k_2}, \eta)$$

and $c_1 \sim c_2$, where $0 \leq k_1 < k_2 \leq n$. But then

$$f^{k_2-k_1}(c_2) = f^{k_2-k_1}(c_1) \in \text{Comp}(f^{k_2}(z), f^n(z), f^{n-k_2}, \eta)$$

so $|f^{k_2-k_1}(c_2) - c_2| < \beta$, contrary to the choice of β. ■

Let $\kappa = \big(\Pi_{c \in \text{Crit}(f) \cap \mathcal{R}} p_c\big)^{-1}$.

Lemma 2.17 *If $z \in J(f)$, $f^n(z) \notin B_e(\Omega(f), \theta)$, then*

$$\mathrm{Mod}\big(\mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \backslash \mathrm{Comp}(z, f^n(z), f^n, \gamma)\big) \geq \kappa \log 2/\#(\mathrm{Crit}(f) \cap \mathcal{R}).$$

Proof. By Lemma 2.16 there exists a geometric annulus

$$R \subset B_e(f^n(z), 2\gamma) \setminus B(f^n(z), \gamma)$$

centered at $f^n(z)$ of modulus $\log 2/\#(\mathrm{Crit}(f) \cap \mathcal{R})$ such that

$$f^{-n}(R) \cap \mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \cap \mathrm{Crit}(f^n) = \emptyset.$$

Since covering maps increase moduli of annuli at most by factors equal to their degrees, we conclude that

$$\begin{aligned} \mathrm{Mod}\big(&\mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \setminus \mathrm{Comp}(z, f^n(z), f^n, \gamma)\big) \\ &\geq \mathrm{Mod}(R_n) \\ &\geq \left(\frac{\log 2}{\#(\mathrm{Crit}(f) \cap \mathcal{R})}\right) \left(\Pi_{c \in \mathrm{Crit}(f) \cap \mathcal{R}} p_c\right)^{-1} \\ &= \frac{\kappa \log 2}{\#(\mathrm{Crit}(f) \cap \mathcal{R})}, \end{aligned} \tag{2.29}$$

where $R_n \subset \mathrm{Comp}(z, f^n(z), f^n, 2\gamma)$ is the connected component of $f^{-n}(B_e(f^n(z), 2\gamma))$ enclosing $\mathrm{Comp}(z, f^n(z), f^n, \gamma)$. ■

As an immediate consequence of this lemma and Koebe's Distortion Theorem, II (Euclidean version) we get the following.

Lemma 2.18 *Suppose that $z \in J(f)$ and $f^n(z) \notin B_e(\Omega(f), \theta)$. If $0 \leq k \leq n$ and $f^k : \mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \to \mathrm{Comp}(f^k(z), f^n(z), f^{n-k}, 2\gamma)$ is univalent, then*

$$\frac{|(f^k)'(y)|}{|(f^k)'(x)|} \leq \mathrm{const}$$

for all $x, y \in \mathrm{Comp}(z, f^n(z), f^n, \gamma)$, where const *is a number depending only on $\#(\mathrm{Crit}(f) \cap \mathcal{R})$ and κ.*

For A, B, any two subsets of a metric space put

$$\mathrm{dist}(A, B) = \inf\{\mathrm{dist}(a, b) : a \in A, b \in B\}$$

and

$$\mathrm{Dist}(A, B) = \sup\{\mathrm{dist}(a, b) : a \in A, b \in B\}.$$

Lemma 2.19 *Suppose that $z \in J(f)$ and $f^n(z) \notin B_e(\Omega(f), \theta)$. Suppose also that $Q^{(1)} \subset Q^{(2)} \subset B(f^n(z), \gamma)$ are connected sets. If $Q_n^{(2)}$ is a connected component of $f^{-n}(Q^{(2)})$ contained in* $\mathrm{Comp}(z, f^n(z), f^n, \gamma)$ *and $Q_n^{(1)}$ is a connected component of $f^{-n}(Q^{(1)})$ contained in $Q_n^{(2)}$, then*

$$\frac{\mathrm{diam}_e\big(Q_n^{(1)}\big)}{\mathrm{diam}_e\big(Q_n^{(2)}\big)} \succeq \frac{\mathrm{diam}_e\big(Q^{(1)}\big)}{\mathrm{diam}_e\big(Q^{(2)}\big)}.$$

Proof. Let $1 \le n_1 \le \ldots \le n_u \le n$ be all the integers k between 1 and n such that

$$\mathrm{Crit}(f) \cap \mathrm{Comp}(f^{n-k}(z), f^n(z), f^k, 2\gamma) \ne \emptyset.$$

Fix $1 \le i \le u$. If $j \in [n_i, n_{i+1}-1]$ (we set $n_{u+1} = n-1$), then by Lemma 2.18 there exists a universal constant $T > 0$ such that

$$\frac{\mathrm{diam}_e(Q_j^{(1)})}{\mathrm{diam}_e(Q_j^{(2)})} \ge T\frac{\mathrm{diam}_e(Q_{n_i}^{(1)})}{\mathrm{diam}_e(Q_{n_i}^{(2)})}. \tag{2.30}$$

Since, in view of Lemma 2.16, $u \le \sharp(\mathrm{Crit}(f) \cap \mathcal{R})$, in order to conclude the proof it is enough to show the existence of a universal constant $E > 0$ such that for every $1 \le i \le u$

$$\frac{\mathrm{diam}_e(Q_{n_i}^{(1)})}{\mathrm{diam}_e(Q_{n_i}^{(2)})} \ge E\frac{\mathrm{diam}_e(Q_{n_i-1}^{(1)})}{\mathrm{diam}_e(Q_{n_i-1}^{(2)})}.$$

Indeed, let c be the critical point in $\mathrm{Comp}(f^{n-n_i}(z), f^n(z), f^{n_i}, 2\gamma)$ and let p_c be its order. Since both sets $Q_{n_i}^{(1)}$ and $Q_{n_i}^{(1)}$ are connected, we get for $j = 1, 2$ that

$$\begin{aligned}\mathrm{diam}_e(Q_{n_i-1}^{(j)}) &\asymp \mathrm{diam}_e(Q_{n_i}^{(j)}) \sup\{|f'(x)| : x \in Q_{n_i}^{(j)}\}\\ &\asymp \mathrm{diam}_e(Q_{n_i}^{(i)})\mathrm{Dist}_e(c, Q_{n_i}^{(i)}).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\mathrm{diam}_e(Q_{n_i}^{(1)})}{\mathrm{diam}_e(Q_{n_i}^{(2)})} &\asymp \frac{\mathrm{diam}_e(Q_{n_i-1}^{(1)})}{\mathrm{Dist}_e(c, Q_{n_i}^{(1)})} \cdot \frac{\mathrm{Dist}_e(c, Q_{n_i}^{(2)})}{\mathrm{diam}_e(Q_{n_i-1}^{(2)})}\\ &\ge \frac{\mathrm{diam}_e(Q_{n_i-1}^{(1)})}{\mathrm{diam}_e(Q_{n_i-1}^{(2)})}.\end{aligned}$$

The proof is finished. ■

2.4 Partial order in $\text{Crit}_c(J(f))$ and stratifications of closed forward-invariant subsets of $J(f)$

In this section all the distances and all closures are understood with respect to the Euclidean metric and the topology on the Euclidean plane $\mathbb{C}$. In particular if $\lim_{n\to\infty} f^n(z) = \infty$ or $z \in \bigcup_{n=1}^{\infty} f^{-n}(\infty)$, then $\omega(z) = \emptyset$. Also $\text{dist}(A, \emptyset) = 0$. What concerns critical points all the proofs in Section 2.4 from [20] used only the fact that $c \notin \omega(c)$ and $\omega(c)$ is a compact subset of $\mathbb{C}$ for every critical point $c \in \text{Crit}(f) \cap J(f)$. All these proofs go through unchanged (only $\text{Crit}(f)$ is replaced by $\text{Crit}_c(f)$ and $\text{PC}(f)$ is replaced by $\overline{\text{PC}_c^0(f)}$ and we collect here the results and their proofs for the sake of completeness and the convenience of the reader. Set

$$\text{Crit}_c(J(f)) = \text{Crit}_c(f) \cap J(f).$$

Lemma 2.20 *The set $\omega(\text{Crit}_c(J(f)))$ is nowhere dense in $J(f)$.*

Proof. Suppose that the interior (relative to $J(f)$) of $\omega(\text{Crit}_c(J(f)))$ is nonempty. Then there exists $c \in \text{Crit}_c(J(f))$ such that $\omega(c)$ has nonempty interior. But then there would exist $n \geq 0$ such that $f^n(\omega(c)) = J(f)$ and, consequently, $\omega(c) = J(f)$. This, however, is a contradiction, as $c \notin \omega(c)$. ■

Now we introduce in $\text{Crit}_c(J(f))$ a relation $<$ which, in view of Lemma 2.21 below, is an ordering relation, by putting

$$c_1 < c_2 \iff c_1 \in \omega(c_2). \tag{2.31}$$

Since $c_2 \sim c_3$ implies $\omega(c_2) = \omega(c_3)$, then if $c_1 < c_2$ and $c_2 \sim c_3$, then $c_1 < c_3$

Lemma 2.21 *If $c_1 < c_2$ and $c_2 < c_3$, then $c_1 < c_3$.*

Proof. Indeed, we have $c_1 \in \omega(c_2) \subset \omega(c_3)$. ■

Lemma 2.22 *There is no infinite, linear subset of the partially ordered set*

$$(\text{Crit}_c(J(f)), <).$$

Proof. Indeed, suppose on the contrary that $c_1 < c_2 < \dots$ is an infinite, linearly ordered subset of $\text{Crit}_c(J(f))$. Since the number of equivalency classes of relation $\sim$ is equal to $\#(\text{Crit}_c(J(f)) \cap \mathcal{R})$ which is finite, there exist two numbers $1 \le i < j$ such that $\omega(c_i) = \omega(c_j)$. But this implies that $c_i \in \omega(c_j) = \omega(c_i)$ and we get a contradiction. The proof is finished. ■

The following observation is a reformulation of the condition that $J(f)$ contains no recurrent critical points.

Lemma 2.23 *If* $c \in \mathrm{Crit}_c(J(f))$*, then it is not the case that* $c < c$*.*

Now define inductively a sequence $\{Cr_i(f)\}$ of subsets of $\mathrm{Crit}_c(J(f))$ by setting $Cr_0(f) = \emptyset$ and

$$Cr_{i+1}(f) = \left\{ c \in \mathrm{Crit}_c(J(f)) \setminus \bigcup_{j=0}^{i} Cr_j(f) : \text{ if } c' < c, \text{ then } c' \in Cr_0(f) \cup \ldots \cup Cr_i(f) \right\}. \tag{2.32}$$

Lemma 2.24 *(a) If* $c \in Cr_i(f)$ *and* $c' \sim c$*, then* $c' \in Cr_i(f)$*.*

(b) The sets $\{Cr_i(f)\}$ *are mutually disjoint.*

(c) $\exists_{p \ge 1}\ \forall_{i \ge p+1}\ \ Cr_i(f) = \emptyset$*.*

(d) $Cr_0(f) \cup \ldots \cup Cr_p(f) = \mathrm{Crit}_c(J(f))$*.*

(e) $Cr_1(f) \ne \emptyset$*.*

Proof. Part (a) follows immediately from the definition of the sets Cr_i and the fact that two equivalent points have the same ω-limit sets. By definition $Cr_{i+1}(f) \cap \bigcup_{j=1}^{i} Cr_j(f) = \emptyset$, so disjointness in (b) is clear. As the number of equivalence classes of the relation $\sim$ is equal to $\#(\mathrm{Crit}(J(f)) \cap \mathcal{R}$, which is finite, (a) and (b) imply (c). Take p to be the minimal number satisfying (c) and suppose that $c \in \mathrm{Crit}_c(J(f)) \setminus \bigcup_{j=1}^{p} Cr_j(f)$. Since $Cr_{p+1}(f) = \emptyset$, there exists $c' \notin \bigcup_{j=1}^{p} Cr_j(f)$ such that $c' < c$. Iterating this procedure, we would obtain an infinite sequence of critical points $c_1 = c > c_2 = c' > c_3 > \ldots$. But this contradicts Lemma 2.22 proving (d). Now part (e) follows from (c) and (2.32). ■

As an immediate consequence of the definition of the sequence $\{Cr_i(f)\}$, we get the following simple lemma.

Lemma 2.25 *If* $c, c' \in Cr_i(f)$*, then it is not the case that* $c < c'$*.*

For each point $z \in J(f)$, define the set

$$\mathrm{Crit}_c(z) = \{c \in \mathrm{Crit}_c(J(f)) : c \in \omega(z)\}.$$

Lemma 2.26 *If $z \in J(f) \setminus I_\infty(f)$, then either $z \in \bigcup_{n\geq 0} f^{-n}(\Omega(f))$ or $\omega(z) \setminus \{\infty\}$ is not contained in $\overline{O_+(f(\mathrm{Crit}_c(z))} \cup \Omega(f)$.*

Proof. Suppose that $z \notin \bigcup_{n\geq 0} f^{-n}(\Omega(f)) \cup I_\infty(f)$. Then by (2.11) the set $\omega(z) \setminus \{\infty\}$ is not contained in $\Omega(f)$. So, if we suppose that

$$\omega(z) \setminus \{\infty\} \subset \overline{O_+(f(\mathrm{Crit}_c(z))} \cup \Omega(f), \tag{2.33}$$

then, as $\omega(z) \setminus \{\infty\} \neq \emptyset$, we conclude that $\mathrm{Crit}(z) \neq \emptyset$. Let $c_1 \in \mathrm{Crit}(z)$. This means that $c_1 \in \omega(z)$; and as $c_1 \notin \Omega(f)$, it follows from (2.33) that there exists $c_2 \in \mathrm{Crit}_c(z)$ such that either $c_1 \in \omega(c_2)$ or $c_1 = f^{n_1}(c_2)$ for some $n_1 \geq 1$. Iterating this procedure, we obtain an infinite sequence $\{c_j\}_{j=1}^{\infty}$ such that for every $j \geq 1$, either $c_j \in \omega(c_{j+1})$ or $c_j = f^{n_j}(c_{j+1})$ for some $n_j \geq 1$. Consider an arbitrary block $c_k, c_{k+1}, \ldots, c_l$ such that $c_j = f^{n_j}(c_{j+1})$ for every $k \leq j \leq l-1$ and suppose that

$$l - (k-1) \geq \#(\mathrm{Crit}(f) \cap \mathcal{R}).$$

Then there are two indexes $k \leq a < b \leq l$ such that $c_a \sim c_b$. Then

$$f^{n_a+n_{a+1}+\ldots+n_{b-1}}(c_a) = f^{n_a+n_{a+1}+\ldots+n_{b-1}}(c_b) = c_a;$$

and consequently, as

$$n_a + n_{a+1} + \ldots + n_{b-1} \geq b - a \geq 1,$$

c_a is a super-attracting periodic point of f. Since $c_a \in J(f)$, this is a contradiction; and in consequence, the length of the block $c_k, c_{k+1}, \ldots, c_l$ is bounded above by $\#(\mathrm{Crit}(f) \cap \mathcal{R})$. Hence there exists an infinite subsequence $\{n_k\}_{k\geq 1}$ such that $c_{n_k} \in \omega(c_{n_k+1})$ for every $k \geq 1$ or, equivalently, $c_{n_k} < c_{n_{k+1}}$ for every $k \geq 1$. This, however, contradicts Lemma 2.22 and we are done. ■

Recall that the integer p was defined in Lemma 2.24. Define for every $i = 0, 1, \ldots, p$,

$$S_i(f) = Cr_0(f) \cup \ldots \cup Cr_i(f) \tag{2.34}$$

and for every $i = 0, 1, \ldots, p-1$, consider $c' \in \bigcup_{c \in Cr_{i+1}(f)} \omega(c) \cap \mathrm{Crit}_c(J(f))$. Then there exists $c \in Cr_{i+1}(f)$ such that $c' \in \omega(c)$, which means that $c' < c$. Thus, by (2.32), we get $c' \in S_i(f)$. So

$$\bigcup_{c \in Cr_{i+1}(f)} \omega(c) \cap (\mathrm{Crit}_c(J(f)) \setminus S_i(f)) = \emptyset. \tag{2.35}$$

Therefore, since the set $\bigcup_{c \in Cr_{i+1}(f)} \omega(c) \subset \mathbb{C}$ is compact and $\mathrm{Crit}_c(J(f)) \setminus S_i(f)$ has no accumulation point in $\mathbb{C}$,

$$\delta_i = \mathrm{dist}_e\big(\bigcup_{c \in Cr_{i+1}(f)} \omega(c), \mathrm{Crit}_c(J(f)) \setminus S_i(f)\big) > 0. \tag{2.36}$$

Set

$$\rho = \frac{1}{2} \min\{\delta_i : i = 0, 1, \ldots, p-1, \quad \mathrm{dist}_e\left(O_+(\mathrm{Crit}_c(f)), \mathrm{Crit}_p(f) \cup \mathrm{Crit}_\infty(f)\right)\}.$$

Fix a closed forward-invariant subset E of $J(f)$ and for every $i = 0, 1, \ldots, p$, define

$$E_i(f) = \{z \in E : \mathrm{dist}_e\big(O_+(z), \mathrm{Crit}_c(J(f)) \setminus S_i(f)\big) \geq \rho\}.$$

Let us now prove the following two lemmas concerning the sets $E_i(f)$. Let $E_i = E_i(f)$, $i = 0, \ldots, p$.

Lemma 2.27 $E_0 \subset E_1 \subset \ldots \subset E_p = E$.

Proof. Since $S_{i+1}(f) \supset S_i(f)$, the inclusions $E_i \subset E_{i+1}$ is obvious. Since $S_p(f) = \mathrm{Crit}_c(J(f))$, it holds $E_p = E$. We are done. ■

Lemma 2.28 *There exists* $l = l(f) \geq 1$ *such that for every* $i = 0, 1, \ldots, p-1$,

$$\bigcup_{c \in Cr_{i+1}(f)} \omega(c) \subset \overline{O_+(f^l(Cr_{i+1}(f)))} \subset \overline{\mathrm{PC}_c(f)_i} \subset \overline{\mathrm{PC}_c^0(f)_i}.$$

Proof. The left-hand inclusion is obvious regardless whatever $l(f) \geq 1$ is. In order to prove the right-hand inclusion, fix $i \in \{0, 1, \ldots, p-1\}$. By the definition of ω-limit sets, there exists $l_i \geq 1$ such that for every $c \in Cr_{i+1}(f)$ we have

$$\mathrm{dist}_e\big(O_+(f^{l_i}(c)), \bigcup_{c \in Cr_{i+1}(f)} \omega(c)\big) < \delta_i/2.$$

Thus, by (2.36),

$$\mathrm{dist}_e\big(\overline{O_+(f^{l_i}(c))}, \mathrm{Crit}_c(J(f)) \setminus S_i(f)\big) > \delta_i/2.$$

Since $\rho \le \delta_i/2$ and since for every $z \in \overline{O_+(f^{l_i}(c))}$ also $O_+(z) \subset \overline{O_+(f^{l_i}(c))}$, we therefore get

$$\overline{O_+(f^l(Cr_{i+1}(f)))} \subset \overline{\mathrm{PC}_c^0(f)_i}.$$

So, putting $l(f) = \max\{l_i : i = 0, 1, \dots, p-1\}$ the proof is completed. ■

2.5 Holomorphic inverse branches

Set

$$\mathrm{Sing}^-(f) = \bigcup_{n\ge 0} f^{-n}\big(\Omega(f) \cup \mathrm{Crit}(J(f)) \cup f^{-1}(\infty)\big)$$

and

$$I_-(f) = \bigcup_{n\ge 1} f^{-n}(\infty).$$

For every $b \in f^{-1}(\infty)$ and every $w \in B_b(2T)$ let $f_{b,w}^{-1} : B_e(f(w), T) \mapsto B_b(T)$ be the inverse branch of f sending $f(w)$ to w. It follows from (2.2) (comp. also (2.5)) that there exists constant $L \ge 1$ and the following properties are satisfied.

$$\mathrm{diam}_e(B_b(T)) \le LT^{-1/q_b} \quad \text{and} \quad B_b(2T) \supset B_e(b, L^{-1}T^{-1/q_b}) \tag{2.37}$$

and for every $R \in (0, T)$ sufficiently small

$$\begin{aligned} B_e\left(w, L^{-1}R|f(w)|^{-\frac{q_b+1}{q_b}}\right) \subset f_{b,w}^{-1}(B_e(f(w), R)) \subset B_e\left(w, LR|f(w)|^{-\frac{q_b+1}{q_b}}\right) \\ \subset B_e(w, R), \end{aligned} \tag{2.38}$$

where the last inclusion was written assuming that $|f(w)| \ge L^{\frac{q_b+1}{q_b}}$. Since there are only finitely many equivalence classes of the relation $\sim$ generated by the poles of f, there exists $R_1 > 0$ so small that $w \in B_e(f^{-1}(\infty), R_1)$ then $|f(w)| \ge L$.

Using now (2.38) and the right-hand side of (2.37) with T replaced by $2T$ a straightforward induction gives the following.

Lemma 2.29 *There exists*

$$R_2 \in (0, \min\{T, (2LT)^{-1}, R_1, \frac{1}{2}\mathrm{dist}_e(f^{-1}(\infty), \mathrm{Crit}(f))\}$$

so small that if $z \in \mathbb{C}$, $n \geq 1$ *and if* $\{f^j(z) : 0 \leq j \leq n-1\} \subset B_e(f^{-1}(\infty), R_2)$ *then there exists a unique holomorphic inverse branch* $f_z^{-n} : B_e(f^n(z), R_2) \to B_e(z, R_2)$ *sending* $f^n(z)$ *to* z.

Now we shall prove the following.

Lemma 2.30 *For every* $\varepsilon > 0$ *there exists* $a = a(\varepsilon) \geq 1$ *such that if* $z \in \mathbb{C} \setminus B_e(f^{-1}(\infty), \varepsilon)$ *then* $z \notin \bigcup_{j=a+1}^{\infty} B_e(f^j(\mathrm{Crit}_\infty(f)), 5)$.

Proof. Suppose on the contrary that there exists $\varepsilon > 0$ for every $a \geq 1$ there exists $z_a \in \bigcup_{j=a+1}^{\infty} B_e(f^j(\mathrm{Crit}_\infty(f)), 5) \setminus B_e(f^{-1}(\infty), \varepsilon)$. Since the sets $f^j(\mathrm{Crit}_\infty(f))$ converge to ∞ when $j \to \infty$, it follows that $\lim_{a\to\infty} z_a = \infty$. But then $z_a \in B_e(f^{-1}(\infty), \varepsilon)$ for all $a \geq 1$ large enough. This contradiction finishes the proof. ■

Since the sets $\mathrm{PC}_c(f)$ and $\mathrm{PC}_p(f)$ are bounded, the number

$$D = \frac{1}{2}\mathrm{Dist}_e(\mathrm{PC}_c(f) \cup \mathrm{PC}_p(f), 0)$$

is finite. The main result of this chapter is the following.

Proposition 2.31 *If* $z \in J(f) \setminus \mathrm{Sing}^-(f)$ *then there exist:*

(a) $\eta(z) > 0$.
(b) $\{n_j\}_{j=1}^{\infty}$, *an increasing sequence of positive integers.*
(c) a sequence $\{x_j(z)\}_{j=1}^{\infty} \subset J(f) \setminus \big(\Omega(f) \cup \omega(\mathrm{Crit}(z))\big)$ *with the following properties:*

  (1) $\mathrm{Comp}(z, f^{n_j}(z), f^{n_j}, \eta(z)) \cap \mathrm{Crit}(f^{n_j}) = \emptyset$.
  (2) $\lim_{j\to\infty} |f^{n_j}(z) - x_j(z)| = 0$.
  (3) If $|x_j(z)| \geq 2D$ *for all* $j \geq 1$, *then* $\eta(z) \geq \min\{2, R_2\}$.
  (4) If the sequence $\{x_j(z)\}_{j=0}^{\infty}$ *is bounded, then it is constant.*
  (5) If $\lim_{j\to\infty} x_j(z) = \infty$ *and* $z \notin I_\infty(f)$ *then* $x_j(z) \sim x_k(z)$ *for all* $j, k \geq 1$.

Proof. If $z \in I_\infty(f)$, equivalently, if $\lim_{n\to\infty} \mathrm{dist}_e(f^n(z), f^{-1}(\infty)) = 0$, then in view of Lemma 2.29 we are done by setting $n_j = j+u$ and $x_j(z) = f^{j+u}(z)$ with some $u \geq 0$ large enough, so that $\mathrm{dist}_e(f^{j+u}(z), f^{-1}(\infty)) < R_2$. So suppose that there exists an $\varepsilon > 0$ such that the set

$$S = \{k \geq 0 : \mathrm{dist}_e(f^k(z), f^{-1}(\infty)) > \varepsilon\}$$

is infinite.

Now, suppose that A_S^d, the set of limits points of $A_S = \{f^k(z) : k \in S\}$ is unbounded. There exists $w \in A_S^d$ and $a(\varepsilon) \geq 1$ ($a(\varepsilon)$ comes from Lemma 2.30) such that

$$B_e(w,5) \cap \left(\bigcup_{j=0}^{a(\varepsilon)} f^j(\mathrm{Crit}_\infty(f) \cup \mathrm{PC}_c(f) \cup \mathrm{PC}_p(f) \right) = \emptyset \tag{2.39}$$

and $|w| \geq 4D$. There also exists an increasing sequence $\{n_j\}_{j=1}^\infty \subset S$ such that $\lim_{j\to\infty} f^{n_j}(z) = w$. Disregarding finitely many elements of this sequence, we may assume without loss of generality that $f^{n_j}(z) \in B_e(w, \min\{1, D\})$ for all $j \geq 1$. In view of the definition of S, Lemma 2.30, and (2.39), we see that for every $j \geq 1$ there exists a holomorphic inverse branch $f_z^{-n_j} : B_e(f^{n_j}(z), 4) \mapsto \mathbb{C}$ sending $f^{n_j}(z)$ to z. Because of the same premises $w \notin \omega(\mathrm{Crit}(f))$ and we are done in this case by setting $x_j(z) = w$ for all $j \geq 1$ and $\eta(z) = 4$.

So, suppose that the set A_S^d is bounded. Assume first that

$$\liminf_{n\to\infty} \mathrm{dist}_e(f^n(z), f^{-1}(\infty)) = 0.$$

Then there exists $\{k_j\}_{j=1}^\infty$, an increasing sequence of positive integers such

$$\mathrm{dist}_e(f^{k_j+1}(z), f^{-1}(\infty)) > \varepsilon \tag{2.40}$$

(i.e. $k_j + 1 \in S$, $j \geq 1$) and we require $f^{k_j}(z)$ to be so close $f^{-1}(\infty)$ (assuming $\varepsilon > 0$ to be sufficiently small) that

$$|f^{k_j+1}(z)| \geq 4D + 2 \tag{2.41}$$

for all $j \geq 1$. Passing to a subsequence, we may assume without loss of generality that the sequence $\{\Pi(f^{k_j+1}(z))\}_{j=1}^\infty$ on the torus $\mathcal{T}$ converges to the same point $y \in \mathcal{T}$, where $\Pi : \mathbb{C} \mapsto \mathcal{T} = \mathbb{C}/\sim$ is the canonical projection from $\mathbb{C}$ onto the torus $\mathcal{T}$. Clearly, there exists a sequence $\{x_j(z)\}_{j=1}^\infty$ such that $\lim_{j\to\infty} |f^{k_j+1}(z) - x_j(z)| = 0$ and $\Pi(x_j(z)) = y$ for all $j \geq 1$.

Assume first that the sequence $\{f^{k_j+1}(z)\}_{j=1}^{\infty}$ is unbounded. Passing to a subsequence we may assume without loss of generality that $\lim_{j\to\infty} f^{k_j+1}(z) = \infty$. Then, applying (2.40), Lemma 2.30, (2.41) and the definition of D, we are done with $n_j = k_j + 1$ and $\eta(z) = 2$. So, assume that the sequence $\{f^{k_j+1}(z)\}_{j=1}^{\infty}$ is bounded. We already know that

$$B_e(f^{k_j+1}(z), 2) \cap \left(\overline{\mathrm{PC}_c(f)} \cup \overline{\mathrm{PC}_p(f)} \cup \overline{\bigcup_{j=a(\varepsilon)+1}^{\infty} f^j(\mathrm{Crit}_\infty(f))} \right) = 0.$$

Since the second component of this intersection is forward-invariant, we conclude that no accumulation point of the sequence $\{f^{k_j-a(\varepsilon)}(z)\}_{j=1}^{\infty}$ belongs to

$$\overline{\mathrm{PC}_c(f)} \cup \overline{\mathrm{PC}_p(f)} \cup \overline{\mathrm{PC}_\infty(f)} = \overline{\mathrm{PC}(f)}.$$

If the sequence $\{f^{k_j-a(\varepsilon)}(z)\}_{j=1}^{\infty}$ is unbounded, we may complete the argument in exactly the same way as above with $k_j + 1$ replaced by $k_j - a(\varepsilon)$. If the sequence $\{f^{k_j-a(\varepsilon)}(z)\}_{j=1}^{\infty}$ is bounded, we are immediately done by passing to a converging subsequence.

So assume that

$$\liminf_{n\to\infty} \mathrm{dist}_e(f^n(z), f^{-1}(\infty)) > 0.$$

Then $\liminf_{n\to\infty} |f^n(z)| < \infty$ and the ω-limit set is compact in the plane $\mathbb{C}$. In view of Lemma 2.26 there exists $x \in \omega(z) \setminus (\Omega(f) \cup \overline{O_+(f(\mathrm{Crit}(z))} \cup \{\infty\})$. The number

$$\eta = \frac{1}{2}\mathrm{dist}_e(x, \Omega(f) \cup \overline{O_+(f(\mathrm{Crit}(z))})$$

is positive since $\omega(\mathrm{Crit}(z))$ is a compact subset of $\mathbb{C}$ and $\Omega(f)$ is finite. Then there exists an infinite increasing sequence $\{m_j\}_{j\geq 1}$ such that

$$\lim_{j\to\infty} f^{m_j}(z) = x \tag{2.42}$$

and

$$B_e(f^{m_j}(z), \eta) \cap \bigcup_{n\geq 1} f^n(\mathrm{Crit}(z)) = \emptyset. \tag{2.43}$$

Now we claim that there exists $\eta(z)$ such that for every $j \geq 1$ large enough

$$\mathrm{Comp}(z, f^{m_j}(z), f^{m_j}, \eta(z)) \cap \mathrm{Crit}(f^{m_j}) = \emptyset. \tag{2.44}$$

Otherwise we would find an increasing to infinity subsequence $\{m_{j_i}\}$ of $\{m_j\}$

and a decreasing to zero sequence of positive numbers η_i such that $\eta_i < \eta$ and

$$\mathrm{Comp}(z, f^{m_{j_i}}(z), f^{m_{j_i}}, \eta_i) \cap \mathrm{Crit}(f^{m_{j_i}}) \neq \emptyset.$$

Let $\tilde{c}_i \in \mathrm{Comp}(z, f^{m_{j_i}}(z), f^{m_{j_i}}, \eta_i) \cap \mathrm{Crit}(f^{m_{j_i}})$. Then there exists $c_i \in \mathrm{Crit}(f)$ such that $f^{p_i}(\tilde{c}_i) = c_i$ for some $0 \leq p_i \leq m_{j_i} - 1$. Since the set $f^{-1}(x)$ is at a positive distance from $\Omega(f)$ and since $\eta_i \to 0$, it follows from Theorem 2.13 that $\lim_{i\to\infty} \tilde{c}_i = z$. Since $z \notin \bigcup_{n\geq 0} f^{-n}(\mathrm{Crit}(f))$, it implies that $\lim_{i\to\infty} p_i = \infty$. But then using Theorem 2.13 again and the formula $f^{p_i}(\tilde{c}_i) = c_i$ we conclude that the set of all accumulation points of the sequence $\{c_i\}$ is contained in $\omega(z)$. Hence, passing to a subsequence, we may assume that the limit $c = \lim_{i\to\infty} c_i$ exists. But since $c \in \omega(z)$, since $\omega(z)$ is a compact subset of $\mathbb{C}$ and since ∞ is the only accumulation point of $\mathrm{Crit}(f)$, we conclude that the sequence c_i is eventually constant. Thus, dropping some finite number of initial terms, we may assume that this sequence is constant. This means that $c_i = c$ for all $i = 1, 2, \ldots$. Since $c = f^{p_i}(\tilde{c}_i)$, we get

$$|f^{m_{j_i}}(z) - f^{m_{j_i}-p_i}(c)| = |f^{m_{j_i}}(z) - f^{m_{j_i}}(\tilde{c}_i)| < \eta_i.$$

Since $\lim_{i\to\infty} \eta_i = 0$ and since $\omega(z)$ is a compact subset of $\mathbb{C}$, we conclude that

$$\lim_{i\to\infty} |f^{m_{j_i}}(z) - f^{m_{j_i}-p_i}(c)| = 0.$$

Since $c \in \mathrm{Crit}(z)$, in view of (2.43) this implies that $m_{j_i} - p_i \leq 0$ for all i large enough. So, we get a contradiction as $0 \leq p_i \leq m_{j_i} - 1$ and (2.44) is proved. ■

In particular for every $j \geq 1$ there exists $f_z^{-n_j} : B_e(x_j(z), \eta(z)) \mapsto \mathbb{C}$. With the notation of this proposition, for every $j \geq 1$, let $T_j : \mathbb{C} \to \mathbb{C}$ be a translation $T_j(w) = w + x_j(z)$.

We shall prove the following.

