STOCHASTICS AND THERMODYNAMICS

FOR

EQUILIBRIUM MEASURES OF HOLOMORPHIC ENDOMORPHISMS ON COMPLEX PROJECTIVE SPACES

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ABSTRACT. It was proved in [UZ] that for every holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of a complex projective space $\mathbb{P}^k$,$k \geq 1$, there exists a positive number $\kappa_f > 0$ such that if $\phi : J \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa_f$ (pressure gap), then ϕ admits a unique equilibrium state μ_ϕ on J. In this paper we prove that the dynamical system (f, μ_ϕ) enjoys exponential decay of correlations of Hölder continuous observables as well as the Central Limit Theorem and the Law of Iterated Logarithm for such class of variables satisfying the natural co-boundary condition. We also show that the topological pressure function $t \mapsto P(t\phi)$ is real–analytic throughout the open set of parameters t for which the potenials $t\phi$ have pressure gaps.

1. INTRODUCTION

Fix an integer $k \geq 1$. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$ of the complex projective space $\mathbb{P}^k$. Denote by $J = J(f)$ the Julia set of the map $f : \mathbb{P}^k \to \mathbb{P}^k$, i. e. the topological support of the measure of maximal

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Date: July 11, 2011.

1991 *Mathematics Subject Classification.* Primary: 37D35; Secondary: 32H50, 37C40, 28D05.

Key words and phrases. Holomorphic dynamics, complex projective spaces, Julia sets, topological pressure, equilibrium states, Hölder continuous potentials, Perron-Frobenius operators, exponential decay of correlations, Central Limit Theorem, the Law of Iterated Logarithm .

The research of the second author supported in part by the NSF Grant DMS 1001874. The research of the first and third author supported in part by the Polish MNiSW Grants N N201 0234 33 and N N201 607940.

entropy. The map $f : \mathbb{P}^k \to \mathbb{P}^k$ is called regular if its exceptional set $E = E(f)$ does not intersect the Julia set $J = J(f)$. Let $\phi : J(f) \to \mathbb{R}$ be a continuous function, in the sequel frequently refered to as a potential. By $P(\phi)$ we denote the (classical) topological pressure of the potential ϕ with respect to the dynamical system $f : J(f) \to J(f)$. Its definition and a systematic account of properties can be found for example in [PU]. If μ is a Borel probability f-invariant measure on $J(f)$, we denote by $\mathrm{h}_\mu(f)$ its Kolmogorov–Sinai metric entropy. The relation between pressure and entropy is given by the following celebrated Variational Principle.

$$\mathrm{P}(\varphi) = \sup\left\{\mathrm{h}_\mu(f) + \int \varphi d\mu\right\}, \tag{1.1}$$

where the supremum is taken over all Borel probability f-invariant measures μ, or equivalently, over all Borel probability f-invariant ergodic measures μ. The measures μ for which

$$\mathrm{h}_\mu(f) + \int \phi\, d\mu = \mathrm{P}(\phi)$$

are called equilibrium states for the potential ϕ. The main theorem proved in [UZ] was this.

Theorem 1.1. *For every regular holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of a complex projective space $\mathbb{P}^k$, $k \geq 1$, there exists a positive number $\kappa_f > 0$ such that if $\phi : J(f) \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa_f$ (we then say that ϕ has a pressure gap), then ϕ admits a unique equilibrium state μ_ϕ on J. This equilibrium state is equivalent to a fixed point of the normalized dual Perron-Frobenius operator. In addition the dynamical system (f, μ_ϕ) is K-mixing, whence ergodic. In the case when the Julia set J does not intersect any periodic irreducible algebraic varieties contained in the critical set of f, we have that $\kappa_f = \log d$.*

The main object of our paper will be the dynamical system (f, μ_ϕ). We shall show in Theorem 7.6 that this system enjoys exponential decay of correlations of Hölder continuous observables as well as the Central Limit Theorem and the Law of Iterated Logarithm for such class of variables satisfying the natural co-boundary condition. We also show in Theorem 6.1 that the topological pressure function $t \mapsto P(t\phi)$ is real–analytic throughout the open set of parameters t for which the potenials $t\phi$ have pressure gaps.

This paper is self-contained in the sense that all notions used are introduced and all the steps leading to the main theorems are explained. Majority of proofs however are exactly the same as those for the 1–dimensional case delt with in [SUZ], and pointing out the particular fragment of [SUZ] we refer the reader to this paper for proofs.

2. Good Holomorphic Inverse Branches

Given an open connected subset W of $\mathbb{P}^k$ and given an integer $n \geq 1$, we denote by $I_n(W)$ the collection of all connected components of $f^{-n}(W)$. If $V \in I_n(W)$ and $f|_V^n : V \to W$ is a bijection (equivalently an injection), we set

$$f_V^{-n} := (f|_V^n)^{-1} : W \to V.$$

We denote the callection of all such components by $\mathrm{PG}_n(W)$ and refere to them as pre-good components. Of course, if $V \in \mathrm{PG}_n(W)$, then the map $f_V^{-n} : W \to V$ is a holomorphic homeomorphism from W to V. We call it the holomorphic inverse branch of f^n from W to V. The main result of this section is the following.

Lemma 2.1. *For every $\gamma \in (0,1)$ and for every $\varepsilon > 0$ there exist some integers $l, q \geq 1$ and real numbers $\eta \in (0,1)$ and $\theta > 0$ such that if B is an arbitrary open ball centered at a point from the Julia set J, and if $(1+\eta)^2 B$ is disjoint from the set PC_l, then for every integer $n \geq 1$ there exists a subset $G_n(B) \subset \mathrm{PG}_{qn}(B)$ with the following properties.*

(a) *For every $V \in G_n(B)$ there exists $W \in \mathrm{PG}_{qn}((1+\eta)B)$ such that $V \subset W$.*

(b) *If $V \in G_{n+1}(B)$, then $f^q(V) \in G_n(B)$.*

(c) *If $V \in G_{n+1}(B)$, then $\mathrm{diam}(V) \leq \gamma^n$.*

(d) $\mu_\phi\big(\bigcup G_n(B)\big) \geq (1-\varepsilon)\mu_\phi(B)$, *where* $\bigcup G_n(B) := \bigcup\{V : V \in G_n(B)\}$

(e) $\mu_\phi\big(\bigcup B_n(B)\big) \leq e^{-\theta q n}$,
where $B_n(B)$ consists of all connected components of the sets $f^{-qn}(V)$, $V \in G_{n-1}(B)$ that do not belong to $G_n(B)$ and $\bigcup G_n(B) := \bigcup\{V : V \in G_n(B)\}$.

Applying Cauchy's formulas for partial derivatives, we directly obtain from Lemma 2.1 the following.

Corollary 2.2. *With the hypotheses of Lemma 2.1 we have that there exists a constant $C_\eta > 0$ such that*

$$||Df_V^{-n}||_\infty \leq C_\eta \gamma^n$$

for all $n \geq 0$ and all $V \in G_n(B)$.

Let

$$\phi_q = \sum_{j=0}^{q-1} \phi \circ f^j$$

The most signifcant consequence of belongin to $G_n(B)$ is this. There exists a constant $C_q \ge 1$ such that for every $n \ge 1$, every $V \in G_n(B)$ and all $x, y \in V$, we have that

$$\frac{\exp\big(S_n\phi_q(x)\big)}{\exp\big(S_n\phi_q(y)\big)} \le C_q, \tag{2.1}$$

where in here S_n refers to Birkhoff's sums corresponding to the dynamical system $f^q : \mathbb{P}^k \to \mathbb{P}^k$.

3. Selection of the Root Ball

Simplyfying the proof of Lemma 9 in [SUZ], we prove the following.

Lemma 3.1. *Let $a < b$ be two real numbers. If μ is a Borel finite measure on $[a, b]$, then for every $\lambda > 1$ large enough there exist a point $c \in (a, b)$ (in fact a measurable set of positive Lebesgue measure of such points c) and $\tilde{\lambda} > 1$ such that*

$$\mu\big((c - \tilde{\lambda}^{-n}, c + \tilde{\lambda}^{-n})\big) \le \lambda^{-n}$$

for all integers $n \ge 1$.

We now keep the setting of Lemma 2.1 with $\varepsilon := 1/2$. As a straightforward consequence of the abstract Lemma 3.1, we shall prove the following.