Lemma 2.32 *With the assumptions and notations from Proposition 2.31 the family* $\mathcal{F}_z := \{f_z^{-n_j} \circ T_j : B_e(0, \eta(z)) \to \mathbb{C},\ j \geq 1\}$ *is normal and all its limit functions are constant.*

Proof. Decreasing $\eta(z)$ if necessary, we can always find a periodic orbit of f of period ≥ 3 disjoint from all the balls $B_e(x_j(z), \eta(z))$. Then this orbit is also disjoint from all the sets

$$f_z^{-n_j} \circ T_j(B_e(0,\eta(z)) = f_z^{-n_j} \circ T_j(B_e(x_j(z),\eta(z))).$$

Hence, by Montel's Theorem, $\mathcal{F}_z$ is a normal family. If there were non-constant limit functions of the family $\mathcal{F}_z$, then there would exists a radius $r > 0$ and an increasing subsequence $\{n_{j_k}\}_{k=1}^{\infty}$ such that

$$T_{j_k}^{-1} \circ f^{n_{j_k}}(B_e(z,r)) \subset B_e(0,\eta(z))$$

or equivalently

$$f^{n_{j_k}}(B_e(z,r)) \subset T_{j_k}(B_e(0,\eta(z))).$$

Passing yet to another subsequence, we may assume without loss of generality that

$$\overline{\mathbb{C}} \setminus \bigcup_{k=1}^{\infty} T_{j_k}(B_e(0,\eta(z)))$$

has a non-empty interior, and consequently contains at least three points. Thus the family

$$\{f^{n_{j_k}} : B_e(z,r) \to \mathbb{C}\}_{k=1}^{\infty}$$

would be normal, contrary to the fact $z \in J(f)$. We are done. ■

As an immediate consequence of this lemma, we get the following.

Corollary 2.33 *If $z \in J(f) \setminus \mathrm{Sing}^-(f)$ and the sequence $\{n_j\}$ is taken from Proposition 2.31, then*

$$\lim_{k\to\infty} |(f^{n_{j_k}})'(z)| = \infty.$$

3
Conformal measures

3.1 Preliminaries from geometric measure theory

Given a subset A of a metric space (X, d), a countable family $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of open balls centered at the set A is said to be a packing of A if and only if for any pair $i \neq j$

$$d(x_i, x_j) > r_i + r_j.$$

Given $t \geq 0$, the t-dimensional outer Hausdorff measure $\mathrm{H}^t(A)$ of the set A is defined as

$$\mathrm{H}^t(A) = \sup_{\epsilon>0} \inf\{\sum_{i=1}^{\infty} r_i^t\},$$

where infimum is taken over all countable covers $\{A_i\}_{i=1}^{\infty}$ by the sets A_i, $i = 1, 2, \ldots$, with diameters $r_i \leq \varepsilon$.

The t-dimensional outer packing measure $\Pi^t(A)$ of the set A is defined as

$$\Pi^t(A) = \inf_{\cup A_i = A} \{\sum_i \Pi_*^t(A_i)\}$$

(A_i are arbitrary subsets of A), where

$$\Pi_*^t(A) = \sup_{\epsilon>0} \sup\{\sum_{i=1}^{\infty} r_i^t\}.$$

Here the second supremum is taken over all packings $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of the set A by open balls centered at A with radii which do not exceed ε. These two outer measures define countable additive measures on Borel σ-algebra of X.

The definition of the Hausdorff dimension HD(A) of the set A is the following

$$\mathrm{HD}(A) = \inf\{t : \mathrm{H}^t(A) = 0\} = \sup\{t : \mathrm{H}^t(A) = \infty\}.$$

Let ν be a Borel probability measure on X which is positive on open sets. Define the function $\rho = \rho_t(\nu) : X \times (0,\infty) \to (0,\infty)$ by

$$\rho(x,r) = \frac{\nu(B(x,r))}{r^t}.$$

The key facts from the geometric measure theory, needed in the sequel, are included in the following two theorems. These are easy consequence of Besicovič covering theorem (see [14]).

Theorem 3.1 *Let $X = \mathbb{R}^n$ for some $n \geq 1$. Then there exists a constant $b(n)$ depending only on n with the following properties. If A is a Borel subset of $\mathbb{R}^n$ and $C > 0$ is a positive constant such that*

(1) for all (but countably many) $x \in A$

$$\limsup_{r\to 0} \rho(x,r) \geq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\mathrm{H}^t(E) \leq b(n)C\nu(E)$ and, in particular, $\mathrm{H}^t(A) < \infty$,
or

(2) for all $x \in A$

$$\limsup_{r\to 0} \rho(x,r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\mathrm{H}^t(E) \geq C\nu(E)$.

Theorem 3.2 *Let $X = \mathbb{R}^n$ for some $n \geq 1$. Then there exists a constant $b(n)$ depending only on n with the following properties. If A is a Borel subset of $\mathbb{R}^n$ and $C > 0$ is a positive constant such that*

(1) for all $x \in A$

$$\liminf_{r\to 0} \rho(x,r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\Pi^t(E) \geq Cb(n)^{-1}\nu(E)$,
or

(2) for all $x \in A$

$$\liminf_{r\to 0} \rho(x,r) \ge C^{-1},$$

then $\Pi^t(E) \le C\nu(E)$ and, consequently, $\Pi^t(A) < \infty$.

(1') If ν is non–atomic then (1) holds under the weaker assumption that the hypothesis of part (1) is satisfied on the complement of a countable set.

3.2 Support of conformal measure

From now on throughout this section and the entire paper H_e^t stands for the t-dimensional Hausdorff measure on $\mathbb{C}$ with respect to the Euclidean metric whereas H_s^t refers to its spherical counterpart. The same convention is applied to the packing measures Π_e^t and Π_s^t. Note that the measures H_e^t and H_s^t as well as Π_e^t and Π_s^t are equivalent with Radon-Nikodym derivative bounded away from zero and ∞ on compact subsets of $\mathbb{C}$. In particular the Hausdorff dimension of any subset A of $\mathbb{C}$ has the same value no matter whether calculated with respect to the Euclidean or spherical metric; it will be denoted in the sequel simply by $\mathrm{HD}(A)$. If H^t or Π^t will be endowed neither with the subscript 'e' nor 's', it will refer to Euclidean as well as spherical measures. We set

$$h = \mathrm{HD}(J(f)).$$

Lemma 3.3 *If m is a t-conformal measure, either Euclidean or spherical, then $t \ge \mathrm{HD}(J(f))$ and $\mathrm{H}^t|_{J(f)}$ is absolutely continuous with respect to m i.e. $\mathrm{H}^t|_{J(f)} \prec m$.*

Proof. Since the measures m_e and m_s are equivalent as well as H_e^t and H_s^t are, it suffices to prove the lemma for Euclidean measures m_e and H_e^t. Now, fix $z \in J(f) \setminus \mathrm{Sing}^-(f)$. Let the sequence $\{n_j\}_{j=1}^\infty$, associated to z, comes from Proposition 2.31. It follows from Koebe's Distortion Theorem, I (Euclidean version) that

$$f_z^{-n_j}(B_e(f^{n_j}(z),\eta(z)) \subset B_e(z, 4^{-1}K\eta(z)|(f^{n_j})'(z)|^{-1}).$$

Applying Koebe's Distortion Theorem, I (Euclidean version) again, Corollary 1.7 and conformality of the measure m, we thus get

$$\begin{aligned} m_e\big(B_e(z,4^{-1}K\eta(z)|(f^{n_j})'(z)|^{-1})\big) &\ge K^{-t}|(f^{n_j})'(z)|^{-t} m_e(B_e(f^{n_j}(z),\eta(z)/4)) \\ &\ge K^{-t}|(f^{n_j})'(z)|^{-t} M(t,\eta(z)/4) \\ &= M(t,\eta(z)/4)K^{-2t}4^t\eta(z)^{-t}(4^{-1}K\eta(z)|(f^{n_j})'(z)|^{-1})^t. \end{aligned}$$

Thus

$$\limsup_{r\to 0} \frac{m_e(B_e(z,r))}{r^t} \geq M(t,\eta(z)/4)(4^{-1}K^2\eta(z))^{-t}$$

and we are done because of Theorem 3.1 (1). ■

The next lemma we will need, proven in [20] as Lemma 3.4, required in fact no assumptions about elliptic function in question (except non-constantness) and we state it below with proof.

Lemma 3.4 *If m is a t-conformal measure for $f:\mathbb{C}\to\overline{\mathbb{C}}$, then*

$$m(I_\infty(f)\setminus I_-(f)) = 0.$$

Even more, there exists $R>0$ such that

$$m(\{z : \liminf_{n\to\infty} |f^n(z)| > R\}) = 0.$$

Proof. It suffices to prove the lemma for the spherical measure m_s. Let b be a pole of $f:\mathbb{C}\to\overline{\mathbb{C}}$. We shall obtain first an upper estimate on $m_s(B_b(R))$ similar to the second inequality in (2.5). And indeed, covering $B_R\setminus\{\infty\}$ by two simply connected domains

$$B_R^+ = \{z\in B_R\setminus\{\infty\} : \mathrm{Im} z > 0\} \quad \text{and} \quad B_R^1 = \{z\in B_R\setminus\{\infty\} : \mathrm{Im} z < 1\}$$

we obtain

$$m_s(B_b(R)\setminus\{b\}) \leq \sum_{j=1}^{q_b}\int_{B_R^+} |(f^{-1}_{b,B_R^+,j})^*|^t dm_s + \sum_{j=1}^{q_b}\int_{B_R^1} |(f^{-1}_{b,B_R^-,j})^*|^t dm_s.$$

Using now (2.3), we obtain

$$\begin{aligned}\int_{B_R^+} |(f^{-1}_{b,B_R^+,j})^*|^t dm_s &\asymp \int_{B_R^+}\left(\frac{1}{1+|b|^2}|z|^{\frac{q_b-1}{q_b}}\right)^t dm_s(z)\\ &= \frac{1}{(1+|b|^2)^t}\int_{B_R^+}|z|^{\frac{q_b-1}{q_b}t} dm_s(z)\\ &\leq (1+|b|^2)^{-t}\int_{B_R^+}|z|^{\frac{q-1}{q}t} dm_s(z).\end{aligned}$$

Looking at the first line of this formula with a pole b of maximal multiplicity q, we see that the integral $\int_{B_R^+}|z|^{\frac{q-1}{q}t} dm_s(z)$ is finite and even more:

$$\lim_{R\to\infty}\int_{B_R^+}|z|^{\frac{q-1}{q}t} dm_s(z) = 0. \tag{3.1}$$

Similarly, the integral $\int_{B_R^1} |z|^{\frac{q-1}{q}t} dm_s(z)$ is finite and it also converges to 0 as $R \to \infty$. Putting

$$\Sigma_R = \max\left\{ \int_{B_R^+} |z|^{\frac{q-1}{q}t} dm_s(z), \int_{B_R^1} |z|^{\frac{q-1}{q}t} dm_s(z) \right\}$$

we therefore conclude that

$$m_s(B_b(R) \setminus \{b\}) \le 2q\Sigma_R(1+|b|^2)^{-t} \le 2q\Sigma_R|b|^{-2t}. \tag{3.2}$$

Now the argument goes essentially in the same way as in [19]. We present it here for the sake of completeness. We take $R_2 \ge R_1$ defined in Section 2.1 so large that

$$LR^{-\frac{1}{q_b}} < R_0 \tag{3.3}$$

for all poles $b \in B_{R_2}$ and all $R \ge R_2$. Given two poles $b_1, b_2 \in B_{2R_2}$ we denote by $f^{-1}_{b_2,b_1,j} : B_e(b_1, R_0) \to \mathbb{C}$ all the holomorphic inverse branches $f^{-1}_{b_2,B(b_1,R_0),j}$. It follows from (2.5) and (3.3) that

$$f^{-1}_{b_2,b_1,j}\big(B_e(b_1,R_0)\big) \subset B_{b_2}(2R_2 - R_0) \subset B_{b_2}(R_2) \subset B_e(b_2, R_0). \tag{3.4}$$

Set

$$I_R(f) = \{z \in \mathbb{C} : \quad \forall_{n\ge 0} \ |f^n(z)| > R\}.$$

Since the series $\sum_{b\in f^{-1}(\infty)\setminus\{0\}} |b|^{-s}$ converges for all $s > 2$ and since by Lemma 3.3 and Theorem 2.1 (comp. Theorem 3 from [19]), $t \ge h > \frac{2q}{q+1}$, there exists $R_3 \ge R_2$ such that

$$q(2A_2)^t \sum_{b\in B_{R_3}\cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t} \le 1/2, \tag{3.5}$$

where A_2 was defined in (2.2) and (2.3). Consider $R \ge 4R_3$. Put

$$I = f^{-1}(\infty) \cap B_{(R/2)}.$$

Since $R/2 + R_0 \le R/2 + R_3 < R/2 + R/2 = R$, it follows from (3.4), (2.5) and (3.3) that for every $l \ge 1$ the family W_l defined as

$$\left\{ f^{-1}_{b_l,b_{l-1},j_l} \circ f^{-1}_{b_{l-1},b_{l-2},j_{l-1}} \circ \ldots \circ f^{-1}_{b_2,b_1,j_2} \circ f^{-1}_{b_1,b_0,j_1}\big(B_{b_0}(R/2) \setminus f^{-1}(\infty)\big) \right\},$$

where $b_i \in I : 1 \le j_i \le q_{b_i}, i = 0, 1, \dots, l$, is well-defined and covers $I_R(f)$. Applying (2.3) and (3.2) we may now estimate as follows.

$$
\begin{aligned}
& m_s(I_R(f)) \le \\
& \le \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} m_s \left(f^{-1}_{b_l, b_{l-1}, j_l} \circ f^{-1}_{b_{l-1}, b_{l-2}, j_{l-1}} \circ \dots \circ f^{-1}_{b_2, b_1, j_2} \circ f^{-1}_{b_1, b_0, j_1} \big(B_{b_0}(R/2) \big) \right) \\
& \le \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} || \big(f^{-1}_{b_l, b_{l-1}, j_l} \circ f^{-1}_{b_{l-1}, b_{l-2}, j_{l-1}} \circ \dots \circ f^{-1}_{b_2, b_1, j_2} \circ f^{-1}_{b_1, b_0, j_1} \big)^* |_{B_{b_0}(R/2)} ||_\infty^t \\
& \times m_s \big(B_{b_0}(R/2) \big) \\
& \le \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} (2A_2)^{lt} \left(\frac{|b_{l-1}|^{\frac{q_{b_l}-1}{q_{b_l}}}}{|b_l|^2} \right)^t \cdot \left(\frac{|b_{l-2}|^{\frac{q_{b_{l-1}}-1}{q_{b_{l-1}}}}}{|b_{l-1}|^2} \right)^t \dots \left(\frac{|b_0|^{\frac{q_{b_1}-1}{q_{b_1}}}}{|b_1|^2} \right)^t \\
& \times (2q\Sigma_R)^t \frac{1}{|b_0|^{2t}} \\
& = (2q\Sigma_R)^t (2A_2)^{lt} \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} |b_l|^{-2t} \big(|b_{l-1}|^{-\frac{q+1}{q}t} \dots |b_0|^{-\frac{q+1}{q}t} \big) \\
& \le (2q\Sigma_R)^t (2A_2)^{lt} \sum_{b_l \in I} \sum_{j_l=1}^{q_{b_l}} \dots \sum_{b_1 \in I} \sum_{j_1=1}^{q_{b_1}} \sum_{b_0 \in I} \big(|b_l|^{-\frac{q+1}{q}t} |b_{l-1}|^{-\frac{q+1}{q}t} \dots |b_0|^{-\frac{q+1}{q}t} \big) \\
& \le (2q\Sigma_R)^t (2A_2)^{lt} \left(\sum_{b \in I} |b|^{-\frac{q+1}{q}t} \right)^l q^l \\
& \le (2q\Sigma_R)^t \left(q(2A_2)^t \sum_{b \in B_{R_3} \cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t} \right)^l .
\end{aligned}
$$

Applying (3.5) we therefore get $m_s(I_R(f)) \le (2q\Sigma_R)^t 2^{-l}$. Letting $l \to \infty$ we therefore get $m_s(I_R(f)) = 0$. Since $m_s \circ f^{-1} \prec m_s$ and since

$$
\{z : \liminf_{n \to \infty} |f^n(z)| > R\} = \bigcup_{j=0}^{\infty} f^{-j}(I_R(f)),
$$

we conclude that

$$
m_s\big(\{z : \liminf_{n \to \infty} |f^n(z)| > R\} \big) = 0.
$$

We are done. ■

4
Hausdorff, packing and conformal measures

In this chapter H_s^h and Π_s^h denote respectively the Hausdorff and packing measures considered with respect to the spherical metric on $\overline{\mathbb{C}}$. Our aim here is to prove first the existence of an h-conformal measure, its atomless and ultimately the following main result.

Theorem 4.1 *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a regular pseudo non-recurrent elliptic function. If $h = \mathrm{HD}(J(f)) = 2$, then $J(f) = \mathbb{C}$. If $h < 2$, then*

(a) $\mathrm{H}_s^h(J(f)) = 0$.
(b) $\Pi_s^h(J(f)) > 0$.
(c) $\Pi_s^h(J(f)) = \infty$ *if and only if* $\Omega(f) \neq \emptyset$.

As an immediate consequence of this theorem, we get the following.

Corollary 4.2 *If $\Omega(f) = \emptyset$, then the Euclidean h-dimensional packing measures Π_e^h is finite on each bounded subsets of $J(f)$.*

4.1 Existence of conformal measures

Now we consider the situation where $H : U_1 \to U_2$ is an analytic map of open subsets U_1, U_2 of the complex plane $\mathbb{C}$. We say that given $t \geq 0$, the Borel measure ν finite on bounded sets of $\mathbb{C}$ is an Euclidean semi t-conformal measure if and only if

$$\nu(H(A)) \geq \int_A |H'|^t \, d\nu$$

for every Borel subset A of U_1 such that $H|_A$ is one-to-one and is called t-conformal if the "$\geq$" sign can be replaced by an "$=$" sign.

Notice that if m_s is a spherical semi t-conformal measure for $f : J(f) \to J(f) \cup \{\infty\}$, then the measure $m_e = (1 + |z|^2)^t m_s$ is Euclidean semi t-conformal, i.e.

$$m_e(f(A)) \ge \int_A |f'|^t dm_e$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is 1-to-1. If m is t-conformal, then so is m_e in the obvious sense. The measure m_e is called the Euclidean version of m. Obviously m_e is equivalent to m and is finite on bounded subsets of $\mathbb{C}$.

Let

$$I_-(f) = \bigcup_{n \ge 1} f^{-n}(\infty).$$

Our main technical concept, also interesting on its own, is a conformal measure. We prove first its existence.

Lemma 4.3 *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a pseudo non-recurrent function, then there exists an h-conformal measure m for f whose atoms are contained in*

$$I_-(f) \cup \bigcup_{n=0}^{\infty} f^{-n}(\mathrm{Crit}(J(f))).$$

Proof. We shall construct an h-conformal measure with required properties by utilizing the methods of $K(V)$ sets developed in [9], comp. [21]. In order to begin, we call $Y \subset \{\infty\} \cup \Omega(f) \cup \bigcup_{n \ge 1} f^n(\mathrm{Crit}(J(f)))$ is a crossing set if Y is finite and the following four conditions are satisfied.

(y1) $\infty \in Y$.
(y2) $Y \cap \{f^n(x) : n \ge 1\}$ is a singleton for all $x \in \mathrm{Crit}(J(f))$.
(y3) $Y \cap \mathrm{Crit}(f) = \emptyset$.
(y4) $\Omega(f) \subset Y$.

Since $f(\mathrm{Crit}(f))$ is finite, crossing sets do exists. Let $V \subset \overline{\mathbb{C}}$ be an open neighborhood of Y such that

$$\mathrm{Crit}(J(f)) \cap \partial V = \emptyset. \tag{4.1}$$

Define

$$K(V) = J(f) \cap \bigcap_{n \ge 0} f^{-n}(\overline{\mathbb{C}} \setminus V) = \{z \in J(f) : \forall n \ge 0 \ f^n(z) \notin V\}.$$

Obviously $f(K(V)) \subset K(V)$. Since $f : \mathbb{C} \to \overline{\mathbb{C}}$ is continuous and V is open, we see that K(V) is a closed subset of $\mathbb{C}$. Since $V \cap K(V) = \emptyset$ and $\infty \in V$, $K(V)$ is bounded. Thus, $K(V)$ is a compact set. Applying Lemma 9.2 from [21] with $X = K(V)$ and $U(f) = \mathbb{C}$, we directly obtain a number $s(V) \in [0, h]$ and a Borel probability measure m_V supported on $K(V)$ such that

$$m_V(f(A)) \geq \int_A |f^*|^{s(V)} dm_V$$

for every special set $A \subset \mathbb{C}$ and

$$m_V(f(A)) = \int_A |f^*|^{s(V)} dm_V$$

for every special set $A \subset \mathbb{C} \backslash \overline{V}$. From now on throughout the entire paper we fix a crossing set Y and we consider an open neighborhood $V \subset \overline{\mathbb{C}}$ of Y such that the closure of V is disjoint from at least one fundamental parallelogram of f. All other neighborhoods of Y considered in this article will be always assumed to be contained in this set V.

An Euclidean semi t-conformal measure m_e is said to be almost t-conformal if

$$m_e(f(A)) = \int_A |f'|^t dm_e$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is 1-to-1 and $A \cap \overline{V} = \emptyset$. Hence for every Borel set A such that $f|_A$ is 1-to-1 and $A \cap \overline{V} = \emptyset$ and for every $w \in \Lambda$, we have

$$\int_A |f'|^t dm_e = m_e(f(A)) = m_e(f(A+w)) \geq \int_{A+w} |f'|^t dm_e, \qquad (4.2)$$

and the last inequality sign becomes an equality either if in addition $(A + w) \cap \overline{V} = \emptyset$ or if m_e is a t-conformal measure, and we assume only that $f|_A$ is 1-to-1. Since f' is periodic with respect to the lattice Λ, all the above statements and assumptions lead to the following.

Lemma 4.4 *For every $w \in \Lambda$, every Borel set $A \subset \mathbb{C}$ such that $A \cap \overline{V} = \emptyset$ and every almost t-conformal measure m*

$$m_e(A + w) \leq m_e(A). \qquad (4.3)$$

If either in addition $(A+w) \cap \overline{V} = \emptyset$ or if m is h-conformal and we assume only that $f|_A$ is 1-to-1, then this inequality becomes an equality. For every

$r > 0$ there exists $M(r) \in (0, \infty)$ independent of any almost t-conformal measure m such that

$$m_e(F) \le M(r) \tag{4.4}$$

for every Borel set $F \subset \mathbb{C}$ with the diameter $\le r$. If in addition m is h-conformal, then for every $R > 0$ there exist constants $Q(R)$ and $Q_h(R)$ such that

$$m_e(B_e(x,r)) \ge Q(R)r^2 \ge Q_h(R)r^h \tag{4.5}$$

for all $x \in J(f)$ and all $r \ge R$.

Proof. Inequality (4.3) as well as its equality counterpart are an immediate application of (4.2). Formula (4.4) follows directly from (4.3) and the fact that V is disjoint from at least one fundamental parallelogram. The second part of formula (4.5) is clear. In order to prove the first one, fix a fundamental parallelogram $\mathcal{R}$ and notice that

$$T(R) = \inf\{m_e(B(z,R)) : z \in J(f) \cap \mathcal{R}\} > 0.$$

Hence, if

$$R \le r \le 4\text{diam}_e(\mathcal{R}),$$

then for any $x \in J(f)$,

$$m_e(B_e(x,r)) \ge m_e(B_e(x,R)) \ge T(R) = \frac{T(R)}{r^2}r^2 \ge \frac{1}{16}\frac{T(R)}{\text{diam}_e^2(\mathcal{R})}r^2,$$

and we are done in this case. So suppose that $r \ge 4\text{diam}(\mathcal{R})$. Then each ball $B_e(x,r)$ contains at least

$$\left(\frac{\sqrt{2}r}{2\text{diam}_e(\mathcal{R})}\right)^2 = \frac{r^2}{2\text{diam}_e^2(\mathcal{R})}$$

non-overlapping Λ-congruent copies of $\mathcal{R}$. Therefore,

$$m_e(B_e(x,r)) \ge \frac{r^2}{2\text{diam}_e^2(\mathcal{R})}m_e(R) = \frac{m_e(\mathcal{R})}{2\text{diam}_e(\mathcal{R})}r^2.$$

We are done. ■

Taking the neighborhood V with sufficiently small diameter, we see that the limit set J_S defined on page 274 of [19] is branchwise (in the sense of Appendix 1 of [21]) and contained in $K(V)$. It therefore follows from

Lemma 9.3 in [21] that $s(V) \geq \mathrm{HD}(J_S)$. Hence, the last inequality in the proof of Theorem 1.1 in [19] gives that $s(V) \geq h_*$, with some $h_* > 1$.

$$1 < h_* < s(V) \leq h. \tag{4.6}$$

Fix from now $(V_n)_{n=1}^{\infty}$, a descending sequence of neighborhood of Y satisfying (4.1) and such that $\mathrm{diam}_s(V_n) \leq \frac{1}{n}$. In view of (4.6), passing to a subsequence, we may assume without loss of generality that the sequence $(s(V_n))_{n=1}^{\infty}$ converges. Denote its limit by $s(Y)$. We then have

$$1 < h_* \leq s(Y) \leq h. \tag{4.7}$$

Passing yet to another subsequence, we may assume, that the sequence $(m_{V_n})_{n=1}^{\infty}$, treated as probability measure on the compact space $\overline{\mathbb{C}}$, converges weakly to a Borel probability measure m_Y on $\overline{\mathbb{C}}$. We shall prove the following.

Lemma 4.5 *If $s(V)$ is an accumulation point of the sequence $s(B_s(Y, \frac{1}{n}))$, then $s(Y) \in (0, h]$ and there exists a Borel probability measure m_Y (an appropriate weak accumulation point on $J(f)$ with the following properties.*

(a) $m_Y(f(A)) \geq \int_A |f^*| dm_Y$ for every special set $A \subset \mathbb{C}$.
(b) $m_Y(f(A)) = \int_A |f^*| dm_Y$ for every special set $A \subset \mathbb{C} \setminus Y$.
(c) $m_Y(\infty) = 0$.

Proof. Following the hints to the proof of Lemma 3.3 in [9] without (a') and (d), and with $\Gamma = Y$, it is not hard to see that conditions (a) and (b) are satisfied. We shall prove that (c) holds. Set

$$m_n^s = m_{V_n}, \quad n \geq 1,$$

and

$$m_n^e = (m_{V_n})_e, \quad s_n = s(V_n).$$

For every $k \geq 0$ consider S_k, the square centered at the origin whose edges are parallel to the coordinates axises are of length 2^k. Since by Lemma 3.7 in [20] and by (4.6), each measure m_n^s is almost conformal with an exponent in $[1, h]$, and since each 'annulus' $A_k = S_{k+1} \setminus S_k$ is a union of a 3×4^k unit squares, it follows from (a) and (b), which say that m_n^s is almost s_n-conformal, and from (4.6) that for all $k \geq 1$ and all $n \geq 1$

$$m_n^e(A_k) \leq 3M(1)4^k.$$

Consequently

$$m_n^s(A_k) \le \frac{3M(1)4^k}{(1+4^k)^{s_n}} \le \frac{3M(1)4^k}{4^{ks_n}} \le \frac{3M(1)4^k}{4^{kh_*}} = 3M(1)4^{(1-h_*)k}.$$

Since $h_* > 1$ (see again (4.6)), we thus get for all $j \ge 1$ and all $n \ge 1$ that

$$m_n^s(S_{j+1}) = m_n^s(\bigcup_{k=j}^{\infty} A_k) = \sum_{k=j}^{\infty} m_n^s(A_k) \le \sum_{k=j}^{\infty} 3M(1)4^{(1-h_*)k}$$

$$= 3M(1)(1-4^{1-h_*})^{-1}4^{(1-h_*)j}.$$

Hence $m_Y(\infty) = 0$ and we are done. ■

Lemma 4.6 *We have $m_Y(\Omega(f))) = 0$.*

Proof. Indeed, fix $\omega \in \Omega(f)$. Take $a \ge 1$ so large that $f^a(\omega) = \omega$ and $(f^a)'(\omega) = 1$. It then follows from Lemma 2.5 and Lemma 2.6 that there exists a compact set $F_\omega \subset (B_e(\omega,\theta)\setminus\{\omega\}) \cap J(f)$ and a constant $C \ge 1$ such that for every $k \ge 1$ and all $z \in F_\omega$, we have

$$C^{-1}k^{-\frac{p(\omega)+1}{p(\omega)}} \le |(f_\omega^{-ak})^*(z)| \le Ck^{-\frac{p(\omega)+1}{p(\omega)}} \tag{4.8}$$

and for every $n \ge 1$, there exists $k_n \ge 1$ such that

$$J(f) \cap B_e\left(\omega, \frac{1}{n}\right) \subset \bigcup_{j=k_n}^{\infty} f_\omega^{-aj}(F_\omega) \quad \text{and} \quad \lim_{n\to\infty} k_n = \infty. \tag{4.9}$$

As in the proof of the previous lemma denote $m_{|V_n}$ by m_n^s and $s(V_n)$ by s_n. It therefore follows from Lemma 4.5, (4.9), (4.8) and (4.6) that for all $n \ge 1$ and all $l \ge 1$.

$$\begin{aligned} m_n^s\left(B_e\left(\omega, \frac{1}{l}\right)\right) &\le \sum_{j=k_l}^{\infty} m_n^s(f_\omega^{-aj}(F_\omega)) \\ &\le C^{\frac{p(\omega)+1}{p(\omega)}s_n} \sum_{j=k_l}^{\infty} j^{-\frac{p(\omega)+1}{p(\omega)}s_n} \\ &\le C^{\frac{p(\omega)+1}{p(\omega)}s(Y)} \sum_{j=k_l}^{\infty} j^{-h_*}. \end{aligned}$$

Consequently

$$m_Y\left(B_e\left(\omega, \frac{1}{l}\right)\right) \le C^{\frac{p(\omega)+1}{p(\omega)}s(Y)} \sum_{j=k_l}^{\infty} j^{-h_*}.$$

Since $\lim_{l\to\infty} k_l = \infty$, we infer $m_Y(\Omega(f)) = 0$. ■

Now, we are in position to complete easily the proof of Lemma 4.3. Let

$$m := m_Y.$$

It follows from the definition of $\Omega(f)$, Lemma 4.5, Lemma 4.6, and Corollary 2.33 that $m_Y(Y) = 0$. Therefore, since in addition, $f(\Omega(f)) = \Omega(f)$, in order to prove $s(Y)$-conformality of the measure m, it suffices to show that

$$m(f(Y \setminus \Omega(f))) = 0.$$

But if $y \in Y \setminus (\Omega(f) \cup \{\infty\})$, then by our definition of Y, $y \notin \mathrm{Sing}^-(f)$; and the formula $m(f(y)) = 0$ immediately follows from Corollary 2.33, the formula

$$m(f^n(f(y)) \geq |(f^n)^*(f(y))|^{s(Y)} m(f(y))$$

and (4.7). Thus the $s(Y)$-conformality of m is proved; and, in addition, all the atoms of m must be contained in $J(f) \setminus \Omega(f)$. In view of Lemma 4.5 and Lemma 3.3, $s(Y) = h$. Applying now Lemma 3.4 and Corollary 2.33 we see that all atoms of m must be contained in

$$I_-(f) \cup \bigcup_{n\geq 0} f^{-n}(\mathrm{Crit}(f)).$$

The proof is complete. ■

4.2 Special facts from the geometric measure theory

We provide in this section some more technical facts taken from Section 2, Section 3 and Section 4 of [34].

Definition 4.7 *Given $r > 0$ and $L > 0$ a point $x \in \mathbb{C}$ is said to be (r, L)-t-upper estimable if $\rho(x, r) \leq L$ and is said to be (r, L)-t-lower estimable if $\rho(x, r) \geq L$. We will frequently abbreviate the notation writing (r, L)-t-u.e. for (r, L)-t-upper estimable and (r, L)-t-l.e. for (r, L)-t-lower estimable. We also say that the point x is t-upper estimable (t-lower estimable) if it is (r, L)-t-upper estimable ((r, L)-t-lower estimable) for some $L > 0$ and all $r > 0$ sufficiently small.*

Definition 4.8 *Given $r > 0$, $\sigma > 0$ and $L > 0$ the point $x \in X$ is said to be (r, σ, L)-t-strongly lower estimable, or shorter (r, σ, L)-t-s.l.e. if $\nu(B_e(y, \sigma r)) \geq Lr^t$ for every $y \in B_e(x, r)$.*

Lemma 4.9 *If z is (r, σ, L)-t-s.l.e., then every point $x \in B_e(z, r/2)$ is $(r/2, 2\sigma, 2^t L)$-t-s.l.e..*

Proof. Let $x \in B_e(z, r/2)$. Then $x \in B_e(z, r)$ and therefore

$$\nu(B_e(x, 2\sigma(r/2))) = \nu(B_e(x, \sigma r)) \geq Lr^t = 2^t L(r/2)^t.$$

■

Lemma 4.10 *If x is (r, σ, L)-t-s.l.e., then for every $0 < u \leq 1$ it is $(ur, \sigma/u, Lu^{-t})$-t-s.l.e..*

Proof. If $y \in B_e(x, ur)$, then $y \in B_e(x, r)$ and therefore

$$\nu(B_e(y, (\sigma/u)ur)) = \nu(B_e(y, \sigma r)) \geq Lr^t = Lu^{-t}(ur)^t.$$

■

We would like to finish this part with the following obvious statement.

Lemma 4.11 *If ν is positive on nonempty open sets, then for every $r > 0$ there exists $E(r) \geq 1$ such that every point $x \in X$ is $(r, E(r))$-t-u.e. and $(r, E(r)^{-1})$-t-l.e..*

The following lemma is a straightforward consequence of Koebe's Distortion Theorem, I (Euclidean version).