Lemma 3.2. *For every point $w \in J \setminus \mathrm{PC}_l$ there exists a ball B centered at w such that $(1 + \eta)B \cap \mathrm{PC}_l = \emptyset$ and*

$$\mu_\phi\big(B(\partial B, \lambda^{-3n})\big) \le \lambda^{-n}.$$

Proof. Take any $R > 0$ so small that $(1+\eta)B(w, R) \cap \mathrm{PC}_l = \emptyset$ and consider the map $P : \overline{B}(w, R) \to [0, R]$ given by the formula $P(z) = ||z - w||$. Applying Lemma 3.1 to the measure $\mu_\phi \circ P^{-1}$ the assertion of Lemma 3.2 immediately follows. □

4. Fine Inducing Scheme

Assume without loss of generality that $\mathrm{P}(\phi) = 0$. Fix an integer $r \geq 1$ so large that

$$C_\eta \gamma^r \leq \frac{1}{4}, \tag{4.1}$$

where the constant C_η comes from Corollary 2.2. More requirements on r will be imposed later. Put

$$h = f^{qr} : \mathbb{P}^k \to \mathbb{P}^k.$$

and

$$\phi_0 := \sum_{j=0}^{qr-1} \phi \circ f^j.$$

Define

$$\mathcal{L}_0 := \mathcal{L}_\phi^{qr} : C(J) \to C(J).$$

So, $\mathcal{L}_0$ is the Perron-Frobenius operator associated to the map $h : J \to J$ and potential $\phi_0 : J \to \mathbb{R}$. It is given by the formula

$$\mathcal{L}_0(g)(x) = \sum_{y \in h^{-1}(x)} g(y) \exp(\phi_0(y)).$$

In exactly the same way as in [UZ] we can prove the following.

Lemma 4.1. *Assume that $Q \subset \mathbb{P}^k$ is a set for which there exists $\beta > 0$ such that*

$$\mathcal{L}_h(1_Q)(x) > \beta \tag{4.2}$$

for almost every $x \in J(f)$. Then there exist $\alpha \in (0,1)$, an integer $n_0 \geq 0$, and $\delta > 0$, all three depending on β only (in particular independent of $r \geq 1$), such that for all $n \geq n_0$ we have that,

$$\mu\left(\left\{x \in J(f) : \#\left\{0 \leq i \leq n : h^i(x) \in Q\right\} \leq \alpha n\right\}\right) < \exp(-\delta n). \tag{4.3}$$

In order to make this lemma useable for us, we need the following.

Lemma 4.2. *For every integer $q \geq 1$ large enough there exists $\beta > 0$ such that for every integer $r \geq 1$, we have for every $z \in J(f)$ that,*

$$\mathcal{L}_0(1\!\!1_Q)(z) \geq \beta,$$

where

$$Q := Q_r := \bigcup \{V : V \in G_{r-1}(B)\}.$$

Proof. As a straightforward consequence of Lemma 2.1, particularly its item (c), and (2.1), we get the following.

$$\mathcal{L}_\phi^{q(r-1)} 1\!\!1_Q(z) \ge (2C_q)^{-1} \tag{4.4}$$

for all $z \in B$. Now, invoking topological exactness of the dynamical system $f : J(f) \to J(f)$, take any $q \ge 1$ so large that $f^q(B) = J(f)$. For every $x \in J$ there then exists $y \in B$ such that $f^q(y) = x$. Apllying (4.4) we can then write

$$\begin{aligned}\mathcal{L}_0(1\!\!1_Q)(x) &\ge \exp\big(S_q\phi(y)\big)\mathcal{L}_\phi^{q(r-1)} 1\!\!1_Q(y) \\ &\ge (2C_q)^{-1}\exp\big(S_q\phi(y)\big) \\ &\ge (2C_q)^{-1}\exp\big(\inf\big(S_q\phi\big)\big).\end{aligned}$$

Setting β to be the last number of this formula finishes the proof. □

We say that a point $z \in B$ has a good pullback of length $n \ge 1$ if $h^n(z) \in B$ and the connected component of $h^{-n}(B)$ containg z belongs to $G_{rn}(B)$. We frequently refer to this component as a good pullback and denote it by $V_n(z)$. We further say that such a good pullback $V_n(z)$ is very good if for every $0 \le j \le n-1$,

$$\operatorname{dist}\big(h^j(V_n(z)), \partial B\big) \ge \tilde{\lambda}^{n-j}.$$

Note that if $r \ge 1$ is so large that

$$\gamma^r < \tilde{\lambda}^{-1},$$

then every good pullback V is entirely contained in B. Now, the proof of Lemma 17 from [SUZ] goes verbatime in our present setting and it asserts this.