Lemma 4.12 *Let ν_e be a Euclidean semi t-conformal measure. Suppose that $D \subset \mathbb{C}$ is an open set, $z \in D$ and $H : D \to \mathbb{C}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B_e(H(z), 2R)$ for some $R > 0$. Then for every $0 \leq r \leq R$*

$$K^{-t}\nu_e(B_e(z, K^{-1}r|H'(z)|^{-1})) \leq |H'(z)|^{-t}\nu_e(B_e(H(z), r)).$$

If, in addition, ν_e is t-conformal, then also

$$|H'(z)|^{-t}\nu_e(B_e(H(z), r)) \leq K^t\nu_e(B_e(z, Kr|H'(z)|^{-1})).$$

Lemma 4.13 *Suppose that ν_e is a Euclidean t-conformal measure. Suppose that $D \subset \mathbb{C}$ is an open set, $z \in D$ and $H : D \to \mathbb{C}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B_e(H(z), 2R)$ for some $R > 0$. If the point $H(z)$ is (r, σ, L)-t-s.l.e., where $r \le R/2$ and $\sigma \le 1$, then the point z is $(K^{-1}|H'(z)|^{-1}r, K^2\sigma, L)$-$t$-s.l.e..*

Proof. In this proof we apply Lemma 4.12 several times without special indicating. Consider

$$x \in B_e(z, K^{-1}r|H'(z)|^{-1}).$$

Then $H(x) \in B_e(H(z), r)$ and therefore $\nu_e(B_e(H(x), \sigma r)) \ge Lr^t$. Since

$$B_e(H(x), \sigma r) \subset B_e(H(z), 2r) \subset B_e(H(z), R)$$

we have

$$H_z^{-1}\big(B_e(H(x), \sigma r)\big) \subset B_e(x, K\sigma r|H'(z)|^{-1}) = B_e(x, K^2\sigma(K^{-1}|H'(z)|^{-1}r)).$$

Thus

$$\nu_e\big(B_e(x, K^2\sigma(K^{-1}|H'(z)|^{-1}r)) \ge K^{-t}|H'(z)|^{-t}Lr^t = L(K^{-1}|H'(z)|^{-1}r)^t.$$

The proof is finished. ■

Lemma 4.14 *Suppose that ν_e is a Euclidean t-conformal measure. Let c be a critical point of an analytic map $H : D \to \mathbb{C}$. If $0 < r \le R(H, c)$ and $H(c)$ is (r, L)-t-l.e., then c is $((Ar)^{1/p_c}, A^{-2t}L)$-t-l.e., where A was defined in (2.25), p_c is the order of H at the critical point c.*

Proof. By Definition 1.1 we get $B_e(H(c), r) = H(\mathrm{Comp}(c, H(c), H, r))$. If

$$x \in \mathrm{Comp}(c, H(c), H, r)$$

then $A^{-1}|x-c|^{p_c} \le |H(x)-H(c)| < r$ which implies that $x \in B_e(c, (Ar)^{1/p_c})$.

Thus $B_e(H(c),r) \subset H(B_e(c,(Ar)^{1/p_c})$ and therefore

$$\begin{aligned} Lr^t &\le \nu_e(B_e(H(c),r)) \\ &\le \nu_e\big(H(B_e(c,(Ar)^{1/p_c}))\big) \\ &\le \int_{B_e(c,(Ar)^{1/p_c})} |H'(z)|^t \, d\nu_e(z) \\ &\le \int_{B_e(c,(Ar)^{1/p_c})} A^t(|z-c|^{p_c-1})^t \, d\nu_e(z) \\ &\le A^t(Ar)^{\frac{p_c-1}{p_c}t}\nu_e(B_e(c,(Ar)^{1/p_c})). \end{aligned}$$

So, $\nu_e(B_e(c,(Ar)^{1/p_c})) \ge A^{-2t}L((Ar)^{1/p_c})^t$. ■

Lemma 4.15 *Let c be a critical point of an analytic map $H : D \to \mathbb{C}$. Let ν_e be a Euclidean semi t-conformal measure such that $\nu_e(c) = 0$. If $0 < r \le R(H,c)$ and $H(c)$ is (s,L)-t-u.e. for all $0 < s \le r$, then c is*

$$\big((A^{-1}r)^{1/p_c}, q(2A^2)^t(2^{t/p_c}-1)^{-1}L\big) - t - u.e.,$$

where A was defined in (2.25), p_c is the order of H at the critical point c.

Proof. Take any $s \le r$. then $H(B_e(c,(A^{-1}s)^{1/p_c})) \subset B_e(H(c),r)$. Therefore, setting $R(c,a,b) = \{z : a \le |z-c| < b\}$, abbreviating

$$R(c, 2^{-1/p_c}(A^{-1}s)^{1/p_c}, (A^{-1}s)^{1/p_c})$$

by $R(c)$ and using the decomposition of $B(c,(A^{-1}s)^{1/p_c})$ described after Definition 1.1 we obtain

$$\begin{aligned} Ls^t &\ge \nu_e(B_e(H(c),s)) \\ &\ge \nu_e\big(H(B_e(c,(A^{-1}s)^{1/p_c}))\big) \\ &= p_c^{-1}\int_{B_e(c,(A^{-1}s)^{1/p_c})} |H'(z)|^t \, d\nu_e(z) \\ &\ge p_c^{-1}\int_{R(c)} |H'(z)|^t \, d\nu_e(z) \\ &\ge p_c^{-1}A^{-t}(2^{-1}A^{-1}s)^{\frac{p_c-1}{p_c}t}\nu_e(R(c)). \end{aligned}$$

So, $\nu_e\big(R(c,2^{-1/p_c}(A^{-1}s)^{1/p_c},(A^{-1}s)^{1/p_c})\big) \le p_c 2^{t(1-\frac{1}{p_c})}A^{2t}L((A^{-1}s)^{1/p_c})^t$ and therefore

$$\begin{aligned}
\nu_e\big(B_e(c,(A^{-1}r)^{1/p_c})\big) &= \nu_e\Big(\bigcup_{n=0}^{\infty} R\big(c, 2^{-\frac{n+1}{p_c}}(A^{-1}r)^{1/p_c}, 2^{-\frac{n}{p_c}}(A^{-1}r)^{1/p_c}\big)\Big) \\
&= \sum_{n=0}^{\infty} \nu_e\big(R(c, 2^{-\frac{1}{p_c}}(A^{-1}2^{-n}r)^{1/p_c}, (A^{-1}2^{-n}r)^{1/p_c})\big) \\
&\le p_c(2^{1-\frac{1}{p_c}}A^2)^t L \sum_{n=0}^{\infty}(A^{-1}2^{-n}r)^{t/p_c} \\
&= p_c(2^{1-\frac{1}{p_c}}A^2)^t \frac{L}{1-2^{-\frac{t}{p_c}}}((A^{-1}r)^{1/p_c})^t \\
&= p_c(2A^2)^t(2^{t/p_c}-1)^{-1}L((A^{-1}r)^{1/p_c})^t.
\end{aligned}$$

The proof is finished. ■

Lemma 4.16 *Suppose that ν_e is a Euclidean t-conformal measure. Let c be a critical point of an analytic map $H: D \to \overline{\mathbb{C}}$. If $0 < r \le \frac{1}{3}R(H,c)$, $0 < \sigma \le 1$ and $H(c)$ is (r,σ,L)-t-s.l.e, then c is $((A^{-1}r)^{1/p_c}, \tilde{\sigma}, \tilde{L})$-$t$-s.l.e, where $\tilde{\sigma} = (2^{p_c+1}KA^2\sigma)^{1/p_c}$, $\tilde{L} = L\min\{K^{-t}, (A^2\sigma)^{\frac{1-p_c}{p_c}t}\}$, A was defined in (2.25) and p_c is the order of H at the critical point c.*

Proof. Let $x \in B_e(c,(A^{-1}r)^{1/p_c})$. If $\tilde{\sigma}(A^{-1}r)^{1/p_c} \ge 2|x-c|$, then

$$\begin{aligned}
B_e(x,\tilde{\sigma}(A^{-1}r)^{1/p_c}) &\supset B_e(c,\tilde{\sigma}(A^{-1}r)^{1/p_c}/2) \\
&= B_e(c,(2K)^{1/p_c}(A\sigma r)^{1/p_c}) \\
&\supset B_e(c,(A\sigma r)^{1/p_c}).
\end{aligned}$$

It follows from assumptions that $H(c)$ is $(\sigma r, \sigma^{-t}L)$-l.e. and therefore, in view of Lemma 4.14 the critical point c is $((A\sigma r)^{1/p_c}, A^{-2t}\sigma^{-t}L)$-l.e.. Thus

$$\begin{aligned}
\nu_e\big(B_e(x,\tilde{\sigma}(A^{-1}r)^{1/p_c})\big) &\ge A^{-2t}\sigma^{-t}L(A\sigma r)^{t/p_c} \\
&= (A^2\sigma)^{\frac{1-p_c}{p_c}t}L((A^{-1}r)^{1/p_c})^t.
\end{aligned} \tag{4.10}$$

So, suppose that

$$\tilde{\sigma}(A^{-1}r)^{1/p_c} < 2|x-c|. \tag{4.11}$$

Since c is a critical point we have

$$|H'(x)| \ge A^{-1}|x-c|^{p_c-1} \ge A^{-1}\tilde{\sigma}^{p_c-1}(A^{-1}r)^{\frac{p_c-1}{p_c}}2^{1-p_c},$$

which means that

$$\begin{aligned}\tilde{\sigma}(A^{-1}r)^{1/p_c} &\geq A^{-1}\tilde{\sigma}_c^p A^{-1} r 2^{1-p_c}|H'(x)|^{-1} \\ &= 4K\sigma r|H'(x)|^{-1} \geq K\sigma r|H'(x)|^{-1}.\end{aligned} \tag{4.12}$$

In view of (4.11)

$$|H(x) - H(c)| \geq A^{-1}|x-c|^{p_c} \geq A^{-1}2^{-p_c}\tilde{\sigma}^{p_c}A^{-1}r = 2K\sigma r \geq 2\sigma r$$

which implies that

$$H(c) \notin B_e(H(x), 2\sigma r). \tag{4.13}$$

Since $|H(x)-H(c)| \leq A|x-c|^{p_c} \leq R/3$, we have $B_e(H(x), 2\sigma r) \subset B_e(H(c), R)$. So, (4.13) implies the existence of a holomorphic inverse branch H_x^{-1} : $B_e(H(x), 2\sigma r) \to \overline{\mathbb{C}}$ of H which sends $H(x)$ to x. Since, by assumptions $H(x)$ is $(\sigma r, \sigma^{-t}L)$-l.e, it follows from Lemma 4.13 that x is $(K\sigma r|H'(x)|^{-1}, (K^2\sigma)^{-t}L)$-l.e.. Thus, using (4.12) we get

$$\begin{aligned}\nu_e\big(B_e(x, \tilde{\sigma}(A^{-1}r)^{1/p_c})\big) &\geq \nu_e\big(B_e(x, K\sigma r|H'(x)|^{-1})\big) \\ &\geq (K^2\sigma)^{-t}L(K\sigma r|H'(x)|^{-1})^t \\ &\geq K^{-t}Lr^tA^{-t}|x-c|^{(1-p_c)t} \\ &\geq K^{-t}L(A^{-1}r)^t(A^{-1}r)^{\frac{1-p_c}{p_c}t} \\ &= K^{-t}L((A^{-1}r)^{1/p_c})^t.\end{aligned}$$

In view of this and (4.10) the proof is completed. ■

Lemma 4.17 *Suppose that ν_e is a Euclidean t-conformal measure. Then for every $R > 0$ and every $0 < \sigma \leq 1$ there exists $L = L(\omega, R, \sigma) > 0$ such that for every $0 < r \leq R$ every point $\omega \in \Omega(f)$ is (r, σ, L)-$\alpha_t(\omega)$-s.l.e. with respect to the measure m_e.*

Proof. Let $z \in B_e(\omega, r)$. If $\sigma r \geq 2|z-\omega|$, then $B_e(z, \sigma r) \supset B_e\big(\omega, \frac{\sigma}{2}r\big)$ and therefore by Lemma 2.7

$$\nu_e(B_e(z, \sigma r)) \geq C(R/2)\left(\frac{\sigma}{2}r\right)^{\alpha_t(\omega)} = \left(\frac{\sigma}{2}\right)^{\alpha_t(\omega)} C(R/2)r^{\alpha_t(\omega)}. \tag{4.14}$$

In order to deal with the opposite case first notice that always

$$B_e(z, \sigma r) \supset B_e(z, \sigma|z-\omega|)$$

and therefore by Lemma 4.6 in [34] we have

$$\nu_e(B_e(z, \sigma r)) \geq C^{-1}(\sigma)|z-\omega|^{\alpha_t(\omega)}.$$

As $\sigma r < 2|z-\omega|$, it implies that

$$\nu_e(B_e(z,\sigma r)) \geq C^{-1}(\sigma)\Big(\frac{\sigma}{2}\Big)^{\alpha_t(\omega)} r^{\alpha_t(\omega)}.$$

So, putting

$$L(\omega,R,\sigma) = (\sigma/2)^{\alpha_t(\omega)} \min\{C(R/2), C^{-1}(\sigma)\}$$

finishes the proof. ■

4.3 Conformal measure and holomorphic inverse branches

Let m be an almost t-conformal measure and let m_e be its Euclidean version. The upper estimability and strongly lower estimability will be considered in this section with respect to the measure m_e. When we speak about lower estimability we assume more, that the measure m is t-conformal. Since the number of parabolic points is finite, passing to an appropriate iteration, we assume in this and the next section without loosing generality that all parabolic points of f are simple. Consider a forward f-invariant closed subset E of $\mathbb{C}$ such that

$$||f'||_E := \sup\{|f'(z)| : z \in E\} < +\infty.$$

Such sets will be called f-pseudo-compact. Obviously, each f-invariant compact subset E of $\mathbb{C}$ is f-pseudo-compact. Recall that θ was defined in (2.10), $\alpha_t(\omega)$ in Lemma 2.7 and that $\tau > 0$ is so small as required in Lemma 2.3.

The proofs of Proposition 4.15 and Proposition 4.16 from [20] carry out verbatim to our case and we bring them here up for the sake of completeness and convenience of the reader.

Proposition 4.18 *Fix an f-pseudo-compact subset E of $J(f)$. Let $z \in E$, $\lambda > 0$ and let $0 < r \leq \tau\theta||f'||_E^{-1}\lambda^{-1}$ be a real number. Suppose that at least one of the following two conditions is satisfied:*

$$z \in E \setminus \bigcup_{n\geq 0} f^{-n}(\mathrm{Crit}(J(f))$$

or

$$z \in E \quad \text{and} \quad r > \tau\theta||f'||_E^{-1}\lambda^{-1}\inf\{|(f^n)'(z)|^{-1} : n = 1,2,\ldots\}.$$

Then there exists an integer $u = u(\lambda,r,z) \geq 0$ such that

$$r|(f^j)'(z)| \leq \lambda^{-1}\theta\tau$$

for all $0 \le j \le u$ *and the following four conditions are satisfied.*

$$\mathrm{diam}_e\big(\mathrm{Comp}(f^j(z), f^u(z), f^{u-j}, r|(f^u)'(z)|)\big) \le \beta \tag{4.15}$$

for every $j = 0, 1, \dots, u$.

For every $\eta > 0$ *there exists a continuous function* $t \mapsto B_t = B_t(\lambda, \eta) > 0$, $t \in [0, \infty)$, *(independent of* z*,* n*, and* r*) and such that if* $f^u(z) \in B_e(\omega, \theta)$ *for some* $\omega \in \Omega(f)$*, then*

$$f^u(z) \;\; is \;\; (\eta r|(f^u)'(z)|, B_t) - \alpha_t(\omega)\text{-}u.e. \tag{4.16}$$

and there exists a function $W_t = W_t(\lambda, \eta) : (0, 1] \to (0, 1]$ *(independent of* z*,* n*, and* r*) such that if* $f^u(z) \in B_e(\omega, \theta)$ *for some* $\omega \in \Omega(f)$*, then for every* $\sigma \in (0, 1]$

$$f^u(z) \;\; is \;\; (\eta r|(f^u)'(z)|, \sigma, W_t(\sigma)) - \alpha_t(\omega)\text{-}s.l.e. \tag{4.17}$$

If $f^u(z) \notin B_e(\Omega(f), \theta)$*, then formulas (4.16) and (4.17) are also true with*

$$\alpha_t(\omega) \;\; replaced \quad by \quad t. \tag{4.18}$$

Proof. Suppose first that

$$\sup\{\lambda r|(f^j)'(z)| : j \ge 1\} > \theta\tau ||f'||_E^{-1}$$

(which implies that $||f'||_E \ge 1$) and let $n = n(\lambda, z, r) \ge 0$ be a minimal integer such that

$$\lambda r|(f^n)'(z)| > \theta\tau \min\{1, ||f'||_E^{-1}\}. \tag{4.19}$$

Then $n \ge 1$ (due to the assumption imposed on r) and also

$$\lambda r|(f^n)'(z)| \le \theta\tau. \tag{4.20}$$

If $f^n(z) \notin B_e(\Omega(f), \theta)$ set $u = u(\lambda, r, z) = n$. The items (4.16), (4.17) and (4.18) are obvious in view of our assumptions imposed on E.

Suppose that $f^n(z) \in B_e(\Omega(f), \theta)$, say $f^n(z) \in B_e(\omega, \theta)$, $\omega \in \Omega(f)$. Let $0 \le k = k(\lambda, z, r) \le n$ be the smallest integer such that $f^j(z) \in B_e(\Omega(f), \theta)$ for every $j = k, k+1, \dots, n$. Consider all the numbers

$$r_i = |f^i(z) - \omega||(f^i)'(z)|^{-1},$$

where $i = k, k+1, \dots, n$. Put $||f'||_E^+ = \big(\min\{1, ||f'||_E^{-1}\}\big)^{-1}$. By (4.19) we have

$$r_n = |f^n(z) - \omega||(f^n)'(z)|^{-1} \le \theta ||f'||_E^+ \theta^{-1}\tau^{-1}\lambda r = ||f'||_E^+ \tau^{-1}\lambda r$$

and therefore there exists a minimal $k \le u = u(\lambda, r, z) \le n$ such that $r_u \le ||f'||_E^+ \tau^{-1}\lambda r$. In other words

$$|f^u(z) - \omega| \le ||f'||_E^+ \tau^{-1}\lambda r|(f^u)'(z)|. \tag{4.21}$$

If $\sup\{\lambda r|(f^j)'(z)| : j \ge 1\} \le \theta\tau||f'||_E^{-1}$, then it follows from Corollary 2.33 that $z \in \bigcup_{j\ge 0} f^{-j}(\Omega(f))$. Define then $u(\lambda, z, r) = k(\lambda, z, r)$ to be the minimal integer $j \ge 0$ such that $f^j(z) \in \Omega(f)$ and put $\omega = f^u(z)$. Notice that in this case formulas (4.20) and (4.21) are also satisfied. Our further considerations are valid in both cases. First note that by (4.21) we have

$$B_e(f^u(z), \eta r|(f^u)'(z)|) \subset B_e(\omega, (1 + ||f'||_E^+ \tau^{-1}\eta^{-1}\lambda)\eta r|(f^u)'(z)|) \tag{4.22}$$

and in view of Lemma 2.7 and (4.20)

$$m_e\big(B_e(f^u(z), \eta r|(f^u)'(z)|)\big) \le C(1 + ||f'||_E^+ \tau^{-1}\eta^{-1}\lambda)^{\alpha_t(\omega)}(\eta r|(f^u)'(z)|)^{\alpha_t(\omega)}.$$

So, item (4.16) is proved. Also applying (4.21), Lemma 4.17, Lemma 4.9 and 4.20) we see that the point $f^u(z)$ is

$$\big(||f'||_E^+ \tau^{-1}\lambda r|(f^u)'(z)|, \sigma\tau||f'||_E^{-1}\eta\lambda^{-1}, 2^{\alpha_t(\omega)}L(\omega, 2||f'||_E^+\theta, \sigma\tau(2||f'||_E^+)^{-1}\eta\lambda^{-1}\big)$$

$-\alpha_t(\omega)$-s.l.e. So, if $||f'||_E^+ \tau^{-1}\lambda \ge \eta$, then by Lemma 4.10, $f^u(z)$ is

$$\big(\eta r|(f^u)'(z)|, \sigma, (2||f'||_E^+ \tau^{-1}\lambda\eta^{-1})^{\alpha_t(\omega)}L(\omega, 2||f'||_E^+\theta, \sigma\tau(2||f'||_E)^{-1}\eta)\lambda^{-1}\big)$$

$-\alpha_t(\omega)$-s.l.e. If instead $||f'||_E \tau^{-1}\lambda \le \eta$, then again it follows from (4.21), Lemma 4.17, Lemma 4.9 and (4.20) that the point $f^u(z)$ is

$$\big(\eta r|(f^u)'(z)|, \sigma, 2^{\alpha_t(\omega)}L(\omega, 2\theta\tau\lambda^{-1}\eta, \sigma/2)\big)$$

$-\alpha_t(\omega)$-s.l.e.. So, part (4.17) is also proved.

In order to prove (4.15) suppose first that $u = k$. In particular this is the case if $z \in \bigcup_{j\ge 0} f^{-j}(\Omega(f))$. Then

$$\mathrm{Comp}(f^{k-1}(z), f^k(z), f, r|(f^u)'(z)|) \subset \mathrm{Comp}(f^{k-1}(z), f^k(z), f, \theta\tau)$$

and by the choice of k and (2.7) we have $f^{k-1}(z) \notin B_e(\Omega(f), \theta)$. Therefore (4.15) follows from the choice of τ (see (2.28) and (2.27)).

If $u > k$ (so the first case holds), then $r_{u-1} > ||f'||_E \tau^{-1}\lambda r$ and by (2.7) we get

$$r_u = \frac{|f^u(z) - \omega|}{|f^{u-1}(z) - \omega|}|f'(f^{u-1}(z))|^{-1} r_{u-1} \ge \|f'\|_E^{-1} r_{u-1} \ge \tau^{-1}\lambda r.$$

So, $\lambda r|(f^u)'(z)| \le \tau|f^u(z) - \omega|$ and applying Lemma 2.4 and (2.7) $u - k$ times we conclude that for every $k \le j \le u$

$$\mathrm{diam}_e\big(\mathrm{Comp}(f^j(z), f^u(z), f^{u-j}, \lambda r|(f^u)'(z)|)\big) \le \theta\tau < \beta.$$

And now for $j = k-1, k-2, \dots, 1, 0$, the same argument applies as in the case $u = k$. ■

Proposition 4.19 *Fix an f-pseudo-compact subset E of $J(f)$. Let ε and λ be both positive numbers such that $\varepsilon < \lambda \min\{1, \tau^{-1}, \theta^{-1}\tau^{-1}\gamma\}$. If $0 < r < \tau\theta||f'||_E^{-1}\lambda^{-1}$ and $z \in E \setminus \mathrm{Crit}(J(f))$, then there exists an integer $s = s(\lambda, \varepsilon, r, z) \ge 1$ with the following three properties.*

$$|(f^s)'(z)| \ne 0. \tag{4.23}$$

Let $u = u(\lambda, r, z)$ be defined in Proposition 4.18. If $u = u(\lambda, r, z)$ is well-defined, then $s \le u(\lambda, r, z)$.
If either u is not defined or $s < u$, then there exists a critical point $c \in Crit(f)$ such that

$$|f^s(z) - c| \le \varepsilon r|(f^s)'(z)|. \tag{4.24}$$

In any case

$$\mathrm{Comp}\big(z, f^s(z), f^s, (KA^2)^{-1}2^{-\#(\mathrm{Crit}(f)\cap\mathcal{R})}\varepsilon r|(f^s)'(z)|\big) \cap \mathrm{Crit}(f^s) = \emptyset, \tag{4.25}$$

where A was defined in (2.25).

Proof. Since $z \notin \mathrm{Crit}(J(f))$ and in view of Proposition 4.18, there exists a minimal number $s = s(\lambda, \varepsilon, r, z)$ for which at least one of the following two conditions is satisfied

$$|f^s(z) - c| \le \varepsilon r|(f^s)'(z)| \tag{4.26}$$

for some $c \in \mathrm{Crit}(J(f))$ or

$$u(\lambda, r, z) \text{ is well-defined} \quad \text{and} \quad s(\lambda, \varepsilon, r, z) = u(\lambda, r, z). \tag{4.27}$$

Since $|(f^s)'(z)| \ne 0$, the parts (4.23) and (4.24) are proved.

In order to prove (4.25) notice first that no matter which of the two numbers s is, in view of Proposition 4.18 we always have

$$\varepsilon r|(f^s)'(z)| \le \varepsilon\lambda^{-1}\theta\tau. \tag{4.28}$$

Let us now argue that for every $0 \le j \le s$

$$\mathrm{diam}_e\big(\mathrm{Comp}(f^{s-j}(z), f^s(z), f^j, \varepsilon r|(f^s)'(z)|)\big) \le \beta. \tag{4.29}$$

Indeed, if $s = u$, it follows immediately from Proposition 4.18 and (4.15) since $\varepsilon \le \lambda$. Otherwise $|f^s(z) - c| \le \varepsilon r|(f^s)'(z)| \le \varepsilon\lambda^{-1}\theta\tau < \theta$ and therefore, by (2.24), $f^s(z) \notin B_e(\Omega(f), \theta)$. Thus (4.29) follows from (2.27).

Now by (4.29) and Lemma 2.16, there exists $0 \le p \le \#(\mathrm{Crit}(f) \cap \mathcal{R})$, an increasing sequence of integers $1 \le k_1 < k_2 < \ldots < k_p \le s$ and mutually distinct critical points $c_1, c_2, \ldots, c_p$ of f such that

$$\{c_l\} = \mathrm{Comp}(f^{s-k_l}(z), f^s(z), f^{k_l}, \varepsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f) \tag{4.30}$$

for every $l = 1, 2, \ldots, p$ and if $j \notin \{k_1, k_2, \ldots, k_p\}$, then

$$\mathrm{Comp}(f^{s-j}(z), f^s(z), f^j, \varepsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f) = \emptyset. \tag{4.31}$$

Setting $k_0 = 0$ we shall show by induction that for every $0 \le l \le p$

$$\mathrm{Comp}(f^{s-k_l}(z), f^s(z), f^{k_l}, (KA^2)^{-1}2^{-l}\varepsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f^{k_l}) = \emptyset. \tag{4.32}$$

Indeed, for $l = 0$ there is nothing to prove. So, suppose that (4.32) is true for some $0 \le l \le p - 1$. Then by (4.31)

$$\mathrm{Comp}(f^{s-(k_{l+1}-1)}(z), f^s(z), f^{k_{l+1}-1}, (KA^2)^{-1}2^{-l}\varepsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f^{k_{l+1}-1}) = \emptyset.$$

So, if

$$c_{l+1} \in \mathrm{Comp}(f^{s-k_{l+1}}(z), f^s(z), f^{k_{l+1}}, (KA^2)^{-1}2^{-(l+1)}\varepsilon r|(f^s)'(z)|)$$

then by Lemma 1.4 applied for holomorphic maps $H = f$, $Q = f^{k_{l+1}-1}$ and the radius $R = (KA^2)^{-1}2^{-(l+1)}\varepsilon r|(f^s)'(z)| < \gamma$ we get

$$\begin{aligned} |f^{s-k_{l+1}}(z) - c_{l+1}| &\le KA^2|(f^{k_{l+1}})'(f^{s-k_{l+1}}(z))|^{-1}(KA^2)^{-1}2^{-(l+1)}\varepsilon r|(f^s)'(z)| \\ &= 2^{-(l+1)}\varepsilon r|(f^{s-k_{l+1}}(z))'| \\ &\le \varepsilon r|(f^{s-k_{l+1}})'(z)| \end{aligned}$$

which contradicts the definition of s and proves (4.32) for $l+1$. In particular it follows from (4.32) that

$$\mathrm{Comp}(z, f^s(z), f^s, (KA^2)^{-1}2^{-\#(\mathrm{Crit}(f)\cap\mathcal{R})}\varepsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f^s) = \emptyset.$$

The proof is finished. ■

We will also need the following similar result.

Lemma 4.20 *Assume $\Omega(f) = \emptyset$. Then there exist two constants $a, \xi > 0$ such that the following holds. Suppose that $z \in J(f) \setminus \bigcup_{n=0}^{\infty} f^{-n}(\{\infty\} \cup \mathrm{Crit}(f))$. Suppose also that $r \in (0, \gamma(a\xi)^{-1})$. Then there exists $s \geq 0$ with the following properties*

(a) $ra\xi|(f^s)'(z)| \geq \gamma$ or
(b) $ra\xi|(f^s)'(z)| < \gamma$
and
(c) there exists a critical point $c \in \mathrm{Crit}(J(f))$ such that $|(f^s)(z) - c| < r\xi|(f^s)'(z)|$ or
(d) there exists a pole $b \in f^{-1}(\infty)$ such that $|(f^s)(z) - b| < r\xi|(f^s)'(z)|$.

In either case

$$\mathrm{Comp}(z, f^s(z), f^s, 2\xi r|(f^s)'(z)|) \cap \mathrm{Crit}(f^s) = \emptyset.$$

Proof. Put $a = 2KA^2 2^{\sharp(\mathrm{Crit}(f)\cap\mathcal{R})}$, where A was defined in (2.25). Fix $\rho \in (0, 1/2)$ so small that for every $w \in (\mathbb{C} \setminus \mathrm{Crit}(f) \cup f^{-1}(\infty))$, the map f restricted to the set

$$B_e(w, 2\rho \mathrm{dist}_e(w, \mathrm{Crit}(f) \cup f^{-1}(\infty))$$

is one-to-one. Set $\xi = 2^{-4}\rho$. Take $\lambda > 0$ in Proposition 4.19 such that $\varepsilon > 0$ appearing there can be taken to be equal to $a\xi$. In view of Corollary 2.33 there exists a least integer $n \geq 0$ such that $ra\xi|(f^n)'(z)| \geq \gamma$, where γ was defined in (2.27). Since $r < \gamma(a\xi)^{-1}$, we see that $n \geq 1$. If there exists an integer $0 \leq j \leq n-1$ satisfying (c) or (d), take s to be the least one. Otherwise take $s = n$. By the definition of n, it follows from (2.27) that

$$\mathrm{diam}_e(\mathrm{Comp}(z, f^k(z), f^k, 2\xi r|(f^k)'(z)|)) < \beta$$

for all $k = 0, \ldots, n-1$. Thus, we see that (4.29) is satisfied if $s \leq n-1$ and the proof of the last formula in our lemma is complete by verbatim repetition of the fragment of the proof of the Lemma 4.19 from "Now by (4.29)" till its end. If $s = n$, the same argument shows that

$$\mathrm{Comp}(z, f^{n-1}(z), f^{n-1}, 2\xi r|(f^{n-1})'(z)|) \cap \mathrm{Crit}(f^{n-1}) = \emptyset. \qquad (4.33)$$

By the choice ξ and the definition of n we also know that the map f^{n-1} restricted to the ball $B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|)$ is injective. Thus by Koebe's $\frac{1}{4}$-Theorem

$$f(B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|) \supset B_e(f^n(z), 4\xi r|(f^n)'(z)|),$$

and therefore

$$\mathrm{Comp}(f^{n-1}(z), f^n(z), f, 2\xi r|(f^{n-1})'(z)|) \subset B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|).$$

Combining this with (4.33) and injectivity of f restricted to

$$B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|),$$

we conclude that

$$\mathrm{Comp}(z, f^n(z), f^n, 2\xi r|(f^n)'(z)|) \cap \mathrm{Crit}(f^n) = \emptyset.$$

We are done. ■

4.4 Conformal measure; uniqueness, ergodicity and conservativity

Let m_s be a Borel probability measure on $\mathbb{C}$ and let m_e be its Euclidean version, i.e.

$$\frac{dm_e}{dm_s}(z) = (1+|z|^2)^t.$$

We prove from the following.

Lemma 4.21 *If $z \in J(f)$, $r_n \searrow 0$ and*

$$\underline{M} \le \liminf_{n\to\infty} r_n^{-t} m_e(B_e(z, r_n)) \le \limsup \liminf_{n\to\infty} r_n^{-t} m_e(B_e(z, r_n)) \le \overline{M},$$

then

$$\limsup_{n\to\infty} \frac{m_s\big(B_s(z, (2(1+|z|^2))^{-1} r_n\big)}{((2(1+|z|^2))^{-1} r_n)^t} \le 2^t \overline{M}$$

and

$$\liminf_{n\to\infty} \frac{m_s\big(B_s(z, 2(1+|z|^2)^{-1} r_n\big)}{(2(1+|z|^2)^{-1} r_n)^t} \ge 2^{-t} \underline{M}.$$

Proof. Since for every $r > 0$ sufficiently small

$$B_e(z, 2^{-1}(1+|z|^2)^{-1} r) \subset B_s(z, r) \subset B_e(z, 2(1+|z|^2) r)$$

and since

$$\lim_{r\searrow 0} \frac{m_e(B_e(z, r))}{m_s(B_e(z, r))} = (1+|z|^2)^t,$$

we get

$$\limsup_{n\to\infty} \frac{m_s\big(B_s(z, (2(1+|z|^2))^{-1} r_n\big)}{((2(1+|z|^2))^{-1} r_n)^t} \le \lim_{n\to\infty} \frac{m_s(B_e(z, r_n))}{2^{-t}(1+|z|^2)^{-t} r_n^t} = 2^t \overline{M}$$

and

$$\liminf_{n\to\infty} \frac{m\big(B_s(z, 2(1+|z|^2)^{-1}r_n\big)}{(2(1+|z|^2)^{-1}r_n)^t} \geq \lim_{n\to\infty} \frac{m_s(B_e(z,r_n))}{2^t(1+|z|^2)^{-t}r_n^t} = 2^{-t}\underline{M}.$$

We are done. ■

Our first goal is to show that the h-conformal measure m proven to exist in Lemma 4.3 is atomless and that $\mathrm{H}_s^h(J(f)) = 0$ if $h < 2$. We will consider almost t-conformal measures ν with $t \geq 1$. The notion of upper estimability introduced in Definition 4.7 is considered with respect to the Euclidean almost t-conformal measure ν_e. Recall that $l = l(f) \geq 1$ is the integer claimed in Lemma 2.28 and put

$$\begin{aligned} R_l(f) &= \inf\{R(f^j, c) : c \in \mathrm{Crit}(f) \text{ and } 1 \leq j \leq l(f)\} \\ &= \min\{R(f^j, c) : c \in \mathrm{Crit}(f) \cap \mathcal{R} \text{ and } 1 \leq j \leq l(f)\} < \infty \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} A_l(f) &= \sup\{A(f^j, c) : c \in \mathrm{Crit}(f) \text{ and } 1 \leq j \leq l(f)\} \\ &= \max\{A(f^j, c) : c \in \mathrm{Crit}(f) \cap \mathcal{R} \text{ and } 1 \leq j \leq l(f)\}, \end{aligned} \tag{4.35}$$

where the numbers $R(f^j, c)$ and $A(f^j, c)$ are defined just after Definition 1.1. Since

$$\overline{O_+(f(\mathrm{Crit}_c(J(f))))}$$

is a compact f-invariant subset of $\mathbb{C}$ (so disjoint from $f^{-1}(\infty)$) and since

$$\overline{\mathrm{PC}_c^0(f)} = \overline{O_+(\mathrm{Crit}_c(J(f)))} = \mathrm{Crit}_c(J(f)) \cup \overline{O_+(f(\mathrm{Crit}_c(J(f))))},$$

we have the following straightforward but useful fact.