Lemma 4.3. *As in Lemma* 4.1 *let* Z_n *be the set of all points* $x \in B$ *for which*

$$\#\big\{0 \le i \le n : h^i(x) \in Q\big\} > \alpha n.$$

Let

$$Y_n := \big\{x \in Z_n : \#\{0 \le j \le n : \text{the pullback } V_j^x \text{ is good}\} < (\alpha/2)n\big\}.$$

Then with $r \ge 1$ *sufficiently large, we have that*

$$\mu_\phi(Y_n) < e^{-\frac{\alpha}{8}nr\theta q}.$$

The proof of the next lemma is also the same as the proof of a lemmma from [SUZ]. This time this is Lemma 18. The only modification is that ∂U is to be replaced by ∂B.

Lemma 4.4. *Let $R_n \subset B$ be the set of points in $x \in Z_n$ that satisfy the following two requirements.*

1. *$x \in B \setminus Y_n$, i.e. the points in R_n have at least $\frac{\alpha}{2}n$ good pullbacks, but*
2. *No good pullback $V_m(x)$ with $m \leq n$, is very good.*

Then

$$\mu_\phi(R_n) \leq (\lambda'')^{-n}$$

with some $\lambda'' > 1$ independent of n.

5. Construction of the induced system

Let

$$X = \bigcup_{n=1}^{\infty} Z_n \setminus (Y_n \cup R_n)$$

It directly follows from Lemma 4.1, Lemma 4.2, Lemma 4.3, and Lemma 4.4 that

$$\mu_\phi(X) = \mu_\phi(B). \tag{5.1}$$

Given $x \in X$ let $n(x)$ be the smallest integer $n \geq 1$ such that $x \in Z_n \setminus (Y_n \cup R_n)$. We define

$$F(x) = h^{n(x)}(x).$$

Keep this $x \in X$ and put $n = n(x)$. Note that, if $y \in V_n(x)$ then this procedure, applied to y leads to the same component V_n. Indeed, by the definition of the induced map, we use the earliest very good pullback. Thus, if $F(y) \neq h^n(y)$ then $F(y) = h^m(y)$ for some $m < n$. Let $V_m(y)$ be the corresponding pullback. Then $V_m(y) \cap V_n(x) \neq \emptyset$ as y belongs to both of these sets, but $V_n(x) \not\subseteq V_m(y)$ since $x \in V_n(x) \setminus V_m(y)$. Let us consider $h^m(V_m(y)) = U$ and $h^m(V_n(x))$. The latter is an element of the pullback chosen for x (a component of $h^{-(n-m)}(B)$) and, since $V_n(x)$ must intersect $\partial V_m(y)$, also $h^m(V_n(x))$ intersects ∂B. But this is impossible by the definition of very good pullbacks. Let $\mathcal{D}$ be the countable family of all sets $V_{n(x)}(x)$, $x \in X$, defined in this way. We have just shown that the function $n : X \to \mathbb{N}$ is constant on each disc $D \in \mathcal{D}$, and so it can and will be treated as a function from $\mathcal{D} \to \mathbb{N}$. In particular, the map

$$F : \bigcup_{D \in \mathcal{D}} \to B$$

is well-defined and its inverse branches $F_D^{-1} : U \to D$, $D \in \mathcal{D}$, form an infinite iterated function systems, which, with a slight abuse of notation, will be also refered to as F. We denote by J_F the limit set of the iterated function system F, i.e.

$$J_F = \bigcap_{n=0}^{\infty} F^{-n}(X). \tag{5.2}$$

The argument leading to (5.1) gives in fact more. Namely:

$$\mu_\phi(J_F) = \mu_\phi(B). \tag{5.3}$$

It immediately follows from the construction of the system F and Lemma 4.4, that

$$m_\phi\left(\bigcup\{D : n(D) = n\}\right) \le (\lambda'')^{-n} \tag{5.4}$$

for all $n \ge 1$. Let us record the following, proved in the same way as Lemma 19 in [SUZ], essential property of this induced system.

Lemma 5.1. *If D_1, D_2 are two domains in $\mathcal{D}$, $F_{|D_1} = h^n$, $F_{|D_2} = h^m$ then for $0 \le s < n$, $0 \le t < m$ either $h^s(D_1) \cap h^t(D_2) = \emptyset$ or the closure of one of these sets is contained in the other set.*

For the sake of Proposition 5.4, we need to extend the potential ϕ beyond the Julia sets $J(f)$.