Lemma 4.22 *The set* $\overline{\mathrm{PC}_c^0(f)}$ *is f-pseudo-compact.*

Recall for the needs of the two next lemmas that the sequence $\{Cr_i(f)\}$ was defined inductively by the formula (2.34) and the sequence $S_i(f)$ was defined by the formula (2.34).

Since the number of equivalence classes of the relation $\sim$ is finite, looking at Lemma 2.28 and Lemma 4.4, the following lemma follows immediately from Lemma 4.15.

Lemma 4.23 *If $R_i^{(u)} > 0$ is a positive constant and $t \mapsto C_{t,i}^{(u)} \in (0,\infty)$, $t \in [1,\infty)$, is a continuous function such that all points $z \in \overline{\mathrm{PC}_c^0(f)_i}$ are $(r, C_{t,i}^{(u)})$-t-u.e. with respect to any Euclidean almost t-conformal measure ν_e (with $t \geq 1$) for all $0 < r \leq R_i^{(u)}$, then there exists a continuous function $t \mapsto \tilde{C}_{t,i}^{(u)} > 0$, $t \in [1,\infty)$, such that all critical points $c \in Cr_{i+1}(f)$ are $(r, \tilde{C}_{t,i}^{(u)})$-t-u.e. with respect to any Euclidean almost t-conformal measure ν_e for all $0 < r \leq A_l^{-1} R_i^{(u)}$.*

In the above lemma the superscript u stands for "upper". In the lemma below it has the same connotation. The number u is also used to denote the value of the function $u(\lambda, r, z)$ defined in Proposition 4.18. This should not cause any confusion.

Lemma 4.24 *If $R_{i,1}^{(u)} > 0$ is a positive constant and $t \mapsto C_{t,i,1}^{(u)} \in (0,\infty)$, $t \in [1,\infty)$, is a continuous function such that all critical points $c \in S_i(f)$ are $(r, C_{t,i,1}^{(u)})$-t-u.e. with respect to any Euclidean almost t-conformal measure ν_e (with $t \geq 1$) for all $0 < r \leq R_{i,1}^{(u)}$, then there exist a continuous function $t \mapsto \tilde{C}_{t,i,1}^{(u)} > 0$, $t \in [1,\infty)$, and $\tilde{R}_{i,1}^{(u)} > 0$ such that all points $z \in \overline{\mathrm{PC}_c^0(f)_i}$ are $(r, \tilde{C}_{t,i,1}^{(u)})$-t-u.e. with respect to any Euclidean almost t-conformal measure ν_e (with $t \geq 1$) for all $0 < r \leq \tilde{R}_{i,1}^{(u)}$.*

Proof. We shall show that one can take

$$\tilde{R}_{i,1}^{(u)} = \min\{\tau\theta\lambda^{-1}||f'||^{-1}_{\overline{\mathrm{PC}_c^0(f)_i}}, R_{i,1}^{(u)}, 1\}$$

and

$$\tilde{C}_{t,i,1}^{(u)} = \max\{K^2 2^t C_{t,i,1}^{(u)}, K^{2t} B_t\}.$$

Indeed, denote $\#(\mathrm{Crit}(f) \cap \mathcal{R})$ by $\#$. Put $\varepsilon = 2K(KA^2)2^{\#}$, A was defined in (2.25), and then choose $\lambda > 0$ so large that

$$\varepsilon < \lambda \min\{1, \tau^{-1}, \theta^{-1}\tau^{-1}\min\{\rho, R_{i,1}^{(u)}/2\}\}. \tag{4.36}$$

Consider $0 < r \leq \tilde{R}_{i,1}^{(u)}$ and $z \in \overline{\mathrm{PC}_c^0(f)_i}$. If $z \in \mathrm{Crit}(J(f))$, then $z \in \mathrm{Crit}_c(J(f))$ and $z \in S_i(f)$, and we are therefore done. Thus, we may assume that $z \notin \mathrm{Crit}(J(f))$. Let $s = s(\lambda, \varepsilon, r, z)$. By the definition of ε,

$$2Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-\#}\varepsilon r|(f^s)'(z)|. \tag{4.37}$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then by item (4.16) in Proposition 4.18 or by item (4.18) in Proposition 4.18, applied with $\eta = 2K$, we see that the point $f^s(z)$ is $(2Kr|(f^s)'(z)|, B_t)$-t-u.e. Using (4.37), we obtain from item (4.25) in Proposition 4.19 and Lemma 4.12 that the point z is $(r, K^{2h}B_t)$-t-u.e..

If either u is not defined or $s < u(\lambda, r, z)$, then in view of item (4.25) in Proposition 4.19, there exists a critical point $c \in \mathrm{Crit}_c(J(f))$ such that $|f^s(z) - c| \le \varepsilon r|(f^s)'(z)|$. Since $s \le u$, by Proposition 4.18 and (4.36) we get

$$2Kr|(f^s)'(z)| \le \varepsilon r|(f^s)'(z)| < \min\{\rho, R_{i,1}^{(u)}/2\}. \tag{4.38}$$

Since $z \in \overline{\mathrm{PC}_c^0(f)_i}$, this implies that $c \in S_i(f)$. Therefore using (4.38), the assumptions of Lemma 4.24, and (4.37) and then applying item (4.25) in Proposition 4.19 (remember that by Lemma 4.22 the set $\overline{\mathrm{PC}_c^0(f)}$ is f-pseudo-compact) and Lemma 4.12, we conclude that z is $(r, K^2 2^t C_{t,i,1}^{(u)})$-$t$-u.e. The proof is complete. ■

For $k \ge 1$ recall that for any pole b of f^k, the number q_b denotes its multiplicity and $B_b^k(R)$ is the connected component of $f^{-k}(B_R)$ containing b. We have proved Lemma 4.21 in [20] with no assumptions on the elliptic functions. In fact, the following more general lemma is true (with the same proofs), where f^{-1} replaced by f^{-k}.

Lemma 4.25 *If $b \in f^{-k}(\infty)$, $k \ge 1$, if ν_e is a Euclidean almost t-conformal measure with $t > \frac{2q_b}{q_b+1}$ such that $\nu_e(b) = 0$, and if m is the h-conformal measure proven to exist in Lemma 4.3, then*

$$\nu_e(B_b^k(R)) \preceq R^{2-\frac{q_b+1}{q_b}t}$$

and

$$m_e(B(b,r)) \succeq r^{(q_b+1)h-2q_b}$$

for all $0 < r \preceq 1$, where $B_b^k(R)$ is the connected component of $f^{-k}(B_R)$.

Proof. It follows from Lemma 4.4 that $m_e(\{z \in \mathbb{C} : R \le |z| < 2R\}) \asymp R^2$ and $\nu_e(\{z \in \mathbb{C} : R \le |z| < 2R\}) \preceq R^2$ for all $R > 0$ large enough. It therefore follows from (2.2) that

$$m_e(B_b^k(R) \setminus \overline{B_b^k(2R)}) \asymp R^2 R^{-\frac{q_b+1}{q_b}h} \tag{4.39}$$

and

$$\nu_e(B_b^k(R) \setminus \overline{B_b^k(2R)}) \preceq R^2 R^{-\frac{q_b+1}{q_b}t}. \tag{4.40}$$

Now fix $r > 0$ so small that $R = (r/L_k)^{-q_b}$ is large enough for formulas (4.39) and (4.40) to hold. Using (2.5) and (4.40), we get

$$\begin{aligned}
\nu_e(B_b^k(R)) =& \nu_e\left(\bigcup_{j\geq 0}(B_b^k(2^jR) \setminus \overline{B_b^k(2^{j+1}R)})\right) \\
&= \sum_{j=0}^{\infty} \nu_e(B_b^k(2^jR) \setminus \overline{B_b^k(2^{j+1}R)}) \\
&\preceq \sum_{j=0}^{\infty} (2^jR)^2 (2^jR)^{-\frac{q_b+1}{q_b}t} \\
&= R^{2-\frac{q_b+1}{q_b}t} \sum_{j=0}^{\infty} 2^{j(2-\frac{q_b+1}{q_b}t)} \\
&= L_k^{q_b(2-\frac{q_b+1}{q_b}t)} r^{(q_b+1)t-2q_b} \sum_{j=0}^{\infty} 2^{j(2-\frac{q_b+1}{q_b}t)} \\
&\asymp r^{(q_b+1)t-2q_b},
\end{aligned}$$

where the last comparability sign holds since $\frac{q_b+1}{q_b}t > 2$. We are done with the first part of our lemma. Now replace ν_e by m_e and t by h (which is greater than $\frac{2q_b}{q_b+1}$ by Theorem 2.1 in the above formula. In this case, the '$\preceq$' sign can by (4.39) be replaced by the comparability sign '$\asymp$'; since the first equality sign becomes '$\geq$' (we do not ruled out the possibility that $m_e(b) > 0$ yet) and $m_e(B(b,r)) \geq \nu(B_b(R))$, we are also done in this case. ∎

From now onwards, in all our considerations, we assume $f : \mathbb{C} \to \overline{\mathbb{C}}$ to be a regular pseudo non-recurrent elliptic function.

We shall prove now the following.

Lemma 4.26 *The h-conformal measure m for $f : J(f) \to J(f) \cup \{\infty\}$ proven to exist in Lemma 4.3 is atomless.*

Proof. By induction on $i = 0, 1, \ldots, p$, it follows from Lemma 4.24 (this lemma provides the base of inductions as $S_0(f) = \emptyset$ and simultaneously contributes to the inductive step), Lemma 4.23 and Lemma 2.27 that there exists a continuous function $t \to C_t \in (0,\infty)$, $t \in [1,\infty)$, such that if ν is an arbitrary almost t-conformal measure on $J(f)$, then

$$\nu_e(B(x,r)) \le C_t r^t \tag{4.41}$$

for all $x \in \overline{\mathrm{PC}_c^0(f)}$ and all $r \le r_0$ for some $r_0 > 0$ sufficiently small. Consider now the almost s_n-conformal measure $m_n^s = m_{V_n}$, introduced in the beginning of the proof of Lemma 4.5, where $s_n = s(V_n)$. Letting $n \to \infty$ and recalling that m_s is a weak limit of measures m_n^s, we see from formula (4.41) that

$$m_e(B(x,r)) \le C_h r^h \tag{4.42}$$

for all $x \in \overline{\mathrm{PC}_c^0(f)}$ and all $r \le r_0$. It now follows from Lemma 4.21 that

$$\limsup_{r \searrow 0} \frac{m_s(B(x,r)}{r^h} \le 2^h C_h \tag{4.43}$$

for all $x \in \overline{\mathrm{PC}_c^0(f)}$. In particular, $m_s(\mathrm{Crit}(f)) = 0$; consequently,

$$m_s\Big(\bigcup_{n\ge 0} f^{-n}(\mathrm{Crit}(f))\Big) = 0. \tag{4.44}$$

Now fix $k \ge 1$, $b \in f^{-k}(\infty)$ and $u \in \left(\frac{2q_b}{q_b+1}, h\right)$. Consider all integers $n \ge 1$ so large that $s_n \ge u$. Since $m_n^e(f^{-k}(\infty)) \le m_n^e(f^{-k}(V_n)) = 0$, it follows from Lemma 4.25 that

$$m_n^e(B_b^k(R)) \preceq R^{2-\frac{q_b+1}{q_b}s_n} \le R^{2-\frac{q_b+1}{q_b}u}.$$

Hence $m_e(b) = 0$. Since m_s and m_e are equivalent on $\mathbb{C}$, this gives $m_s(b) = 0$. Consequently $m_s(f^{-1}(\infty)) = 0$ and $m_s(\mathrm{Crit}_p(f)) = 0$. Since $s_n \nearrow h$ and since $h_- < h$ (h_- was defined in (2.18)), disregarding finitely many $j's$, we may assume without loss of generality that

$$s_j > h_- \tag{4.45}$$

for all $j \ge 1$. Fix $c \in Crit_\infty(f)$. Fix also $j \ge 1$ and put $t := s_j$. Since $\lim_{n\to\infty} f^n(c) = \infty$, there exists $k \ge 1$ such that $q_{b_n} \le q_c$ (where $b_n \in f^{-1}(\infty)$ is near $f^n(c)$, q_c was defined in (2.16)) and

$$|f^n(c)| > \max\{1, 2\mathrm{Dist}_e(0, f(\mathrm{Crit}(f))\} \tag{4.46}$$

for all $n \ge k$. Put $a = f^k(c)$ (we may need in the course of the proof $k \ge 1$ to be bigger). We recall that κ_c was defined in (2.19).

We shall prove the following.

Claim 1. *There exists a constant $c_1 \geq 1$, independent of j, such that*

$$m_j^e(B_e(a,r)) \leq c_1 r^{\kappa_c}$$

for all $r > 0$ small enough independently of j.

Proof. Put $q = q_c$. In view of (4.46) for every $n \geq 1$ there is a well-defined holomorphic inverse branch $f_n^{-1} : B_e(f^n(a), \frac{1}{2}|f^n(a)|) \to \mathbb{C}$ of sending $f^n(a)$ to $f^{n-1}(a)$. Let $b_n \in f^{-1}(\infty)$ be the unique pole (assuming $k \neq 1$ is large enough) such that

$$|f^n(a) - b_n| \leq \Delta \ll 1.$$

Then, by Lemma 1.2

$$\begin{aligned} f_n^{-1}\left(B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right)\right) &\subset B_e\left(f^{n-1}(a), \frac{K}{4}|f^n(a)||f'(f^{n-1}(a))|^{-1}\right) \\ &\subset B_e(f^{n-1}(a), C|f^n(a)||f^n(a)|^{-\frac{q+1}{q}}) \\ &= B_e(f^{n-1}(a), C|f^n(a)|^{-\frac{1}{q}}) \\ &\subset B_e\left(f^{n-1}(a), \frac{1}{2}|f^{n-1}(a)|\right) \end{aligned}$$

with some $C > 0$, where the last inclusion was written assuming that $|f^{n-1}(a)| \geq 2c_1|f^n(a)|^{-\frac{1}{q}}$ which we can assume that to hold for all $n \geq k$ large enough. So, the composition

$$f_a^{-n} = f_1^{-1} \circ f_2^{-1} \circ \ldots \circ f_n^{-1} : B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right) \to \mathbb{C}$$

sending $f^n(a)$ to a is well-defined and this is a holomorphic branch of f^{-n}. Take $0 < r < \frac{1}{16}|a|$ and let $n+1 \geq 1$ be the least integer such that

$$r|(f^{n+1})'(a)| \geq \frac{1}{16}|f^{n+1}(a)|.$$

Such integer exists since $|f'(z)| \asymp |f(z)|^{\frac{q_b+1}{q_b}}$ if z is near a pole b. By definition $n \geq 0$, and since $r < \frac{1}{16}|a|$, we have

$$r|(f^n)'(a)| < \frac{1}{16}|(f^n)(a)|.$$

Then by $\frac{1}{4}$-Koebe's Distortion Theorem

$$B_e(a,r) \subset f_a^{-n}(B_e(f^n(a), 4r|(f^n)'(a)|)). \tag{4.47}$$

Now we consider three cases determined by the value of $r|(f^n)'(a)|$.

Case 1. $\Delta \le r|(f^n)'(a)| < \frac{1}{16}|f^n(a)|$.

In view of (4.47) and Koebe's Distortion Theorem along with almost conformality of the measure m_j^e, we get that

$$\begin{aligned} m_j^e(B_e(a,r)) &\le K^t|(f^n)'(a)|^{-t} m_j^e(B_e(f^n(a), 4r|(f^n)'(a)|)) \\ &\preceq K^t|(f^n)'(a)|^{-t}(4r|(f^n)'(a)|)^2 \\ &\asymp r^2|(f^n)'(a)|^{2-t}. \end{aligned} \tag{4.48}$$

Put

$$q_n = q_{b_n}.$$

Since $t > h_-$ (see formula (4.45)) and $q_n \le q_c$, it follows from (2.20) that

$$\left(\frac{t-\kappa_c}{2-\kappa_c}\right)\left(\frac{q_n+1}{q_n}\right) > 1.$$

Hence,

$$\begin{aligned} |f^n(a)| <& |f^n(a)|^{\frac{t-\kappa_c}{2-\kappa_c}\frac{q_n+1}{q_n}} \\ \asymp& |f'(f^{n-1}(a))|^{\frac{t-\kappa_c}{2-\kappa_c}} \\ \preceq& |(f^n)'(a)|^{\frac{t-\kappa_c}{2-\kappa_c}} \\ =& |(f^n)'(a)||(f^n)'(a)|^{\frac{t-2}{2-\kappa_c}}. \end{aligned}$$

Combining this and Case 1 assumption, we get

$$r < \frac{1}{16}|(f^n)'(a)|^{-1}|f^n(a)| \preceq |(f^n)'(a)|^{\frac{t-2}{2-\kappa_c}}.$$

Therefore $r^{2-\kappa_c} \preceq |(f^n)'(a)|^{t-2}$, or equivalently $r^2|(f^n)'(a)|^{2-t} \preceq r^{\kappa_c}$. Together with (4.48), we obtain

$$m_j^e(B(a,r)) \preceq r^{\kappa_c}.$$

Case 2. $|f^n(a) - b_n| \le 32A^{\frac{q_{min}+1}{q_{min}}} r|(f^n)'(a)| < 32A^{\frac{q_{min}+1}{q_{min}}}\Delta$, *where* A *was defined in (2.25)* and $q_{\min}$ is the minimal order of all critical points and poles.

Put $c = 32A^{\frac{q_{min}+1}{q_{min}}}$. Then

$$B_e(f^n(a), 4r|(f^n)'(a)|) \subset B_e(b_n, (4+c)r|(f^n)'(a)|) \subset B_e(b_n, (4+c)\Delta)$$

and it follows from Lemma 4.25 that

$$m_j^e(B_e(f^n(a), 4r|(f^n)'(a)|)) \preceq (4r|(f^n)'(a)|)^{(q_n+1)t-2q_n}.$$

Thus

$$\begin{aligned} m_j^e(B_e(a,r)) &\leq K^t|(f^n)'(a)|^{-t}(4r|(f^n)'(a)|)^{(q_n+1)t-2q_n} \\ &\asymp r^{(q_n+1)t-2q_n}|(f^n)'(a)|^{(t-2)q_n} \\ &\leq r^{(q_n+1)t-2q_n}. \end{aligned}$$

But, as $q_n \leq q_c$ and $t > h_-$, it follows from (2.19) that

$$(q_n+1)t - 2q_n \geq (q_n+1)t - 2q_c > \kappa_c$$

and therefore

$$m_j^e(B(a,r)) \leq r^{\kappa_c}.$$

It remains to consider

Case 3. $r|(f^n)'(a)| < \frac{1}{32}A^{-\frac{q_{\min}+1}{q_{\min}}}|f^n(a) - b_n|$.

But then

$$\begin{aligned} r|(f^{n+1})'(a)| &= r|(f^n)'(a)||f'(f^n(a))| \\ &< \frac{1}{32}A^{-\frac{q_{\min}+1}{q_{\min}}}|f^n(a) - b_n|(A|f^{n+1}(a)|)^{\frac{q_n+1}{q_n}} \\ &\leq \frac{1}{32}A^{-\frac{q_{\min}+1}{q_{\min}}}A^{\frac{1}{q_n}+1}|f^{n+1}(a)| \\ &\leq \frac{1}{32}|f^{n+1}(a)| \\ &\leq \frac{1}{16}|f^{n+1}(a)| \end{aligned}$$

contrary to the definition of n. So, Claim 1 is proved.

The last step of our proof is to demonstrate the following.

Claim 2. *There exist $c_2 \geq 1$ and $R > 0$, all independent of j, such that*

$$m_j^e(B_e(c,r)) \leq c_2 r^{p_c\kappa_c + h(1-p_c)}$$

for all $j \geq 1$ and for all $r \leq R$, where p_c is the order of critical point c of the map f^k.

Proof. Let $p := p_c \geq 2$. There exists $R > 0$ so small that

$$f^k(B_e(c), R) \subset B_e(f^k(c), 2^{-4}|f^k(c)|)$$

and that there exists $M \geq 1$ such that

$$M^{-1}|z-c|^p \leq |f^k(z) - f^k(c)| \leq M|z-c|^p$$

and

$$M^{-1}|z-c|^{p-1} \leq |(f^k)'(z)| \leq M|z-c|^{p-1}$$

for all $z \in B_e(c, R)$. Thus, for all $k \geq 0$ and all $r \leq R$

$$f^k(A(c, 2^{-(l+1)}r, 2^{-l}r)) \subset A(f^k(c); M^{-1}r^p 2^{-p(l+1)}, Mr^p 2^{-pl}).$$

Since the map $f^k_{|B_e(c,R)}$ is p-to-one, using almost conformality of the measure m^e_j and the right-hand side of (2.19), we get that

$$\begin{aligned} m^e_j(A(f^k(c); M^{-1}r^p 2^{-p(l+1)}, Mr^p 2^{-pl})) &\\ \geq \frac{1}{p} M^{-h}(2^{-(l+1)}r)^{t(p-1)} m^e_j(A(c, 2^{-(l+1)}r, 2^{-l}r)) &\\ \geq p^{-1}M^{-h}(2^{-(l+1)}r)^{h(p-1)} m^e_j(A(c, 2^{-(l+1)}r, 2^{-l}r)). & \end{aligned}$$

Applying Claim 1, we therefore get

$$\begin{aligned} m^e_j(B_e(c, r)) &= \sum_{l=0}^{\infty} m^e_j(A(c, 2^{-(l+1)}r, 2^{-l}r)) \\ &\leq pM^h r^{h(1-p)} \sum_{l=0}^{\infty} 2^{h(p-1)(l+1)} m^e_j(A(f^k(c); M^{-1}r^p 2^{-p(l+1)}, Mr^p 2^{-pl})) \\ &\leq pM^h c_1 2^{h(p-1)} r^{h(1-p)} \sum_{p=0}^{\infty} 2^{h(p-1)l}(Mr^p 2^{-pl})^{\kappa_c} \\ &= p2^{h(p-1)} c_1 M^{h+\kappa_c} r^{h(1-p)+p\kappa_c} \sum_{l=0}^{\infty} 2^{(h(p-1)-p\kappa_c)l} \\ &= p2^{h(p-1)} c_1 M^{h+\kappa_c}(1 - 2^{h(p-1)-p\kappa_c})^{-1} r^{p\kappa_c + h(1-p)}, \end{aligned}$$

where writing the last equality sign we used the fact that $p\kappa_c + h(1-p) > 0$ equivalent to the left-hand side of (2.19). Repeating again that $p\kappa_c + h(1-p) > 0$, Claim 2 implies that the limiting measure m does not charge the critical point c, and we are done. ■

The argument from the beginning of the proof of this lemma, based on Lemma 4.24 and Lemma 4.23 gives the following,

Lemma 4.27 *The set* $\overline{\mathrm{PC}_c^0(f)}$ *is uniformly* h*-upper estimable.*

Denote by $\mathrm{Tr}(f) \subset J(f)$ the set of all transitive points of f, that is the set of points in $J(f)$ such that $\overline{O_+(z)} = J(f)$.

Theorem 4.28 *There exists a unique atomless* t*-conformal measure* m *for* $f : J(f) \to J(f) \cup \{\infty\}$*. Then* $t = h$*,* m *is ergodic conservative and all other conformal measures are purely atomic, supported on* $\mathrm{Sing}^-(f)$ *with exponents larger than* h*. Consequently* $m(\mathrm{Tr}(f)) = 1$*.*

Proof. In view of Lemma 4.26 there exists an atomless h-conformal measure m for $f : J(f) \to J(f) \cup \{\infty\}$. Suppose that ν_e is an arbitrary Euclidean t-conformal measure for f and some $t \geq 0$. By Lemma 3.3, $t \geq h$. Fix $z \in J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))$. Then in view of Proposition 2.31 there exist points $x_j = x_j(z) \in J(f)$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $x(z) = \lim_{k\to\infty} f^{n_k}(z)$ and $x_j \sim x_k$ for all $j, k \geq 1$. Define for every $l \geq 1$

$$Z_l = \{z \in J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)) : |x(z)| \leq l \text{ and } \eta(z) \geq 1/l\},$$

fix $l \geq 1$ and $z \in Z_l$. Consider for k large enough the sets

$$f_z^{-n_k}\left(B_e\left(x, \frac{1}{4l}\right)\right) \quad \text{and} \quad f_z^{-n_k}\left(B_e\left(x, \frac{1}{4Kl}\right)\right),$$

where $f_z^{-n_k}$ is the holomorphic inverse branch of f^{n_k} defined on $B_e(x, \frac{1}{2l})$ and sending $f^{n_k}(z)$ to z. Using conformality of the measure ν along with Koebe's Distortion Theorem, I (Euclidean version) we easily deduce now that

$$B(\nu_e, l)^{-1} c|(f^{n_k})'(z)|^{-h} \leq \nu_e\big(B_e(z, c|(f^{n_k})'(z)|^{-1})\big) \leq B(\nu_e, l) c|(f^{n_k})'(z)|^{-h} \tag{4.49}$$

for all $k \geq 1$ large enough, where $K \geq 1$ is the constant appearing in the Koebe's Distortion theorem and ascribed to the scale $1/2$ and $c > 0$ is some constant comparable with 1. Fix now E, an arbitrary bounded Borel set contained in Z_l. Since m_e is regular, for every $x \in E$ there exists a radius $r(x) > 0$ of the form from (4.49) such that

$$m_e\Big(\bigcup_{x\in E} B_e(x, r(x)) \setminus E\Big) < \varepsilon. \tag{4.50}$$

Now by the Besicovič theorem (see [14]) we can choose a countable subcover

$$\{B_e(x_i, r(x_i))\}_{i=1}^{\infty},$$

$r(x_i) \le \varepsilon$, from the cover $\{B_e(x, r(x))\}_{x\in E}$ of E, of multiplicity bounded by some constant $C \ge 1$, independent of the cover. Therefore by (4.49) and (4.50), we obtain

$$\begin{aligned}
\nu_e(E) &\le \sum_{i=1}^{\infty} \nu_e(B_e(x_i, r(x_i))) \le B(\nu_e, l) \sum_{i=1}^{\infty} r(x_i)^t \\
&\le B(\nu_e, l) B(m_e, l) \sum_{i=1}^{\infty} r(x_i)^{t-h} m_e(B_e(x_i, r(x_i))) \\
&\le B(\nu_e, l) B(m_e, l) C\varepsilon^{t-h} m_e\Big(\bigcup_{i=1}^{\infty} B_e(x_i, r(x_i))\Big) \\
&\le C B(\nu_e, l) B(m_e, l) \varepsilon^{t-h} (\varepsilon + m_e(E)).
\end{aligned} \tag{4.51}$$

In the case when $t > h$, letting $\varepsilon \searrow 0$ we obtain $\nu_e(Z_l) = 0$. Since

$$J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)) = \bigcup_{l=1}^{\infty} Z_l,$$

we therefore get

$$\nu_e\big(J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))\big) = 0$$

which by Lemma 3.4 implies that $\nu_e(\mathrm{Sing}^-(f)) = 1$ and the last part of our theorem is proved . Suppose now that $t = h$. Since, in view of Lemma 3.4,

$$\nu_e(I_\infty(f) \setminus I_-(f)) = m_e(I_\infty(f)) = 0,$$

using (4.51) and letting $l \nearrow \infty$, we conclude that

$$\nu_e|_{J(f)\setminus \mathrm{Sing}^-(f)} \prec m_e|_{J(f)\setminus \mathrm{Sing}^-(f)} \asymp m_s|_{J(f)\setminus \mathrm{Sing}^-(f)}.$$

Exchanging the roles of m_e and ν_e we infer that the measures $\nu_{e|J(f)\setminus \mathrm{Sing}^-(f)}$ and $m_{s|_{J(f)\setminus \mathrm{Sing}^-(f)}}$ are equivalent. Suppose that $\nu_e(\mathrm{Sing}^-(f)) > 0$. Then there exists

$$y \in \mathrm{Crit}(J(f)) \cup \Omega(f) \cup f^{-1}(\infty)$$

such that $m_s(y) > 0$. But then

$$\sum_{\xi \in y^-} |(f^{n(\xi)})^*(\xi)|^{-h} < \infty,$$

where $y^- = \bigcup_{n\geq 0} f^{-n}(y)$ and for every $\xi \in y^-$, $n(\xi)$ is the least integer $n \geq 0$ such that $f^n(\xi) = y$. Hence,

$$\nu_y = \frac{\sum_{\xi\in y^-} |(f^{n(\xi)})^*(\xi)|^{-h}\delta_\xi}{\sum_{\xi\in y^-} |(f^{n(\xi)})^*(\xi)|^{-h}}$$

is a spherical h-conformal measure supported on $y^- \subset \mathrm{Sing}^-(f)$. This contradicts the proven fact that the measures $\nu_y|_{J(f)\setminus \mathrm{Sing}^-(f)}$ and $m_{s|_{J(f)\setminus \mathrm{Sing}^-(f)}}$ are equivalent and $m_s(J(f)\setminus \mathrm{Sing}^-(f)) = 1$. Thus ν_e and m_s are equivalent.

Let us now prove that any h-conformal measure ν_s is ergodic. Indeed, suppose to the contrary that $f^{-1}(G) = G$ for some Borel set $G \subset J(f)$ with $0 < \nu_s(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(f)\setminus G}$

$$\nu_G(B) = \frac{\nu_s(B\cap G)}{\nu_s(G)}, \quad \nu_{J(f)\setminus G}(B) = \frac{\nu_s(B\cap J(f)\setminus G)}{\nu_s(J(f)\setminus G)}$$

would be h-conformal and mutually singular; a contradiction.

If now ν_s is again an arbitrary spherical h-conformal measure, then by a simple computation based on the definition of conformal measures we see that the Radon-Nikodym derivative $\phi = d\nu_s/dm_s$ is constant on grand orbits of f. Therefore by ergodicity of m_s we conclude that ϕ is constant m_s-almost everywhere. As both m_s and ν_s are probability measures, it implies that $\phi = 1$ a.e., hence $\nu_s = m_s$.

Let us show now that m is conservative. We shall prove first that every forward invariant ($f(E) \subset E$) subset E of $J(f)$ is either of measure 0 or 1. Indeed, suppose to the contrary that $0 < m(E) < 1$. Since

$$m(I_\infty(f) \cup \mathrm{Sing}^-(f)) = 0,$$

it suffices to show that

$$m(E \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))) = 0.$$

Denote by Z the set of all points $z \in E \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)))$ such that

$$\lim_{r\to 0} \frac{m_e(B(z,r) \cap (E \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))))}{m_e(B(z,r))} = 1. \tag{4.52}$$

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [12]), $m_e(Z) = m_e(E)$. Since $m_e(E) > 0$ we find at least one point $z \in Z$. Since

$$z \in J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)),$$

let $x \in J(f)$, $\eta(z) > 0$, and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ be given by Proposition 2.31. Put

$$\delta = \eta(z)/8.$$

Suppose that $m_e(B_e(x,\delta) \setminus E) = 0$. By conformality of m_e, $m_e(f(Y)) = 0$ for all Borel sets Y such that $m_e(Y) = 0$. Hence,

$$\begin{aligned} 0 = m_e\big(f^n(B_e(x,\delta) \setminus E)\big) &\geq m_e\big(f^n(B_e(x,\delta)) \setminus f^n(E)\big) \\ &\geq m_e\big(f^n(B_e(x,\delta)) \setminus E\big) \geq m_e\big(f^n(B_e(x,\delta)\big) - m_e(E) \end{aligned} \tag{4.53}$$

for all $n \geq 0$. Since $J(f) = \overline{\bigcup_{n\geq 1} f^{-n}(\infty)}$, for some $p \geq 2$, the image $f^{p-1}(B_e(x,\delta))$ contains an open neighborhood of ∞. Thus, it contains at least one (in fact infinitely many) copy of the fundamental parallelogram $\mathcal{R}$ and consequently $f^p(B_e(x,\delta)) = \overline{\mathbb{C}}$. In particular $m_e\big(f^p(B_e(x,\delta))\big) = 1$. Then (4.53) implies that $0 \geq 1 - m_e(E)$ which is a contradiction. Consequently $m_e(B_e(x,\delta) \setminus E) > 0$. Hence for every $j \geq 1$ large enough,

$$m_e\big(B_e(f^{n_j}(z), 2\delta) \setminus E\big) \geq m_e\big(B_e(x,\delta) \setminus E\big) > 0.$$

Therefore, as $f^{-1}(J(f) \setminus E) \subset J(f) \setminus E$, the standard application of Koebe's Distortion Theorem, I (Euclidean version) and Lemma 4.12 shows that

$$\limsup_{r\to 0} \frac{m_e(B(z,r) \setminus E)}{m_e(B(z,r))} > 0$$

which contradicts (4.52). Thus either $m_e(E) = 0$ or $m_e(E) = 1$.

Now conservativity is straightforward. One needs to prove that for every Borel set $B \subset J(f)$ with $m(B) > 0$ one has $m(G) = 0$, where

$$G = \{x \in J(f) : \sum_{n\geq 0} \chi_B(f^n(x)) < +\infty\}.$$

Indeed, suppose that $m(G) > 0$ and for all $n \geq 0$ let

$$\begin{aligned} G_n =& \{x \in J(f) : \sum_{k\geq n} \chi_B(f^n(x)) = 0\} \\ =& \{x \in J(f) : f^k(x) \notin B \ \text{ for all } \ k \geq n\}. \end{aligned}$$

Since $G = \bigcup_{n\geq 0} G_n$, there exists $k \geq 0$ such that $m(G_k) > 0$. Since all the sets G_n are forward invariant we conclude that $m(G_k) = 1$. But on the other hand all the sets $f^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_k. This contradiction finishes the proof of conservativity of m. Consequently $m(\mathrm{Tr}(f)) = 1$. We are done. ■

4.5 Hausdorff measure

The proof of part (a) of Theorem 4.1. Let m be the unique h-conformal atomless measure proven to exist in Theorem 4.28. Consider an arbitrary point $z \in \mathrm{Tr}(f)$. Since f is regular pseudo non-recurrent, there exists

$$b \in \bigcup_{j=1}^{\infty} f^{-j}(\infty) \setminus \overline{O_+(\mathrm{Crit}(f))},$$

say $b \in f^{-k}(\infty) \setminus \overline{O_+(\mathrm{Crit}(f))}$. Hence, there exists $\gamma > 0$ such that

$$B_e(b, \gamma) \cap O_+(\mathrm{Crit}(f)) = \emptyset. \tag{4.54}$$

Since $z \in \mathrm{Tr}(f)$, there exists an infinite increasing sequence $\{n_j\}_{j=0}^{\infty}$ such that

$$\lim_{j \to \infty} f^{n_j}(z) = b \ \text{ and } |f^{n_j}(z) - b| < \gamma/4 \tag{4.55}$$

for every $j \geq 1$. It follows from this and (4.54) that for every $j \geq 1$ there exists a holomorphic inverse branch

$$f_z^{-n_j} : B_e(f^{n_j}(z), 3\gamma/4) \to \mathbb{C}$$

of f^{n_j} sending $f^{n_j}(z)$ to z. Using now Koebe's Distortion Theorem I, (Euclidean version), Lemma 4.12 and Lemma 4.25, we conclude that

$$\begin{aligned}
m_e\big(z, B_e\big(K|(f^{n_j})'(z)|^{-1}2|f^{n_j}(z) - b|\big)\big) & \\
&\geq m_e\big(B_e\big(f^{n_j}(z), 2|f^{n_j}(z) - b|\big)\big)|(f^{n_j})'(z)|^{-h} \\
&\geq m_e\big(B_e(b, |f^{n_j}(z) - b|)\big)|(f^{n_j})'(z)|^{-h} \\
&\succeq |f^{n_j}(z) - b|^{(q_b+1)h - 2q_b}|(f^{n_j})'(z)|^{-h} \\
&= \big(K|(f^{n_j})'(z)|^{-1}|f^{n_j}(z) - b|\big)^h K^{-h}|f^{n_j}(z) - b|^{q_b(h-2)}.
\end{aligned}$$

Since $h < 2$, using (4.55), this implies that

$$\overline{\lim_{r \to 0}} r^{-h} m_e(B_e(z, r)) = \infty.$$

Hence

$$\mathrm{H}_e^h(\mathrm{Tr}(f)) = 0$$

in view of Theorem 3.1 (1). Since by Theorem 4.28 $m_e(J(f) \setminus \mathrm{Tr}(f)) = 0$, it follows from Lemma 3.3 that $\mathrm{H}_e^h(J(f) \setminus \mathrm{Tr}(f)) = 0$. In conclusion

$$\mathrm{H}_e^h(J(f)) = 0,$$

which completes the proof. ■

4.6 Packing measure I

Proposition 4.29 *The conformal measure m is absolutely continuous with respect to the packing measure Π^h and moreover, the Radon-Nikodym derivative $dm_s/d\Pi^h$ is uniformly bounded away from infinity. In particular*

$$\Pi^h(J(f)) > 0.$$

Proof. Since

$$J(f) \cap \omega\big(\mathrm{Crit}(f) \setminus \mathrm{Crit}(J(f))\big) = \Omega(f),$$

we conclude from Lemma 2.20 that there exists $y \in J(f)$ at a positive distance, say 8η, from $O_+(\mathrm{Crit}(f))$. Fix $z \in \mathrm{Tr}(f)$. Then there exists an infinite sequence $n_j \geq 1$ of increasing integers such that $f^{n_j}(z) \in B_e(y, \eta)$. Therefore

$$B_e(f^{n_j}(z), 4\eta) \cap O_+(\mathrm{Crit}(f)) = \emptyset$$

and consequently

$$\mathrm{Comp}(z, f^{n_j}(z), f^{n_j}, \eta/2) \cap \mathrm{Crit}(f^{n_j}) = \emptyset.$$

Hence, it follows from Lemma 1.2 and Lemma 4.12 that

$$\liminf_{r \to 0} \frac{m_e(B_e(z, r))}{r^h} \leq B$$

for some constant $B \in (0, \infty)$ and all $z \in \mathrm{Tr}(f)$. Applying Lemma 4.21 we therefore get that

$$\liminf_{r \to 0} \frac{m_s(B_s(z, r))}{r^h} \leq 2^h B.$$

Hence, by Theorem 3.2(1), the measure $m_s|_{\mathrm{Tr}(f)}$ is absolutely continuous with respect to $\Pi_s^h|_{\mathrm{Tr}(f)}$. Since, by Theorem 4.28, $m_s(J(f) \setminus \mathrm{Tr}(f)) = 0$, we are done. ■

Lemma 4.30 *If $\Omega(f) \neq \emptyset$, then $\Pi_s^h(J(f)) = +\infty$.*

Proof. Fix $\xi \in \Omega$. Since

$$\bigcup_{n \geq 0} f^{-n}(\xi)$$

is dense in $J(f)$ and, by Lemma 2.20, $\omega(\mathrm{Crit}(f))$ is nowhere dense in $J(f)$, there exist an integer $s > 0$, a real number $\eta > 0$, and a point

$$y \in f^{-s}(\xi) \setminus B_e\Big(\bigcup_{n \geq 0} f^n(\mathrm{Crit}(f)), \eta\Big).$$

Since by Theorem 2.1, $h > 1$, it follows from Lemma 2.7 and Lemma 4.15 (y may happen to be a critical point of f^s!) that

$$\liminf_{r\to 0} \frac{m_e(B_e(y,r))}{r^h} = 0. \tag{4.56}$$

Consider now a transitive point $z \in J(f)$, i.e. $z \in \mathrm{Tr}(f)$. Then there exists an infinite increasing sequence $n_j = n_j(z) \geq 1$ of positive integers such that

$$\lim_{j\to\infty} |f^{n_j}(z) - y| = 0 \quad \text{and} \quad r_j = |f^{n_j}(z) - y| < \eta/7$$

for every $j = 1, 2, \ldots$. By the choice of y, for all $j \geq 1$ there exist holomorphic inverse branches

$$f_z^{-n_j} : B_e(f^{n_j}(z), 6r_j) \to \overline{\mathbb{C}}$$

sending $f^{n_j}(z)$ to z. So, applying Lemma 1.2 and Lemma 4.12 with $R = 3r_j$, we conclude from (4.56) that

$$\liminf_{r\to 0} \frac{m_e(B_e(z,r))}{r^h} = 0.$$

Applying Lemma 4.21, we conclude that the same formulas remain true with m_e replaced by m_s and $B_e(z,r)$ by $B_s(z,r)$. Therefore, it follows from Theorem 4.28 ($m_s(\mathrm{Tr}(f)) = 1$) and Theorem 3.2(1) that $\Pi_s^h(J(f)) = +\infty$. We are done. ■

4.7 Packing measure II

From now on let m denote the unique atomless h-conformal measure proven to exist in Theorem 4.28. Our aim in this section is to show that in the absence of parabolic points the h-dimensional Euclidean packing measure is finite on bounded subsets of $J(f)$ and $\Pi_s^h(J(f)) < \infty$.

Recall that the numbers $R_l(f)$ and $A_l(f)$ have been defined by formulas (4.34) and (4.35) respectively. Since the number of equivalence classes of the relation $\sim$ is finite, looking at Lemma 2.28 and Lemma 4.4, the following lemma (where the superscript l indicates that we mean the "lower" estimates) follows immediately from Lemma 4.16.

Lemma 4.31 *If $C_i^{(l)} > 0$, $0 < R_i^{(l)} \leq R_l(f)/3$, and $0 < \sigma \leq 1$ are three real numbers such that all points $z \in \overline{\mathrm{PC}_c^0(f)_i}$ are $(r, \sigma, C_i^{(l)})$-h-s.l.e. with respect to the measure m_e, then there exists $\tilde{C}_i^{(l)} > 0$ such that all critical points*

$c \in Cr_{i+1}(f)$ are $(r, \tilde{\sigma}, \tilde{C}_i^{(l)})$-h-s.l.e. with respect to the measure m_e for all $0 < r \le A_l(f)^{-1} R_i^{(l)}$, where $\tilde{\sigma}$ was defined in Lemma 4.16.

Let us prove the following.

Lemma 4.32 *Suppose that $\Omega(f) = \emptyset$. Assume that $C_{i,1}^{(l)} > 0$, $R_{i,1}^{(l)} > 0$ and $0 < \sigma \le 1$ are three real numbers such that all critical points $c \in S_i(f)$ are $(r, \sigma, C_{i,1}^{(l)})$-h-s.l.e. with respect to the measure m_e for all $0 < r \le R_{i,1}^{(l)}$. Then there exist $\tilde{C}_{i,1}^{(l)} > 0$, $\tilde{R}_{i,1}^{(l)} > 0$ and such that all points $z \in \overline{\mathrm{PC}_c^0(f)_i}$ are $(r, 8K^3A^2 2^{\#(\mathrm{Crit}(f)\cap\mathcal{R})}\sigma, \tilde{C}_{i,1}^{(l)})$-h-s.l.e. with respect to the measure m_e for all $0 < r \le \tilde{R}_{i,1}^{(l)}$, where A was defined in (2.25).*

Proof. Recall that by Lemma 4.22 the set $\overline{\mathrm{PC}_c^0(f)}$ is f-pseudo-compact. We shall show that this time one can take

$$\tilde{R}_{i,1}^{(l)} = \min\{\tau\theta||f'||^{-1}\lambda^{-1}, R_{i,1}^{(l)}, 1\} \quad \text{and} \quad \tilde{C}_{i,1}^{(l)} = \left(8(KA^2)2^{\#}\right)^h C_{i,1}^{(l)},$$

where $||f'|| = ||f'||_{\overline{\mathrm{PC}_c^0(f)_i}}$. Indeed, denote again $\#(\mathrm{Crit}(f)\cap\mathcal{R})$ by $\#$. Take $\varepsilon = 4K(KA^2)2^{\#}$ and then choose $\lambda > 0$ so large that

$$\varepsilon < \lambda \min\left\{1, \tau^{-1}, \theta^{-1}\tau^{-1}\min\{\rho, R_{i,1}^{(l)}/2\}\right\}. \tag{4.57}$$

Consider $0 < r \le \tilde{R}_{i,1}^{(l)}$ and $z \in \overline{\mathrm{PC}_c^0(f)}_i$. If $z \in \mathrm{Crit}_c(J(f))$, then $z \in S_i(f)$ and we are done. Thus, we may assume that $z \notin \mathrm{Crit}_c(J(f))$, then $z \notin \mathrm{Crit}(J(f))$.

Let $s = s(\lambda, \varepsilon, r, z)$. By the definition of ε

$$4Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-\#}\varepsilon r|(f^s)'(z)|. \tag{4.58}$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then by item (4.17) in Proposition 4.18, applied with $\eta = K$, we see that the point

$$f^s(z) \text{ is } (Kr|(f^s)'(z)|, \sigma/K^2, W_h(\sigma/K^2)) - h\text{-s.l.e..}$$

Using (4.58) it follows from item (4.25) in Proposition 4.19 and Lemma 4.13 that the point z is $(r, \sigma, W_h(\sigma/K^2))$-$h$-s.l.e.. If either u is not defined or $s \le u(\lambda, r, z)$, then in view of item (4.24) in Proposition 4.19, there exists a critical point $c \in \mathrm{Crit}(f)$ such that

$$|f^s(z) - c| \le \varepsilon r|(f^s)'(z)|.$$

Since $s \le u$, by Proposition 4.19 and (4.57) we get

$$|f^s(z) - c| \le \varepsilon r |(f^s)'(z)| < \min\{\rho, R_{i,1}^{(l)}/2\}. \tag{4.59}$$

Since $z \in \overline{\mathrm{PC}_c^0(f)_i}$ it implies that $c \in S_i(f)$. Therefore, by the assumptions of Lemma 4.32 and by (4.59) we conclude that c is $(2\varepsilon r|(f^s)'(z)|, \sigma, C_{i,1}^{(l)})$-$h$-s.l.e.. Consequently, in view of Lemma 4.9, the point $f^s(z)$ is $(\varepsilon r|(f^s)'(z)|, 2\sigma, 2^h C_{i,1}^{(l)})$-$h$-s.l.e.. So, by Lemma 4.10 this point is

$$(Kr|(f^s)'(z)|, 2\sigma\varepsilon/K, (2\varepsilon K^{-1})^h C_{i,1}^{(l)}) - h\text{-s.l.e.}$$

Using now formula (4.58) and item (4.25) in Proposition 4.19 along with the fact that $K\varepsilon^{-1} < 1$ we have from Lemma 4.13 that the point z is $(r, 2K\varepsilon\sigma, (2\varepsilon K^{-1})^h C_{i,1}^{(l)})$-$h$-s.l.e.. The proof is completed. ■

As a fairly straightforward consequence of these two lemmas we get the following.

Lemma 4.33 *With some $R > 0$ and some $G > 0$ each point of $\overline{\mathrm{PC}_c^0(f)_i}$ (in particular each point of $\mathrm{Crit}_c(f)$) is $(r, 1/2, G)$-h-s.l.e. for every $r \in [0, R]$.*

Proof. Since $S_0(f) = \emptyset$, starting with $\sigma > 0$ as small as we wish, it immediately follow from Lemma 4.32, Lemma 4.31 and Lemma 2.27 by induction on $i = 0, 1, \ldots, p$ that all the points of $S_i(f)$ and $\overline{\mathrm{PC}_c^0(f)_i}$ are $(r, 1/2, G)$-h-s.l.e. with same $G, R > 0$ and all $r \in [0, R]$. We are done. ■

This lemma and Lemma 4.25, taken together, give the following.

Lemma 4.34 *Every point of the set $\mathrm{Crit}(J(f)) \cup f^{-1}(\infty)$ is h-s.l.e. with $\sigma \in (0, 1)$ arbitrary.*

Fix $c \in Crit_\infty(f)$. Since $\lim_{n\to\infty} f^n(c) = \infty$, there exists $k \ge 1$ such that $q_{b_n} \le q_c$ (where $b_n \in f^{-1}(\infty)$ is near $f^n(c)$, q_c was defined in (2.16)) and

$$|f^n(c)| > \max\{1, 2\mathrm{Dist}_e(0, f(\mathrm{Crit}(f))\} \tag{4.60}$$

for all $n \ge k$. Put $a = f^k(c)$ (we may need in the course of the proof $k \ge 1$ to be bigger).

We shall prove the following.

Lemma 4.35 *There exists a constant $c_1 \geq 1$, such that*

$$m_e(B_e(a,r)) \geq c_1^{-1} r^h$$

for all $r > 0$ small enough.

Proof. Put $q = q_c$. In view of (4.60) for every $n \geq 1$ there is a well-defined holomorphic inverse branch

$$f_n^{-1} : B_e(f^n(a), \frac{1}{2}|f^n(a)|) \to \mathbb{C}$$

of sending $f^n(a)$ to $f^{n-1}(a)$. Let $b_n \in f^{-1}(\infty)$ be the unique pole (assuming $k \neq 1$ is large enough) such that

$$|f^n(a) - b_n| \leq \Delta \ll 1.$$

Then, by Koebe's Distortion Theorem, I (Euclidean version)

$$\begin{aligned} f_n^{-1}\left(B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right)\right) &\subset B_e\left(f^{n-1}(a), \frac{K}{4}|f^n(a)||f'(f^{n-1}(a))|^{-1}\right) \\ &\subset B_e(f^{n-1}(a), c|f^n(a)||f^n(a)|^{-\frac{q+1}{q}}) \\ &= B_e(f^{n-1}(a), c|f^n(a)|^{-\frac{1}{q}}) \\ &\subset B_e\left(f^{n-1}(a), \frac{1}{4}|f^{n-1}(a)|\right), \end{aligned}$$

where $c > 0$ is a constant and the last inclusion was written assuming that

$$|f^{n-1}(a)| \geq 4c|f^n(a)|^{-\frac{1}{q}}$$

which we can assume that to hold for all $n \geq k$ large enough. So, the composition

$$f_a^{-n} = f_1^{-1} \circ f_2^{-1} \circ \ldots \circ f_n^{-1} : B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right) \to \mathbb{C}$$

sending $f^n(a)$ to a is well-defined and this is holomorphic branch of f^{-n}. Take $0 < r < \frac{K}{8}|a|$ and let $n+1 \geq 1$ be the least integer such that

$$r|(f^{n+1})'(a)| \geq \frac{K}{8}|f^{n+1}(a)|.$$

Such integer exists since $|f'(z)| \asymp |f(z)|^{\frac{q_b+1}{q_b}}$ if z is near a pole b. By definition $n \geq 0$ and, since $r < \frac{K}{8}|a|$, we have

$$r|(f^n)'(a)| < \frac{K}{8}|(f^n)(a)|.$$

Then by Koebe's Distortion Theorem, I (Euclidean version)

$$B_e(a,r) \supset f_a^{-n}(B_e(f^n(a), K^{-1}|(f^n)'(a)|). \tag{4.61}$$

Now we consider three cases determined by the value of $r|(f^n)'(a)|$.

Case 1. $\Delta \le r|(f^n)'(a)| < \frac{K}{8}|f^n(a)|$.

In view of (4.60) and Koebe's Distortion Theorem, I (Euclidean version) along with almost conformality of the measure m_e, we get that

$$\begin{aligned} m_e(B(a,r)) &\ge K^{-h}|(f^n)'(a)|^{-h} m_e(B_e(f^n(a), 4r|(f^n)'(a)|)) \\ &\succeq K^{-h}|(f^n)'(a)|^{-h}(4r|(f^n)'(a)|)^2 \\ &\succeq |(f^n)'(a)|^{-h}(4r|(f^n)'(a)|)^h \\ &= 4^h r^h. \end{aligned} \tag{4.62}$$

and we are done in this case.

Case 2. $|f^n(a) - b_n| \le 32A^{\frac{q_{min}+1}{q_{min}}} r|(f^n)'(a)| < 32A^{\frac{q_{min}+1}{q_{min}}}\Delta$, *where* A *was defined in (2.25).*

It follows from Lemma 4.25 that

$$m_e(B_e(f^n(a), K^{-1}|(f^n)'(a)|)) \succeq (K^{-1}|(f^n)'(a)|)^h.$$

Thus

$$m_e(B_e(a,r)) \ge K^{-h}|(f^n)'(a)|^{-h}(K^{-1}|(f^n)'(a)|)^h \asymp r^h.$$

And we are done in this case.

It remains to consider

Case 3. $r|(f^n)'(a)| < \frac{1}{8}A^{-\frac{q_{\min}+1}{q_{\min}}} K|f^n(a) - b_n|$.

But then

$$\begin{aligned} r|(f^{n+1})'(a)| &= r|(f^n)'(a)||f'(f^n(a))| \\ &< \frac{K}{8}A^{-\frac{q_{\min}+1}{q_{\min}}}|f^n(a)-b_n|(A|f^{n+1}(z)|)^{\frac{q_n+1}{q_n}} \\ &\le \frac{K}{8}A^{-\frac{q_{\min}+1}{q_{\min}}}A^{\frac{1}{q_n}+1}|f^{n+1}(a)| \\ &\le \frac{K}{8}|f^{n+1}(a)| \end{aligned}$$

contrary to the definition of n. So this case is ruled out and lemma is proved. ■

We are ready to prove the following.

Theorem 4.36 *If $\Omega(f)=\emptyset$ then the h-dimensional packing measure Π_e^h of every bounded Borel subset of $J(f)$ is finite and $\Pi_s^h(J(f))<\infty$.*

Proof. Consider arbitrary point

$$z \in J(f) \setminus \bigcup_{n=0}^{\infty} f^{-n}(\{\infty\}\cup \mathrm{Crit}(f))$$

and a radius $r\in(0,\gamma(a\xi)^{-1})$. Let $s\ge 0$, associated to the point z and the radius r/ξ, come from Lemma 4.20. If the case (a) of this lemma holds, then we have from Lemma 1.2 and Lemma 4.12 that

$$\begin{aligned} m_e(B_e(z,r)) &\ge K^{-h}|(f^s)'(z)|^{-h}m_e(B_e(f^s(z),K^{-1}r|(f^s)'(z)|) \\ &\succeq K^{-h}|(f^s)'(z)|^{-h}(K^{-1}r|(f^s)'(z)|)^2 \\ &\asymp r^h(r|(f^s)'(z)|^{2-h}\succeq r^h. \end{aligned} \tag{4.63}$$

If the case (b) of this lemma holds, then applying this lemma along with Lemma 4.34 (with $\sigma\le K^{-1}\xi$), we get that

$$\begin{aligned} m_e(B_e(z,r)) &\ge K^{-h}|(f^s)'(z)|^{-h}m_e(B_e(f^s(z),K^{-1}r|(f^s)'(z)|) \\ &\succeq K^{-4}|(f^s)'(z)|^{-h}(K^{-1}r|(f^s)'(z)|)^h\asymp r^h. \end{aligned}$$

Combining this and (4.63), completes the proof of the first part because of Theorem 3.2(a). Since $\Pi_e^h(A)=\Pi_e^h(A+w)$ for every $\omega\in\Lambda$ and since

$\frac{d\Pi_s^h}{d\Pi_e^h}(z) = (1+|z|^2)^{-h}$, we get with $R = 4\text{diam}(\mathcal{R})$ that

$$\begin{aligned}\Pi_s^h(J(f)) &= \sum_{k=0}^{\infty} \Pi_s^h(A(0, 2^k R, 2^{k+1}R)) + \Pi_s^h(B_e(0,R)) \\ &\preceq \Pi_e^h(B_e(0,R)) + \sum_{k=0}^{\infty} \Pi_e^h(A(0,2^kR,2^{k+1}R))R^{-2h}4^{-hk} \\ &\preceq \Pi_e^h(B_e(0,R)) + \sum_{k=0}^{\infty}(2^kR)^2R^{-2h}4^{-hk} \\ &= \Pi_e^h(B_e(0,R)) + R^{(2(1-h)}\sum_{k=0}^{\infty}4^{(1-h)k} < +\infty.\end{aligned}$$

since $h > 1$. We are done. ■

Proposition 4.37 *If* $\text{HD}(J(f)) = 2$*, then* $J(f) = \mathbb{C}$.

Proof. Since Π_e^2 and l_2, the two-dimensional Lebesgue measure on $\mathbb{C}$ coincide up to a multiplicative constant, it follows from (already proved) Theorem 4.1 (b) that if $h = 2$, then $l_2(J(f)) > 0$. So, in order to prove our proposition it suffices to show that if $J(f) \subsetneq \mathbb{C}$, then $l_2(J(f)) = 0$. So, suppose that $J(f) \neq \mathbb{C}$. By Corollary 1.3 in [19], $l_2(I_\infty(f)) = 0$. Thus, we are to show that

$$l_2(J(f)) \setminus (I_\infty(f) \cup \text{Sing}^-(f)) = 0.$$

Let for any $l \geq 1$ the set Z_l have the same meaning as in the proof of Theorem 4.28. Since $J(f)$ is a nowhere dense subset of $\mathbb{C}$, there exists $\varepsilon > 0$ such that for every $y \in B_e(0, 2l)$ there exists $y_\varepsilon \in B_e(y, \frac{1}{2l})$ such that

$$B_e(y_\varepsilon, \varepsilon) \subset B_e\left(y, \frac{1}{2l}\right) \setminus J(f). \tag{4.64}$$

Keep the notation from the proof of Theorem 4.28. Fix arbitrary $z \in Z_l$. Disregarding finitely many iterates we may assume without loss of generality that $f^{n_k}(z) \in B_e(0, 2l)$ for al $k \geq 1$. By Koebe's Distortion Theorem (Euclidean version), Koebe's $\frac{1}{4}$-Theorem and (4.64), we have

$$\begin{aligned}f_z^{-n_k}(B_e(f^{n_k(z)}, \varepsilon)) &\subset f_z^{-n_k}(B_e(f^{n_k(z)}, (2l)^{-1}) \setminus J(f)) \\ &\subset B_e(z, K|(f^{n_k})'(z)|^{-1}(2l)^{-1}) \setminus J(f)\end{aligned}$$

and

$$f_z^{-n_k}(B_e(f^{n_k}(z)_\varepsilon, \varepsilon)) \supset B_e(f_z^{-n_k}(f^{n_k}(z)_\varepsilon), \frac{1}{4}\varepsilon|(f^{n_k(z)})'(z)|^{-1}).$$

Therefore, we see that

$$\frac{l_2(B_e(z, K|(f^{n_k})'(z)|^{-1}(2l)^{-1}) \setminus J(f))}{l_2(B_e(z, K|(f^{n_k})'(z)|^{-1}(2l)^{-1}))} \geq \left(\frac{\varepsilon l}{2K}\right) > 0.$$

So, z is not a Lebesgue's density point for Z_l, and therefore $l_2(Z_l) = 0$. Hence

$$l_2(J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)) = l_2(\bigcup_{l=1}^{\infty} Z_l) = 0,$$

and we are done. ■

Theorem 4.1 is now a logical consequence of Chapter 4.5, Proposition 4.29, Lemma 4.30 and Theorem 4.36.

5
Invariant measures

In this chapter we deal with σ-finite invariant measures equivalent to the conformal measure m. We prove their existence, ergodicity, conservativity and we detect the points around which these measures are finite or infinite. This allows us to provide sufficient conditions for their finiteness.

In order to prove Theorem 5.2 below we apply a general sufficient condition for the existence of σ-finite absolutely continuous invariant measure proven in [25]. In order to formulate this condition suppose that X is a σ-compact metric space, ν is a Borel probability measure on X, positive on open sets, and that a measurable map $f : X \to X$ is given with respect to which measure ν is quasi-invariant, i.e.

$$\nu \circ f^{-1} \prec \nu.$$

Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure ν. We also assume that

$$\nu(X \setminus \bigcup_{n\geq 0} A_n) = 0,$$

and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$\nu(f^{-k}(A_m) \cap A_n) > 0,$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [25] reads as follows.

Theorem 5.1 *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to* *the measure ν. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1} \frac{\nu(A)}{\nu(A_n)} \leq \frac{\nu(f^{-k}(A))}{\nu(f^{-k}(A_n))} \leq K_n \frac{\nu(A)}{\nu(A_n)},$$

then T has a σ-finite T-invariant measure μ that is absolutely continuous with respect to ν. In addition, μ is equivalent to ν, conservative and ergodic, and unique up to a multiplicative constant. Moreover, for every Borel set $A \subset X$

$$\mu(A) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \nu(f^{-k}(A))}{\sum_{k=0}^{n} \nu(f^{-k}(A_0))}.$$

From now on throughout the chapter m_s stands for the unique h-conformal measure.

The first result of this chapter is the following.

Theorem 5.2 *There exists a σ-finite f-invariant measure μ that is absolutely continuous with respect to the h-conformal measure m_s. In addition, μ is equivalent to m_s, ergodic and conservative.*

Proof. Let $\xi \in \mathbb{C}$ be a periodic point of f with some period $p \geq 3$. We put

$$P_3(f) = \overline{O_+(f(\mathrm{Crit}(f)))} \cup \{\xi, f(\xi), \dots, f^{p-1}(\xi)\}.$$

Since $\overline{O_+(f(\mathrm{Crit}(f))}$ is a forward-invariant nowhere-dense subset of $J(f)$ and since the h-conformal measure m_s is positive on nonempty open subsets of $J(f)$, it follows from ergodicity and conservativity of m_s (see Theorem 4.28) that

$$m_s(\overline{O_+(f(\mathrm{Crit}(f)))}) = 0.$$

Since m_s has no atoms (see Theorem 4.28) we therefore obtain that

$$m_s(P_3(f)) = 0.$$

We shall now construct the partition α of the set $J(f) \setminus P_3(f)$. We shall check next that it satisfies the assumptions of Theorem 5.1. We first define the family of balls

$$\left\{ B_e\left(z, \frac{1}{2}\mathrm{dist}_e(z, P_3(f))\right)\right\}_{z \in J(f)\setminus P_3(f)}.$$

This family obviously covers $J(f) \setminus P_3(f)$. Since $J(f) \setminus P_3(f)$ is an open set, it is a Lindelöf space, and therefore we can choose a countable subcover of $J(f) \setminus P_3(f)$, which we denote by

$$\left\{ B_e\left(z_i, \frac{1}{2}\mathrm{dist}_e(z_i, P_3(f))\right)\right\}_{i=1}^{\infty}.$$

We inductively define a partition $\mathcal{A} = \{A_i\}_{i=0}^{\infty}$ of $J(f) \setminus P_3(f)$ as follows. Let

$$A_0 = B_e\left(z_0, \frac{1}{2}\mathrm{dist}_e(z_0, P_3(f))\right).$$

Assume that we have defined the sets $A_1, \ldots, A_n$ such that

$$A_j \subset B_e\left(z_j, \frac{1}{2}\mathrm{dist}_e(z_j, P_3(f))\right)$$

and

$$\mathrm{Int} A_j \neq \emptyset.$$

Then A_{n+1} we define as

$$A_{n+1} = B_e\left(z_{n+1}, \frac{1}{2}\mathrm{dist}_e(z_{n+1}, P_3(f))\right) \setminus \bigcup_{j=1}^{n} A_j.$$

The set A_{n+1} is disjoint from the sets $A_1, \ldots, A_n$ and

$$A_{n+1} \subset B_e\left(z_{n+1}, \frac{1}{2}\mathrm{dist}_e(z_{n+1}, P_3(f))\right) \setminus \bigcup_{j=1}^{n} B_e\left(z_j, \frac{1}{2}\mathrm{dist}_e(z_j, P_3(f))\right).$$

Thus either $A_{n+1} = \emptyset$ or $\mathrm{Int} A_{n+1} \neq \emptyset$ and we remove all the empty sets.

We shall now check that the partition is irreducible. And indeed, it follows from the construction of the sets $\{A_i\}_{i=0}^{\infty}$ and continuity of the measure m that it suffices to demonstrate that if $z \in J(f)$, $r > 0$ and $K \subset \mathbb{C}$ is a compact set, then there exists $n \geq 1$ such that

$$f^n\left(B_e(z, r) \setminus \bigcup_{k \geq 0} f^{-k}(\infty)\right) \supset K \setminus \bigcup_{k \geq 0} f^{-k}(\infty).$$

Since the set of repelling periodic points is dense in the Julia set ([2], comp. [6]), there thus exists a periodic point $x \in B_e(z, r)$, say of period $q \geq 1$. Since x is repelling there exists $s > 0$ so small that $B_e(x, s) \subset B_e(z, r)$ and $f^q(B_e(x, s)) \supset B_e(x, s)$. Since

$$\bigcup_{j \geq 1} f^{qj}(B_e(x, s)) \supset \mathbb{C},$$

since K is a compact subset of $\mathbb{C}$ and since $\{f^{qj}(B_e(x,s))\}_{j=1}^{\infty}$ is an increasing family of open sets, there thus exists $k \geq 1$ such that $f^{qk}(B_e(x,s)) \supset K$.

Let us check now the distortion assumption of Theorem 5.1. And indeed, in view of Koebe's Distortion Theorem, I (spherical version) there exists a constant $\hat{K} \geq 1$ such that if

$$f_*^{-n} : B_e\big(z_i, \mathrm{dist}_e(z_i, P_3(f))\big) \to \mathbb{C}$$

is a holomorphic branch of f^{-n}, then for every $k \geq 0$ and all $x, y \in A_k$ we have

$$\frac{|(f_*^{-n})^*(y)|}{|(f_*^{-n})^*(x)|} \leq \hat{K}, \tag{5.1}$$

where $A_k \subset B_e\left(z_i, \frac{1}{2}\mathrm{dist}_e\left(z_i, P_3(f)\right)\right)$. We therefore obtain for all Borel sets $A, B \subset A_k$ with $m_s(B) > 0$ and all $n \geq 0$ that

$$\begin{aligned}
\frac{m_s(f_*^{-n}(A))}{m_s(f_*^{-n}(B))} &= \frac{\int_A |(f_*^{-n})^*|^h dm_s}{\int_A |(f_*^{-n})^*|^h dm_s} \\
&\leq \frac{\sup_{A_k}\{|(f_*^{-n})^*|^h\} m_s(A)}{\inf_{A_k}\{|(f_*^{-n})^*|^h\} m_s(B)} \\
&\leq \hat{K}^h \frac{m_s(A)}{m_s(B)},
\end{aligned}$$

and similarly

$$\frac{m_s(f_*^{-n}(A))}{m_s(f_*^{-n}(B))} \geq \hat{K}^{-h} \frac{m_s(A)}{m_s(B)}.$$

Since by Theorem 4.28 the measure is conservative ergodic, all the assumptions of Theorem 5.1 have been checked and we are done. ■

The following lemma easily follows from Theorem 5.1.

Lemma 5.3 *For every $n \geq 0$ we have $0 < \mu(A_n) < \infty$.*

We say that the f-invariant measure μ produced in Theorem 5.2 is of finite condensation at $x \in J(f)$ if and only if there exists an open neighborhood V of x such that $\mu(V) < \infty$. Otherwise μ is said to be of infinite condensation at x. We respectively say that x is a point of finite or infinite condensation of μ. We end this chapter with the following obvious results.

Lemma 5.4 *If x is a point of infinite condensation of μ, then each point of the closure $\overline{\{f^n(x) : n \geq 0\}}$ is also of infinite condensation of μ.*

Lemma 5.5 *The set of points of infinite condensation of the measure μ is contained in the union $\overline{O_+(\text{Crit}(f))} \cup \Omega(f) \cup \{\infty\}$.*

Proof. If

$$z \notin \overline{O_+(\text{Crit}(f))} \cup \Omega(f) \cup \{\infty\},$$

then by local finiteness of the family $\{A_n : n \geq 0\}$ there exist an open neighborhood V of z and an integer $k \geq 0$ such that $m_s\big(V \setminus \bigcup_{j=0}^k A_j\big) = 0$. Hence, in view of Lemma 5.3 and Theorem 5.2 ($\mu \prec m_s$) we get

$$\mu(V) \leq \sum_{j=0}^{k} \mu(A_j) < \infty.$$

The proof is finished. ■

Fix $w \in J(f) \setminus B_e(\Omega(f), \theta)$ and an open Jordan domain $Q \subset B_e(w, 2\gamma)$. A sequence $\{Q_n\}_{n=0}^{\infty}$ of connected components of inverse images of $f^{-n}(Q)$ is called ω-nested if $f(Q_{n+1}) = Q_n$ for all $n \geq 0$. We start with the following simple fact.

Lemma 5.6 *All the sets $Q_n, n \geq 0$, are open Jordan domains.*

Proof. Suppose that the lemma is not true and let $n \geq 0$ be the least integer such that Q_n is not a Jordan domain. Then $n \geq 1$ and Q_{n-1} is a Jordan domain. Since Q_n is a connected component of $f^{-1}(Q_{n-1})$, the set Q_n must contain a critical point of f, say c. Since $Q_n \subset B_e(c, \eta)$ and

$$B_e(c, \eta) \cap f^{-1}(\infty) = \emptyset,$$

it follows from the Maximal Modulus Principle that Q_n is simply-connected. This finishes the proof. ■

Let $\text{Crit}_h(f)$ be the set of all critical points of f that are h-upper estimable. The key fact for what will follow is this.