Lemma 5.2. *The function ϕ can be extended in a Hölder continuous manner, with the same Hölder exponent, to the whole projective space $\mathbb{P}^k$.*

This lemma is well-known; a proof can be found in [UZ]. From now on, we assume that the potential ϕ is defined and Hölder continuous in the whole Riemann sphere $\hat{\mathbb{C}}$.

As we have passed to induced system, we shall modify the potential ϕ accordingly to this inducing process. First, if $D \in \mathcal{D}$ is one of discs on which F is defined, then we put, for all $x \in D$, that

$$\hat{\phi}(x) = \sum_{k=0}^{n(D)-1} S_{qr}\phi(h^k(x)).$$

Then, for all Borel sets $A \subset D_e$ we have that,

$$\begin{aligned} m_\phi(F(A)) = m_\phi(h^{n(D)}(A)) &= \int_A \exp\left(-\sum_{k=0}^{n(D)-1} S_{qr}\phi \circ h^k\right) dm_\phi \\ &= \int_A \exp\bigl(-\hat{\phi}(x)\bigr) dm_\phi(x). \end{aligned}$$

Along with (5.3) this entails the following.

Lemma 5.3. *The probability measure* m_ϕ *is* $\exp(-\hat{\phi})$*-conformal for the map* $F : J_F \to J_F$.

Having Lemmas 5.3 and 5.2, the general theory of infinite iterated function systems, as developed in [MU2] along with [MU3], gives the following.

Proposition 5.4. *There exists a unique probability* F*-invariant measure* $\mu_{\hat{\phi}}$ *which is equivalent to* m_ϕ*. Moreover the Radon-Nikodym derivative* $\hat{\rho} := \frac{d\mu_{\hat{\phi}}}{dm_\phi}$ *is bounded above and separated below from zero. This Radon-Nikodym derivative* $\hat{\rho}$ *has a continuous extension* $\hat{\rho} : B \to (0; +\infty)$ *to the whole ball* B *and this extension is a fixed point of the following transfer operator.*

$$\mathcal{L}_{\hat{\phi}}(v)(x) = \sum_{y \in F^{-1}(x)} \exp \hat{\phi}(y) v(y).$$

This is a bounded linear operator acting on the Banach space $C_b(B)$ *of all bounded real-valued continuous functions defined on* B*, and it is easy to see that this operator is almost periodic.*

6. REAL ANALYTYICITY OF TOPOLOGICAL PRESSURE

For every Hölder continuous potential $\phi : J(f) \to \mathbb{R}$, let

$$\Delta_\phi = \left\{ t \in \mathbb{R} : \sup_{n \geq 1} \left(P(t\phi) - \frac{1}{n} \sup(S_n(t\phi)) \right) > 0 \right\}.$$

Obviously, Δ_ψ is an open subset of $\mathbb{R}$. Having all the material of the previous sections i.e. Section 4 and Section 5, particularly formula (5.4), we can repeat verbatime Section 6, Real Analytyicity of the Pressure Function, from [SUZ] to get the following.

Theorem 6.1. *The topological pressure function*

$$\Delta_\phi \ni t \mapsto P(t\phi) \in \mathbb{R}$$

is real–analytic.

7. Stochastic properties of the equilibrium measure μ_ϕ

In this section we obtain strong transparent stochastic properties of the dynamical system (f, μ_ϕ). We deduce them from the corresponding properties of the induced system $(F, \mu_{\hat{\phi}})$. We follow the scheme worked out in [LSY] in the way it was presented in [SUZ]. We recall it briefly now. We do this in an abstract context. Let $(\Delta_0, \mathcal{B}_0, m_0)$ be a measure space with a finite measure m_0, let $\mathcal{P}_0$ be a countable measurable partition of Δ_0 and let $T_0 : \Delta_0 \to \Delta_0$ be a measurable map such that, for every $\Delta' \in \mathcal{P}_0$ the map $T_0 : \Delta' \to \Delta_0$ is a bijection onto Δ_0. Moreover, we assume that the partition $\mathcal{P}_0$ is generating, i.e. for every $x, y \in \Delta_0$ there exists $s \geq 0$ such that $T_0^s(x), T_0^s(y)$ are in different elements of the partition $\mathcal{P}_0$. We denote by $s = s(x, y)$ the smallest integer with this property and we call it a separation time for the pair x, y. We assume also that for each $\Delta' \in \mathcal{P}_0$ the map $(T_{0|\Delta'})^{-1}$ is measurable and that the Jacobian $Jac_{m_0}(T_0)$ with respect to the measure m_0 is well–defined and positive a.e. in Δ'. The following distortion property is assumed to be satisfied.