Lemma 5.7 *For every $\xi > 0$ there exists a constant $T_\xi \geq 1$ with the following properties. Let $w \in J(f) \setminus B_e(\Omega(f), \theta)$, let $Q \subset B_e(w, \gamma)$ be a Jordan domain with $\mathrm{diam}_e(Q) \geq \xi$, and let $\{Q_n\}_{n=0}^{\infty}$ be a w-nested sequence of connected components of the sets $f^{-n}(Q)$. For every $n \geq 0$ let W_n be the connected component of $f^{-n}(B_e(w, 2\gamma))$ containing Q_n. If*

$$(\mathrm{Crit}(f) \setminus \mathrm{Crit}_h(f)) \cap \bigcup_{n=1}^{\infty} W_n = \emptyset,$$

then

$$m_e(Q_n) \leq T_\xi^h \mathrm{diam}_e^h(Q_n)$$

for all $n \geq 0$.

Proof. Since $M_{2\gamma} = \sup\{m_e(B_e(x, 2\gamma)) : x \in J(f)\} < \infty$, we get

$$m_e(Q) \leq M_{2\gamma} = \frac{M_{2\gamma}}{\xi^h} \xi^h \leq \frac{M_{2\gamma}}{\xi^h} \mathrm{diam}_e^h(Q). \tag{5.2}$$

Fix $k \geq 0$ and $n \geq 0$. Suppose that W_{k+n} contains no critical points of f^n. It then follows from Koebe's Distortion Theorem, II (Euclidean version) and (2.29) (note that also $Q \subset (B_e(w, \gamma))$ that

$$\begin{aligned}
m_e(Q_{k+n}) &\leq \sup\{|(f^n)'(z)|^{-h} : z \in Q_{k+n}\} m_e(Q_k) \\
&\leq K_*^h \inf\{|(f^n)'(z)|^{-h} : z \in Q_{k+n}\} m_e(Q_k) \\
&\leq K_*^h \frac{\mathrm{diam}_e^h(Q_{k+n})}{\mathrm{diam}_e^h(Q_k)} m_e(Q_k) \\
&\leq K_*^h \frac{m_e(Q_{k+n})}{\mathrm{diam}_e^h(Q_k)} \mathrm{diam}_e^h(Q_{k+n}),
\end{aligned} \tag{5.3}$$

with an appropriate universal $K_*^h \geq 1$. Now suppose that W_{k+1} contains a critical point c of f. By (2.27) and Lemma 2.16, c is the only critical point of f in W_{k+1}. Suppose first that

$$\mathrm{dist}_e(f(c), Q_k) \geq 4\mathrm{diam}_e(Q_k).$$

Fix $z \in Q_k$. Then $Q_k \subset B_e(z, \mathrm{diam}_e(Q_k))$,

$$f(\mathrm{Crit}(f)) \cap B_e(z, 2\mathrm{diam}(Q_k)) = \emptyset$$

(assuming that $\gamma, \eta > 0$ sufficiently small), which makes other (finitely many) critical values lying sufficiently far apart from $f(c)$. Hence denoting by

$$f_*^{-1} : B_e(z, 2\gamma\mathrm{diam}_e(Q)) \to \mathbb{C}$$

the holomorphic inverse branches of f whose image covers Q_k, using Koebe's Distortion Theorem, II (Euclidean version) we estimate similarly as above

$$\begin{aligned} m_e(Q_{k+1}) &\le \sup\{|(f_*^{-1})'(x)|^h : x \in Q_{k+1}\} m_e(Q_k) \\ &\le K^h \inf\{|(f_*^{-1})'(x)|^h : x \in Q_{k+1}\} m_e(Q_k) \\ &\le K^h \frac{\operatorname{diam}_e^h(Q_{k+1})}{\operatorname{diam}_e^h(Q_k)} m_e(Q_k) \\ &\le K^h \frac{m_e(Q_k)}{\operatorname{diam}_e^h(Q_k)} \operatorname{diam}_e^h(Q_{k+1}). \end{aligned} \tag{5.4}$$

Now, assume that

$$\operatorname{dist}_e(f(c), Q_k) \le 4\operatorname{diam}_e(Q_k). \tag{5.5}$$

We thus get that $Q_k \subset B_e(f(c), 5\operatorname{diam}_e(Q_k))$, and therefore

$$Q_{k+1} \subset B_e(c, A(5\operatorname{diam}_e(Q_k))^{1/p_c}).$$

Hence, making use of h-upper estimability of the point c, we get that

$$m_e(Q_{k+1}) \le L(A(5\operatorname{diam}_e(Q_k))^{1/p_c})^h. \tag{5.6}$$

It follows from (5.5) that

$$\begin{aligned} \operatorname{Dist}_e(c, Q_{k+1}) &\le A(\operatorname{Dist}_e(f(c), Q_k))^{1/p_c} \\ &\le A(\operatorname{dist}_e(f(c), Q_k) + \operatorname{diam}_e(Q_k))^{1/p_c} \\ &\le A(5\operatorname{diam}_e(Q_k))^{1/p_c}. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{diam}_e(Q_k) &\le \operatorname{diam}_e(Q_{k+1}) A(\operatorname{Dist}_e(c, Q_{k+1}))^{p_c - 1} \\ &\le A^2 5^{1/p_c} \operatorname{diam}_e(Q_{k+1}) \operatorname{diam}_e^{\frac{p_c-1}{p_c}}(Q_k). \end{aligned}$$

Thus

$$\operatorname{diam}_e^{1/p_c}(Q_k) \le A^2 5^{1/p_c} \operatorname{diam}_e(Q_{k+1}).$$

Inserting this to (5.6), we get that

$$m_e(Q_{k+1}) \le L(25)^{h/p_c} A^{3h/p_c} \operatorname{diam}_e^h(Q_{k+1}).$$

Applying this, (5.2) (5.3), (5.4) and making use of Lemma 2.16 along with (2.27), a straightforward inductive argument yields that for every $j \ge 1$

$$m_e(Q_j) \le [L((25A^3)^{1/p_c} K K_*)^h]^\sharp M_{2\gamma} \xi^{-h} \operatorname{diam}_e^h(Q_j).$$

We are done. ■

As an immediate consequence of this lemma, Lemma 2.19 and Koebe's Distortion Theorem, II (Euclidean version) we obtain the following.

Lemma 5.8 *Let $w \in J(f) \setminus B_e(\Omega(f), \theta)$, let $V \subset B_e(w, \gamma)$ be a Jordan domain with* $\mathrm{diam}_e(V) \geq \xi$*, let U be a Jordan domain contained in V. Let $\{V_n\}_{n=0}^{\infty}$ be a w-nested sequence of connected components of $f^{-n}(V)$ and let $\{U_n\}_{n=0}^{\infty}$, with $U_n \subset V_n$, be a w-nested sequence of connected components of $f^{-n}(U)$. For every $n \geq 0$ let W_n be the connected component of $f^{-n}(B_e(w, 2\gamma))$ containing V_n. Suppose that*

$$(\mathrm{Crit}(f) \setminus \mathrm{Crit}_h(f)) \cap \bigcup_{n=1}^{\infty} W_n = \emptyset$$

and that there exists a Jordan domain $\tilde{U}$ such that $\overline{U} \subset \tilde{U} \subset V$ and $\tilde{U} \cap \mathrm{PC}(f) = \emptyset$. Then

$$m_e(U_n) \geq (2\gamma T_\xi)^{-h} \mathrm{diam}_e^h(U),$$

and the same inequality remains true (perhaps with a larger constant on the right-hand side) with m_e replaced by m_s since the diameter of the sets are bounded by β.

Now we can take the first fruit of this lemma.

Proposition 5.9 *All the points of the set $\overline{\mathrm{PC}_c^0(f)} \setminus \Omega(f)$ are of finite condensation with respect to the invariant measure μ.*

Proof. Take an arbitrary point $w \in \overline{\mathrm{PC}_c^0(f)} \setminus \Omega(f)$. Assuming $\theta > 0$ to be small enough, we will have $w \notin B_e(\Omega(f), \theta)$. Fix $V \subset B_e(w, \gamma)$, a round open neighborhood of w disjoint from $\mathrm{PC}_p(f) \cup \mathrm{PC}_\infty(f)$. Since $\overline{\mathrm{PC}_c^0(f)}$ is a nowhere dense subset of $J(f)$, we may assume without loss of generality that A_0, the set coming from Theorem 5.2 is an open ball centered at the point of $J(f) \setminus \overline{\mathrm{PC}_c^0(f)}$, such that $2A_0 \subset V$ and $2A_0 \cap \mathrm{PC}(f) = \emptyset$. Invoking Lemma 4.27 it immediately from Lemma 5.8 that

$$m_s(f^{-n}(A_0) \cap V_n) \geq \Lambda_\xi m_s(V_n)$$

for every $n \geq 0$, where V_n is a connected component of $f^{-n}(V)$ and

$$\Lambda_\xi \succeq (2\gamma T_\xi)^{-h} \mathrm{diam}_e^h(A_0). \tag{5.7}$$

Therefore, summing over all connected components V_n of $f^{-n}(V)$, we obtain

$$m_s(f^{-n}(A_0)) = m_s(f^{-n}(A_0) \cap f^{-n}(V)) \geq \Lambda_\xi m_s(f^{-n}(V)).$$

Consequently

$$\frac{\sum_{n=0}^k m_s(f^{-n}(V))}{\sum_{n=0}^k m_s(f^{-n}(A_0))} \leq \Lambda_\xi^{-1},$$

and invoking the formula for the measure μ, stated in Theorem 5.1 we conclude that $\mu(V) < \infty$, which finishes the proof. ■

Lemma 5.10 *All the points of the set* $\mathrm{PC}_p^0(f) \cup \mathrm{PC}_\infty^0(f)$ *are of finite condensation with respect to the measure* μ.

Proof. Fix a point $w \in \mathrm{PC}_p^0(f) \cup \mathrm{PC}_\infty^0(f)$. There exists $j \geq 0$ so large that

$$f^{-j}(w) \cap (\overline{O_+(\mathrm{Crit}(f))} \cup \Omega(f)) = \emptyset.$$

Therefore, taking $\theta > 0$ and $\gamma > 0$ small enough, there exists open disk V centered at w with the following properties:

a) For every $z \in f^{-j}(w)$, $\mathrm{dist}_e(z, \Omega(f)) > \theta$.
b) For every $z \in f^{-j}(w)$, if V_z is the connected component of $f^{-j}(V)$ containing z, then V_z is Jordan domain and $V_z \subset (B_e(z, \gamma))$.
c) $\bigcup_{z \in f^{-j}(w)} B_e(z, 2\gamma) \cap \overline{\mathrm{PC}(f)} = \emptyset$.

We may assume without loss of generality A_0 to be a round disk centered (at the point of $J(f)$) such that $2A_0 \subset V$ and $2A_0 \cap \overline{\mathrm{PC}(f)} = \emptyset$. It follows from condition (c) that

$$\mathrm{Crit}(f) \cap \bigcup_{z \in f^{-j}(w)} \bigcup_{n=0}^{\infty} f^{-n}(B_e(z, 2\gamma)) = \emptyset.$$

So, we may apply Lemma 5.8 to the pairs (U_z, V_z), $z \in f^{-j}(w)$, where U_z are the connected components of $f^{-j}(A_0)$ contained in V_z, to get, similarly as in the proof of Proposition 5.9, that for every $z \in f^{-j}(w)$ and every $n \geq 0$,

$$\sum_{i=0}^n m_s(f^{-i}(V_z)) \leq \Lambda_w^{-1} \sum_{i=0}^n m_s(f^{-i}(\hat{U}_z)),$$

where $\hat{U}_z$, $z \in f^{-j}(w)$, is the union of all components of $f^{-j}(A_0)$ contained in V_z, and $\Lambda_w \le 1$ is a universal constant depending on w. Summing over all $z \in f^{-j}(w)$, we thus get

$$\sum_{i=j}^{j+n} m_s(f^{-i}(V)) = \sum_{i=0}^{n} m_s(f^{-i}(f^{-j}((V)))$$
$$\le \Lambda_w^{-1} \sum_{i=j}^{j+n} m_s(f^{-i}(A_0)).$$

Since both $\sum_{i=0}^{j-1} m_s(f^{-i}(A_0))$ and $\sum_{i=0}^{j-1} m_s(f^{-i}(V))$ are finite, we thus get

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} m_s(f^{-i}(V))}{\sum_{i=0}^{n} m_s(f^{-i}(A_0))} \le \Lambda_w^{-1}.$$

So $\mu(V) < \infty$, and we are done. ■

As an immediate consequence of this lemma, Proposition 5.9 and Lemma 5.5 we get the following.

Theorem 5.11 *The set of points of infinite condensation of the measure μ is contained in $\Omega(f) \cup \{\infty\}$.*

Now we will deal with point ∞. We shall prove the following.

Proposition 5.12 *If $\mathrm{Crit}_\infty(f) \subset \mathrm{Crit}_h(f)$, then ∞ is a point of finite condensation of the invariant measure μ.*

Proof. Fix $R > 0$ so large that for each $f^{-1}(B_R) \cap \overline{\mathrm{PC}_c^0(f)} = \emptyset$ and for every pole $b \in f^{-1}(\infty)$, V_b, the connected component of $f^{-1}(B_R)$ containing b, is a Jordan domain disjoint from $B_e(\Omega(f), \theta)$ and is contained in $B_e(b, \gamma)$. Assume without loss of generality that A_0 is a round disk centered at a point of $J(f)$ such that $2A_0 \subset B_R$ and $2A_0 \cap \overline{\mathrm{PC}(f)} = \emptyset$. Notice that

$$\xi := \inf\{\mathrm{diam}_e(V_b) : b \in f^{-1}(\infty)\} > 0.$$

For every $b \in f^{-1}(\infty)$ let A_0^b be a connected component of $f^{-1}(A_0)$ contained in V_b and let $2A_0^b$ be the connected component of $f^{-1}(2A_0)$ containing A_0^b. Since the set $P_1 := \overline{\mathrm{PC}(f)} \cap f^{-1}(\infty)$ is finite, proceeding as in the second part of the proof of Lemma 5.10 with the pair (A_0, V) replaced by (A_0^b, V_b), we see that there exists $k \geq 0$ such that for all $n \geq 0$

$$\sum_{b\in P_1}\sum_{j=k}^{k+n} m_s(f^{-j}(V_b)) \leq \Lambda_\xi^{-1}\sum_{b\in P_1}\sum_{j=k}^{n} m_s(f^{-j}(A_0^b)). \tag{5.8}$$

Where Λ_ξ was defined in (5.7). Since $\mathrm{Crit}_\infty(f) \subset \mathrm{Crit}_h(f)$, it directly follows from Lemma 5.8 that for all $j \geq 0$,

$$m_s(f^{-j}(A_0^b) \cap V_{b,j}) \geq \Lambda_\xi m_s(V_{b,j})$$

for all $b \in P_2 := f^{-1}(\infty) \setminus P_1$. Summing over all connected components $V_{b,j}$ of $f^{-1}(V_b)$, we thus get that $m_s(f^{-j}(A_0^b)) \geq \Lambda_\xi m_s(f^{-j}(V_b))$. Hence, for all $n \geq 0$,

$$\sum_{b\in P_2}\sum_{j=0}^{n} m_s(f^{-j}(V_b)) \leq \Lambda_\xi^{-1}\sum_{b\in P_2}\sum_{j=0}^{n} m_s(f^{-j}(A_0^b)).$$

Adding this inequality and (5.8) side by side, we get that

$$\sum_{j=0}^{n} m_s(f^{-j}(B_R)) - F \leq \Lambda_\xi^{-1}\sum_{j=0}^{n} m_s(f^{-j}(A_0)) - G$$

with positive numbers F and G independent of n. Thus, the formula for the invariant measure μ stated in Theorem 5.1, yields that $\mu(B_R) \leq \Lambda_\xi^{-1} < \infty$. We are done. ■

Corollary 5.13 *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically pseudo non-recurrent elliptic function and* $\mathrm{Crit}_\infty(f) \cup \Omega(f) = \emptyset$*, then the invariant measure μ, equivalent to the conformal measure m_s (which in this case coincide with the packing measure Π^h), is finite.*

and

Corollary 5.14 *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically pseudo non-recurrent elliptic function whose Julia set is equal to the entire complex plane $\mathbb{C}$, then there exists a unique Borel probability f-invariant measure μ equivalent to the planar Lebesgue measure on $\mathbb{C}$.*

Proof. Since $\Omega(f) = \emptyset$ and $\mathrm{Crit}_2(f) = \mathrm{Crit}(f)$, there existence of μ follows immediately from Theorem 5.11 and Proposition 5.12. Uniqueness is guaranteed by Theorem 5.2. ■

6
Rigidity

In this chapter we extend the rigidity theorem proved in [30] for tame rational function and in [33] for holomorphic expanding repellers, to the class of regular pseudo non-recurrent elliptic functions.

Our ultimate goal in this chapter is to prove the following rigidity theorem.

Theorem 6.1 *Suppose that f and g are two regular pseudo non-recurrent elliptic functions. Let $h : J(f) \to J(g)$ be a homeomorphism conjugating f to g, namely $h \circ f = g \circ h$ on $J(f)$. Then the following conditions (1)-(6) are equivalent.*

1) *h extends to an affine conjugacy from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ between $f : \mathbb{C} \to \overline{\mathbb{C}}$ and $g : \mathbb{C} \to \overline{\mathbb{C}}$.*
2) *h extends to a conformal homeomorphism conjugating f and g on neighborhoods of $J(f)$ and $J(g)$ in $\mathbb{C}$.*
3) *h extends to a real-analytic diffeomorphism conjugating f and g on neighborhoods of $J(f)$ and $J(g)$ in $\mathbb{C}$.*
4) *The homeomorphisms $h : J(f) \to J(g)$ and $h^{-1} : J(g) \to J(f)$ are Lipschitz continuous.*
5) *For every periodic point z of f, say of period p, $|(f^p)'(z)| = |(g^p)'(h(z))|$.*
6) *The measure class of m_f is transported under h to the measure class of m_g.*

For this we need to know that the Jacobian $D_\mu = \frac{d\mu \circ f}{d\mu}$ is real analytic. This in turn requires to know that the Radon-Nikodym derivative $\frac{d\mu}{dm_e}$ (equivalently $\frac{d\mu}{dm_s}$) is real-analytic, the fact interesting itself. This is our aim now. The first step toward this end is to project the dynamics as well as conformal and invariant measures to the torus $\mathbb{C}/\sim$. We shall describe this procedure now. Indeed, let $\mathcal{T} = \mathbb{C}/\sim$ be the quotient space of $\mathbb{C}$ with respect to the equivalence relation $\sim$, i.e. the torus generated by the lattice Λ. Let $\Pi : \mathbb{C} \to \mathcal{T}$ be the corresponding canonical projection. Put $\hat{\mathcal{T}} = \Pi(\mathbb{C} \setminus f^{-1}(\infty))$. Since the elliptic function $f : \mathbb{C} \setminus f^{-1}(\infty) \to \mathbb{C}$ respects the equivalence relation $\sim$, it induces a holomorphic map $\hat{f} : \hat{\mathcal{T}} \to \mathcal{T}$ and the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} \setminus f^{-1}(\infty) & \xrightarrow{f} & \mathbb{C} \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \hat{\mathcal{T}} & \xrightarrow{\hat{f}} & \mathcal{T}. \end{array}$$

Let $\hat{J} = \Pi(J(f)) \subset \mathcal{T}$. Define the probability measure $\hat{m}$ on $\hat{J}$ by the formula

$$\hat{m}(A) := m_e(\Pi^{-1}(A) \cap \mathcal{R}).$$

This definition is in fact independent of the choice of the fundamental region $\mathcal{R}$, and the measure $\hat{m}$ is clearly h-conformal with respect to the map $\hat{f}$. Because the above diagram commutes the Borel probability measure

$$\hat{\mu} := \mu \circ \Pi^{-1}$$

($\hat{\mu}(J(f)) = 1$) is $\hat{f}$-invariant, and since μ is equivalent to m_e, $\hat{\mu}$ is equivalent to $\hat{m}$. We shall prove the following.

Lemma 6.2 *The Radon-Nikodym derivative $\hat{\rho} = \frac{d\hat{\mu}}{d\hat{m}}$ has a real analytic extension on a neighborhood of $\hat{J} \setminus \Pi(\overline{\mathrm{PC}(f)})$ in $\mathcal{T}$.*

Proof. Since the measure m_e is ergodic and conservative (Theorem 4.28), so is the measure $\hat{m}$. Since we have bounded distortion on the complement of $\Pi(\overline{\mathrm{PC}(f)})$ by Koebe's Distortion Theorem, the assumption of Theorem 5.1 are satisfied for the dynamical system $\hat{f} : \hat{J} \to \hat{J}$ and the conformal measure $\hat{m}$. Therefore,

$$\hat{\rho}(z) = \lim_{n\to\infty} a_n^{-1} \sum_{k=0}^{n} \sum_{\xi \in \hat{f}^{-k}(z)} |(\hat{f}^k)'(\xi)|^{-h} \tag{6.1}$$

for every $z \in \hat{J} \setminus \Pi(\overline{\mathrm{PC}(f)})$, where $a_n = \sum_{k=0}^{n} \hat{m}(\hat{f}^{-k}(A_0))$ with some set A_0 as required in Theorem 5.1. Fix such an arbitrary point $z \in \hat{J} \setminus \Pi(\overline{\mathrm{PC}(f)})$ and take $r = r(z) > 0$ so small that

$$B(z, 2r) \cap \Pi(\overline{\mathrm{PC}(f)}) = \emptyset.$$

We can assume without loss of generality that $A_0 \subset B(z,r)$. For every $k \geq 0$ and every $\xi \in \hat{f}^{-k}(z)$, let $\hat{f}_\xi^{-k} : B(z,2r) \to \mathcal{T}$ be the unique holomorphic inverse branch of $\hat{f}^k$ defined on $B(z,2r)$ and determined by the requirement $\hat{f}_\xi^{-k}(z) = \xi$. Now embed $\mathbb{C}$ into $\mathbb{C}^2$ by the formula $x + iy \to (x,y) \in \mathbb{C}^2$. For every $\xi \in \hat{f}^{-k}(z)$ consider the map $g_\xi : B(z,2r) \to \mathbb{C}$ defined as follows

$$g_\xi(w) = \frac{(\hat{f}_\xi^{-k})'(w)}{(\hat{f}_\xi^{-k})'(z)}.$$

Since the ball $B(z,2r)$ is simply connected, since the Jacobian g_ξ nowhere vanishes on $B(z,2r)$ and since $g_\xi(z) = 1$, there exists $\log g_\xi : B(z,2r) \to \mathbb{C}$, a unique holomorphic branch of logarithm g_ξ such that $\log g_\xi(z) = 0$. By the Koebe's Distortion Theorem, I (Euclidean version) and Koebe's Distortion Theorem for arguments ([18] vol. II, Cor. p. 353), there exists a constant $\hat{K}$ such that $|\log g_\xi| \leq \hat{K}$ throughout $B(z,r)$. Expand $\log g_\xi$ into its Taylor series

$$\log g_\xi = \sum_{n=0}^{\infty} u_n (w-z)^n.$$

By Cauchy's estimates

$$|u_n| \leq \hat{K}/r^n, \quad n \geq 0. \tag{6.2}$$

For every point $x + iy \in B(z,2r)$, we can write

$$\begin{aligned}
\operatorname{Re}\log g_\xi(x+iy) &= \operatorname{Re}\left(\sum_{n=0}^{\infty} u_n\left((x - \operatorname{Re}z) + i(y - \operatorname{Im}z)\right)^n\right) \\
&= \sum_{p,q=0}^{\infty} \operatorname{Re}\left(u_{p+q}\binom{p+q}{q} i^q\right)(x - \operatorname{Re}z)^p (y - \operatorname{Im}z)^q \\
&= \sum c_{p,q}(x - \operatorname{Re}z)^p (y - \operatorname{Im}z)^q.
\end{aligned}$$

In view of (6.2) we can have

$$|c_{p,q}| \leq \hat{K} r^{-(p+q)} 2^{p+q}.$$

Hence $\operatorname{Re}\log g_\xi$ extends, by the same power series expansion

$$\sum c_{p,q}(x - \operatorname{Re}z)^p (y - \operatorname{Im}z)^q,$$

to the polydisc $\mathbb{D}_{\mathbb{C}^2}(z, r/3)$ and its modulus is bounded there from above by $4\hat{K}$. Denote this extension by $\hat{\operatorname{Re}}\log g_\xi$. Now, for every $n \geq 0$ consider the function $b_n : B(z, 2r) \to \mathbb{C}$ given by the formula

$$b_n(w) = a_n^{-1} \sum_{k=0}^{n} \sum_{\xi \in \hat{f}^{-k}(z)} |(\hat{f}_\xi^{-k})'(w)|^h.$$

Each function b_n extends to a holomorphic function $B_n : \mathbb{D}_{\mathbb{C}^2}(z, 2r) \to \mathbb{C}$ as follows

$$B_n = a_n^{-1} \sum_{k=0}^{n} \sum_{\xi \in \hat{f}^{-k}(z)} |(\hat{f}^k)'(\xi)|^{-h} \exp(h \hat{\mathrm{Re}} \log g_\xi).$$

Since $A_0 \subset B(z, 2r)$, it follows from (6.1) and Koebe's Distortion Theorem, I (Euclidean version) that $L = \sup_{n \geq 0}\{b_n(z)\} < +\infty$. Therefore, for every $w \in \mathbb{D}_{\mathbb{C}^2}(z, r/3)$, we get

$$\begin{aligned}
|B_n(z)| &\leq a_n^{-1} \sum_{k=0}^{n} \sum_{\xi \in \hat{f}^{-k}(z)} |(\hat{f}^k)'(\xi)|^{-h} |\exp(h \hat{\mathrm{Re}} \log g_\xi(w))| \\
&\leq a_n^{-1} \sum_{k=0}^{n} \sum_{\xi \in \hat{f}^{-k}(z)} |(\hat{f}^k)'(\xi)|^{-h} \exp(h |\hat{\mathrm{Re}} \log g_\xi(w)|) \\
&\leq \exp(4h\hat{K}) a_n^{-1} \sum_{k=0}^{n} \sum_{\xi \in \hat{f}^{-k}(z)} |(\hat{f}^k)'(\xi)|^{-h} \\
&= \exp(4h\hat{K}) b_n(z) \\
&\leq L \exp(4h\hat{K}).
\end{aligned}$$

Hence, applying Cauchy's Integral Formula (in $\mathbb{D}_{\mathbb{C}^2}(z, r/2)$), we see that the family $\{B_n\}_{n=0}^{\infty}$ is equicontinuous on $\mathbb{D}_{\mathbb{C}^2}(z, r/4)$. Thus, we can chose from $\{B_n\}_{n=0}^{\infty}$ a subsequence uniformly convergent on $\mathbb{D}_{\mathbb{C}^2}(z, r/5)$. Its limit function

$$G_z : \mathbb{D}_{\mathbb{C}^2}(z, r_z/5) \to \mathbb{C}$$

is analytic and

$$G_{z|\hat{J} \cap \mathbb{D}_{\mathbb{C}^2}(z, r_z/5)} = \hat{\rho}_{|\hat{J} \cap \mathbb{D}_{\mathbb{C}^2}(z, r_z/5)}.$$

So $G_{z|B(z, r_z/5)}$ is a real-analytic extension of $\hat{\rho}_{|\hat{J} \cap \mathbb{D}_{\mathbb{C}^2}(z, r_z/5)}$. Now, if

$$B(z, r_z/10) \cap B(z', r_z'/10) \neq \emptyset,$$

$z, z' \in \hat{J} \setminus \Pi(\overline{\mathrm{PC}(f)})$, then chose a point

$$v \in B(z, r_z/10) \cap B(z', r_z'/10).$$

We may assume without loss of generality that $r_z \le r'_z$. Then

$$z \in B\left(z', \frac{r_z}{10} + \frac{r'_z}{10}\right) \subset B(z', r'_z/5).$$

So, $z \in B(z, r_z/5) \cap B(z', r'_z/5)$; in particular

$$\hat{J} \cap B(z, r_z/5) \cap B(z', r'_z/5) \neq \emptyset.$$

Since this intersection is not contained in any real-analytic curve (its Hausdorff dimension is larger than 1), we thus conclude that

$$G_{z|B(z,r_z/5)\cap B(z',r'_z/5)} = G_{z'|B(z,r_z/5)\cap B(z',r'_z/5)}.$$

In particular,

$$G_{z|B(z,r_z/10)\cap B(z',r'_z/10)} = G_{z'|B(z,r_z/10)\cap B(z',r'_z/10)}.$$

So, the formula $G(w) = G_z(w)$ if $z \in \hat{J} \setminus \Pi(\overline{\mathrm{PC(f)}})$ and $w \in B(z, r_z/10)$ provides a well-defined real-analytic function on $\bigcup_{z\in\hat{J}\setminus\Pi(\overline{\mathrm{PC(f)}})} B(z, r_z/10)$ which coincides with $\hat{\rho}$ on $\hat{J} \setminus \Pi(\overline{\mathrm{PC}(f)})$. We are done. ■

Now, we can prove one of the main results of this chapter.

Theorem 6.3 *The Radon-Nikodym derivative* $\frac{d\mu}{dm_e}$ *has a real-analytic extension on a neighborhood of* $J(f) \setminus \overline{\mathrm{PC}(f)}$ *in* $\mathbb{C}$.

Proof. Fix a point $z \in J(f) \setminus \overline{\mathrm{PC}(f)}$ and put $R_z = \frac{1}{2}\mathrm{dist}(z, \overline{\mathrm{PC}(f)})$. Fix also a point $\xi \in f^{-1}(z)$. Then for every Borel set $A \subset B(z, R_z)$, we have

$$\begin{aligned}
\frac{\mu(A)}{m_e(A)} &= \frac{\mu(f^{-1}(A))}{m_e(f_\xi^{-1}(A))}\frac{m_e(f_\xi^{-1}(A))}{m_e(A)} \\
&= \frac{\sum_{\omega\in\Lambda}\mu(\omega + f_\xi^{-1}(A))}{m_e(f_\xi^{-1}(A))}\frac{m_e(f_\xi^{-1}(A))}{m_e(A)}. \\
&= \frac{\hat{\mu}(\Pi(f^{-1}(A)))}{\hat{m}(\Pi(f_\xi^{-1}(A)))}\frac{m_e(f_\xi^{-1}(A))}{m_e(A)}.
\end{aligned}$$

Hence, passing to Radon-Nikodym derivatives, we get

$$\frac{d\mu}{dm_e}(w) = \frac{d\hat{\mu}}{d\hat{m}}(\Pi(f_\xi^{-1}(w)))|(f_\xi^{-1})'(w)|^h$$

for all $w \in B(z, R_z) \cap J(f)$. Since by Lemma 6.2, the function

$$w \to \frac{d\hat{\mu}}{d\hat{m}}(\Pi(f_\xi^{-1}(w)))|(f_\xi^{-1})'(w)|^h$$

is real analytic on some ball $B(z, \hat{R}_z)$, $0 < \hat{R}_z \le R_z$, we conclude, exactly as in the proof of this lemma, that the formula

$$w \to \frac{d\hat{\mu}}{d\hat{m}}(\Pi(f_\xi^{-1}(w)))|(f_\xi^{-1})'(w)|^h$$

for $z \in J(f) \setminus \overline{\mathrm{PC}(f)}$ and $w \in B(z, \hat{R}_z/2)$ gives a real analytic extension of $\frac{d\mu}{dm_e}$ onto $\bigcup_{z \in J(f)\setminus\overline{\mathrm{PC}(f)}} B(z, \hat{R}_z/2$. ■

Since

$$\frac{d\mu \circ f}{d\mu} = \frac{d\mu}{dm_e} \circ f|f'|^h \left(\frac{d\mu}{dm_e}\right)^{-1},$$

as an immediate consequence of Theorem 6.3, we get the following.

Corollary 6.4 *The Jacobian* $D_\mu f = \frac{d\mu \circ f}{d\mu}$ *has a real analytic extension on*
a neighborhood of $J(f) \setminus (\overline{\mathrm{PC}(f)} \cup f^{-1}(\infty))$ *in* $\mathbb{C}$.

We recall that the sets $\mathrm{PC}(f)$ and $\mathrm{PC}^0(f)$ were defined in (2.15). Let

$$\mathrm{PS}^0(f) := \mathrm{PC}^0(f) \cup f^{-1}(\infty), \qquad \mathrm{PS}_-(f) := \bigcup_{n=0}^{\infty} f^{-n}\big(\mathrm{PS}^0(f)\big).$$

Given a set $A \subset \mathbb{C}$ and $r > 0$ let $B(A, r) = \{z \in C : \mathrm{dist}(z, A) < r\}$. Let $\overline{B}(A, r)$ denote its closure. For A being a point this is the closed ball centered at A and of radius r.

If Y is a subset of $\mathbb{C}$, then we say that $u : Y \to S^1$ is an invariant line field on Y if $u(f(x)) = \left(\frac{f'(x)}{|f'(x)|}\right)^2 u(x)$ for all $x \in Y \cap f^{-1}(Y)$.

Theorem 6.5 *If* $f : \mathbb{C} \to \overline{\mathbb{C}}$ *is a critically pseudo non-recurrent elliptic function, then no of the following statement is true.*

- *(a) The Jacobian* $D_{\mu_f} : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to (0, +\infty)$ *is locally constant.*
- *(b) The function* $\log|f'| : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to \mathbb{R}$ *is cohomologous to a locally constant function on* $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ *in the class of continuous functions on* $J(f) \setminus \overline{\mathrm{PS}^0(f)}$.
- *(c) There exists a continuous* f*-invariant line field on* $J(f) \setminus \overline{\mathrm{PS}^0(f)}$.

(d) For every $n \geq 1$ and every point $z \in J(f) \setminus \mathrm{PS}_-(f)$,

$$\det(\nabla(D_{\mu_f} \circ f^n)(z), \nabla(D_{\mu_f})(z)) = 0.$$

Proof. The structure of the proof is to establish the following implications
(a) $\Longrightarrow$ (b), (d) $\Longrightarrow$(a)$\vee$(c) and to show that both (b) and (c) lead to a contradiction.

(a) $\Longrightarrow$(b). Let $\rho = \frac{d\mu}{dm_e}$. Since $D_{\mu_f} = \rho \circ f |f'|^h \rho^{-1}$, we get that

$$\log D_{\mu_f} = \log \rho \circ f + h \log |f'| - \log \rho.$$

Hence

$$\log |f'| = h^{-1} \log D_{\mu_f} + h^{-1} \log \rho - h^{-1} \log \rho \circ f.$$

Since $\rho : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to (0, +\infty)$ is by Theorem 6.3 continuous and since $\log D_{\mu_f}$ is locally constant by (a), (b) follows.