$$\left| \frac{\mathrm{Jac}_{m_0} T_0(x)}{\mathrm{Jac}_{m_0} T_0(y)} - 1 \right| \leq C \beta^{s_0(T_0(x), T_0(y))}. \tag{7.1}$$

We have also a function $R : \Delta_0 \to \mathbb{N}$ ("return time") which is constant on each element of the partition $\mathcal{P}_0$. We assume that the greatest common divisor of the values of R is equal to 1. Finally, let

$$\Delta = \{(z, n) \in \Delta_0 \times \mathbb{N} \cup \{0\} : 0 \leq n < R(z)\}$$

and each point $z \in \Delta_0$ is identified with $(z, 0)$. The map T acts on Δ as

$$T(z, n) = \begin{cases} (z, n+1) \text{ if } n+1 < R(z) \\ (T_0(z), 0) \text{ if } n+1 = R(z) \end{cases}$$

The measure m_0 is spread over the whole space Δ by putting

$$\tilde{m}_{|\Delta_0} = m_0 \quad \text{and} \quad \tilde{m}_{|\Delta' \times \{j\}} = m_{0|\Delta'} \circ \pi_j^{-1}, \ \Delta' \in \mathcal{P},$$

where $\pi_j(z, 0) = (z, j)$. Thus, the measure $\tilde{m}$ is finite iff $\int_{\Delta_0} R dm_0 < \infty$. The separation time $s((x, n), (y, m))$ is defined to be equal to $s(x, y)$ if $n = m$ and x, y are in the same set of the partition $\mathcal{P}$. Otherwise we set $s(x, y) = 0$. Given $\beta > 0$ we define the space

$$C_\beta(\Delta) = \{\varphi : \Delta \to \mathbb{R} : \exists\ C_\varphi \text{ such that } |\varphi(x) - \varphi(y)| < C_\varphi \beta^{s(x,y)} \ \forall x, y \in \Delta\}.$$

We refer to the pentapol $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R, m_0)$ as a Young tower. The first three items of the following basic result have been proved in [LSY] while the fourth item was proved in [SUZ].

Theorem 7.1. *If $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R, m_0)$ is a Young tower and $\int R dm_0 < \infty$ then the following hold.*

(1) *There exists a unique probability T–invariant measure ν, absolutely continuous with respect to $\tilde{m}$. The Radon-Niokodym derivative $d\nu/d\tilde{m}$ is bounded from below by a positive constant. The dynamical system (T, ν) is exact, thus ergodic.*

(2) *If $m_0(R > n) = O(\theta^n)$ for some $0 < \theta < 1$, then there exists $0 < \tilde{\theta} < 1$ such that for all functions $\psi \in L^\infty$ and we have $g \in C_\beta$,*

$$Cov(\psi \circ T^n, g) = \left| \int (\psi \circ T^n) g d\nu - \int \psi d\nu \int g d\nu \right| = O(\tilde{\theta}^n) \tag{7.2}$$

(3) *If $m_0(R > n) = O(n^{-\alpha})$ with some $\alpha > 1$ (in particular, if $m_0(R > n) = O(\theta^n)$), then the Central Limit Theorem is satisfied for all functions $g \in C_\beta$, that are not cohomologous to a constant in $L^2(\nu)$.*

(4) *If $m_0(R > n) = O(n^{-\alpha})$ with some $\alpha > 4$ (in particular, if $m_0(R > n) = O(\theta^n)$), then the Law of Iterated Logarithm holds for all functions $g \in C_\beta$, that are not cohomologous to a constant in $L^2(\nu)$. This means that there exists a real positive number A_g such that such that ν almost everywhere*

$$\limsup_{n\to\infty} \frac{S_n g - n \int g d\mu}{\sqrt{n \log\log n}} = A_g.$$

Passing to our holomorphic dynamical system (f, μ_ϕ) we shall check that the assumptions of this theorem are satisfied for our induced system (F, m_ϕ). The space Δ_0 is now J_F, the limit set of the iterated function system F. The partition $\mathcal{P}_0$ consists of the sets $D \cap J_F$, $D \in \mathcal{D}$. The measure m_0 is the conformal measure m_ϕ, restricted to J_F. The map T_0 is, in our setting, the map F. The function R, the return time, is, naturally, defined as $R(D) = n(D)$. We shall check that the pentapol $\mathcal{Y}_\phi = (J_F, F, \mathcal{P}, n, m_\phi)$ is a Young Tower, i.e. it satisfies the hypotheses of Theorem 7.1. We start with the following.