(d) $\Longrightarrow$ (a) $\vee$ (c). Suppose first that $\nabla D_{\mu_f} \equiv 0$ on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$. This of course equivalently means that D_{μ_f} is locally constant on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ giving (a). So, suppose that there exists $v \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$ such that $\nabla D_{\mu_f}(v) \neq 0$. Since the gradient ∇D_{μ_f} is locally real-analytic, there thus exists a simply-connected neighborhood $W \subset \mathbb{C} \setminus \overline{\mathrm{PS}^0(f)}$ of $v \in \mathbb{C}$ on which the gradient nowhere vanishes. Then there exists a continuous function $l : W \to S^1$ such that $l(z)$ is orthogonal to $\nabla D_{\mu_f}(z)$ at every point $z \in W$. Now, for every $z \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$ there exists $n \geq 0$ and $\xi \in W \cap f^{-n}(z)$. Then define

$$l(z) = (f^n)'(\xi) l(\xi).$$

We want to show first that the function l is well-defined on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ i.e. that if $\zeta \in f^{-m}(z) \cap W$, $m \geq 0$, then

$$(f^n)'(\xi) l(\xi) = (f^m)'(\zeta) l(\zeta). \tag{6.3}$$

Suppose on the contrary that (6.3) fails with some z, ξ, ζ as above. Then there exists a point $x \in (J(f) \setminus \overline{\mathrm{PS}^0(f)}) \cap W$, $k \geq 0$, and $w \in f^{-k}(x)$ so close to z that there are two points $y_1 \in f^{-n}(w)$ and $y_2 \in f^{-m}(w)$ so close respectively to ξ and ζ that

$$(f^n)'(y_1) l(y_1) \neq (f^m)'(y_2) l(y_2).$$

Hence,

$$(f^{k+n})'(y_1)l(y_1) \neq (f^{k+m})'(y_2)l(y_2).$$

So, either

$$(f^{k+n})'(y_1)l(y_1) \neq l(x) \quad \text{or} \quad (f^{k+m})'(y_2)l(y_2) \neq l(x). \tag{6.4}$$

Suppose without loss of generality that the first of these two inequalities holds. Consider now gradients as horizontal vectors and vectors parallel to l as vertical ones. The standard inner product becomes then the product of matrices (horizontal or vertical). Let t be a unit vector parallel to $l(x)$. Since, by the Chain Rule,

$$\nabla(D_{\mu_f} \circ f^{k+n})(y_1) = \nabla D_{\mu_f}(f^{k+n}(y_1))(f^{k+n})'(y_1) = \nabla D_{\mu_f}(x)(f^{k+n})'(y_1)$$

and since the matrix $((f^{k+n})'(y_1))^{-1}$ is symmetric, we get

$$\begin{aligned}
&< \nabla(D_{\mu_f} \circ f^{k+n})(y_1), ((f^{k+n})'(y_1))^{-1}t >= \\
&< \nabla(D_{\mu_f} \circ f^{k+n})(y_1)((f^{k+n})'(y_1))^{-1}, t >= \\
&< \nabla D_{\mu_f}(x), t >= 0.
\end{aligned}$$

Combining this and (6.4) we see that $l(y_1)$ is not perpendicular to

$$\nabla(D_{\mu_f} \circ f^{k+n})(y_1).$$

This means that $\nabla D_{\mu_f}(y_1)$ and $\nabla(D_{\mu_f} \circ f^{k+n})(y_1)$ are not parallel, or equivalently,

$$det(\nabla(D_{\mu_f} \circ f^{k+n})(y_1), \nabla D_{\mu_f}(y_1)) \neq 0,$$

contrary to (d) since $y_1 \notin \mathrm{PS}_-(f)$. Thus $l : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to S^1$ is well-defined and the invariance of the line field $u(z) = \left(\frac{l(z)}{|l(z)|}\right)^2$ defined on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ is immediate from the definition of l. The implication (d) $\Longrightarrow$(a)$\vee$ (c) is thus established.

Item (c) leads to a contradiction. By item (c) there exists a continuous function $l : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to S^1$ such that

$$l(f(z)) = l(z)\left(\frac{f'(z)}{|f'(z)|}\right)^2 \tag{6.5}$$

for all $z \in J(f) \setminus (\overline{\mathrm{PS}^0(f)} \cup f^{-1}(\overline{\mathrm{PS}^0(f)}))$. Fix a pole $b \in f^{-1}(\infty)$. Let $q \geq 1$

be the order of b. Taking $R > 0$ sufficiently small, there exists $r > 0$ and a holomorphic function

$$A : B(b, r) \to \{z \in \mathbb{C} : r < |z| < r^{-1}\}$$

such that

$$f(z) = A(z)(z - b)^{-q}$$

for all $z \in B(b, R)$. Consequently

$$f(z) = A(z - w)(z - w - b)^{-q}$$

for all $w \in \Lambda$ and for all $z \in B(b + w, R)$. So,

$$f'(z) = (z - w - b)^{-q-1}(A'(z - w)(z - w - b) - qA(z)).$$

Therefore,

$$\frac{f'(z)}{|f'(z)|} = \frac{|z - w - b|^{q+1}}{(z - w - b)^{q+1}} \frac{A'(z - w)(z - w - b) - qA(z - w)}{|A'(z - w)(z - w - b) - qA(z - w)|}. \tag{6.6}$$

Now note that for every $w \in \Lambda$ with sufficiently large modulus, there exists $z_w \in B(w + b, R)$ with a fixed point of f, i.e.

$$f(z_w) = z_w. \tag{6.7}$$

Note also that $\lim_{|w|\to\infty} |z_w - (b + w)| = 0$. Therefore,

$$\lim_{|w|\to+\infty} \frac{A'(z_w - w)(z_w - w - b) - qA(z_w - w)}{|A'(z_w - w)(z_w - w - b) - qA(z_w - w)|} = -\frac{A(b)}{|A(b)|}. \tag{6.8}$$

If $|w|$ is large enough, then

$$z_w \in J(f) \setminus (\overline{\mathrm{PS}^0(f)} \cup f^{-1}(\overline{\mathrm{PS}^0(f)})),$$

and it therefore follows from (6.5) that

$$\big(f'(z_w)/|f'(z_w)|\big)^2 = 1.$$

Hence, combining (6.6) and (6.8), we get that

$$\lim_{|w|\to\infty} \left(\frac{(z_w - w - b)^{q+1}}{|z_w - w - b|^{q+1}}\right)^2 = \left(\frac{A(b)}{|A(b)|}\right)^2.$$

Thus,

$$\lim_{|w|\to\infty} \frac{(z_w - w - b)^{2(q+1)}}{|z_w - w - b|^{2(q+1)}} = \left(\frac{A(b)}{|A(b)|}\right)^2. \tag{6.9}$$

By (6.7), $A(z_w - w)(z_w - w - b)^{-q} = z_w$, or equivalently, $(z_w - b - w)^q = A(z_w - w)z_w^{-1}$. Hence,

$$\frac{(z_w - w - b)^{2q(q+1)}}{|z_w - w - b|^{2q(q+1)}} = \left(\frac{A(z_w - w)}{|A(z_w - w)|}\right)^{2(q+1)} \left(\frac{|z_w|}{z_w}\right)^{2(q+1)}.$$

Since $\lim_{|w|\to\infty}(z_w - w) = b$, inserting this to (6.9), we get that

$$\lim_{|w|\to\infty}\left(\frac{z_w}{|z_w|}\right)^{2(q+1)} = \left(\frac{A(b)}{|A(b)|}\right)^2.$$

This is a contradiction since the set of accumulation points of the sequence $\left(\frac{z_w}{|z_w|}\right)_{w\in\Lambda}$ is the entire unit circle S^1. We are done.

Item (b) leads to a contradiction. Let $N := \sharp(f(\mathrm{Crit}(f))$. Since $\mathrm{HD}(J(f)) > 1$, in fact the Hausdorff dimension of every non-empty open subset of $J(f)$ is equal to $\mathrm{HD}(J(f)) > 1$, we conclude that there are closed polygonal arcs $\gamma_1, \hat{\gamma}_1, \dots, \gamma_N, \hat{\gamma}_N, \gamma_{N+1}, \hat{\gamma}_{N+1}$ consisting of finitely many straight line segments with the following properties

- (i) There exists $x \in \mathbb{C} \setminus \bigcup_{n\geq 0}^\infty f^{-n}(\overline{\mathrm{PS}^0(f)})$ such that for all different $i, j \in \{1, \dots, N+1\}$, $\gamma_i \cap \gamma_j = \hat{\gamma}_i \cap \hat{\gamma}_j = \{x\}$.
- (ii) The arcs $\gamma_1, \hat{\gamma}_1, \dots, \gamma_N, \hat{\gamma}_N$ are compact.
- (iii) The arcs γ_{N+1} and $\hat{\gamma}_{N+1}$ are unbounded and $\gamma_{N+1} \cap \hat{\gamma}_{N+1} = \{x\}$.
- (iv) For every $1 \leq j \leq N$ the arcs γ_j and $\hat{\gamma}_j$ have the same endpoints, one of which belongs to $f(\mathrm{Crit}(f))$, and the intersection $\gamma_j \cap \hat{\gamma}_j$ is a doubleton. In particular, $\gamma_j \cup \hat{\gamma}_j$ is a closed Jordan curve.
- (v) If $Q = \bigcup_{j=1}^{N+1} \gamma_j$ and $\hat{Q} = \bigcup_{j=1}^{N+1} \hat{\gamma}_j$, then each connected component of $\mathbb{C} \setminus (Q \cup \hat{Q})$ intersects $J(f)$.
- (vi) $f(\mathrm{Crit}(f))$ is contained in the set of endpoints of $\gamma_j, j = 1, \dots, N$ (and also $\hat{\gamma}_j, j = 1, \dots, N$).

Take a fixed (with respect to f) point $w \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$ (note that there are infinitely many of such points). Fix also an arbitrary point

$$\xi_0 \in J(f) \setminus (Q \cup \overline{\mathrm{PS}^0(f)})$$

and an arbitrary radius $R > 0$ so large, say $R \geq R^*$, that $B(\xi_0, R) \setminus Q$ and $B(\xi_0, R) \setminus \hat{Q}$ are open topological disks. For every $\xi_1 \in f^{-1}(\xi_0)$ there then exists a unique holomorphic inverse branch $f_{\xi_1}^{-1} : B(\xi_0, R) \setminus Q \to \mathbb{C}$ of f sending ξ_0 to ξ_{-1}. Note that all the sets $f_{\xi_1}^{-1}(B(\xi_0, R) \setminus Q)$ are uniformly bounded, have piecewise smooth boundaries and

$$\mathrm{dist}(f_{\xi_1}^{-1}(B(\xi_0, R) \setminus Q), f^{-1}(\infty)) > 0.$$

Recalling also that $\mathrm{PC}_c(f)$ is bounded and $\mathrm{PC}_p(f)$ is bounded (even finite), we thus deduce that for all $\xi_1 \in f^{-1}(\xi_0)$ with sufficiently large modulus (depending on R),

$$\overline{f_{\xi_1}^{-1}(B(\xi_0, R) \setminus Q)} \cap \overline{\mathrm{PS}^0(f)} = \emptyset. \tag{6.10}$$

Denote by V the unbounded connected component of

$$\mathbb{C} \setminus \overline{f_{\xi_1}^{-1}(B(\xi_0, R) \setminus Q)}.$$

Obviously V is an open simply connected set whose boundary is a piecewise smooth Jordan curve contained in $\partial(f_{\xi_1}^{-1}(B(\xi_0, R) \setminus Q))$. Fixing ξ_1 with sufficiently large modulus, we will also have,

$$\overline{\mathbb{C} \setminus V} \cap \overline{\mathrm{PC}(f)} = \emptyset.$$

Then there exists $r > 0$ such that, for every $s \in (0, r]$,

$$W_s = B(\mathbb{C} \setminus V, s) \supset B(f_{\xi_1}^{-1}(B(\xi_0, R) \setminus Q), s)$$

is a topological disk disjoint from $\mathrm{PC}(f)$. Extend now ξ_0 and ξ_1 to a sequence $\xi = \{\xi_n\}_0^\infty$ such that $f(\xi_{n+1}) = \xi_n$ for all $n \geq 0$ and

$$\lim_{n\to\infty} \xi_n = w. \tag{6.11}$$

For every $n \geq 2$, let $f_n^{-(n-1)} : W_r \to \mathbb{C}$, be the unique holomorphic inverse branch of f^{n-1} sending ξ_1 to ξ_n. For every $z \in W_r$ put $z_n = f_n^{-(n-1)}(z)$, $n \geq 2$. We now shall show that the series

$$\sum_{n=2}^{\infty} (\log|f'(\xi_n)| - \log|f'(z_n)|) \tag{6.12}$$

converges uniformly on $W_{r/2}$. Indeed, take arbitrary $1 \leq k \leq l$. Then by the Chain Rule

$$\left| \sum_{j=k+1}^{l} \big(\log|f'(\xi_j)| - \log|f'(z_j)|\big) \right| = \left| \log \left| \frac{(f_{k,l}^{-(l-k)})'(z_k)}{(f_{k,l}^{-(l-k)})'(\xi_k)} \right| \right|,$$

where $f_{k,l}^{-(l-k)} : f_k^{-(k-1)}(W_s) \to \mathbb{C}$, the unique holomorphic inverse branch sending ξ_k to ξ_l, is equal to $f_l^{-(l-1)} \circ f^{k-1}$. Since for all $k \geq 1$ large enough $f_{k,l}^{-(l-k)}$ extends univalently (and holomorphically) to $B(\xi_k, \frac{1}{2}\mathrm{dist}(w, \mathrm{PC}(f)))$ and since $\lim_{n\to\infty} |z_n - \xi_n| = 0$ uniformly on $W_{r/2}$, we conclude from Koebe's

Distortion Theorem, II (Euclidean version) that

$$\lim_{k\to\infty} \sup_{z\in W_{r/2}} \sup_{l\ge k} \left\{ \left| \sum_{j=k+1}^{l} \big(\log|f'(\xi_j)| - \log|f'(z_j)|\big) \right| \right\} = 0. \tag{6.13}$$

This means that the sequence of partial sums of the series (6.12) is uniformly Cauchy (fundamental), and it therefore converges uniformly to a harmonic function. Thus the function

$$\begin{aligned} u_R(z) &= u(\xi_0) + \log|f'(\xi_1)| - \log|f'(f_{\xi_1}^{-1}(z))| \\ &\quad + \sum_{n=2}^{\infty} \Big(\log|f'(\xi_n)| - \log|f'((f_{\xi_1}^{-1}(z))_n)|\Big) \end{aligned} \tag{6.14}$$

is well defined and harmonic on $B(\xi_0, R) \setminus Q$. By the assumption (b) there exists a continuous function $u : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to \mathbb{R}$ and locally constant function $c : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to \mathbb{R}$ such that

$$\log|f'(z)| = c(z) + u(z) - u(f(z)) \tag{6.15}$$

for all $z \in J(f) \setminus (\mathrm{PS}^0(f) \cup f^{-1}(\mathrm{PS}^0(f)))$. Consider now the set $E \subset J(f)$ consisting of all those points y for which $f^{-1}(y)$ is not a subset of $\overline{\mathrm{PS}^0(f)}$. Note that $J(f) \setminus E \subset f(\mathrm{Crit}(f))$. Fix $x \in f^{-1}(y) \setminus \overline{\mathrm{PS}^0(f)}$. Making use of (6.15), for points z near x, we deduce that u extends continuously to E. Noting also that if $z \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$, then $f(z) \in E$, we further deduce that (6.15) holds for all $z \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$, i.e.

$$\log|f'(z)| = c(z) + u(z) - u(f(z)). \tag{6.16}$$

Now using (6.10) and (6.11) we conclude also that there exists $\hat{R}_Q > 0$ so small that the function $c : J(f) \setminus \overline{\mathrm{PS}^0(f)} \to \mathbb{R}$ is constant on each set

$$f_n^{-(n-1)}(B(\xi_1, \hat{R}_Q)) \cap (J(f) \setminus \overline{\mathrm{PS}^0(f)})$$

and on the set

$$B(\xi_1, \hat{R}_Q) \cap (J(f) \setminus \overline{\mathrm{PS}^0(f)}).$$

Taking $R_Q = R_{Q,\xi} \in (0, R)$ so small that

$$\overline{B(\xi_0, R_Q)} \subset B(\xi_0, R) \setminus Q$$

and

$$f_{\xi_1}^{-1}(B(\xi_0, R_Q)) \subset B(\xi_1, \hat{R}_Q),$$

and recalling that

$$\lim_{n\to\infty} u((f_{\xi_1}^{-1}(z))_n) = u(w) = \lim_{n\to\infty} u(\xi_n)$$

for all $z \in B(\xi_0, R) \setminus Q$, uniformly on $B(\xi_0, R_Q)$ (apply (6.10) along with Koebe's Distortion Theorem, I (Euclidean version) and the standard normality argument), we conclude from (6.16) and (6.14) that

$$u(z) = u_R(z) \tag{6.17}$$

for all $z \in B(\xi_0, R_Q) \cap (J(f) \setminus \overline{\mathrm{PS}^0(f)})$ and $u(z) - u_R(z)$ is locally constant throughout

$$(B(\xi_0, R) \setminus Q) \cap (J(f) \setminus \overline{\mathrm{PS}^0(f)}).$$

Suppose now that $0 < R^* \le R_1 \le R_2$. Since, by Theorem 2.1, $\mathrm{HD}(J(f)) > 1$ and $\overline{\mathrm{PS}^0(f)}$ is a nowhere dense subset of $J(f)$, $J(f)\setminus\overline{\mathrm{PS}^0(f)})$ is not contained in any countable union of real-analytic curves. It therefore follows from (6.17) that u_{R_2} restricted to $B(\xi_0, R_1) \setminus Q$ coincides with u_{R_1}. Thus, we can define a harmonic function $\tilde{u}_Q : \mathbb{C} \setminus Q \to \mathbb{R}$ by the formula $\tilde{u}_Q(z) = u_{|z|+1}(z)$ and it holds that

$$\tilde{u}_Q(z) = u(z) \tag{6.18}$$

for all $z \in B(\xi_0, R_Q) \cap (J(f) \setminus \overline{\mathrm{PS}^0(f)})$ and

Claim 1. $u(z) - \tilde{u}_Q(z)$ is locally constant throughout $(\mathbb{C} \setminus Q) \cap (J(f) \setminus \overline{\mathrm{PS}^0(f)})$.

Using condition (v), we thus conclude that

Claim 2. The function $\tilde{u}_{\hat{Q}} - \tilde{u}_Q$ is constant on each connected component of $\mathbb{C} \setminus (Q \cup \hat{Q})$.

Combining Claim 1 again and (6.16), we conclude that $\tilde{u}_Q - \tilde{u}_Q \circ f - \log|f'|$ is locally constant and harmonic on

$$(\mathbb{C} \setminus (Q \cup f^{-1}(Q))) \cap (J(f) \setminus \overline{\mathrm{PS}^0(f)}).$$

Using the fact that $\overline{\mathrm{PS}^0(f)}$ is nowhere dense in $J(f)$, we get the following.

Claim 3. The function $\tilde{u}_Q - \tilde{u}_Q \circ f - \log|f'|$ is constant on each connected component of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$ that intersects $J(f)$).

Our nearest goal now is to extend this claim to all connected components of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$. And indeed, consider S, a connected component of

$$\mathbb{C} \setminus f^{-1}(Q) = f^{-1}(\mathbb{C} \setminus Q).$$

Two connected components S_1 and S_2 of

$$\mathbb{C} \setminus (Q \cup f^{-1}(Q))$$

contained in S are called equivalent if $\overline{S_1} \cap \overline{S_2}$ is a non-degenerate segment of Q (since S is simply-connected, the other possibilities are that either $\overline{S_1} \cap \overline{S_2} = \emptyset$ or $\overline{S_1} \cap \overline{S_2}$ is a singleton contained in Q). A connected component S' of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$ is called tame if the function

$$\tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'|$$

is constant on S'. We shall prove the following.

Claim 4. Suppose S is a connected compact component of $\mathbb{C} \setminus f^{-1}(Q)$. If S_1 and S_2 are two arbitrary equivalent connected components of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$ contained in S and S_1 is tame, then so is S_2.

Proof. Let $\Delta = \bar{S_1} \cap \bar{S_2} \subset Q$. Perturbing Δ a little bit we can replace it by a closed segment $\hat{\Delta}$ with the following properties:

- $\hat{\Delta}$ has the same endpoints as Δ.
- $\mathrm{Int}\hat{\Delta} \subset S_2$.
- If $\hat{Q}$ is obtained from Q by replacing Δ by $\hat{\Delta}$, then there exists S_3, a connected component of $\mathbb{C} \setminus (\hat{Q} \cup f^{-1}(\hat{Q}))$ that has non-empty intersections with S_1 and S_2.

Consider $S_{3,1}$, a connected component of $S_1 \cap S_3$. Then

$$S_{3,1} \cup f(S_{3,1}) \subset \mathbb{C} \setminus (Q \cup \hat{Q}),$$

and it follows from Claim 2 that $\tilde{u}_{\hat{Q}} - \tilde{u}_Q$ and $\tilde{u}_{\hat{Q}} \circ f - \tilde{u}_Q \circ f$ are both constant on $S_{3,1}$. Since S_1 is tame, we thus conclude that

$$\tilde{u}_{\hat{Q}} - \tilde{u}_{\hat{Q}} \circ f - \log |f'|$$

is constant on $S_{3,1}$, and hence, on S_3. Let $S_{3,2}$ be a connected component of $S_3 \cap S_2$. As above, $\tilde{u}_Q - \tilde{u}_{\hat{Q}}$ and $\tilde{u}_Q \circ f - \tilde{u}_{\hat{Q}} \circ f$ are both constant on $S_{3,2}$. Therefore

$$\tilde{u}_Q - \tilde{u}_Q \circ f - \log |f'|$$

is constant on $S_{3,2}$ and hence, S_2 is tame. The proof of Claim 4 is complete.

Since each connected component S of

$$\mathbb{C} \setminus f^{-1}(Q) = f^{-1}(\mathbb{C} \setminus Q)$$

intersects the Julia set $J(f)$, the component S contains at least one tame, connected component of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$. Since in addition, any two connected components S' and S'' of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$ contained in S can be connected by a sequence $S' = S_1, S_2, \dots, S_k = S''$ of components of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$ contained in S such that any two consecutive are equivalent, we thus conclude from Claim 4 the following.

Claim 5. The function $\tilde{u}_Q - \tilde{u}_Q \circ f - \log|f'|$ is constant on each connected component of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$.

Define $\mathrm{Sing}_+(\tilde{u}_Q)$ and $\mathrm{Sing}_-(\tilde{u}_Q)$, the sets of singularities of $\tilde{u}_Q$ as follows.

$$\mathrm{Sing}_+(\tilde{u}_Q) = \{w \in \mathbb{C} : \limsup_{z \to w} \tilde{u}_Q(z) = +\infty\}.$$

and

$$\mathrm{Sing}_-(\tilde{u}_Q) = \{w \in \mathbb{C} : \liminf_{z \to w} \tilde{u}_Q(z) = -\infty\}.$$

It immediately follows from Claim 2 that

$$\mathrm{Sing}_+(\tilde{u}_Q) \cup \mathrm{Sing}_-(\tilde{u}_Q) \subset \{x\} \cup f(\mathrm{Crit}(f)). \tag{6.19}$$

Since the family of closures of connected components of $\mathbb{C} \setminus (Q \cup f^{-1}(Q))$ is locally finite, as an immediate consequence of Claim 5, we get the following.

$$f((\mathrm{Sing}_+(\tilde{u}_Q) \setminus f^{-1}(\infty)) \cup (\mathrm{Crit}(f) \setminus \mathrm{Sing}_-(\tilde{u}_Q))) \subset \mathrm{Sing}_+(\tilde{u}_Q) \tag{6.20}$$

and

$$f^{-1}(\mathrm{Sing}_+(\tilde{u}_Q)) \subset \mathrm{Sing}_+(\tilde{u}_Q) \cup \mathrm{Crit}(f). \tag{6.21}$$

Now, since the set $f(\mathrm{Crit}(f))$ is finite, for every point $w \in f(\mathrm{Crit}(f))$ there exists

$$z \in (\mathrm{Crit}(f) \setminus (\{x\} \cup f(\mathrm{Crit}(f)))) \cap \bigcup_{n=1}^{\infty} f^{-n}(w).$$

It then follows from (6.19) that $z \notin \mathrm{Sing}_-(\tilde{u}_Q)$. Hence, using (6.20), we see that $f(z) \in \mathrm{Sing}_+(\tilde{u}_Q)$. Since there is $n \geq 1$ such that $w = f^n(z)$, applying

(6.20) $n-1$ times, we conclude that $w \in \mathrm{Sing}_+(\tilde{u}_Q)$. So, we have proved the following.

$$f(\mathrm{Crit}(f)) \subset \mathrm{Sing}_+(\tilde{u}_Q). \tag{6.22}$$

Claim 6. The set $\bigcup_{n=0}^{\infty} f^n(f(\mathrm{Crit}(f))$ is finite.

Proof. Suppose on the contrary that this union is infinite. Since the set $f(\mathrm{Crit}(f))$ is finite, there thus exists $w \in f(\mathrm{Crit}(f))$ such that all the points $f^n(w)$, $n \geq 0$, are mutually distinct. If then follows from (6.22) and (6.20) that

$$\{f^n(w)\}_{n=0}^{\infty} \subset \mathrm{Sing}_+(\tilde{u}_Q).$$

Since, by (6.19), the set $\mathrm{Sing}_+(\tilde{u}_Q)$ is finite, we get a contradiction which finishes the proof.

Due to Non-Wandering Theorem [4] and classification of periodic components of the Fatou set $F(f)$ given in [3], as an immediate consequence of this claim, we get the following.

Claim 7. Either f has a superattracting periodic orbit or else $J(f) = \mathbb{C}$.

Consider now two cases.

Case 1. The elliptic map $f : \mathbb{C} \to \overline{\mathbb{C}}$ has a critical periodic point c.

By (6.22) and (6.20) both points c and $f(c)$ belong to $\mathrm{Sing}_+(\tilde{u}_Q)$ (the whole forward orbit of c does). Since $\mathrm{Sing}_+(\tilde{u}_Q) \cup \mathrm{Sing}_-(\tilde{u}_Q)$ is finite by (6.19), since $[c]$, the equivalence class of c with respect to the relation $\sim$, is infinite, there exists

$$w \in [c] \setminus (\mathrm{Sing}_+(\tilde{u}_Q) \cup \mathrm{Sing}_-(\tilde{u}_Q)).$$

There then exist two constants $C_1, C_2 \in \mathbb{R}$, $\lambda \in \Lambda$, and a sequence $\{w_n\}_{n=1}^{\infty}$ with the following properties:

(α) $\forall n \geq 1$ $w_n \notin \mathrm{Sing}_+(\tilde{u}_Q) \cup \mathrm{Sing}_-(\tilde{u}_Q)$,
(β) $\lim_{n\to\infty} w_n = w$ and $\lim_{n\to\infty}(w_n + \lambda) = c$,
(γ) $\tilde{u}_Q(w_n) - \tilde{u}_Q(f(w_n)) - \log|f'(w_n)| = C_1$, (Claim 5)
(δ) $\tilde{u}_Q(w_n + \lambda) - \tilde{u}_Q(f(w_n + \lambda)) - \log|f'(w_n + \lambda)| = C_2$, (Claim 5)
(ε) $\lim_{n\to\infty} \tilde{u}_Q(w_n + \lambda) = \infty$.

Since $f(w_n + \lambda) = f(w_n)$ and $f'(w_n + \lambda) = f'(w_n)$, (γ) and (δ) imply that $\tilde{u}_Q(w_n + \lambda) = \tilde{u}_Q(w_n) + C_2 - C_1$. Since $w \notin \mathrm{Sing}_+(\tilde{u}_Q)$, we conclude that

$$\overline{\lim}_{n\to\infty}\tilde{u}_Q(w_n + \lambda) = C_2 - C_1 + \overline{\lim}_{n\to\infty}\tilde{u}_Q(w_n) < \infty,$$

contrary to (ε). The Case 1 is ruled out.

Case 2. $J(f) = \mathbb{C}$

By Claim 6 the set $\mathbb{C} \setminus \overline{\mathrm{PS}^0(f)}$ is connected, and therefore, the locally constant function $c : \mathbb{C} \setminus \overline{\mathrm{PS}^0(f)} \to \mathbb{R}$ is constant, say equal to c. By Claim 6 there exists a sequence $\{z_n\}_1^\infty$ of points in $\mathbb{C} \setminus \overline{\mathrm{PS}^0(f)}$ with the following properties

$$\lim_{n\to\infty} z_n = \infty$$

and $f(z_n) = z_n$ for all $n \geq 1$. Since $\lim_{n\to\infty} f(z_n) = \infty$, it follows that

$$\lim_{n\to\infty} \mathrm{dist}(z_n, f^{-1}(\infty)) = 0,$$

and therefore $\lim_{n\to\infty} \log|f'(z_n)| = \infty$, contrary to (6.16) from which follows that $\log|f'(z_n)| = c(z_n) = c$. We are done. ■

We will prove the chain of implications (**2**) $\Rightarrow$ (**3**) $\Rightarrow$ (**4**)$\ldots$ $\Rightarrow$ (**6**) of Theorem 6.1.

We will frequently apply the following easy fact.

Lemma 6.6 *Every homeomorphism $h : X \to Y$, where X and Y are closed subsets of $\mathbb{C}$, is uniformly continuous with respect to the spherical metric on $\mathbb{C}$.*

Proof. If one of the sets X or Y is compact, then so is the other, and we are done. So, we may assume that neither X nor Y are compact. It suffices to show that the map $\hat{h} : X \cup \{\infty\} \to Y \cup \{\infty\}$, determined by the requirements $\hat{h}_{|X} = h$ and $\hat{h}(\infty) = \infty$, is continuous at ∞. To prove this suppose for the contrary that $\hat{h}$ is not continuous at ∞. This means that there exists a sequence $(x_n)_{n=0}^\infty \subset X$ converging to ∞ such that $(\hat{h}(x_n))_{n=0}^\infty$ does not converge to ∞. Passing to a subsequence, we may assume without lost of generality that the sequence $(\hat{h}(x_n))_{n=0}^\infty$ is bounded. But then its closure $\left(\overline{h(x_n)}\right)_{n=0}^\infty$ is compact. So, the set $h^{-1}\big((\overline{h(x_n)})_{n=0}^\infty\big)$ is compact, thus bounded. Since $(x_n)_{n=0}^\infty \subset h^{-1}(h(x_n)_{n=0}^\infty)$, the sequence $(x_n)_{n=0}^\infty$ is bounded, contrary to the assumption that it converges to ∞. ■

Using this lemma we get the following.

Proposition 6.7 *If a homeomorphism $h : J(f) \to J(g)$ satisfying condition (5) of Theorem 6.1 conjugates two critically pseudo non-recurrent elliptic functions f and g, then $h(\mathrm{Crit}(J(f))) = \mathrm{Crit}(J(g))$, $h(\Omega(f)) = \Omega(g)$, $h(f^{-1}(\infty)) = g^{-1}(\infty)$ and $h(\mathrm{PC}(f)) = \mathrm{PC}(g)$.*

The implications (**2**) $\Rightarrow$ (**3**) and (**3**) $\Rightarrow$ (**4**) are obvious. For the sake of completeness we provide now an easy proof of the implication (**4**) $\Rightarrow$ (**5**). Suppose that $x \in J(f)$ is a periodic point of period p and

$$|(f^p)'(x)| \neq |(g^p)'(h(x))|.$$

Without losing generality we may assume that

$$|(f^p)'(x)| < |(g^p)'(h(x))|.$$

Fix

$$|(f^p)'(x)| < \mu < \lambda < |(g^p)'(h(x))|.$$

Let U be a neighborhood of x such that both continuous inverse branches $f_x^{-p} : U \to U$ of f^p and $g_{h(x)}^{-p} : h(U \cap J(f)) \to h(U \cap J(f))$ of g^p sending respectively x to x and $h(x)$ to $h(x)$ are well defined. Taking U sufficiently small we may assume that $|f_x^{-pn}(z) - x| \geq \mu^{-n}$ and $|g_{h(x)}^{-pn}(w) - h(x)| \leq \lambda^{-n}$ for all $n \geq 1$, $z \in U$ and $w \in h(U)$. Hence

$$\frac{|f_x^{-pn}(z) - x|}{|h(f_x^{-pn}(z)) - h(x)|} \geq \frac{\mu^{-n}}{\lambda^{-n}} = \left(\frac{\lambda}{\mu}\right)^n \to \infty$$

if $n \to \infty$. The implication (4) $\Rightarrow$ (5) is proved.

In order to show that (**5**) $\Rightarrow$ (**6**) we need first the following version of the closing lemma (or shadowing lemma).