Lemma 7.2. *There exists a constant $C > 0$ such that if $D \in \mathcal{D}$ and $x, y \in D$, then*

$$\mathrm{dist}(x, y) \le C 4^{-s(x,y)}$$

Proof. The assertion follows immediately from Corollary 2.2, formula (4.1), and the definition of the separation time s. ◻

As a fairly straightforward consequence of this lemma, we get the following.

Lemma 7.3. *If $D \in \mathcal{D}$ and and $x, y \in D$, then*

$$\operatorname{dist}(h^j(x), h^j(y)) \le 2C2^{-s(x,y)}$$

for all $0 \le j \le n(D)$.

Proof. Note that $s(x,y) \ge 1$. Fix $0 \le j \le n(D)$. It follows from Corollary 2.2 that $\operatorname{dist}(h^j(x), h^j(y)) \le \operatorname{dist}(h^{n(D)}(x), h^{n(D)}(y))$. But $s(h^{n(D)}(x), h^{n(D)}(y)) = s(x,y) - 1$, it therefore follows from the previous lemma, that

$$\operatorname{dist}(h^j(x), h^j(y)) \le \operatorname{dist}(h^{n(D)}(x), h^{n(D)}(y)) \le C2^{-s(h^{n(D)}(x), h^{n(D)}(y))} = 2C2^{-s(x,y)}.$$

We are done. □

We are now ready to prove the following.

Lemma 7.4. *The pentapol $\mathcal{Y}_\phi = (J_F, F, \mathcal{P}, n, m_\phi)$ is a Young Tower, i.e. it satisfies the hypotheses of Theorem 7.1. In addition, $\tilde{m}_\phi(\Delta) < +\infty$.*

Proof. First, we need to show that the formula (7.1) holds. To do this fix an arbitrary disc $D \in \mathcal{D}$ and aribtrary two points $x, y \in J_F \cap D$. Recalling that the function $S_{qr}\phi : J(f) \to \mathbb{R}$ is Hölder continuous with some exponent $\alpha > 0$, and using Lemma 7.3, we can write as follows.

$$\begin{aligned}
\left|\log \frac{\operatorname{Jac}_{m_\phi} F(y)}{\operatorname{Jac}_{m_\phi} F(y)}\right| &= \left|\sum_{j=0}^{n(D)-1} S_{qr}\phi(h^j(y)) - \sum_{j=0}^{n(D)-1} S_{qr}\phi(h^j(x))\right| \\
&\le \sum_{j=0}^{n(D)-1} \left|S_{qr}\phi(h^j(y)) - S_{qr}\phi(h^j(x))\right| \\
&\le \sum_{j=0}^{n(D)-1} C_1 \operatorname{dist}^\alpha(h^j(y), h^j(x)) \\
&\le 2CC_1 \sum_{j=0}^{n(D)-1} 2^{-s(x,y)} = 2CC_1 n(D) 4^{-s(x,y)} \\
&\le 2CC_1 s(x,y) 4^{-s(x,y)} \\
&\le C_2 3^{-s(x,y)}
\end{aligned}$$

with appropriate positive constants C, C_1, and C_2.

We also need to take care of the last assumption in Theorem 7.1 requiring that the greatest common divisor of all the values of $n(D)$, $D \in \mathcal{D}$, is equal to 1. If for our induced system this value is equal to some integer $s > 1$, then we replace the map

h by its iterate h^s. The return times are now equal to $n(D)/s$, $D \in \mathcal{D}$, and their greatest common divisor equals 1.

The finiteness of $\tilde{m}_\phi(\Delta)$ follows immediately from (5.4) and the definition of the return time R. □

Now consider $\pi : \Delta \to \mathbb{P}^k$, the natural projection from the abstract tower Δ to the projective space $\mathbb{P}^k$ given by the formula

$$\pi(z,n) = h^n(z).$$

Then

$$\pi \circ T = h \circ \pi, \tag{7.3}$$

$$\tilde{m_\phi}\big|_{J_F} \circ \pi^{-1} = m_0 = m_\phi,$$

and

$$\tilde{m_\phi}_{D\times\{n\}} \circ \pi^{-1} = m_{\phi_{D\times\{0\}}} \circ h^{-n} = m_{0|D} \circ h^{-n}$$

for all $D \in \mathcal{D}$ and all $0 \le n \le n(D)$. Now, the measure $\tilde{m}_{\phi_{D\times\{n\}}} \circ \pi^{-1}$ is absolutely continuous with respect to m_ϕ with the Radon-Nikodym derivative equal to $J_{D,n} := \mathrm{Jac}_{m_\phi}(h^{-n})$ in $h^n(D)$ and zero elsewhere. Therefore,

$$\begin{aligned}
\int_{\mathbb{P}^k} \sum_{D\in\mathcal{D}} \sum_{n=0}^{n(D)-1} J_{D,n}\, dm_\phi &= \sum_{D\in\mathcal{D}} \sum_{n=0}^{n(D)-1} \int_{\mathbb{P}^k} J_{D,n}\, dm_\phi = \sum_{D\in\mathcal{D}} \sum_{n=0}^{n(D)-1} \int_{h^n(D)} J_{D,n}\, dm_\phi \\
&= \sum_{D\in\mathcal{D}} \sum_{n=0}^{n(D)-1} \tilde{m}_{\phi_{D\times\{n\}}} \circ \pi^{-1}(h^n(D)) \\
&= \sum_{D\in\mathcal{D}} \sum_{n=0}^{n(D)-1} \tilde{m}_{\phi_{D\times\{n\}}} \circ \pi^{-1}(\mathrm{P}^k) \\
&= \tilde{m}_\phi \circ \pi^{-1}(\mathbb{P}^k) = \tilde{m}_\phi(\Delta) \\
&< +\infty,
\end{aligned}$$

where in writing the inequality sign we used the last assertion of Lemma 7.4. Thus, the function $\sum_{D\in\mathcal{D}} \sum_{0\le n<n(D)} J_{D,n}$ is integrable with respect to the measure m_ϕ. This implies immediately that the measure $\tilde{m}_\phi \circ \pi^{-1}$ is absolutely continuous with respect to the measure m_ϕ with the Radon–Nikodyma derivative equal to $\sum_{D\in\mathcal{D}} \sum_{0\le n<n(D)} J_{D_i,n}$. Hence, the measure $\nu \circ \pi^{-1}$ is also absolutely continuous with respect to m_ϕ. Since ν is F–invariant and $\pi \circ T = h \circ \pi$, the measure $\nu \circ \pi^{-1}$ is h–invariant. But the measure μ_ϕ is h–invariant ergodic and equivalent with the conformal measure m_ϕ. Hence, $\nu \circ \pi^{-1}$ is absolutely continuous with respect to the ergodic measure μ_ϕ. Innvariance and ergodicity of $\nu \circ \pi^{-1}$ yield this.

Lemma 7.5.

$$\nu \circ \pi^{-1} = \mu_\phi.$$

Having this, we can prove in exactly the same way as Theorem 56 in [SUZ], the following.

Theorem 7.6. *For the dynamical system* (f, μ_ϕ) *the following hold.*

(1) *For every* $\alpha \leq 1$*, every* α*–Hölder continuous function* $g : J(f) \to \mathbb{R}$ *and every bounded measurable function* $\psi : J(f) \to \mathbb{R}$*, we have that*

$$\left| \int \psi \circ f^n \cdot g d\mu_\phi - \int g dmu_\phi \int \psi d\mu_\phi \right| = O(\theta^n)$$

for some $0 < \theta < 1$ *depending on* α*.*

(2) *The Central Limit Theorem holds for every Hölder continuous function* $g : J(f) \to \mathbb{R}$ *that is not cohomologous to a constant in* $L^2(\mu_\phi)$*, i.e. for which there is no square integrable function* η *for which* $g = \mathrm{const} + \eta \circ f - \eta$*. Precisely this means that there exists* $\sigma > 0$ *such that*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ f^j \to \mathcal{N}(0, \sigma)$$

in distribution.

(3) *The Law of Iterated Logarithm holds for every Hölder continuous function* $g : J(f) \to \mathbb{R}$ *that is not cohomologous to a constant in* $L^2(\mu_\phi)$*. This means that there exists a real positive constant* A_g *such that such that* μ_ϕ *almost everywhere*

$$\limsup_{n\to\infty} \frac{S_n g - n \int g d\mu}{\sqrt{n \log\log n}} = A_g.$$

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