Lemma 6.8 *Fix $s > 0$. Then for all $0 < \rho_2 < s$ there exists $\rho_1 > 0$ an integer $n_1 \geq 1$ such that for every $n \geq n_1$ if $f^n(x) \in J(f) \setminus B_s(\overline{\mathrm{PC}(f)}, s)$ and if $f^n(x) \in B_s(x, \rho_1)$, then there exists $y \in J(f)$ such that $f^n(y) = y$ and $|f^j(x) - f^j(y)|_* \leq \rho_2$ for all $0 \leq j \leq n - n_1$ and $|y - f^n(x)|_* < \rho_2$.*

Proof. It easily follows from Corollary 2.15, Koebe's Distortion Theorem, I (spherical version) and the normal family argument that

$$\lim_{n\to\infty} \sup\{\text{diam}_s(P_n)\} = 0,$$

where P_n ranges over all connected components of $f^{-n}(\overline{B_s(z,\rho_2)})$, $z \in J(f) \setminus B_s(\overline{\text{PC}(f)}, s)$. Take n_1 so large that $\text{diam}_s(P_n) < \rho_2/2$ for all $n \geq n_1$. Take $\rho_1 < \rho_2/2$. Let B_n, $n \geq n_1$, be the connected component of $f^{-n}(\overline{B_s(f^n(x),\rho_2)})$ containing x. Let $f_x^{-n} : \overline{B_s(f^n(x),\rho_2)} \to B_n$ be the holomorphic inverse branch of f^n sending $f^n(x)$ to x. We then have

$$f_x^{-n}(\overline{B_s(f^n(x),\rho_2)}) \subset \overline{B_s(x,\rho_2/2)} \subset \overline{B_s(f^n(x),\rho_1+\rho_2/2)} \subset \overline{B_s(f^n(x),\rho_2)}.$$

Hence by the Brower fixed point theorem there exists $y \in \overline{B_s(f^n(x),\rho_2)}$ such that $f^{-n}(y) = y$ which implies that $f^n(y) = y$. Finally note that

$$|f_x^{-j}(f^n(x)) - f_x^{-j}(f^n(y))|_* = |f_x^{-j}(f^n(x)) - f_x^{-j}(f^n(y))|_* \leq \rho_2/2 \leq \rho_2$$

for all $j \geq n_1$. ■

By topological exactness of f the set of transitive points of f is dense in $J(f)$. Choose one such a point, say x. For every $z \in J(f)\setminus(\text{Crit}(f)\cup f^{-1}(\infty))$ define

$$\eta(z) = \log|g'(h(z))| - \log|f'(z)|.$$

Note that $x \notin \text{Crit}(f) \cup f^{-1}(\infty)$ and for every $n \geq 1$, set

$$u(f^n(x)) = \sum_{j=0}^{n-1} \eta(f^j(x)). \tag{6.23}$$

Lemma 6.9 *Suppose that condition (5) of Theorem 6.1 holds. Then for every $s > 0$ the function u restricted to the set $(J(f)\setminus B_s(\overline{\text{PC}(f)}, s))\cap\{f^n(x) : n \geq 0\}$ is uniformly continuous with respect to the spherical metric.*

Proof. By Lemma 6.6 and Proposition 6.7 there exists $s' > 0$ such that

$$h(J(f) \setminus B_s(\overline{\text{PC}(f)}, s)) \subset J(g) \setminus B_s(\overline{\text{PC}(g)}, s').$$

Fix $\delta \in (0, \min\{s, s'\})$. By Lemma 6.6 there exists $0 < \rho_2 < \delta$ so small that $|h(z) - h(w)|_* \leq \delta$ whenever $|z - w|_* \leq \rho_2$. Choose ρ_1 and n_1 according to Lemma 6.8 applied to the function f. Consider two points $f^n(x)$ and $f^m(x)$ in $J(f) \setminus B_s(\overline{\text{PC}(f)}, s)$ such that $n \geq m$ and $|f^n(x) - f^m(x)|_* < \rho_1$. Then in view of Lemma 6.8 there exists a point $y \in J(f)$ such that $f^{n-m}(f^m(y)) = f^m(y)$,

$$|f^{m+j}(x) - f^{m+j}(y)|_* < \rho_2$$

for all $j = 0, 1, \dots, n-m-n_1$, and $|f^n(x) - f^n(y)|_* < \rho_2$. Since by the assumption $\sum_{j=m}^{n-1} \eta(f^j(y)) = 0$, we therefore get

$$\begin{aligned} u(f^n(x)) - u(f^m(x)) &= \sum_{j=m}^{n-1} \eta(f^j(x)) = \sum_{j=m}^{n-1} \big(\eta(f^j(x)) - \eta(f^j(y))\big) \\ &= \sum_{j=m}^{n-1} (\log|g'(h(f^j(x)))| - \log|g'(h(f^j(y)))|) \\ &\quad - (\log|f'(f^j(x))| - \log|f'(f^j(y))|) \\ &= \log\left|\frac{(g^{n-m})'(h(f^m(x)))}{(g^{n-m})'(h(f^m(y)))}\right| - \log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|. \end{aligned}$$

We want $u(f^n(x))$ and $u(f^m(x))$ to be close one to the other if $f^n(x)$ and $f^m(x)$ are. For this it suffices to know that each term $\log\left|\frac{(g^{n-m})'(h(g^m(x)))}{(g^{n-m})'(h(g^m(y)))}\right|$ and $\log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|$ is small. But for the later the term this follows from the Koebe's Distortion Theorem, I (spherical version) since $|f^m(x) - f^m(y)|_* < \rho_2 < \delta$ and the inverse branch $f^{-(n-m)}_{f^m(x)}$ sending $f^n(x)$ to $f^m(x)$ is defined on $B_s(f^n(x), s)$. For the former term, since $|f^m(x) - f^m(y)|_* < \rho_2$ we have

$$|g^m(h(x)) - g^m(h(x))|_* = |h(f^m(x)) - h(f^m(y))|_* \le \delta,$$

and note that the inverse branch $g^{-(n-m)}_{g^m(h(x))}$ sending $g^n(x)$ to $g^m(x)$ is defined on $B_s(g^n(x), s')$. The proof is finished. ■

Consequently the function u extends continuously to each set

$$J(f) \setminus B_s(\overline{\mathrm{PC}(f)}, s), \quad s > 0,$$

and therefore to the set $J(f) \setminus \overline{\mathrm{PC}(f)}$.

Lemma 6.10 *The functions* $\log|f'(z)|$ *and* $\log|g'(h(z))|$ *are cohomologous in the class of continuous functions on* $(J(f) \setminus \overline{\mathrm{PC}(f)}) \cap f^{-1}(J(f) \setminus \overline{\mathrm{PC}(f)})$. *More precisely there exists a continuous function* $u : J(f) \setminus \overline{\mathrm{PC}(f)} \to \mathbb{R}$ *such that*

$$\log|g'(h(z))| - \log|f'(z)| = u(f(z)) - u(z)$$

for all $z \in (J(f) \setminus \overline{\mathrm{PC}(f)}) \cap f^{-1}(J(f) \setminus \overline{\mathrm{PC}}(f))$.

Proof. From (6.23) we get

$$\eta(f^n(x)) = u(f^n(x)) - u(f^{n-1}(x)).$$

Since the set $\{f^n(x) : n \geq 1\}$ is dense in

$$(J(f) \setminus \overline{\mathrm{PC}(f)}) \cap (J(f) \setminus f^{-1}(\overline{\mathrm{PC}(f)}))$$

and all the functions η, u, $u \circ f$ are continuous in this set, the lemma is proved. ■

Proof of the implication (**5**) ⇒ (**6**). Let

$$T(f) = \{z \in J(f) : \limsup_{j\to\infty} \mathrm{dist}_s(f^j(z), \overline{\mathrm{PC}(f)}) > 0\}$$

and

$$T_n(f) = \{z \in J(f) : \limsup_{j\to\infty} \mathrm{dist}_s(f^j(z), \overline{\mathrm{PC}(f)}) > 2/n\},\ n \in \mathbb{N}.$$

Then $T_n(f) \subset T(f)$. By Lemma 6.6 and Proposition 6.7 there exists $k_n \geq 1$ such that $\mathrm{dist}_s(x, \overline{\mathrm{PC}(f)}) \geq 2/n$ implies $\mathrm{dist}_s(h(x), \overline{\mathrm{PC}(f)}) \geq 2/k_n$. In particular

$$h(T_n(f)) \subset T_{k_n}(g).$$

Fix now $z \in T_n(f)$. Then there exists an infinite sequence $n_j = n_j(z)$ such that $\mathrm{dist}_s(f^{n_j}(z), \overline{\mathrm{PC}(f)} \geq 2/n$ for all $j \geq 1$. Applying now Lemma 6.9 and Lemma 6.10 we see that there exists a constant $Q_n \geq 1$ such that

$$Q_n^{-1} \leq \frac{|(g^{n_j})'(h(z))|}{|(f^{n_j})'(z)|} \leq Q_n. \tag{6.24}$$

In view of uniform continuity of h there exists $\gamma_n \leq 1/n$ such that

$$h(J(f) \cap B_s(x, \gamma_n)) \subset B_s(h(x), 1/k_n)$$

for all $x \in J(f)$. By the choice of the sequence $n_j = n_j(z)$ for every $j \geq 1$ there exists an inverse branch

$$f_z^{-n_j} : B_s(f^{n_j}(z), 2/n) \to \overline{\mathbb{C}}$$

sending $f^{n_j}(z)$ to z. Set $r_j(z) = K^{-1}|(f_z^{-n_j})^*(f^{n_j}(z))|\gamma_n$. Then by Lemma 1.3 and Corollary 2.15,

$$B_s(z, r_j(z)) \supset f_z^{-n_j}(B_s(f^{n_j}(z), K^{-2}\gamma_n)),$$

and therefore

$$\begin{aligned} m_f(B_s(z, r_j(z))) &\ge K^{-h_f}|(f_z^{-n_j})^*(f^{n_j}(z))|^{h_f} m_f(B_s(f^{n_j}(z), K^{-2}\gamma_n) \\ &\ge M_n K^{-h_f}|(f_z^{-n_j})^*(f^{n_j}(z))|^{h_f}, \end{aligned} \tag{6.25}$$

where $M_n = \inf\{m_f(B_s(x, K^{-2}\gamma_n) : x \in J(f)\} > 0$. Similarly, by Lemma 1.3 and Corollary 2.15, $B_s(z, r_j(z)) \subset f_z^{-n_j}(B_s(f^{n_j}(z), \gamma_n))$. Hence

$$\begin{aligned} h(J(f) \cap B_s(z, r_j(z))) &\subset h(f_z^{-n_j}(B_s(f^{n_j}(z), \gamma_n) \cap J(f))) \\ &= g_{h(z)}^{-n_j}(h(B_s(f^{n_j}(z), \gamma_n) \cap J(f))) \\ &\subset g_{h(z)}^{-n_j}(B_s(h(f^{n_j}(z)), 1/k_n))) \\ &= g_{h(z)}^{-n_j}(B_s(g^{n_j}(h(z)), 1/k_n)). \end{aligned}$$

Therefore using (6.24), (6.25) and the Koebe's Distortion Theorem II (spherical version), we get

$$\begin{aligned} m_g \circ h(B_s(z, r_j(z))) &\le m_g(g_{h(z)}^{-n_j}(B_s(g^{n_j}(h(z)), 1/k_n))) \\ &\le K^{h_g}|(g_{h(z)}^{-n_j})^*(g^{n_j}(h(z))|^{h_g} m_g(B_s(g^{n_j}(h(z)), 1/k_n)) \\ &\le K^{h_g}|(g^{n_j})^*(h(z))|^{-h_g} \\ &\le K^{h_g} Q_n^{h_g}|(f^{n_j})^*(z)|^{-h_f+(h_f-h_g)} \\ &\le M_n^{-1} K^{h_f+h_g} Q_n^{h_g} m_f(B_s(z, r_j(z)))|(f^{n_j})^*(z)|^{h_f-h_g}. \end{aligned}$$

If $h_f - h_g < 0$, then it would follow from Lemma 1.5 and the fact that $\lim_{j\to\infty}|(f^{n_j})'(z)| = \infty$ (it implies that $\lim_{j\to\infty}|(f^{n_j})^*(z)| = \infty$), that $m_g \circ h(T_n(f)) = 0$ for every $n \ge 1$. Since

$$\bigcup_{n\ge 1} T_n(f) = T(f),$$

it would imply that $m_g(h(T(f)) = 0$. But since $h(\overline{\mathrm{PC}(f)}) = \overline{\mathrm{PC}(g)}$, using uniform continuity of h and h^{-1}, we conclude that $h(T(f)) = T(g)$. Hence $m_g(T(g)) = 0$ which would contradict Theorem 4.28. Thus for every $n \ge 1$ and every $z \in T_n(f)$

$$m_g \circ h(B(z, r_j(z))) \le M_n^{-1} K^{2h_f} Q_n^{h_g} m_f(B_s(z, r_j(z)).$$

Therefore, applying Lemma 1.5 we conclude that $m_g \circ h|_{T_n(f)}$ is absolutely continuous with respect to $m_f|_{T_n(f)}$ for every $n \ge 1$. Since

$$\bigcup_{n\ge 1} T_n(f) = T(f),$$

this implies that $m_g \circ h|_{T(f)}$ is absolutely continuous with respect $m_f|_{T(f)}$. Since

$$m_g \circ h(\overline{\mathrm{PC}(f)}) = m_g(\overline{\mathrm{PC}(f)}) = 0$$

and

$$m_f(\overline{\mathrm{PC}(f)}) = 0,$$

we obtain that $m_g \circ h$ is absolutely continuous with respect to m_f. By symmetry $m_f \circ h^{-1}$ is absolutely continuous with respect to m_g and consequently the measure $m_g \circ h$ and m_f are equivalent. The proof of the implication (5)⇒(6) is finished. ■

We are left to establish the implication (**6**) ⇒ (**1**). As the first step, we shall prove the following

Lemma 6.11 *If condition (6) of Theorem 6.1 is satisfied, then* $h : J(f) \to J(g)$ *extends to a real-analytic endomorphism from a neighborhood of* $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ *onto a neighborhood of* $J(g) \setminus \overline{\mathrm{PS}^0(g)}$.

Proof. In view of Theorem 6.5 (d) there exist $n \geq 1$ and $z \in J(g) \setminus \mathrm{PS}_-(g)$ such that

$$det(\nabla(D_{\mu_g} \circ g^n)(z), \nabla(D_{\mu_g})(z)) \neq 0.$$

Therefore, using Corollary 6.4, we conclude that there exists an open set $W \subset \mathbb{C} \setminus \bigcup_{j=0}^{n} g^{-j}(\infty)$ containing z and such that

$$det(\nabla(D_{\mu_g} \circ g^n)(w), \nabla(D_{\mu_g})(w)) \neq 0 \tag{6.26}$$

for all $w \in W$. Since the measures $m_f \circ h^{-1}$ and m_g are equivalent, the ergodic measures $\mu_f \circ h^{-1}$ and μ_g coincide up to a multiplicative constant. Thus

$$D_{\mu_f} \circ f^k = D_{\mu_g} \circ g^k \circ h \tag{6.27}$$

throughout $J(f) \setminus \bigcup_{j=0}^{k} f^{-j}(\overline{\mathrm{PS}^0(f)})$ for every $k \geq 0$. Define

$$F(x) = (D_{\mu_f}(x), D_{\mu_f} \circ f^k(x))$$

and

$$G(y) = (D_{\mu_g}(y), D_{\mu_g} \circ g^k(y))$$

for $x \in U$, an open neighborhood of $J(f) \setminus \bigcup_{j=0}^{k} f^{-j}(\overline{\mathrm{PS}^0(f)})$ and $y \in V$, an open neighborhood of $J(g) \setminus \bigcup_{j=0}^{k} g^{-j}(\overline{\mathrm{PS}^0(g)})$. We may of course assume that $W \subset U$ (as a matter of fact W had to be chosen after U was chosen) and, because of (6.26) that G is invertible on W. By (6.27), $F(h^{-1}(z)) = G(z)$, and therefore, there exists $U_z \subset U$, an open neighborhood of $h^{-1}(z)$, such that $F(U_z) \subset G(W)$. Hence, the map $G^{-1} \circ F$ is well-defined on U_z, and, because of (6.27) again,

$$G^{-1} \circ F(x) = h(x)$$

for all $x \in J(f) \cap U_z$. Now we consider ξ, an arbitrary point in $J(f) \setminus \overline{\mathrm{PC}(f)}$. Since $f : J(f) \to J(f)$ is topologically transitive, there exists $k \geq 0$ and $\hat{\xi} \in U_z \cap f^{-k}(\xi)$. Then there exists $r_\xi > 0$ so small that $r_\xi < \frac{1}{2}\mathrm{dist}(\xi, \overline{\mathrm{PC}(f)})$ and $f_{\hat{\xi}}^{-k}(B(\xi, r_\xi)) \subset U_z$. So, the map

$$g^k \circ (G^{-1} \circ F) \circ f_{\hat{\xi}}^{-k}$$

is well-defined on $B(\xi, r_\xi)$, real analytic, and since (h conjugates f and g)

$$g^k \circ (G^{-1} \circ F) \circ f_{\hat{\xi}}^{-k}$$

restricted to $J(f) \cap B(\xi, r_\xi)$ coincides with h. Now, since no open subset of $J(f)$ is contained in a countable union of real-analytic curves, the same argument as in the end of the proof of Lemma 6.2, shows that all the maps

$$g^k \circ (G^{-1} \circ F) \circ f_{\xi}^{-k}$$

glue together on the balls $B(\xi, r_\xi/2)$ to form a real-analytic map from $\bigcup_{z \in J(f) \setminus \overline{\mathrm{PS}^0(f)}} B(z, r_z/2)$ onto an open neighborhood of $J(g) \setminus \overline{\mathrm{PS}^0(g)}$. This map restricted to $J(f)$ coincides with h. ■

Now we shall prove the next step.

Lemma 6.12 *If the topological conjugacy $h : J(f) \to J(g)$ has a real- analytic extension on an open neighborhood of $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ in $\mathbb{C}$, then it has a holomorphic conformal extension on an open neighborhood of $J(f) \setminus \overline{\mathrm{PS}^0(f)}$.*

Proof. Let $H : U \to \mathbb{C}$ be a real analytic extension of h on an open neighborhood U of $J(f) \setminus \overline{\mathrm{PS}^0(f)}$. Hence, the complex dilatation $\mu_H = \frac{\overline{\partial} H}{\partial H}$ is well-defined throughout U (decreasing it if necessary). Since $H \circ f = g \circ H$, we get $H = g \circ H \circ f_z^{-1}$ on some ball $B(f(z), r_z)$ and all

$$z \in J(f) \setminus (\overline{\mathrm{PS}^0(f)} \cup f^{-1}(\overline{\mathrm{PS}^0(f)})).$$

Hence, since f and g are conformal,

$$\mu_H(f(z)) = \mu_{g \circ H}(z) \left(\frac{f'(z)}{|f'(z)|} \right)^2 = \mu_H(z) \left(\frac{f'(z)}{|f'(z)|} \right)^2. \qquad (6.28)$$

It follows from this equation that if $\mu_H(w) = 0$ at some point $w \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$, then μ_H vanishes everywhere on $f^{-1}(w)$ and $f^{-1}(w) \cap \overline{\mathrm{PS}^0(f)} = \emptyset$. So, by induction, μ_H vanishes on $\bigcup_{n=0}^\infty f^{-n}(w)$. Since, by transitivity of f, this set is dense in $J(f)$ and since μ_H is continuous on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$, we thus conclude that μ_H vanishes throughout $J(f) \setminus \overline{\mathrm{PS}^0(f)}$. So, if $\mu_H(z) \neq 0$ at some point $z \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$, then μ_H does not vanish anywhere on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$. It then firstly follows from (6.28) that the modulus of μ_H is constant on orbits of f, and secondly, that

$$\frac{\mu_H}{|\mu_H|}(f(z)) = \frac{\mu_H}{|\mu_H|}(z)\left(\frac{f'(z)}{|f'(z)|}\right)^2, \quad z \in J(f) \setminus (\overline{\mathrm{PS}^0(f)} \cup f^{-1}(\overline{\mathrm{PS}^0(f)})).$$

Thus $\frac{\mu_H}{|\mu_H|}$ is a continuous invariant line field on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$, contrary to Theorem 6.5 (c). So, $\mu_H(z) = 0$ for all $z \in J(f) \setminus \overline{\mathrm{PS}^0(f)}$. Since μ_H is real-analytic on U and since no non-empty open subset of $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ is contained in a countable union of real-analytic curves, we conclude that μ_H vanishes on some neighborhood of $J(f) \setminus \overline{\mathrm{PS}^0(f)}$, meaning that H is conformal on this neighborhood. We are done. ■

Lemma 6.13 *Suppose that the topological conjugacy $h : J(f) \to J(g)$ has a holomorphic conformal extension on a neighborhood $U_f \subset \mathbb{C} \setminus \overline{\mathrm{PS}^0(f)}$ of $J(f) \setminus \overline{\mathrm{PS}^0(f)}$, where U_f has the property that for each point $z \in U_f$ there exists $r_z > 0$ such that $J(f) \cap B_e(z, r_z) \neq \emptyset$ and $B_e(z, r_z) \subset U_f$, and $h^{-1} : J(g) \to J(f)$ has a conformal extension on a neighborhood $U_g \subset \mathbb{C} \setminus \overline{\mathrm{PS}^0(g)}$ of $J(g) \setminus \overline{\mathrm{PS}^0(g)}$, where U_g has the property that for each point $z \in U_g$ there exists $r_z > 0$ such that $J(g) \cap B_e(z, r_z) \neq \emptyset$ and $B_e(z, r_z) \subset U_g$, then h extends to an affine map $(az + b)$ on $\mathbb{C}$ and this is a conjugacy between $f : \mathbb{C} \to \overline{\mathbb{C}}$ and $g : \mathbb{C} \to \overline{\mathbb{C}}$.*

Proof. Let $H : U_f \to \mathbb{C}$ be the holomorphic extension of the conjugacy $h : J(f) \to J(g)$ coming from the hypothesis of the lemma. Shrinking U_f if necessary we may assume without loss of generality that H maps bounded subsets of U_f onto bounded subsets of $\mathbb{C}$. We shall prove the following.

Claim. If B is a round open ball contained in U_f and $f_*^{-n} : B \to \mathbb{C}$ is a holomorphic inverse branch of f^n such that $f_*^{-n}(B) \subset U_f$, then

$$g^n \circ H \circ f_*^{-n} : B \to \hat{\mathbb{C}}$$

coincides with H.

Proof. Fix $z \in B$ and let $r_z > 0$ come from the hypothesis of our lemma. We have $w = f_*^{-n}(z) \in U_f$ and $B_e(z, r_z) \cap \overline{\text{PS}^0(f)} = \emptyset$. There thus exists a unique holomorphic inverse branch $f_w^{-n} : B_e(z, r_z) \to \mathbb{C}$ of f^n sending z to w. Since f_*^{-n} and f_w^{-n} agree on $B \cap B_e(z, r_z)$, they glue together to a holomorphic function $F : B \cup B_e(z, r_z) \to \mathbb{C}$. Since

$$(J(f) \setminus \overline{\text{PS}^0(f)}) \cap B_e(z, r_z) \neq \emptyset,$$

there exists a round open ball $\hat{B} \subset B_e(z, r_z)$ such that

$$\hat{B} \cap (J(f) \setminus \overline{\text{PS}^0(f)}) \neq \emptyset$$

and $f_w^{-n}(\hat{B}) \subset U_f$. Therefore,

$$g^n \circ H \circ f_w^{-n}|_{\hat{B} \cap J(f)} = h|_{\hat{B} \cap J(f)}.$$

Consequently $g^n \circ H \circ f_w^{-n}|_{\hat{B}} = h|_{\hat{B}}$. Thus $F|_{\hat{B}} = h|_{\hat{B}}$. Since

$$\hat{B} \subset B_e(z, r_z) \subset B_e(z, r_z) \cup B \subset U_f,$$

we therefore conclude that $F = H|_{B \cup B_e(z, r_z)}$. So,

$$g^n \circ H \circ f_*^{-n} = F|_B = H|_B.$$

We are done. ■

Consider the family Φ of all pairs (V, ϕ) with the following properties:

(a) V is an open subset of $\mathbb{C}$ containing U and $\phi : V \to \mathbb{C}$ is a holomorphic function mapping bounded subsets of V onto bounded subsets of $\mathbb{C}$.
(b) $\phi|_{U_f} = H$.
(c) For all $z \in \mathbb{C}$, $r > 0$, and integers $n \geq 0$ if $B_e(f^n(z), r) \subset V$,

$$B_e(f^n(z), r) \cap \bigcup_{j=1}^{n} f^j(\text{Crit}(f)) = \emptyset,$$

and $f_z^{-n}(B_e(f^n(z), r)) \subset U_f$, then ϕ and $g^n \circ H \circ f_z^{-n}$ coincide on $B_e(f^n(z), r)$.

First we shall argue that $(U_f, H) \in \Phi$. Conditions (a) and (b) are obvious. Condition (c) follows directly from the Claim. Thus $(U_f, H) \in \Phi$ and, in particular, $\Phi \neq \emptyset$. We partially order the set Φ by declaring that $(V, \phi) \leq (W, \psi)$ if and only if $V \subset W$ and $\psi|_V = \phi$. Now, if $(V_\lambda, \phi_\lambda)_{\lambda \in \Lambda}$ is a linearly ordered subset of Φ, then setting $V := \bigcup_{\lambda \in \Lambda} V_\lambda$, the formula $\phi(z) = \phi_\lambda(z)$

if $z \in V_\lambda$, defines well a holomorphic map $\phi : V \to \mathbb{C}$. It is easy to see that $(V, \phi) \in \Phi$ and $(V_\lambda, \phi_\lambda) \le (V, \phi)$ for all $\lambda \in \Lambda$. So, Kuratowski-Zorn Lemma applies to give a maximal element $(W, \psi) \in \Phi$.

We shall show that $W \supset \mathbb{C} \setminus \Omega(f)$. Indeed, suppose on the contrary that

$$W \not\supset \mathbb{C} \setminus \Omega(f).$$

Then $\partial W \setminus \Omega(f) \ne \emptyset$ and $(\partial W \cap W) \setminus \Omega(f) = \emptyset$. Fix an arbitrary point $y \in \partial W \setminus \Omega(f)$. Since $\mathrm{PC}_c(f)$ is bounded, since the points of $\mathrm{Crit}_\infty(f)$ escapes to ∞, since $f(\mathrm{Crit}(f))$ is finite and since the set of the repelling fixed points of f (lying in the Julia set of f) is unbounded, there exists a, a repelling fixed point of f belonging to $J(f) \setminus \overline{\mathrm{PS}^0(f)}$. Take $\varepsilon > 0$ so small that $B_e(a, 2\varepsilon) \subset U_f$. If $y \in J(f)$, take $\delta > 0$ ascribed to $\varepsilon > 0$ according to Theorem 2.13. If $y \notin J(f)$ take $\delta > 0$ so small that

$$B_e(y, 2\delta) \cap J(f) = \emptyset$$

and for every $k \ge 0$ the Euclidean diameter of each connected component of the set $f^{-k}(B_e(y, 2\delta))$ is less than ε (this is possible since f has no Siegel disks, nor Herman rings). Since $U_f \cap J(f) \ne \emptyset$, there exists $n \ge 0$ such that $f^{-n}(y) \cap U_f \ne \emptyset$. Fix $x \in U_f \cap f^{-n}(y)$. Since the set

$$\bigcup_{j=1}^{n} f^j(\mathrm{Crit}(f))$$

is finite, there exists $R \in (0, \delta]$ so small that

$$B_e(y, R) \cap \bigcup_{j=1}^{n} f^j(\mathrm{Crit}(f)) \subset \{y\}.$$

Take $r \in (0, R]$ so small that U_x, the connected component of $f^{-n}(B_e(y, r))$ containing x, is contained in U_f. Let Δ_1 be a closed ray emanating from y. Fix $f_1^{-n} : B_e(y, r) \setminus \Delta_1 \to \mathbb{C}$, an arbitrary holomorphic inverse branch of f^n whose range is contained in U_x. Then the composition

$$g^n \circ H \circ f_1^{-n} : B_e(y, r) \setminus \Delta_1 \to \mathbb{C}$$

is a well-defined meromorphic map with a closed (relative to $B_e(y, r) \setminus \Delta_1$) countable set being the closure of essential singularities. It follows from condition (c) that $g^n \circ H \circ f_1^{-n}$ coincides with ψ on each connected component of $W \cap \big(B_e(y, r) \setminus \Delta_1\big)$. Now, let Δ_2 be a different closed ray emanating from y and let

$$f_2^{-n} : B_e(y, r) \setminus \Delta_2 \to \mathbb{C}$$

be the corresponding inverse branch. So, $g^n \circ H \circ f_1^{-n}$, $g^n \circ H \circ f_2^{-n}$ and ψ, all coincide on connected components of

$$W \cap B_e(y,r) \setminus \big(\Delta_1 \cup \Delta_2\big).$$

Consequently, $g^n \circ H \circ f_1^{-n}$ and $g^n \circ H \circ f_2^{-n}$ glue to a holomorphic function ψ_y on $B_e(y,r) \setminus E$ that coincides with ψ on $B_e(y,r) \cap W$ and where E is a relatively closed countable subset of $B_e(y,r)$ such that

$$\overline{\lim}_{a \to b} |\psi_y(a)| = +\infty$$

for all $b \in E$, where a converges to b in $B_e(y,r) \setminus E$. If $\partial W \cap B_e(y,r) \subset E$, then the set $\partial W \cap B_e(y,r)$ is a countable and closed (as E is). We can therefore find an isolated point $\xi_1 \in \partial W \cap B_e(y,r)$. Consequently, there exists $r_2 > 0$ so small that $B_e(\xi_1, r_2) \setminus \{\xi_1\} \cap W \neq \emptyset$. So, the set $\psi_y(B_e(\xi_1, r_2) \cap W)$ is bounded, contrary to the fact that $\xi_1 \in E$. Thus $\partial W \cap B_e(y,r) \setminus E \neq \emptyset$, then

$$\xi \in \partial W \cap B_e(y,r) \setminus E$$

and let $r_1 > 0$ be so small that $B_e(\xi, 2r_1) \subset B_e(y,r)$ and $B_e(\xi, 2r_1) \cap E = \emptyset$. Thus $\psi_y|_{B_e(\xi,2r_1)}$ is holomorphic and $\psi_y(B_e(\xi, r_1))$ is bounded. Set then $G = B_e(\xi, r_1)$. Thus, we get a holomorphic function $\hat{\psi}$ defined on $W \cup G$ whose restriction to W coincides with ψ and whose restriction to G equals ψ_y. We claim that $(W \cup G, \hat{\psi}) \in \Phi$. Indeed, since $\hat{\psi}(G) = \psi_y(G)$ is a bounded set and since ψ maps bounded subsets of W onto bounded subsets, conditions (a) and (b) are clearly satisfied.

In order to prove condition (c) suppose z, r_3 (corresponding to r) and n_3 (corresponding to n) are taken as required in this condition. We may assume without loss of generality that $G \cap B_e(f^{n_3}(z), r_3)) \neq \emptyset$. Fix $k \geq 0$ so large that $f^k(B_e(a,\varepsilon))$ contains the connected component of $f^{-n_3}(G \cup B_e(f^{n_3}(z), r_3))$ containing z. Since the set

$$\bigcup_{j=1}^{n_3+k} f^j(\mathrm{Crit}(f))$$

is finite, there exists an open simply-connected $B \subset G$ such that $B \cap W \neq \emptyset$, $B \cap B_e(f^{n_3}(z), r_3) \neq \emptyset$ and

$$B \cap \bigcup_{j=1}^{n_3+k} f^j(\mathrm{Crit}(f)) = \emptyset.$$

There thus exists $f_*^{-(n_3+k)}: B \to \mathbb{C}$, a holomorphic branch of $f^{-(n_3+k)}$ such that

$$f_*^{-(n_3+k)}(B) \cap B_e(a, \varepsilon) \neq \emptyset$$

and $f^k \circ f_*^{-(n_3+k)}$ restricted to $B \cap B_e(f^{n_3}(z) \cap r_3)$ coincides with $f_z^{-n_3}$ restricted to the same intersection. Since $B \subset G \subset B_e(f^n(y), \delta)$, $\operatorname{diam}(f_*^{-(n+k)}(B)) < \varepsilon$, and therefore

$$f_*^{-(n_3+k)}(B) \subset B_e(a, 2\varepsilon) \subset U_f.$$

It therefore follows from condition (c) applied to the pair (W, ψ) that

$$g^{n_3+k} \circ H \circ f_*^{-(n_3+k)}$$

coincides with ψ (hence $\hat{\psi}$) on $B \cap W$. So,

$$g^{n_3+k} \circ H \circ f_*^{-(n_3+k)}$$

and $\hat{\psi}$ coincide on B. Now, since

$$f_z^{-n_3}(B \cap B_e(f^{n_3}(z), r_3)) \subset U_f$$

and since $f_z^{-n_3}(B) \subset U_f$, we conclude from the Claim that

$$g^k \circ H \circ f_*^{-(n_3+k)} \circ f^{n_3}$$

coincides with H on $f_z^{-n_3}(B \cap B_e(f^{n_3}(z), r_3))$. Thus,

$$\begin{aligned}
&\hat{\psi}|_{B \cap B_e(f^{n_3}(z), r_3)} = g^{n_3+k} \circ H \circ f_*^{-(n_3+k)}|_{B \cap B_e(f^{n_3}(z), r_3)} \\
&= g^{n_3} \circ g^k \circ H \circ f_*^{-(n_3+k)} \circ f^{n_3}|_{f_z^{-n_3}(B \cap B_e(f^{n_3}(z), r_3))} \circ f_z^{-n_3}|_{B \cap B_e(f^{n_3}(z), r_3)} \\
&= g^{n_3} \circ H \circ f_z^{-n_3}|_{B \cap B_e(f^{n_3}(z), r_3)}.
\end{aligned}$$

Hence $\hat{\psi}|_{B_e(f^{n_3}(z), r_3)} = g^{n_3} \circ H \circ f_z^{-n_3}|_{B_e(f^{n_3}(z), r_3)}$. Thus the pair $(W \cup G, \hat{\psi})$ satisfies condition (c), and consequently, $(W \cup G, \hat{\psi}) \in \Phi$. Since, obviously $(W, \psi) \leq (W \cup G, \hat{\psi})$ and $(W, \psi) \neq (W \cup G, \hat{\psi})$, we have contradiction with maximality of (W, ψ). Thus $W \supset \mathbb{C} \setminus \Omega(f)$.

Since ψ maps bounded sets onto bounded sets and $\Omega(f)$ is finite, ψ extends to a holomorphic function $A_f : \mathbb{C} \to \mathbb{C}$ and

$$A_f|_{J(f) \setminus \overline{\mathrm{PS}^0(f)}} = h.$$

By the symmetry of the situation there exists a holomorphic function $A_g : \mathbb{C} \to \mathbb{C}$ such that

$$A_g|_{J(g) \setminus \overline{\mathrm{PS}^0(g)}} = h^{-1}.$$

Consequently $A_g \circ A_f$ and $A_f \circ A_g$ are identities respectively on $J(f) \setminus \overline{\mathrm{PS}^0(f)}$ and $J(g) \setminus \overline{\mathrm{PS}^0(g)}$. Therefore, there are identities on $\mathbb{C}$ also, and so, both $A_f : \mathbb{C} \to \mathbb{C}$ and $A_g : \mathbb{C} \to \mathbb{C}$ are homographes. The equality $A_f \circ f = g \circ A_g$ clearly both holds on $\mathbb{C}$. The implication (6)⇒(1) now readily follows from Lemma 6.11, Lemma 6.12 and Lemma 6.13. We are done. ■

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